OPTIMAL PRIMAL-DUAL METHODS FOR A CLASS OF SADDLE POINT PROBLEMS

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Abstract. We present a novel accelerated primal-dual (APD) method for solving a class of deterministic and stochastic saddle point problems (SPP). The basic idea of this algorithm is to incorporate a multi-step acceleration scheme into the primal-dual method without smoothing the objective function. For deterministic SPP, the APD method achieves the same optimal rate of convergence as Nesterov's smoothing technique. Our stochastic APD method exhibits an optimal rate of convergence for stochastic SPP not only in terms of its dependence on the number of the iteration, but also on a variety of problem parameters. To the best of our knowledge, this is the first time that such an optimal algorithm has been developed for stochastic SPP in the literature. Furthermore, for both deterministic and stochastic SPP, the developed APD algorithms can deal with the situation when the feasible region is unbounded, as long as a saddle point exists. In the unbounded case, we incorporate the modified termination criterion introduced by Monteiro and Svaiter in solving SPP problem posed as monotone inclusion, and demonstrate that the rate of convergence of the APD method depends on the distance from the initial point to the set of optimal solutions.

1. Introduction. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the finite-dimensional vector spaces equipped with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$ be given closed convex sets. The basic problem of interest in this paper is the saddle-point problem (SPP) given in the form of:

$$\min_{x \in X} \left\{ f(x) := \max_{y \in Y} G(x) + \langle Kx, y \rangle - J(y) \right\}. \tag{1.1}$$

Here, $G(x)$ is a general smooth convex function such that, for some $L_G \geq 0$,

$$G(y) - G(x) - \langle \nabla G(x), y - x \rangle \leq \frac{L_G}{2} \|y - x\|^2, \quad \forall x, y \in X, \tag{1.2}$$

$K : \mathcal{X} \to \mathcal{Y}$ is a linear operator with induced norm $L_K = \|K\|$, and $J : Y \to \mathbb{R}$ is a relatively simple, proper, convex, lower semi-continuous (l.s.c.) function (i.e., problem (2.5) is easy to solve). In particular, if J is the convex conjugate of some convex function F and $Y \equiv \mathcal{Y}$, then (1.1) is equivalent to the primal problem:

$$\min_{x \in X} G(x) + F(Kx). \tag{1.3}$$

Problems of these types have recently found many applicaitons in data analysis, especially in imaging processing and machine learning. In many of these applications, $G(x)$ is a convex data fidelity term, while $F(Kx)$ is a certain regularization, e.g., total variation [41], low rank tensor [17, 43], overlapped group lasso [14, 24], and graph regularization [14, 42].

This paper focuses on first-order methods for solving both determinisitc SPP, where exact first-order information on f is available, and stochastic SPP, where we only have access to inexact information about f. Let us start by reviewing a few existing first-order methods in both cases.

1.1. Deterministic SPP. Since the objective function f defined in (1.1) is nonsmooth in general, traditional nonsmooth optimization methods, e.g., subgradient methods, would exhibit an $\mathcal{O}(1/\sqrt{N})$ rate of convergence when applied to (1.1) [30], where N denotes the number of iterations. However, following the breakthrough paper by Nesterov [35], much research effort has been devoted to the development of more efficient methods for solving problem (1.1).

(1) *Smoothing techniques.* In [35], Nesterov proposed to approximate the nonsmooth objective function f in (1.1) by a smooth one with Lipschitz-continuous gradient. Then, the smooth approximation function is

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minimized by an accelerated gradient method in [33, 34]. Nesterov demonstrated in [35] that, if X and Y are compact, then the rate of convergence of this smoothing scheme applied to (1.1) can be bounded by:

$$\mathcal{O}\left(\frac{L_G}{N^2} + \frac{L_K}{N}\right), \tag{1.4}$$

which significantly improves the previous bound $\mathcal{O}(1/\sqrt{N})$. It can be seen that the rate of convergence in (1.4) is actually optimal, based on the following observations:

   a) There exists a function G with Lipschitz continuous gradients, such that for any first-order method, the rate of convergence for solving $\min_{x \in X} G(x)$ is at most $\mathcal{O}\left(L_G/N^2\right)$ [34].
   b) There exists $b \in Y$, where Y is a convex compact set of $\mathbb{R}^m$ for some $m > 0$, and a linear bounded operator K, such that for any first-order method, the rate of convergence for solving $\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle - J(y) := \min_{x \in X} \max_{y \in Y} \langle Kx - b, y \rangle$ is at most $\mathcal{O}\left(L_K/N\right)$ [31, 28].

Nesterov's smoothing technique has been extensively studied, see, e.g., [32, 2, 21, 7, 36, 44, 3, 20]). Observe that in order to properly apply these smoothing technqiues, we need to assume either X or Y to be bounded.

(2) *Primal-dual methods.* While Nesterov's smoothing scheme or its variants rely on a smooth approximation to the orginal problem (1.1), primal-dual methods work directly with the original saddle-point problem. This type of method was first presented by Arrow et al. [1] and named as the primal-dual hybrid gradient (PDHG) method in [45]. The results in [45, 6, 9] showed that the PDHG algorithm, if employed with well-chosen stepsize policies, exhibit very fast convergence in practice, especially for some imaging applications. Recently Chambolle and Pork [6] presented a unified form of primal-dual algorithms, and demonstrated that, with a properly specified stepsize policy and averaging scheme, these algorithms can also achieve the $\mathcal{O}(1/N)$ rate of convergence. They also discussed possible ways to extend primal-dual algorithms to deal with the case when either X and Y are unbounded. In the original work of Chambolle and Pork, they assume G to be relatively simple so that the subproblems can be solved efficiently. With little additional effort, one can show that, by linearizing G at each step, their method can also be applied for a general smooth convex funtion G and the rate of convergence of this modified algorithm is given by

$$\mathcal{O}\left(\frac{L_G + L_K}{N}\right). \tag{1.5}$$

It should be noted, however, that although both bounds in (1.4) and (1.5) are $\mathcal{O}(1/N)$, the one in (1.4) has a significantly better dependence on L_G. More specifically, Nesterov's smoothing scheme would allow a very large Lipschitz constant L_G (as big as $\mathcal{O}(N)$) without affecting the rate of convergence (up to a constant factor of 2). This is desirable in many data analysis applications (e.g., image processing), where L_G is usually significantly bigger than L_K. Note that the primal-dual methods are also related to the Douglas-Rachford splitting method [8] and a pre-conditioned version of the alternating direction method of multipliers [10].

(3) *Extragradient methods for variation inequality (VI) reformulation.* Motivated by Nesterov's work, Nemirovski presented a mirror-prox method, by modifying Korpelevich's extragradient algorithm [18], for solving a more general class of variational inequalities [28] (see also [15]). Similar to the primal-dual methods mentioned above, the extragradient methods update iterates on both the primal space $\mathcal{X}$ and dual space $\mathcal{Y}$, and do not require any smoothing technique. The difference is that each iteration of the extragradient methods requires an extra gradient descent step. Nemirovski's method, when specialized to (1.1), also exhibits a rate of convergence given by (1.5), which, in view of our previous discussion, is not optimal in terms of its dependence on L_G. It can be shown that, in some special cases (e.g., G is quadratic), one can write explicitly the (strongly concave) dual function of $G(x)$ and obtain a result similar to (1.4), e.g., by applying an improved algorithm in [15]. However, this approach would increase the dimension of the problem and cannot be applied for a general smooth function G. It should be noted that, while Nemirovski's initial work only considers the case when both X and Y are bounded, Monteiro and Svaiter [27] recently showed that extragradient methods can deal with unbounded sets X and Y by using a slightly modified termination criterion.

1.2. Stochastic SPP. While determinisitc SPP has been extensively explored, the study on stochastic first-order methods for stochastic SPP is still quite limited. In the stochastic setting, we assume that there exists a *stochastic oracle* ($\mathcal{SO}$) that can provide unbiased estimators to the gradient operators $\nabla G(x)$ and $(-Kx, K^T y)$. More specifically, at the i-th call to $\mathcal{SO}$, $(x_i, y_i) \in X \times Y$ being the input, the oracle will output the *stochastic gradient* $(\hat{\mathcal{G}}(x_i), \hat{\mathcal{K}}_x(x_i), \hat{\mathcal{K}}_y(y_i)) \equiv (\mathcal{G}(x_i, \xi_i), \mathcal{K}_x(x_i, \xi_i), \mathcal{K}_y(y_i, \xi_i))$ such that

$$\mathbb{E}[\hat{\mathcal{G}}(x_i)] = \nabla G(x_i), \quad \mathbb{E}\left[\begin{pmatrix} -\hat{\mathcal{K}}_x(x_i) \\ \hat{\mathcal{K}}_y(y_i) \end{pmatrix}\right] = \begin{pmatrix} -Kx_i \\ K^T y_i \end{pmatrix}. \tag{1.6}$$

Here $\{\xi_i \in \mathbb{R}^d\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables. In addition, we assume that, for some $\sigma_{x,G}, \sigma_y, \sigma_{x,K} \geq 0$, the following assumption hold:

A1. $\mathbb{E}[\|\hat{\mathcal{G}}(x_i) - \nabla G(x_i)\|_*^2] \leq \sigma_{x,G}^2$, $\mathbb{E}[\|\hat{\mathcal{K}}_x(x_i) - Kx_i\|_*^2] \leq \sigma_y^2$ and $\mathbb{E}[\|\hat{\mathcal{K}}_y(y_i) - K^T y_i\|_*^2] \leq \sigma_{x,K}^2$.

Sometimes we simply denote $\sigma_x := \sqrt{\sigma_{x,G}^2 + \sigma_{x,K}^2}$ for the sake of notational convenience. Stochastic SPP often appears in machine learning applications. For example, for problems given in the form of (1.3), $G(x)$ (resp. $F(Kx)$) can be used to denote a smooth (resp. nonsmooth) expected convex loss function. It should also be noted that deterministic SPP is a special case of the above setting with $\sigma_x = \sigma_y = 0$.

In view of the classic complexity theory for convex programming [30, 16], a lower bound on the rate of convergence for solving stochastic SPP is given by

$$\Omega\left(\frac{L_G}{N^2} + \frac{L_K}{N} + \frac{\sigma_x + \sigma_y}{\sqrt{N}}\right), \tag{1.7}$$

where the first two terms follow from the discussion after (1.4) and the last term follows from Section 5.3 and 6.3 of [30]. However, to the best of our knowledge, there does not exist an optimal algorithm in the literature which exhibits exactly the same rate of convergence as in (1.7), although there are a few general-purpose stochastic optimization algorithms which possess different nearly optimal rates of convergence when applied to above stochastic SPP.

(1) *Mirror-descent stochastic approximation (MD-SA).* The MD-SA method developed by Nemirovski et al. in [29] originates from the classical stochastic approximation (SA) of Robbins and Monro [40]. The classical SA mimics the simple gradient descent method by replacing exact gradients with stochastic gradients, but can only be applied to solve strongly convex problems (see also Polyak [38] and Polyak and Juditsky [39], and Nemirovski et al. [29] for an account for the earlier development of SA methods). By properly modifying the classical SA, Nemirovski et al. showed in [29] that the MD-SA method can optimally solve general nonsmooth stochastic programming problems. The rate of convergence of this algorithm, when applied to the stochastic SPP, is given by (see Section 3 of [29])

$$\mathcal{O}\left\{(L_G + L_K + \sigma_x + \sigma_y)\frac{1}{\sqrt{N}}\right\}.$$

However, the above bound is significantly worse than the lower bound in (1.7) in terms of its dependence on both L_G and L_K.

(2) *Stochastic mirror-prox (SMP).* In order to improve the convergence of the MD-SA method, Juditsky et al. [16] developed a stochastic counterpart of Nemirovski's mirror-prox method for solving general variational inequalities. The stochastic mirror-prox method, when specialized to the above stochastic SPP, yields a rate of convergence given by

$$\mathcal{O}\left\{\frac{L_G + L_K}{N} + \frac{\sigma_x + \sigma_y}{\sqrt{N}}\right\}.$$

Note however, that the above bound is still significantly worse than the lower bound in (1.7) in terms of its dependence on L_G.

(3) *Accelerated stochastic approximation (AC-SA).* More recently, Lan presented in [19] (see also [12, 11]) a unified optimal method for solving smooth, nonsmooth and stochastic optimization by developing a stochas-

tic verstion of Nesterov's method [33, 34]. The developed AC-SA algorithm in [19], when applied to the aforementioned stochastic SPP, possesses the rate of convergence given by

$$\mathcal{O}\left\{\frac{L_G}{N^2} + (L_K + \sigma_x + \sigma_y)\frac{1}{\sqrt{N}}\right\}.$$

However, since the nonsmooth term in f of (1.1) has certain special structure, the above bound is still significantly worse than the lower bound in (1.7) in terms of its dependence on L_K. It should be noted that some improvement for AC-SA has been made by Lin et al. [23] by applying the smoothing technique to (1.1). However, such an improvement works only for the case when Y is bounded and $\sigma_y = \sigma_{x,K} = 0$. Otherwise, the rate of convergence of the AC-SA algorithm will depend on the "variance" of the stochastic gradients computed for the smooth approximation problem, which is usually unknown and difficult to characterize (see Section 3 for more discussions).

Therefore, none of the stochastic optimization algorithms mentioned above could achieve the lower bound on the rate of convergence in (1.7).

1.3. Contribution of this paper. Our contribution in this paper mainly consists of the following three aspects. Firstly, we present a new primal-dual type method, namely the accelerated primal-dual (APD) method, that can achieve the optimal rate of convergence in (1.4) for deterministic SPP. The basic idea of this algorithm is to incorporate a multi-step acceleration scheme into the primal-dual method in [6]. We demonstrate that, without requiring the application of the smoothing technique, this method can also achieve the same optimal rate of convergence as Nesterov's smoothing scheme when applied to (1.1). We also show that the cost per iteration for APD is comparable to that of Nesterov's smoothing scheme. Hence our method can efficiently solve problems with a big Lipschtiz constant L_G.

Secondly, in order to solve stochastic SPP, we develop a stochastic counterpart of the APD method, namely stochastic APD and demonstrate that it can actually achieve the lower bound on the rate of convergence in (1.7). Therefore, this algorithm exhibits an optimal rate of convergence for stochastic SPP not only in terms of its dependence on N, but also on a varity of problem parameters including, L_G, L_K, σ_x and σ_y. To the best of our knowledge, this is the first time that such an optimal algorithm has been developed for stochastic SPP in the literature. In addition, we investigate the stochastic APD method in more details, e.g., by developing the large-deviation results associated with the rate of convergence of the stochastic APD method.

Finally, for both deterministic and stochastic SPP, we demonstrate that the developed APD algorithms can deal with the situation when either X or Y is unbounded, as long as a saddle point of problem (1.1) exists. We incorporate into the APD method the termination criterion employed by Monteiro and Svaiter [25] for solving variational inequalities, and generalize it for solving stochastic SPP. In both deterministic and stochastic cases, the rate of convergence of the APD algorithms will depend on the distance from the initial point to the set of optimal solutions.

1.4. Organization of the paper. We present the APD methods and discuss their main convergence properties for solving deterministic and stochastic SPP problems, respectively, in Sections 2 and 3. In order to facilitate the readers, we put the proofs of our main results in Section 4. Some brief concluding remarks are made in Section 5.

2. Accelerated Primal-Dual Methods for Deterministic SPP. Our goal in this section is to present an accelerated primal-dual method for deterministic SPP and discuss its main convergence properties.

The study on first-order primal-dual method for nonsmooth convex optimization has been mainly motivated by solving total variation based image processing problems (e.g. [45, 9, 37, 6, 4, 13]). Algorithm 1 shows a primal-dual method summarized in [6] for solving a special case of problem (1.1), where $Y = R^m$ for some $m > 0$, and $J(y) = F^*(y)$ is the convex conjugate of a convex and l.s.c. function F.

The convergence of the sequence $\{(x_t, y_t)\}$ in Algorithm 1 has been studied in [37, 9, 6, 4, 13] for various choices of θ_t, and under different conditions on the stepsizes τ_t and η_t. However, the rate of convergence for this algorithm has only been discussed by Chambolle and Pock in [6]. More specifically, they assume that the constant stepsizes are used, i.e., $\tau_t = \tau$, $\eta_t = \eta$ and $\theta_t = \theta$ for some $\tau, \eta, \theta > 0$ for all $t \geq 1$. If $\tau\eta L_K^2 < 1$, where

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Algorithm 1 Primal-dual method for solving deterministic SPP
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1: Choose $x_1 \in X$, $y_1 \in Y$. Set $\bar{x}_1 = x_1$.
2: For $t = 1, \ldots, N$, calculate

$$y_{t+1} = \operatorname*{argmin}_{y \in Y} \langle -K\bar{x}_t, y \rangle + J(y) + \frac{1}{2\eta_t} \|y - y_t\|^2, \tag{2.1}$$

$$x_{t+1} = \operatorname*{argmin}_{x \in X} G(x) + \langle Kx, y_{t+1} \rangle + \frac{1}{2\tau_t} \|x - x_t\|^2, \tag{2.2}$$

$$\bar{x}_{t+1} = \theta_t(x_{t+1} - x_t) + x_{t+1}. \tag{2.3}$$

3: Output $x^N = \frac{1}{N} \sum_{t=1}^{N} x_t$, $y^N = \frac{1}{N} \sum_{t=1}^{N} y_t$.
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Algorithm 2 Accelerated primal-dual method for deterministic SPP
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1: Choose $x_1 \in X, y_1 \in Y$. Set $x_1^{ag} = x_1, y_1^{ag} = y_1, \bar{x}_1 = x_1$.
2: For $t = 1, 2, \ldots, N - 1$, calculate

$$x_t^{md} = (1 - \beta_t^{-1})x_t^{ag} + \beta_t^{-1}x_t, \tag{2.4}$$

$$y_{t+1} = \operatorname*{argmin}_{y \in Y} \langle -K\bar{x}_t, y \rangle + J(y) + \frac{1}{\tau_t} V_Y(y, y_t), \tag{2.5}$$

$$x_{t+1} = \operatorname*{argmin}_{x \in X} \langle \nabla G(x_t^{md}), x \rangle + \langle x, K^T y_{t+1} \rangle + \frac{1}{\eta_t} V_X(x, x_t), \tag{2.6}$$

$$x_{t+1}^{ag} = (1 - \beta_t^{-1})x_t^{ag} + \beta_t^{-1}x_{t+1}, \tag{2.7}$$

$$y_{t+1}^{ag} = (1 - \beta_t^{-1})y_t^{ag} + \beta_t^{-1}y_{t+1}, \tag{2.8}$$

$$\bar{x}_{t+1} = \theta_{t+1}(x_{t+1} - x_t) + x_{t+1}. \tag{2.9}$$

3: Output x_N^{ag}, y_N^{ag}.
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$L_K = \|K\|$, then the output (x^N, y^N) possesses a rate of convergence of $\mathcal{O}(1/N)$ for $\theta = 1$, and of $O(1/\sqrt{N})$ for $\theta = 0$, in terms of partial duality gap (duality gap in a bounded domain, see (2.14) below).

One possible limitation of [6] is that both G and J need to be simple enough so that the two subproblems (2.1) and (2.2) in Algorithm 1 are easy to solve. To make Algorithm 1 applicable to more practical problems we consider more general cases, where J is simple, but G may not be so. In particular, we assume that G is a general smooth convex function satisfying (1.1). In this case, we can replace G in (2.2) by its linear approximation $G(x_t) + \langle \nabla G(x_t), x - x_t \rangle$. Then (2.2) becomes

$$x_{t+1} = \operatorname*{argmin}_{x \in X} \langle \nabla G(x_t), x \rangle + \langle Kx, y_{t+1} \rangle + \frac{1}{2\tau} \|x - x_t\|^2. \tag{2.10}$$

In the following context, we will refer to this modified algorithm as the "linearized version" of Algorithm 1. By some extra effort we can show that, if for $t = 1, \ldots, N$, $0 < \theta_t = \tau_{t-1}/\tau_t = \eta_{t-1}/\eta_t \leq 1$, and $L_G\tau_t + L_K^2\eta_t\tau_t \leq 1$, then (x^N, y^N) has an $\mathcal{O}((L_G + L_K)/N)$ rate of convergence in the sense of the partial duality gap.

As discussed in Section 1, the aforementioned rate of convergence for the linearized version of Algorithm 1 is the same as that proved in [6], and not optimal in terms of its dependence on L_G (see (1.5)). However, this algorithm solves the problem (1.1) directly without smoothing the nonsmooth objective function. Considering the primal-dual method as an alternative to Nestrov's smoothing method, and inspired by his idea of using

accelerated gradient descent algorithm to solve the smoothed problem [33, 34, 35], we propose the following accelerated primal-dual algorithm that integrates the accelerated gradient descent algorithm into the linearized version of Algorithm 1.

Our accelerated primal-dual (APD) method is presented in Algorithm 2. Observe that in this algorithm, the superscript "ag" stands for "aggregated", and "md" stands for "middle". The functions $V_X(\cdot, \cdot)$ and $V_Y(\cdot, \cdot)$ are Bregman divergences defined as

$$V_X(x, u) := d_X(x) - d_X(u) - \langle \nabla d_X(u), x - u \rangle, \forall x, u \in X, \tag{2.11}$$

$$V_Y(y, v) := d_Y(y) - d_Y(v) - \langle \nabla d_Y(v), y - v \rangle, \forall y, v \in Y, \tag{2.12}$$

where $d_X(\cdot)$ and $d_Y(\cdot)$ are strongly convex functions with strong convexity parameters α_X and α_Y. For example, under the Euclidean setting, we can simply set $V_X(x, x_t) := \|x - x_t\|^2/2$ and $V_Y(y, y_t) := \|y - y_t\|^2/2$, and $\alpha_X = \alpha_Y = 1$. We assume that $J(y)$ is a simple convex function, so that the optimization problem in (2.5) can be solved efficiently.

Note that if $\beta_t = 1$ for all $t \geq 1$, then $x_t^{md} = x_t$, $x_{t+1}^{ag} = x_{t+1}$, and Algorithm 2 is the same as the linearized version of Algorithm 1. However, by specifying a different selection of β_t (e.g., $\beta_t = O(t)$), we can significantly improve the rate of convergence of Algorithm 2 in terms of its dependence on L_G. It should be noted that the iteration cost for the APD algorithm is about the same as that for the linearized version of Algorithm 1.

In order to analyze the convergence of Algorithm 2, it is necessary to introduce a notion to characterize the solutions of (1.1). Specifically, denoting $Z = X \times Y$, for any $\tilde{z} = (\tilde{x}, \tilde{y}) \in Z$ and $z = (x, y) \in Z$, we define

$$Q(\tilde{z}, z) := [G(\tilde{x}) + \langle K\tilde{x}, y \rangle - J(y)] - [G(x) + \langle Kx, \tilde{y} \rangle - J(\tilde{y})]. \tag{2.13}$$

It can be easily seen that $\tilde{z}$ is a solution of problem (1.1), if and only if $Q(\tilde{z}, z) \leq 0$ for all $z \in Z$. Therefore, if Z is bounded, it is suggestive to use the gap function

$$g(\tilde{z}) := \max_{z \in Z} Q(\tilde{z}, z) \tag{2.14}$$

to assess the quality of a feasible solution $\tilde{z} \in Z$. In fact, we can show that $f(\tilde{x}) - f^* \leq g(\tilde{z})$ for all $\tilde{z} \in Z$, where f^* denotes the optimal value of problem (1.1). However, if Z is unbounded, then $g(\tilde{z})$ is not well-defined even for a nearly optimal solution $\tilde{z} \in Z$. Hence, in the sequel, we will consider the bounded and unbounded case separately, by employing a slightly different error measure for the latter situation.

The following theorem describes the convergence properties of Algorithm 2 when Z is bounded.

THEOREM 2.1. *Suppose that for some* $\Omega_X, \Omega_Y > 0$,

$$\sup_{x_1, x_2 \in X} V_X(x_1, x_2) \leq \Omega_X^2 \quad and \quad \sup_{y_1, y_2 \in Y} V_Y(x_1, x_2) \leq \Omega_Y^2. \tag{2.15}$$

Also assume that the parameters $\beta_t, \theta_t, \eta_t, \tau_t$ *in Algorithm 2 are chosen such that for all* $t \geq 1$,

$$\beta_1 = 1, \ \beta_{t+1} - 1 = \beta_t \theta_{t+1}, \tag{2.16}$$

$$0 < \theta_t \leq \min\{\frac{\eta_{t-1}}{\eta_t}, \frac{\tau_{t-1}}{\tau_t}\}, \tag{2.17}$$

$$\frac{\alpha_X}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2 \tau_t}{\alpha_Y} \geq 0. \tag{2.18}$$

Then for all $t \geq 1$,

$$g(z_{t+1}^{ag}) \leq \frac{1}{\beta_t \eta_t} \Omega_X^2 + \frac{1}{\beta_t \tau_t} \Omega_Y^2. \tag{2.19}$$

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There are various options for choosing the parameters β_t, η_t, τ_t and θ_t such that (2.16)–(2.18) hold. Below we provide such an example.

COROLLARY 2.2. *Suppose that (2.15) holds. In Algorithm 2, if the parameters are set to*

$$\beta_t = \frac{t+1}{2}, \ \theta_t = \frac{t-1}{t}, \ \eta_t = \frac{\alpha_X t}{2L_G + tL_K D_Y/D_X} \ and \ \tau_t = \frac{\alpha_Y D_Y}{L_K D_X}, \tag{2.20}$$

where $D_X := \Omega_X \sqrt{2/\alpha_X}$ and $D_Y := \Omega_Y \sqrt{2/\alpha_Y}$, then for all $t \geq 2$,

$$g(z_t^{ag}) \leq \frac{2L_G D_X^2}{t(t-1)} + \frac{2L_K D_X D_Y}{t}. \tag{2.21}$$

Proof. It suffices to verify that the parameters in (2.20) satisfies (2.16)–(2.18) in Theorem 2.1. It is easy to check that (2.16) and (2.17) hold. Furthermore,

$$\frac{\alpha_X}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2 \tau_t}{\alpha_Y} = \frac{2L_G + tL_K D_Y/D_X}{t} - \frac{2L_G}{t+1} - \frac{L_K D_Y}{D_X} \geq 0,$$

so (2.18) holds. Therefore, by (2.19), for all $t \geq 1$ we have

$$g(z_t^{ag}) \leq \frac{1}{\beta_{t-1}\eta_{t-1}}\Omega_X^2 + \frac{1}{\beta_{t-1}\tau_{t-1}}\Omega_Y^2 = \frac{4L_G + 2(t-1)L_K D_Y/D_X}{\alpha_X t(t-1)} \cdot \frac{\alpha_X}{2} D_X^2 + \frac{2L_K D_X/D_Y}{\alpha_Y t} \cdot \frac{\alpha_Y}{2} D_Y^2$$

$$= \frac{2L_G D_X^2}{t(t-1)} + \frac{2L_K D_X D_Y}{t}.$$

□

Clearly, in view of (1.4), the rate of convergence of Algorithm 2 applied to problem (1.1) is optimal when the parameters are chosen according to (2.20). Also observe that we need to estimate D_Y/D_X to use these parameters. However, it should be pointed out that replacing the ratio D_Y/D_X in (2.20) by any positive constant only results an increase in the RHS of (2.21) by a constant factor.

Now, we study the convergence properties of the APD algorithm for the case when $Z = X \times Y$ is unbounded, by using a perterbation-based termination criterion recently employed by Monteiro and Svaiter and applied to SPP [27, 26, 25]. This termination criterion is based on the enlargement of a maximal monotone operator, which is first introduced in [5]. One advantage of using this criterion is that its definition does not depend on the boundedness of the domain of the operator. More specifically, as shown in [25, 27], there always exists a purterbation vector v such that

$$\tilde{g}(\tilde{z}, v) := \max_{z \in Z} Q(\tilde{z}, z) - \langle v, \tilde{z} - z \rangle \tag{2.22}$$

is well-defined, although the value of $g(\tilde{z})$ in (2.14) may be unbounded if Z is unbounded. In the following result, we show that the APD algorithm can compute a nearly optimal solution $\tilde{z}$ with a small residue $\tilde{g}(\tilde{z}, v)$, for a small purterbation vector v (i.e., $\|v\|$ is small). In addition, our derived iteration complexity bounds are proportional to the distance from the initial point to the solution set.

THEOREM 2.3. *Let $\{z_t^{ag}\} = \{(x_t^{ag}, y_t^{ag})\}$ be the iterates generated by Algorithm 2 with $V_X(x, x_t) = \|x - x_t\|^2/2$ and $V_Y(y, y_t) = \|y - y_t\|^2/2$. Assume that the parameters $\beta_t, \theta_t, \eta_t$ and τ_t satisfy (2.16),*

$$\theta_t = \frac{\eta_{t-1}}{\eta_t} = \frac{\tau_{t-1}}{\tau_t}, \tag{2.23}$$

$$\frac{\alpha_X}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2 \tau_t}{p\alpha_Y} \geq 0, \tag{2.24}$$

for all $t \geq 1$ and for some $0 < p < 1$, then there exists a perturbation vector v_{t+1} such that

$$\tilde{g}(z_{t+1}^{ag}, v_{t+1}) \leq \frac{(2-p)D^2}{\beta_t \eta_t (1-p)} =: \varepsilon_{t+1} \tag{2.25}$$

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for any $t \geq 1$. Moreover, we have

$$\|v_{t+1}\| \leq \frac{1}{\beta_t \eta_t}\|\hat{x} - x_1\| + \frac{1}{\beta_t \tau_t}\|\hat{y} - y_1\| + \left[\frac{1}{\beta_t \eta_t}\left(1 + \sqrt{\frac{\eta_1}{\tau_1(1-p)}}\right) + \frac{2L_K}{\beta_t}\right]D, \qquad (2.26)$$

where $(\hat{x}, \hat{y})$ is a pair of solutions for problem (1.1) and

$$D := \sqrt{\|\hat{x} - x_1\|^2 + \frac{\eta_1}{\tau_1}\|\hat{y} - y_1\|^2}. \qquad (2.27)$$

Below we suggest a specific parameter setting which satisfies (2.16), (2.23) and (2.24).

COROLLARY 2.4. *In Algorithm 2, if N is given and the parameters are set to*

$$\beta_t = \frac{t+1}{2}, \ \theta_t = \frac{t-1}{t}, \ \eta_t = \frac{t+1}{2(L_G + NL_K)}, \text{ and } \tau_t = \frac{t+1}{2NL_K} \qquad (2.28)$$

then there exists v_N that satisfies (2.25) with

$$\varepsilon_N \leq \frac{10L_G\hat{D}^2}{N^2} + \frac{10L_K\hat{D}^2}{N} \quad \text{and} \quad \|v_N\| \leq \frac{15L_G\hat{D}}{N^2} + \frac{16L_K\hat{D}}{N}, \qquad (2.29)$$

where $\hat{D} = \sqrt{\|\hat{x} - x_1\|^2 + \|\hat{y} - y_1\|^2}$.

Proof. For the parameters β_t, γ_t, η_t, τ_t in (2.28), it is clear that (2.16), (2.23) holds. Furthermore, let $p = 1/4$, for any $t = 1, \ldots, N-1$, we have

$$\frac{1}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2\tau_t}{p} = \frac{2L_G + 2L_KN}{t+1} - \frac{2L_G}{t+1} - \frac{2L_K^2(t+1)}{L_KN} \geq \frac{2L_KN}{t+1} - \frac{2L_K(t+1)}{N} \geq 0,$$

thus (2.24) holds. By Theorem 2.3, inequalities (2.25) and (2.26) hold. Noting that $\eta_t \leq \tau_t$, in (2.25) and (2.26) we have $D \leq \hat{D}$, $\|\hat{x} - x_1\| + \|\hat{y} - y_1\| \leq \sqrt{2}\hat{D}$, hence

$$\|v_{t+1}\| \leq \frac{\sqrt{2}\hat{D}}{\beta_t \eta_t} + \frac{(1 + \sqrt{4/3})\hat{D}}{\beta_t \eta_t} + \frac{2L_K\hat{D}}{\beta_t},$$

$$\varepsilon_{t+1} \leq \frac{(2-p)\hat{D}^2}{\beta_t \eta_t(1-p)} = \frac{7\hat{D}^2}{3\beta_t \eta_t}.$$

Also note that by (2.28),

$$\frac{1}{\beta_{N-1}\eta_{N-1}} = \frac{4(L_G + L_KN)}{N^2} = \frac{4L_G}{N^2} + \frac{4L_K}{N}.$$

Using the above three relations and the definition of β_t in (2.28), we obtain (2.29) after simpliying the constants.
□

It is interesting to notice that, if the parameters in Algorithm 2 are set to (2.28), then both residues ε_N and $\|v_N\|$ in (2.29) reduce to zero with approximately the same rate of convergence (up to a factor of $\hat{D}$). Also observe that in Theorem 2.3 and Corollary 2.4, we fix $V_X(\cdot, \cdot)$ and $V_Y(\cdot, \cdot)$ to be regular distance functions rather than more general Bregman divergences. This is due to fact that we need to apply the Triagular inequality associated with $\sqrt{V_X(\cdot, \cdot)}$ and $\sqrt{V_Y(\cdot, \cdot)}$, while such an inequality does not necessarily hold for Bregman divergences in general.

Algorithm 3 Stochastic APD method for stochastic SPP

Modify (2.5) and (2.6) in Algorithm 2 to

$$y_{t+1} = \operatorname*{argmin}_{y \in Y} \ \langle -\hat{\mathcal{K}}_x(\bar{x}_t), y \rangle + J(y) + \frac{1}{\tau_t} V_Y(y, y_t) \tag{3.1}$$

$$x_{t+1} = \operatorname*{argmin}_{x \in X} \ \langle \hat{\mathcal{G}}(x_t^{md}), x \rangle + \langle x, \hat{\mathcal{K}}_y(y_{t+1}) \rangle + \frac{1}{\eta_t} V_X(x, x_t) \tag{3.2}$$

---

3. Stochastic APD Methods for Stochastic SPP. Our goal in this section is to present a stochastic APD method for stochastic SPP (i.e., problem (1.1) with a stochastic oracle) and demonstrate that it can actually achieve the lower bound in (1.7) on the rate of convergence for stochastic SPP.

The stochastic APD method is a stochastic counterpart of the APD algorithm in Section 2, obtained by simply replacing the gradient operators $-K\bar{x}_t$, $\nabla G(x_t^{md})$ and $K^T y_{t+1}$, used in (2.5) and (2.6), with the stochastic gradient operators computed by the $\mathcal{SO}$, i.e., $-\hat{\mathcal{K}}_x(\bar{x}_t)$, $\hat{\mathcal{G}}(x_t^{md})$ an $\hat{\mathcal{K}}_y(y_{t+1})$, respectively. This algorithm is formally described as in Algorithm 3.

A few more remarks about the development of the above stochastic APD method are in order. Firstly, observe that, although primal-dual methods have been extensively studied for solving deterministic saddle-point problems, it seems that these types of methods have not yet been generalized for stochastic SPP in the literature. Secondly, as noted in Section 1, one possible way to solve stochastic SPP is to apply the AC-SA algorithm in [19] to a certain smooth approximation of (1.1) by Nesterov [35]. However, the rate of convergence of this approach will depend on the variance of the stochastic gradients computed for the smooth approximation problem, which is usually unkown and difficult to characterize. On the other hand, the stochastic APD method described above works directly with the original problem without requring the application of the smoothing technique, and its rate of convergence will depend on the variance of the stochastic gradient operators computed for the original problem, i.e., $\sigma_{x,G}^2$, σ_y^2 and $\sigma_{x,K}^2$ in A1. We will show that it can achieve exactly the lower bound in (1.7) on the rate of convergence for stochastic SPP.

Similarly to Section 2, we use the two gap functions $g(\cdot)$ and $\tilde{g}(\cdot, \cdot)$, respectively, defined in (2.14) and (2.22) as the termination criteria for the stochastic APD algorithm, depending on whether the feasible set $Z = X \times Y$ is bounded or not. Since the algorithm is stochastic in nature, for both cases we establish its expected rate of convergence in terms of $g(\cdot)$ or $\tilde{g}(\cdot, \cdot)$, i.e., the "average" rate of convergence over many runs of the algorithm. In addition, we show that if Z is bounded, then the convergence of the APD algorithm can be strengthened under the following "light-tail" assumption on $\mathcal{SO}$.

A2. $\quad \mathbb{E}\left[\exp\{ \|\nabla G(x) - \hat{\mathcal{G}}(x)\|_*^2 / \sigma_{x,G}^2 \} \right] \leq \exp\{1\}, \quad \mathbb{E}\left[\exp\{ \|Kx - \hat{\mathcal{K}}_x(x)\|_*^2 / \sigma_y^2 \} \right] \leq \exp\{1\}$

$\quad$ and $\ \mathbb{E}\left[\exp\{ \|K^T y - \hat{\mathcal{K}}_y(y)\|_*^2 / \sigma_{x,K}^2 \} \right] \leq \exp\{1\}.$

It is easy to see that A2 implies A1 by Jensen's inequality.

Theorem 3.1 below summarizes the convergence properties of Algorithm 3 when Z is bounded. Note that the following quanity will be used in the statement of this result and the convergence analysis of the APD algorithms (see Section 4):

$$\gamma_t = \begin{cases} 1, & t = 1, \\ \theta_t^{-1} \gamma_{t-1}, & t \geq 2. \end{cases} \tag{3.3}$$

THEOREM 3.1. *Suppose that (2.15) holds for some* $\Omega_X, \Omega_Y > 0$. *Also assume that for all* $t \geq 1$, *the parameters* $\beta_t, \theta_t, \eta_t$ *and* τ_t *in Algorithm 3 satisfy (2.16), (2.17), and*

$$\frac{q\alpha_X}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2 \tau_t}{p\alpha_Y} \geq 0 \tag{3.4}$$

for some $p, q \in (0, 1)$. *Then,*

(a). *Under assumption* **A1**, *for all* $t \geq 1$,

$$\mathbb{E}[g(z_{t+1}^{ag})] \leq \mathcal{Q}_0(t), \tag{3.5}$$

where

$$\mathcal{Q}_0(t) := \frac{1}{\beta_t \gamma_t} \left\{ \frac{2\gamma_t}{\eta_t} \Omega_X^2 + \frac{2\gamma_t}{\tau_t} \Omega_Y^2 \right\} + \frac{1}{2\beta_t \gamma_t} \sum_{i=1}^t \left\{ \frac{(2-q)\eta_i \gamma_i}{(1-q)\alpha_X} \sigma_x^2 + \frac{(2-p)\tau_i \gamma_i}{(1-p)\alpha_Y} \sigma_y^2 \right\}. \tag{3.6}$$

(b). *Under assumption* **A2**, *for all* $\lambda > 0$ *and* $t \geq 1$,

$$Prob\{g(z_{t+1}^{ag}) > \mathcal{Q}_0(t) + \lambda \mathcal{Q}_1(t)\} \leq 3\exp\{-\lambda^2/3\} + 3\exp\{-\lambda\}, \tag{3.7}$$

where

$$\mathcal{Q}_1(t) := \frac{1}{\beta_t \gamma_t} \left(\frac{\sqrt{2}\sigma_x \Omega_X}{\sqrt{\alpha_X}} + \frac{\sigma_y \Omega_Y}{\sqrt{\alpha_Y}} \right) \sqrt{2\sum_{i=1}^t \gamma_i^2} + \frac{1}{2\beta_t \gamma_t} \sum_{i=1}^t \left\{ \frac{(2-q)\eta_i \gamma_i}{(1-q)\alpha_X} \sigma_x^2 + \frac{(2-p)\tau_i \gamma_i}{(1-p)\alpha_Y} \sigma_y^2 \right\}. \tag{3.8}$$

We provide below a specific choice of the parameters β_t, θ_t, η_t and τ_t for the stochastic APD method for the case when Z is bounded.

COROLLARY 3.2. *Suppose that (2.15) holds and let D_X and D_Y be defined in Corolloary 2.2. In Algorithm 3, if $N \geq 1$ is given and the parameters are set to*

$$\beta_t = \frac{t+1}{2}, \ \theta_t = \frac{t-1}{t}, \ \eta_t = \frac{2\alpha_X D_X t}{6L_G D_X + 3L_K D_Y(N-1) + 3\sigma_x N\sqrt{N-1}} \ and$$

$$\tau_t = \frac{2\alpha_Y D_Y t}{3L_K D_X(N-1) + 3\sigma_y N\sqrt{N-1}}. \tag{3.9}$$

Then under Assumption **A1**, *we have*

$$E[g(z_N^{ag})] \leq \frac{6L_G D_X^2}{N(N-1)} + \frac{6L_K D_X D_Y}{N} + \frac{4(\sigma_x D_X + \sigma_y D_Y)}{\sqrt{N-1}} =: \mathcal{C}_0(N). \tag{3.10}$$

If in addition, Assumption **A2** *holds, then for all $\lambda > 0$, we have*

$$Prob\{g(z_N^{ag}) > \mathcal{C}_0(N) + \lambda \mathcal{C}_1(N)\} \leq 3\exp\{-\lambda^2/3\} + 3\exp\{-\lambda\}, \tag{3.11}$$

where

$$\mathcal{C}_1(N) = \frac{3(\sigma_x D_X + \sigma_y D_Y)}{\sqrt{N-1}}. \tag{3.12}$$

Proof. First we check that the parameters in (3.9) satisfy the conditions in Theorem 3.1. The inequalities (2.16) and (2.17) can be checked easily. Furthermore, for all $t = 1, \ldots, N-1$, setting $p = q = 2/3$ we have

$$\frac{q\alpha_X}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2 \tau_t}{p\alpha_Y} \geq \frac{2L_G D_X + L_K D_Y(N-1)}{D_X t} - \frac{2L_G}{t+1} - \frac{L_K^2 D_Y t}{L_K D_X(N-1)} \geq 0,$$

thus (3.4) hold, and hence Theorem 3.1 holds.

To show (3.10) and (3.11), it suffices to show that $\mathcal{C}_0(N) \geq \mathcal{Q}_0(N-1)$ and $\mathcal{C}_1(N) \geq \mathcal{Q}_1(N-1)$. Observe that by (3.3) and (3.9), we have $\gamma_t = t$. Also, observe that $\sum_{i=1}^{N-1} i^2 \leq (N-1)N^2/3$, thus

$$\frac{1}{\gamma_{N-1}} \sum_{i=1}^{N-1} \eta_i \gamma_i \leq \frac{2\alpha_X D_X}{3(N-1)^{3/2}N\sigma_x} \sum_{i=1}^{N-1} i^2 \leq \frac{2\alpha_X D_X N}{9\sigma_x \sqrt{N-1}},$$

$$\frac{1}{\gamma_{N-1}} \sum_{i=1}^{N-1} \tau_i \gamma_i \leq \frac{2\alpha_Y D_Y}{3(N-1)^{3/2}N\sigma_y} \sum_{i=1}^{N-1} i^2 \leq \frac{2\alpha_Y D_Y N}{9\sigma_y \sqrt{N-1}}.$$

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Apply the above bounds to (3.5) and (3.8), we get

$$\mathcal{Q}_0(N-1)$$

$$\leq \frac{2}{N}\left(\frac{6L_G D_X + 3L_K D_Y(N-1) + 3N\sqrt{N-1}\sigma_x}{\alpha_X D_X(N-1)} \cdot \frac{\alpha_X}{2}D_X^2 + \frac{3L_K D_X(N-1) + 3N\sqrt{N-1}\sigma_y}{\alpha_Y D_Y(N-1)} \cdot \frac{\alpha_Y}{2}D_Y^2\right.$$

$$\left. + \frac{2\sigma_x^2}{\alpha_X} \cdot \frac{2\alpha_X D_X N}{9\sigma_x\sqrt{N-1}} + \frac{2\sigma_y^2}{\alpha_Y} \cdot \frac{2\alpha_Y D_Y N}{9\sigma_y\sqrt{N-1}}\right) \leq \mathcal{C}_0(N),$$

$$\mathcal{Q}_1(N-1)$$

$$\leq \frac{2}{N(N-1)}\left(\sigma_x D_X + \frac{\sigma_y D_Y}{\sqrt{2}}\right)\sqrt{\frac{2(N-1)N^2}{3}} + \frac{4\sigma_x^2}{\alpha_X N} \cdot \frac{2\alpha_X D_X N}{9\sigma_x\sqrt{N-1}} + \frac{4\sigma_y^2}{\alpha_Y N} \cdot \frac{2\alpha_Y D_Y N}{9\sigma_y\sqrt{N-1}} \leq \mathcal{C}_1(N),$$

so (3.10) and (3.11) holds. □

Comparing the rate of convergence established in (3.10) with the lower bound in (1.7), we can clearly see that the stochastic APD algorithm is an optimal method for solving the stochastic saddle-point problems. More specifically, in view of (3.10), this algorithm allows us to have very large Lipschitz constants L_G (as big as $\mathcal{O}(N^{\frac{3}{2}})$) and L_K (as big as $\mathcal{O}(\sqrt{N})$) without significantly affecting its rate of convergence.

We now present the convergence results for the stochastic APD method applied to stochastic saddle-point problems with possibly unbounded feasible set Z. It appears that the solution methods of these types of problems have not been well-studied in the literature.

THEOREM 3.3. *Let $\{z_t^{ag}\} = \{(x_t^{ag}, y_t^{ag})\}$ be the iterates generated by Algorithm 2 with $V_X(x, x_t) = \|x - x_t\|^2/2$ and $V_Y(y, y_t) = \|y - y_t\|^2/2$. Assume that the parameters $\beta_t, \theta_t, \eta_t$ and τ_t in Algorithm 3 satisfy (2.16), (2.23) and (3.4) for all $t \geq 1$ and some $p, q \in (0, 1)$, then there exists a perturbation vector v_{t+1} such that*

$$\mathbb{E}[\tilde{g}(z_{t+1}^{ag}, v_{t+1})] \leq \frac{1}{\beta_t \eta_t}\left(\frac{6-4p}{1-p}D^2 + \frac{5-3p}{2-2p}C^2\right) =: \varepsilon_{t+1} \tag{3.13}$$

for any $t \geq 1$. Moreover, we have

$$\mathbb{E}[\|v_{t+1}\|] \leq \frac{2\|\hat{x} - x_1\|}{\beta_t \eta_t} + \frac{2\|\hat{y} - y_1\|}{\beta_t \tau_t} + \sqrt{2D^2 + C^2}\left[\frac{2}{\beta_t \eta_t} + \frac{1}{\beta_t \tau_t}\sqrt{\frac{\tau_1}{\eta_1}}\left(\sqrt{\frac{1}{1-p}} + 1\right) + \frac{2L_K}{\beta_t}\right], \tag{3.14}$$

where $(\hat{x}, \hat{y})$ is a pair of solutions for problem (1.1), D is defined in (2.27) and

$$C := \sqrt{\sum_{i=1}^{t}\frac{\eta_i^2\sigma_x^2}{1-q} + \sum_{i=1}^{t}\frac{\eta_i\tau_i\sigma_y^2}{1-p}}. \tag{3.15}$$

Below we specialize the results in Theorem 3.3 by choosing a set of parameters satisfying (2.16), (2.23) and (3.4).

COROLLARY 3.4. *In Algorithm 3, if N is given and the parameters are set to*

$$\beta_t = \frac{t+1}{2}, \ \theta_t = \frac{t-1}{t}, \ \eta_t = \frac{3t}{4\eta}, \ and \ \tau_t = \frac{t}{\eta}, \tag{3.16}$$

where

$$\eta = 2L_G + 2L_K(N-1) + N\sqrt{N-1}\sigma/\tilde{D} \ for \ some \ \tilde{D} > 0, \ \sigma = \sqrt{\frac{9}{4}\sigma_x^2 + \sigma_y^2}, \tag{3.17}$$

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then there exists v_N that satisfies (3.13) with

$$\varepsilon_N \leq \frac{36L_G D^2}{N(N-1)} + \frac{36L_K D^2}{N} + \frac{\sigma D\left(18D/\tilde{D} + 3\tilde{D}/D\right)}{\sqrt{N-1}}, \tag{3.18}$$

$$\mathbb{E}[\|v_N\|] \leq \frac{50L_G D}{N(N-1)} + \frac{L_K D(55 + 3\tilde{D}/D)}{N} + \frac{\sigma(6 + 25D/\tilde{D})}{\sqrt{N-1}}, \tag{3.19}$$

where D is defined in (2.27).

Proof. For the parameters in (3.16), it is clear that (2.16) and (2.23) hold. Furthermore, let $p = 1/4$, $q = 3/4$, then for all $t = 1, \ldots, N-1$, we have

$$\frac{q}{\eta_t} - \frac{L_G}{\beta_t} - \frac{L_K^2 \tau_t}{p} = \frac{\eta}{t} - \frac{2L_G}{t+1} - \frac{4L_K^2 t}{\eta} \geq \frac{2L_G + 2L_K(N-1)}{t} - \frac{2L_G}{t} - \frac{2L_K^2 t}{L_K(N-1)} \geq 0,$$

thus (3.4) holds. By Theorem 3.3, we get (3.13) and (3.14). Note that $\eta_t/\tau_t = 3/4$, and

$$\frac{1}{\beta_{N-1}\eta_{N-1}}\|\hat{x} - x_1\| \leq \frac{1}{\beta_{N-1}\eta_{N-1}}D, \quad \frac{1}{\beta_{N-1}\tau_{N-1}}\|\hat{y} - y_1\| \leq \frac{1}{\beta_{N-1}\eta_{N-1}} \cdot \frac{\eta_{N-1}}{\tau_{N-1}} \cdot \sqrt{\frac{4}{3}}D = \frac{\sqrt{3/4}D}{\beta_{N-1}\eta_{N-1}},$$

so in (3.13) and (3.14) we have

$$\varepsilon_{t+1} \leq \frac{1}{\beta_{N-1}\eta_{N-1}}\left(\frac{20}{3}D^2 + \frac{17}{6}C^2\right), \tag{3.20}$$

$$\mathbb{E}[\|v_{t+1}\|] \leq \frac{(2+\sqrt{3})D}{\beta_{N-1}\eta_{N-1}} + \frac{\sqrt{2D^2 + C^2}\left(3 + \sqrt{3/4}\right)}{\beta_{N-1}\eta_{N-1}} + \frac{2L_K\sqrt{2D^2 + C^2}}{\beta_{N-1}}. \tag{3.21}$$

By (3.15) and the fact that $\sum_{i=1}^{N-1} i^2 \leq N^2(N-1)/3$, we have

$$C = \sqrt{\sum_{i=1}^{N-1}\frac{9\sigma_x^2 i^2}{4\eta^2} + \sum_{i=1}^{N-1}\frac{\sigma_x^2 i^2}{\eta^2}} \leq \sqrt{\frac{1}{3\eta^2}N^2(N-1)\left(\frac{9\sigma_x^2}{4} + \sigma_x^2\right)} = \frac{\sigma N\sqrt{N-1}}{\sqrt{3}\eta}$$

Applying the above bound to (3.20) and (3.21), and using the fact that $\sqrt{2D^2 + C^2} \leq \sqrt{2}D + C$, we obtain

$$\eta = 2L_G + 2L_K(N-1) + N\sqrt{N-1}\sigma/\tilde{D},$$

$$\varepsilon_N \leq \frac{8\eta}{3N(N-1)}\left(\frac{20}{3}D^2 + \frac{17\sigma^2 N^2(N-1)}{18\eta^2}\right) = \frac{8}{3N(N-1)}\left(\frac{20}{3}\eta D^2 + \frac{17\sigma^2 N^2(N-1)}{18\eta}\right)$$

$$\leq \frac{320L_G D^2}{9N(N-1)} + \frac{320L_K(N-1)D^2}{9N(N-1)} + \frac{160N\sqrt{N-1}\sigma D^2/\tilde{D}}{9N(N-1)} + \frac{68\sigma^2 N^2(N-1)}{27N^2(N-1)^{3/2}\sigma/\tilde{D}}$$

$$\leq \frac{36L_G D^2}{N(N-1)} + \frac{36L_K D^2}{N} + \frac{\sigma D\left(18D/\tilde{D} + 3\tilde{D}/D\right)}{\sqrt{N-1}},$$

$$\mathbb{E}[\|v_N\|] \leq \frac{1}{\beta_{N-1}\eta_{N-1}}\left(2D + \sqrt{3}D + 3\sqrt{2}D + \sqrt{6}D/2 + 3C + \sqrt{3}C/2\right) + \frac{2\sqrt{2}L_K D}{\beta_{N-1}} + \frac{2L_K C}{\beta_{N-1}}$$

$$\leq \frac{16L_G + 16L_K(N-1) + 8N\sqrt{N-1}\sigma/\tilde{D}}{3N(N-1)}\left(2 + \sqrt{3} + 3\sqrt{2} + \sqrt{6}/2\right)D$$

$$+ \frac{8\sigma}{3\sqrt{N-1}}\left(\sqrt{3} + 1/2\right) + \frac{4\sqrt{2}L_K D}{N} + \frac{4L_K \sigma N\sqrt{N-1}}{N\sqrt{3}N\sqrt{N-1}\sigma/\tilde{D}}$$

$$\leq \frac{50L_G D}{N(N-1)} + \frac{L_K D(55 + 3\tilde{D}/D)}{N} + \frac{\sigma(6 + 25D/\tilde{D})}{\sqrt{N-1}}.$$

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⬚

Observe that the parameter settings in (3.16)-(3.17) are more complicated than the ones in (2.28) for the deterministic unbounded case. In particular, for the stochastic unbounded case, we need to choose a parameter $\tilde{D}$ which is not required for the deterministic case. Clearly, the optimal selection for $\tilde{D}$ minimizing the RHS of (3.18) is given by $\sqrt{6}D$. Note however, that the value of D will be very difficult to estimate for the unbounded case and hence one often has to resort to a suboptimal selection for $\tilde{D}$. For example, if $\tilde{D} = 1$, then the RHS of (3.18) and (3.19) will become $\mathcal{O}(L_G D^2/N^2 + L_K D^2/N + \sigma D^2/\sqrt{N})$ and $\mathcal{O}(L_G D/N^2 + L_K D/N + \sigma D/\sqrt{N})$, respectively.

4. Convergence analysis. Our goal in this section is to prove the main results presented in Section 2 and 3, namely, Theorems 2.1, 2.3, 3.1 and 3.3.

4.1. Convergence analysis for the deterministic APD algorithm. In this section, we prove Theorems 2.1 and 2.3 which, respectively, describe the convergence properties for the deterministic APD algorithm for the bounded and unbounded SPPs.

Before proving Theorem 2.1, we first prove two technical results: Proposition 4.1 shows some important properties for the function $Q(\cdot, \cdot)$ in (2.13) and Lemma 4.2 establishes a bound on $Q(x_t^{ag}, z)$.

PROPOSITION 4.1. *Assume that $\beta_t \geq 1$ for all t. If $z_{t+1}^{ag} = (x_{t+1}^{ag}, y_{t+1}^{ag})$ is generated by Algorithm 2, then for all $z = (x,y) \in Z$,*

$$
\begin{aligned}
&\beta_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)Q(z_t^{ag}, z) \\
&\leq \langle \nabla G(x_t^{md}), x_{t+1} - x \rangle + \frac{L_G}{2\beta_t}\|x_{t+1} - x_t\|^2 + [J(y_{t+1}) - J(y)] + \langle Kx_{t+1}, y \rangle - \langle Kx, y_{t+1} \rangle.
\end{aligned}
\tag{4.1}
$$

Proof. By equations (2.4) and (2.7), $x_{t+1}^{ag} - x_t^{md} = \beta_t^{-1}(x_{t+1} - x_t)$. Using this observation and the convexity of $G(\cdot)$, we have

$$
\begin{aligned}
\beta_t G(x_{t+1}^{ag}) &\leq \beta_t G(x_t^{md}) + \beta_t \langle \nabla G(x_t^{md}), x_{t+1}^{ag} - x_t^{md} \rangle + \frac{\beta_t L_G}{2}\|x_{t+1}^{ag} - x_t^{md}\|^2 \\
&\leq \beta_t G(x_t^{md}) + \beta_t \langle \nabla G(x_t^{md}), x_{t+1}^{ag} - x_t^{md} \rangle + \frac{L_G}{2\beta_t}\|x_{t+1} - x_t\|^2 \\
&= \beta_t G(x_t^{md}) + (\beta_t - 1)\langle \nabla G(x_t^{md}), x_t^{ag} - x_t^{md} \rangle + \langle \nabla G(x_t^{md}), x_{t+1} - x_t^{md} \rangle + \frac{L_G}{2\beta_t}\|x_{t+1} - x_t\|^2 \\
&= (\beta_t - 1)\left[G(x_t^{md}) + \langle \nabla G(x_t^{md}), x_t^{ag} - x_t^{md} \rangle\right] + \left[G(x_t^{md}) + \langle \nabla G(x_t^{md}), x_{t+1} - x_t^{md} \rangle\right] + \frac{L_G}{2\beta_t}\|x_{t+1} - x_t\|^2 \\
&= (\beta_t - 1)\left[G(x_t^{md}) + \langle \nabla G(x_t^{md}), x_t^{ag} - x_t^{md} \rangle\right] + \left[G(x_t^{md}) + \langle \nabla G(x_t^{md}), x - x_t^{md} \rangle\right] + \langle \nabla G(x_t^{md}), x_{t+1} - x \rangle \\
&\quad + \frac{L_G}{2\beta_t}\|x_{t+1} - x_t\|^2 \\
&\leq (\beta_t - 1)G(x_t^{ag}) + G(x) + \langle \nabla G(x_t^{md}), x_{t+1} - x \rangle + \frac{L_G}{2\beta_t}\|x_{t+1} - x_t\|^2.
\end{aligned}
\tag{4.2}
$$

Moreover, by (2.8) and the convexity of $J(\cdot)$, we have

$$
\begin{aligned}
\beta_t J(y_{t+1}^{ag}) - \beta_t J(y) &\leq (\beta_t - 1)J(y_t^{ag}) + J(y_{t+1}) - \beta_t J(y) \\
&= (\beta_t - 1)\left[J(y_t^{ag}) - J(y)\right] + J(y_{t+1}) - J(y).
\end{aligned}
$$

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By (2.13), (2.7), (2.8) and the above two inequalities above, we obtain

$$\beta_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)Q(z_t^{ag}, z)$$
$$= \beta_t \left\{ \left[G(x_{t+1}^{ag}) + \langle Kx_{t+1}^{ag}, y \rangle - J(y) \right] - \left[G(x) + \langle Kx, y_{t+1}^{ag} \rangle - J(y_{t+1}^{ag}) \right] \right\}$$
$$\quad - (\beta_t - 1) \left\{ [G(x_t^{ag}) + \langle Kx_t^{ag}, y \rangle - J(y)] - [G(x) + \langle Kx, y_t^{ag} \rangle - J(y_t^{ag})] \right\}$$
$$= \beta_t G(x_{t+1}^{ag}) - (\beta_t - 1)G(x_t^{ag}) - G(x) + \beta_t \left[J(y_{t+1}^{ag}) - J(y) \right]$$
$$\quad - (\beta_t - 1) \left[J(y_t^{ag}) - J(y) \right] + \langle K(\beta_t x_{t+1}^{ag} - (\beta_t - 1)x_t^{ag}), y \rangle - \langle Kx, \beta_t y_{t+1}^{ag} - (\beta_t - 1)y_t^{ag} \rangle$$
$$\leq \langle \nabla G(x_t^{md}), x_{t+1} - x \rangle + \frac{L_G}{2\beta_t} \|x_{t+1} - x_t\|^2 + J(y_{t+1}) - J(y) + \langle Kx_{t+1}, y \rangle - \langle Kx, y_{t+1} \rangle.$$

▢

Lemma 4.2 establishes a bound for $Q(z_{t+1}^{ag}, z)$ for all $z \in Z$, which will be used in the proof of both Theorems 2.1 and 2.3.

LEMMA 4.2. *Let $z_{t+1}^{ag} = (x_{t+1}^{ag}, y_{t+1}^{ag})$ be the iterates generated by Algorithm 2. Assume that the parameters $\beta_t, \theta_t, \eta_t,$ and τ_t satisfy (2.16), (2.17) and (2.18). Then, for any $z \in Z$, we have*

$$\beta_t \gamma_t Q(z_{t+1}^{ag}, z) \leq \mathcal{B}_t(z, z_{[t]}) + \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \gamma_t \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2, \qquad (4.3)$$

where γ_t is defined in (3.3), $z_{[t]} := \{(x_i, y_i)\}_{i=1}^{t+1}$ and

$$\mathcal{B}_t(z, z_{[t]}) := \sum_{i=1}^{t} \left\{ \frac{\gamma_i}{\eta_i} \left[V_X(x, x_i) - V_X(x, x_{i+1}) \right] + \frac{\gamma_i}{\tau_i} \left[V_Y(y, y_i) - V_Y(y, y_{i+1}) \right] \right\}. \qquad (4.4)$$

Proof. First of all, we explore the optimality conditions in iterations (2.5) and (2.6). Apply Lemma 2 in [11] to (2.5), we have

$$\langle -K\bar{x}_t, y_{t+1} - y \rangle + J(y_{t+1}) - J(y) \leq \frac{1}{\tau_t} V_Y(y, y_t) - \frac{1}{\tau_t} V_Y(y_{t+1}, y_t) - \frac{1}{\tau_t} V_Y(y, y_{t+1})$$
$$\leq \frac{1}{\tau_t} V_Y(y, y_t) - \frac{\alpha_Y}{2\tau_t} \|y_{t+1} - y_t\|^2 - \frac{1}{\tau_t} V_Y(y, y_{t+1}), \qquad (4.5)$$

where the last inequality follows from the fact that, by the strong convexity of $d_Y(\cdot)$ and (2.12),

$$V_Y(y_1, y_2) \geq \frac{\alpha_Y}{2} \|y_1 - y_2\|^2, \text{ for all } y_1, y_2 \in \mathcal{Y}. \qquad (4.6)$$

Similarly, from (2.6) we can derive that

$$\langle \nabla G(x_t^{md}), x_{t+1} - x \rangle + \langle x_{t+1} - x, K^T y_{t+1} \rangle \leq \frac{1}{\eta_t} V_X(x, x_t) - \frac{\alpha_X}{2\eta_t} \|x_{t+1} - x_t\|^2 - \frac{1}{\eta_t} V_X(x, x_{t+1}). \qquad (4.7)$$

Our next step is to establish a crucial recursion of Algorithm 2. It follows from (4.1), (4.5) and (4.7) that

$$\beta_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)Q(z_t^{ag}, z)$$
$$\leq \langle \nabla G(x_t^{md}), x_{t+1} - x \rangle + \frac{L_G}{2\beta_t} \|x_{t+1} - x_t\|^2 + [J(y_{t+1}) - J(y)] + \langle Kx_{t+1}, y \rangle - \langle Kx, y_{t+1} \rangle$$
$$\leq \frac{1}{\eta_t} V_X(x, x_t) - \frac{1}{\eta_t} V(x, x_{t+1}) - \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 \qquad (4.8)$$
$$\quad + \frac{1}{\tau_t} V_Y(y, y_t) - \frac{1}{\tau_t} V(y, y_{t+1}) - \frac{\alpha_Y}{2\tau_t} \|y_{t+1} - y_t\|^2$$
$$\quad - \langle x_{t+1} - x, K^T y_{t+1} \rangle + \langle K\bar{x}_t, y_{t+1} - y \rangle + \langle Kx_{t+1}, y \rangle - \langle Kx, y_{t+1} \rangle.$$

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Also observe that by (2.9), we have

$$- \langle x_{t+1} - x, K^T y_{t+1} \rangle + \langle K\bar{x}_t, y_{t+1} - y \rangle + \langle Kx_{t+1}, y \rangle - \langle Kx, y_{t+1} \rangle$$
$$= \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \theta_t \langle K(x_t - x_{t-1}), y - y_{t+1} \rangle$$
$$= \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \theta_t \langle K(x_t - x_{t-1}), y - y_t \rangle - \theta_t \langle K(x_t - x_{t-1}), y_t - y_{t+1} \rangle.$$

Multiplying both sides of (4.8) by γ_t, using the above identity and the fact that $\gamma_t \theta_t = \gamma_{t-1}$ due to (3.3), we obtain

$$\beta_t \gamma_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)\gamma_t Q(z_t^{ag}, z)$$
$$\leq \frac{\gamma_t}{\eta_t} V_X(x, x_t) - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} V_Y(y, y_t) - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1})$$
$$+ \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \gamma_{t-1} \langle K(x_t - x_{t-1}), y - y_t \rangle \qquad (4.9)$$
$$- \gamma_t \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 - \frac{\alpha_Y \gamma_t}{2\tau_t} \|y_{t+1} - y_t\|^2 - \gamma_{t-1} \langle K(x_t - x_{t-1}), y_t - y_{t+1} \rangle.$$

Now, applying Cauchy-Schwartz inequality to the last term in (4.9), using the notation $L_K = \|K\|$ and noticing that $\gamma_{t-1}/\gamma_t = \theta_t \leq \min\{\eta_{t-1}/\eta_t, \tau_{t-1}/\tau_t\}$ from (2.17), we have

$$- \gamma_{t-1} \langle K(x_t - x_{t-1}), y_t - y_{t+1} \rangle \leq \gamma_{t-1} \|K(x_t - x_{t-1})\|_* \|y_t - y_{t+1}\|$$
$$\leq L_K \gamma_{t-1} \|x_t - x_{t-1}\| \|y_t - y_{t+1}\| \leq \frac{L_K^2 \gamma_{t-1}^2 \tau_t}{2\alpha_Y \gamma_t} \|x_t - x_{t-1}\|^2 + \frac{\alpha_Y \gamma_t}{2\tau_t} \|y_t - y_{t+1}\|^2 \qquad (4.10)$$
$$\leq \frac{L_K^2 \gamma_{t-1} \tau_{t-1}}{2\alpha_Y} \|x_t - x_{t-1}\|^2 + \frac{\alpha_Y \gamma_t}{2\tau_t} \|y_t - y_{t+1}\|^2.$$

Noting that $\theta_{t+1} = \gamma_t/\gamma_{t+1}$, so by (2.16) we have $(\beta_{t+1} - 1)\gamma_{t+1} = \beta_t \gamma_t$. Combining the above two relations with inequality (4.9), we get the following recursion for Algorithm 2.

$$(\beta_{t+1} - 1)\gamma_{t+1} Q(z_{t+1}^{ag}, z) - (\beta_t - 1)\gamma_t Q(z_t^{ag}, z) = \beta_t \gamma_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)\gamma_t Q(z_t^{ag}, z)$$
$$\leq \frac{\gamma_t}{\eta_t} V_X(x, x_t) - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} V_Y(y, y_t) - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1})$$
$$+ \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \gamma_{t-1} \langle K(x_t - x_{t-1}), y - y_t \rangle$$
$$- \gamma_t \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 + \frac{L_K^2 \gamma_{t-1} \tau_{t-1}}{2\alpha_Y} \|x_t - x_{t-1}\|^2, \forall t \geq 1.$$

Applying the above inequality inductively and assuming that $x_0 = x_1$, we conclude that

$$(\beta_{t+1} - 1)\gamma_{t+1} Q(z_{t+1}^{ag}, z) - (\beta_1 - 1)\gamma_1 Q(z_1^{ag}, z) \leq \mathcal{B}_t(z, z_{[t]}) + \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle$$
$$- \gamma_t \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 - \sum_{i=1}^{t-1} \gamma_i \left(\frac{\alpha_X}{2\eta_i} - \frac{L_G}{2\beta_i} - \frac{L_K^2 \tau_i}{2\alpha_Y} \right) \|x_{i+1} - x_i\|^2,$$

which, in view of (2.18) and the facts that $\beta_1 = 1$ and $(\beta_{t+1} - 1)\gamma_{t+1} = \beta_t \gamma_t$ by (2.16), implies (4.3). □

We are now ready to prove Theorem 2.1, which follows as an immediate consequence of Lemma 4.2.

Proof of Theorem 2.1. Let $\mathcal{B}_t(z)$ be defined in (4.4). First note that by the definition of γ_t in (3.3) and relation (2.17), we have $\theta_t = \gamma_{t-1}/\gamma_t \leq \eta_{t-1}/\eta_t$ and hence $\gamma_{t-1}/\eta_{t-1} \leq \gamma_t/\eta_t$. Using this observation and

(2.15), we conclude that

$$\mathcal{B}_t(z, z_{[t]}) = \frac{\gamma_1}{\eta_1} V_X(x, x_1) - \sum_{i=1}^{t-1} \left(\frac{\gamma_i}{\eta_i} - \frac{\gamma_{i+1}}{\eta_{i+1}} \right) V_X(x, x_{i+1}) - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1})$$

$$+ \frac{\gamma_1}{\tau_1} V_Y(y, y_1) - \sum_{i=1}^{t-1} \left(\frac{\gamma_i}{\tau_i} - \frac{\gamma_{i+1}}{\tau_{i+1}} \right) V_Y(y, y_{i+1}) - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1})$$

$$\leq \frac{\gamma_1}{\eta_1} \Omega_X^2 - \sum_{i=1}^{t-1} \left(\frac{\gamma_i}{\eta_i} - \frac{\gamma_{i+1}}{\eta_{i+1}} \right) \Omega_X^2 - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) \qquad (4.11)$$

$$+ \frac{\gamma_1}{\tau_1} \Omega_Y^2 - \sum_{i=1}^{t-1} \left(\frac{\gamma_i}{\tau_i} - \frac{\gamma_{i+1}}{\tau_{i+1}} \right) \Omega_Y^2 - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1})$$

$$= \frac{\gamma_t}{\eta_t} \Omega_X^2 - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} \Omega_Y^2 - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1}).$$

Now applying Cauchy-Schwartz inequality to the inner product term in (4.3), we get

$$\gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle \leq L_K \gamma_t \|x_{t+1} - x_t\| \|y - y_{t+1}\| \leq \frac{L_K^2 \gamma_t \tau_t}{2\alpha_Y} \|x_{t+1} - x_t\|^2 + \frac{\alpha_Y \gamma_t}{2\tau_t} \|y - y_{t+1}\|^2. \qquad (4.12)$$

Using the above two relations, (2.18), (4.3) and (4.6), we have

$$\beta_t \gamma_t Q(z_{t+1}^{ag}, z) \leq \frac{\gamma_t}{\eta_t} \Omega_X^2 - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} \Omega_Y^2 - \frac{\gamma_t}{\tau_t} \left(V_Y(y, y_{t+1}) - \frac{\alpha_Y}{2} \|y - y_{t+1}\|^2 \right)$$

$$- \gamma_t \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} - \frac{L_K^2 \tau_t}{2\alpha_Y} \right) \|x_{t+1} - x_t\|^2 \leq \frac{\gamma_t}{\eta_t} \Omega_X^2 + \frac{\gamma_t}{\tau_t} \Omega_Y^2, \quad \forall z \in Z, \qquad (4.13)$$

which together with (2.14), then clearly imply (2.19). □

Our goal in the remaining part of this subsection is to prove Theorem 2.3, which summarizes the convergence properties of Algorithm 2 when X or Y is unbounded. We will first prove a technical result which specializes the results in Lemma 4.2 for the case when (2.16), (2.23) and (2.24) hold.

LEMMA 4.3. *Let* $\hat{z} = (\hat{x}, \hat{y}) \in Z$ *be a saddle point of* (1.1). *If* $V_X(x, x_t) = \|x - x_t\|^2/2$ *and* $V_Y(y, y_t) = \|y - y_t\|^2/2$ *in Algorithm 2, and the parameters* $\beta_t, \theta_t, \eta_t$ *and* τ_t *satisfy* (2.16), (2.23) *and* (2.24), *then*

(a).
$$\|\hat{x} - x_{t+1}\|^2 + \frac{\eta_t(1-p)}{\tau_t} \|\hat{y} - y_{t+1}\|^2 \leq \|\hat{x} - x_1\|^2 + \frac{\eta_t}{\tau_t} \|\hat{y} - y_1\|^2, \text{ for all } t \geq 1. \qquad (4.14)$$

(b).
$$\tilde{g}(z_{t+1}^{ag}, v_{t+1}) \leq \frac{1}{2\beta_t \eta_t} \|x_{t+1}^{ag} - x_1\|^2 + \frac{1}{2\beta_t \tau_t} \|y_{t+1}^{ag} - y_1\|^2 =: \delta_{t+1}, \text{ for all } t \geq 1, \qquad (4.15)$$

where $\tilde{g}(\cdot, \cdot)$ *is defined in* (2.22) *and*

$$v_{t+1} = \left(\frac{1}{\beta_t \eta_t} (x_1 - x_{t+1}), \frac{1}{\beta_t \tau_t} (y_1 - y_{t+1}) + \frac{1}{\beta_t} K(x_{t+1} - x_t) \right). \qquad (4.16)$$

Proof. It is easy to check that the conditions in Lemma 4.2 are satisfied. By (2.23), (4.3) in Lemma 4.2 becomes

$$\beta_t Q(z_{t+1}^{ag}, z) \leq \frac{1}{2\eta_t} \|x - x_1\|^2 - \frac{1}{2\eta_t} \|x - x_{t+1}\|^2 + \frac{1}{2\tau_t} \|y - y_1\|^2 - \frac{1}{2\tau_t} \|y - y_{t+1}\|^2$$

$$+ \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \left(\frac{1}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2. \qquad (4.17)$$

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To prove (4.14), *observe that*

$$\langle K(x_{t+1} - x_t), y - y_{t+1}\rangle \leq \frac{L_K^2 \tau_t}{2p}\|x_{t+1} - x_t\|^2 + \frac{p}{2\tau_t}\|y - y_{t+1}\|^2 \tag{4.18}$$

where p is the constant in (2.24). *By* (2.24) *and the above two inequalities, we get*

$$\beta_t Q(z_{t+1}^{ag}, z) \leq \frac{1}{2\eta_t}\|x - x_1\|^2 - \frac{1}{2\eta_t}\|x - x_{t+1}\|^2 + \frac{1}{2\tau_t}\|y - y_1\|^2 - \frac{1-p}{2\tau_t}\|y - y_{t+1}\|^2.$$

Letting $z = \hat{z}$ in the above, and using the fact that $Q(z_{t+1}^{ag}, \hat{z}) \geq 0$, we obtain (4.14).

Now we prove (4.15). *Noting that*

$$\begin{aligned}
\|x - x_1\|^2 - \|x - x_{t+1}\|^2 &= 2\langle x_{t+1} - x_1, x\rangle + \|x_1\|^2 - \|x_{t+1}\|^2 \\
&= 2\langle x_{t+1} - x_1, x - x_{t+1}^{ag}\rangle + 2\langle x_{t+1} - x_1, x_{t+1}^{ag}\rangle + \|x_1\|^2 - \|x_{t+1}\|^2 \\
&= 2\langle x_{t+1} - x_1, x - x_{t+1}^{ag}\rangle + \|x_{t+1}^{ag} - x_1\|^2 - \|x_{t+1}^{ag} - x_{t+1}\|^2,
\end{aligned} \tag{4.19}$$

we conclude from (2.24) *and* (4.17) *that for any $z \in Z$,*

$$\beta_t Q(z_{t+1}^{ag}, z) - \langle K(x_{t+1} - x_t), y_{t+1}^{ag} - y\rangle - \frac{1}{\eta_t}\langle x_1 - x_{t+1}, x_{t+1}^{ag} - x\rangle - \frac{1}{\tau_t}\langle y_1 - y_{t+1}, y_{t+1}^{ag} - y\rangle$$

$$\leq \frac{1}{2\eta_t}\left(\|x_{t+1}^{ag} - x_1\|^2 - \|x_{t+1}^{ag} - x_{t+1}\|^2\right) + \frac{1}{2\tau_t}\left(\|y_{t+1}^{ag} - y_1\|^2 - \|y_{t+1}^{ag} - y_{t+1}\|^2\right)$$

$$\quad + \langle K(x_{t+1} - x_t), y_{t+1}^{ag} - y_{t+1}\rangle - \left(\frac{1}{2\eta_t} - \frac{L_G}{2\beta_t}\right)\|x_{t+1} - x_t\|^2$$

$$\leq \frac{1}{2\eta_t}\left(\|x_{t+1}^{ag} - x_1\|^2 - \|x_{t+1}^{ag} - x_{t+1}\|^2\right) + \frac{1}{2\tau_t}\left(\|y_{t+1}^{ag} - y_1\|^2 - \|y_{t+1}^{ag} - y_{t+1}\|^2\right)$$

$$\quad \frac{p}{2\tau_t}\|y_{t+1}^{ag} - y_{t+1}\|^2 - \left(\frac{1}{2\eta_t} - \frac{L_G}{2\beta_t} - \frac{L_K^2\tau_t}{2p}\right)\|x_{t+1} - x_t\|^2$$

$$\leq \frac{1}{2\eta_t}\|x_{t+1}^{ag} - x_1\|^2 + \frac{1}{2\tau_t}\|y_{t+1}^{ag} - y_1\|^2.$$

The result in (4.15) *and* (4.16) *immediately follows from the above inequality and* (2.22).
□

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. We have established the expression of v_{t+1} and δ_{t+1} in Lemma 4.3. It suffices to estimate the bound on $\|v_{t+1}\|$ and δ_{t+1}. It follows from the definition of D, (2.23) and (4.14) that for all $t \geq 1$,

$$\|\hat{x} - x_{t+1}\| \leq D \text{ and } \|\hat{y} - y_{t+1}\| \leq D\sqrt{\frac{\tau_1}{\eta_1(1-p)}}.$$

Now by (4.16), we have

$$\begin{aligned}
\|v_{t+1}\| &\leq \frac{1}{\beta_t\eta_t}\|x_1 - x_{t+1}\| + \frac{1}{\beta_t\tau_t}\|y_1 - y_{t+1}\| + \frac{L_K}{\beta_t}\|x_{t+1} - x_t\| \\
&\leq \frac{1}{\beta_t\eta_t}\left(\|\hat{x} - x_1\| + \|\hat{x} - x_{t+1}\|\right) + \frac{1}{\beta_t\tau_t}\left(\|\hat{y} - y_1\| + \|\hat{y} - y_{t+1}\|\right) + \frac{L_K}{\beta_t}\left(\|\hat{x} - x_{t+1}\| + \|\hat{x} - x_t\|\right) \\
&\leq \frac{1}{\beta_t\eta_t}\left(\|\hat{x} - x_1\| + D\right) + \frac{1}{\beta_t\tau_t}\left(\|\hat{y} - y_1\| + D\sqrt{\frac{\tau_1}{\eta_1(1-p)}}\right) + \frac{2L_K}{\beta_t}D \\
&= \frac{1}{\beta_t\eta_t}\|\hat{x} - x_1\| + \frac{1}{\beta_t\tau_t}\|\hat{y} - y_1\| + D\left[\frac{1}{\beta_t\eta_t}\left(1 + \sqrt{\frac{\eta_1}{\tau_1(1-p)}}\right) + \frac{2L_K}{\beta_t}\right].
\end{aligned}$$

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To estimate the bound of δ_{t+1}, consider the sequence $\{\gamma_t\}$ defined in (3.3). Using the fact that $(\beta_{t+1}-1)\gamma_{t+1} = \beta_t\gamma_t$ due to (2.16) and (3.3), and applying (2.7) and (2.8) inductively, we have

$$x_{t+1}^{ag} = \frac{1}{\beta_t\gamma_t}\sum_{i=1}^{t}\gamma_i x_{t+1}, \ \ y_{t+1}^{ag} = \frac{1}{\beta_t\gamma_t}\sum_{i=1}^{t}\gamma_i y_{t+1} \text{ and } \ \ \frac{1}{\beta_t\gamma_t}\sum_{i=1}^{t}\gamma_i = 1. \tag{4.20}$$

Thus x_{t+1}^{ag} and y_{t+1}^{ag} are convex combinations of sequences $\{x_{i+1}\}_{i=1}^{t}$ and $\{y_{i+1}\}_{i=1}^{t}$. Using these relations and (4.14), we have

$$\begin{aligned}
\delta_{t+1} &= \frac{1}{2\beta_t\eta_t}\|x_{t+1}^{ag} - x_1\|^2 + \frac{1}{2\beta_t\tau_t}\|y_{t+1}^{ag} - y_1\|^2 \\
&\leq \frac{1}{\beta_t\eta_t}\left(\|\hat{x} - x_{t+1}^{ag}\|^2 + \|\hat{x} - x_1\|^2\right) + \frac{1}{\beta_t\tau_t}\left(\|\hat{y} - y_{t+1}^{ag}\|^2 + \|\hat{y} - y_1\|^2\right) \\
&= \frac{1}{\beta_t\eta_t}\left(D^2 + \|\hat{x} - x_{t+1}^{ag}\|^2 + \frac{\eta_t(1-p)}{\tau_t}\|\hat{y} - y_{t+1}^{ag}\|^2 + \frac{\eta_t p}{\tau_t}\|\hat{y} - y_{t+1}^{ag}\|^2\right) \\
&\leq \frac{1}{\beta_t\eta_t}\left[D^2 + \frac{1}{\beta_t\gamma_t}\sum_{i=1}^{t}\gamma_i\left(\|\hat{x} - x_{i+1}\|^2 + \frac{\eta_t(1-p)}{\tau_t}\|\hat{y} - y_{i+1}\|^2 + \frac{\eta_t p}{\tau_t}\|\hat{y} - y_{i+1}\|^2\right)\right] \\
&\leq \frac{1}{\beta_t\eta_t}\left[D^2 + \frac{1}{\beta_t\gamma_t}\sum_{i=1}^{t}\gamma_i\left(D^2 + \frac{\eta_t p}{\tau_t}\cdot\frac{\tau_1}{\eta_1(1-p)}D^2\right)\right] = \frac{(2-p)D^2}{\beta_t\eta_t(1-p)}.
\end{aligned}$$

□

4.2. Convergence analysis for the stochastic APD algorithm. In this subsection, we prove Theorems 3.1 and 3.3 which descirbe the convergence properties of the stochastic APD algorithm presented in Section 3.

Let $\hat{\mathcal{G}}(x_i^{md})$, $\hat{\mathcal{K}}_x(\bar{x}_i)$ and $\hat{\mathcal{K}}_y(y_{i+1})$ be the output from the $\mathcal{SO}$ at the t-th iteration of Algorithm 3. Throughout this subsection, we denote

$$\Delta_{x,G}^t := \hat{\mathcal{G}}(x_i^{md}) - \nabla G(x_t^{md}), \ \ \Delta_{x,K}^t := \hat{\mathcal{K}}_y(y_{i+1}) - K^T y_{t+1}, \ \ \Delta_y^t := -\hat{\mathcal{K}}_x(\bar{x}_i) + K\bar{x}_t,$$

$$\Delta_x^t := \Delta_{x,G}^t + \Delta_{x,K}^t \ \ \text{and} \ \ \Delta^t := (\Delta_x^t, \Delta_y^t).$$

Moreover, for a given $z = (x,y) \in Z$, let us denote $\|z\|^2 = \|x\|^2 + \|y\|^2$ and its associate dual norm for $\Delta = (\Delta_x, \Delta_y)$ by $\|\Delta\|_*^2 = \|\Delta_x\|_*^2 + \|\Delta_y\|_*^2$. We also define the Bregman divergence $V(z, \tilde{z}) := V_X(x, \tilde{x}) + V_Y(y, \tilde{y})$ for $z = (x, y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$.

Before proving Theorem 3.1, we first estimate a bound on $Q(z_{t+1}^{ag}, z)$ for all $z \in Z$. This result is analogous to Lemma 4.2 for the deterministic APD method.

LEMMA 4.4. *Let $z_t^{ag} = (x_t^{ag}, y_t^{ag})$ be the iterates generated by Algorithm 3. Assume that the parameters $\beta_t, \theta_t, \eta_t$ and τ_t satisfy (2.16), (2.17) and (3.4). Then, for any $z \in Z$, we have*

$$\beta_t\gamma_t Q(z_{t+1}^{ag}, z) \leq \mathcal{B}_t(z, z_{[t]}) + \gamma_t\langle K(x_{t+1} - x_t), y - y_{t+1}\rangle - \gamma_t\left(\frac{q\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t}\right)\|x_{t+1} - x_t\|^2 + \sum_{i=1}^{t}\Lambda_i(z), \tag{4.21}$$

where γ_t and $\mathcal{B}_t(z, z_{[t]})$, respectively, are defined in (3.3) and (4.4), $z_{[t]} = \{(x_i, y_i)\}_{i=1}^{t+1}$ and

$$\Lambda_i(z) := -\frac{(1-q)\alpha_X\gamma_i}{2\eta_i}\|x_{i+1} - x_i\|^2 - \frac{(1-p)\alpha_Y\gamma_i}{2\tau_i}\|y_{i+1} - y_i\|^2 - \gamma_i\langle\Delta^i, z_{i+1} - z\rangle. \tag{4.22}$$

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Proof. Similar to (4.5) *and* (4.7)*, we conclude from the optimality conditions of* (3.1) *and* (3.2) *that*

$$\langle -\hat{\mathcal{K}}_x(\bar{x}_t), y_{t+1} - y \rangle + J(y_{t+1}) - J(\hat{y}) \leq \frac{1}{\tau_t} V_Y(\hat{y}, y_t) - \frac{\alpha_Y}{2\tau_t} \|y_{t+1} - y_t\|^2 - \frac{1}{\tau_t} V_Y(\hat{y}, y_{t+1}),$$

$$\langle \hat{\mathcal{G}}(x_t^{md}), x_{t+1} - x \rangle + \langle x_{t+1} - x, \hat{\mathcal{K}}_y(y_{t+1}) \rangle \leq \frac{1}{\eta_t} V_X(x, x_t) - \frac{\alpha_X}{2\eta_t} \|x_{t+1} - x_t\|^2 - \frac{1}{\eta_t} V_X(x, x_{t+1}).$$

Now we establish an important recursion for Algorithm 3. Observing that Proposition 4.1 also holds for Algorithm 3, and applying the above two inequalities to (4.1) *in Proposition 4.1, similar to* (4.9)*, we have*

$$\begin{aligned}
&\beta_t \gamma_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)\gamma_t Q(z_t^{ag}, z) \\
&\leq \frac{\gamma_t}{\eta_t} V_X(x, x_t) - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} V_Y(y, y_t) - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1}) \\
&\quad + \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \gamma_{t-1} \langle K(x_t - x_{t-1}), y - y_t \rangle \\
&\quad - \gamma_t \left(\frac{\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 - \frac{\alpha_Y \gamma_t}{2\tau_t} \|y_{t+1} - y_t\|^2 - \gamma_{t-1} \langle K(x_t - x_{t-1}), y_t - y_{t+1} \rangle \\
&\quad - \gamma_t \langle \Delta_{x,G}^i + \Delta_{x,K}^i, x_{t+1} - x \rangle - \gamma_t \langle \Delta_y^i, y_{t+1} - y \rangle, \quad \forall z \in Z.
\end{aligned} \tag{4.23}$$

By Cauchy-Schwartz inequality and (2.17)*, for all* $p \in (0, 1)$*,*

$$\begin{aligned}
&- \gamma_{t-1} \langle K(x_t - x_{t-1}), y_t - y_{t+1} \rangle \leq \gamma_{t-1} \|K(x_t - x_{t-1})\|_* \|y_t - y_{t+1}\| \\
&\leq L_K \gamma_{t-1} \|x_t - x_{t-1}\| \|y_t - y_{t+1}\| \leq \frac{L_K^2 \gamma_{t-1}^2 \tau_t}{2p\alpha_Y \gamma_t} \|x_t - x_{t-1}\|^2 + \frac{p\alpha_Y \gamma_t}{2\tau_t} \|y_t - y_{t+1}\|^2 \\
&\leq \frac{L_K^2 \gamma_{t-1} \tau_{t-1}}{2p\alpha_Y} \|x_t - x_{t-1}\|^2 + \frac{p\alpha_Y \gamma_t}{2\tau_t} \|y_t - y_{t+1}\|^2.
\end{aligned} \tag{4.24}$$

By (2.16)*,* (4.22)*,* (4.23) *and* (4.24)*, we can develop the following recursion for Algorithm 3:*

$$\begin{aligned}
&(\beta_{t+1} - 1)\gamma_{t+1} Q(z_{t+1}^{ag}, z) - (\beta_t - 1)\gamma_t Q(z_t^{ag}, z) = \beta_t \gamma_t Q(z_{t+1}^{ag}, z) - (\beta_t - 1)\gamma_t Q(z_t^{ag}, z) \\
&\leq \frac{\gamma_t}{\eta_t} V_X(x, x_t) - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} V_Y(y, y_t) - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1}) \\
&\quad + \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \gamma_{t-1} \langle K(x_t - x_{t-1}), y - y_t \rangle \\
&\quad - \gamma_t \left(\frac{q\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 + \frac{L_K^2 \gamma_{t-1} \tau_{t-1}}{2p\alpha_Y} \|x_t - x_{t-1}\|^2 + \Lambda_t(x), \quad \forall z \in Z.
\end{aligned}$$

Applying the above inequality inductively and assuming that $x_0 = x_1$*, we obtain*

$$\begin{aligned}
&(\beta_{t+1} - 1)\gamma_{t+1} Q(z_{t+1}^{ag}, z) - (\beta_1 - 1)\gamma_1 Q(z_1^{ag}, z) \\
&\leq \mathcal{B}_t(z, z_{[t]}) + \gamma_t \langle K(x_{t+1} - x_t), y - y_{t+1} \rangle - \gamma_t \left(\frac{q\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} \right) \|x_{t+1} - x_t\|^2 \\
&\quad - \sum_{i=1}^{t-1} \gamma_i \left(\frac{q\alpha_X}{2\eta_i} - \frac{L_G}{2\beta_i} - \frac{L_K^2 \tau_i}{2p\alpha_Y} \right) \|x_{i+1} - x_i\|^2 + \sum_{i=1}^{t} \Lambda_i(x), \quad \forall z \in Z.
\end{aligned} \tag{4.25}$$

Relation (4.21) *then follows immediately from the above inequality,* (2.16) *and* (3.4)*.* □

We also need the following technical result whose proof is based on Lemma 2.1 of [29].

LEMMA 4.5. *Let* η_i, τ_i *and* γ_i*,* $i = 1, 2, \ldots$*, be given positive constants. For any* $z_1 \in Z$*, if we define* $z_1^v = z_1$ *and*

$$z_{i+1}^v = \underset{z=(x,y)\in Z}{\operatorname{argmin}} \left\{ -\eta_i \langle \Delta_x^i, x \rangle - \tau_i \langle \Delta_y^i, y \rangle + V(z, z_i^v) \right\}, \tag{4.26}$$

then

$$\sum_{i=1}^{t} \gamma_i \langle -\Delta_i, z_i^v - z \rangle \leq \mathcal{B}_t(z, z_{[t]}^v) + \sum_{i=1}^{t} \frac{\eta_i \gamma_i}{2\alpha_X} \|\Delta_x^i\|_*^2 + \sum_{i=1}^{t} \frac{\tau_i \gamma_i}{2\alpha_Y} \|\Delta_y^i\|_*^2, \tag{4.27}$$

where $z_{[t]}^v := \{z_i^v\}_{i=1}^{t}$ and $\mathcal{B}_t(z, z_{[t]}^v)$ is defined in (4.4).

Proof. Noting that (4.26) implies $z_{i+1}^v = (x_{i+1}^v, y_{i+1}^v)$ where $x_{i+1}^v = \operatorname*{argmin}_{x = \in X} \left\{ -\eta_i \langle \Delta_x^i, x \rangle + V_X(x, x_i^v) \right\}$ and
$y_{i+1}^v = \operatorname*{argmin}_{y \in Y} \left\{ -\tau_i \langle \Delta_y^i, y \rangle + V(y, y_i^v) \right\}$, from Lemma 2.1 of [29] we have

$$V_X(x, x_{i+1}^v) \leq V_X(x, x_i^v) - \eta_i \langle \Delta_x^i, x - x_i \rangle + \frac{\eta_i^2 \|\Delta_x^i\|_*^2}{2\alpha_X},$$

$$V_Y(y, y_{i+1}^v) \leq V_Y(y, y_i^v) - \tau_i \langle \Delta_y^i, y - y_i \rangle + \frac{\tau_i^2 \|\Delta_y^i\|_*^2}{2\alpha_Y},$$

for all $i \geq 1$. Thus

$$\frac{\gamma_i}{\eta_i} V_X(x, x_{i+1}^v) \leq \frac{\gamma_i}{\eta_i} V_X(x, x_i^v) - \gamma_i \langle \Delta_x^i, x - x_i \rangle + \frac{\gamma_i \eta_i \|\Delta_x^i\|_*^2}{2\alpha_X},$$

$$\frac{\gamma_i}{\eta_i} V_Y(y, y_{i+1}^v) \leq \frac{\gamma_i}{\eta_i} V_Y(y, y_i^v) - \gamma_i \langle \Delta_y^i, y - y_i \rangle + \frac{\gamma_i \tau_i \|\Delta_y^i\|_*^2}{2\alpha_Y}.$$

Adding the above two inequalities together, and summing up them from $i = 1$ to t we get

$$0 \leq \mathcal{B}_t(z, z_{[t]}^v) - \gamma_i \langle \Delta^i, z - z_i \rangle + \frac{\gamma_i \eta_i \|\Delta_x^i\|_*^2}{2\alpha_X} + \frac{\gamma_i \tau_i \|\Delta_y^i\|_*^2}{2\alpha_Y},$$

so (4.27) holds. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1 Firstly, applying the bounds in (4.11) and (4.12) to (4.21), we get

$$\beta_t \gamma_t Q(z_{t+1}^{ag}, z) \leq \frac{\gamma_t}{\eta_t} \Omega_X^2 - \frac{\gamma_t}{\eta_t} V_X(x, x_{t+1}) + \frac{\gamma_t}{\tau_t} \Omega_Y^2 - \frac{\gamma_t}{\tau_t} V_Y(y, y_{t+1}) + \frac{\alpha_Y \gamma_t}{2\tau_t} \|y - y_{t+1}\|^2$$

$$- \gamma_t \left(\frac{q\alpha_X}{2\eta_t} - \frac{L_G}{2\beta_t} - \frac{L_K^2 \tau_t}{2\alpha_Y} \right) \|x_{t+1} - x_t\|^2 + \sum_{i=1}^{t} \Lambda_i(z) \tag{4.28}$$

$$\leq \frac{\gamma_t}{\eta_t} \Omega_X^2 + \frac{\gamma_t}{\tau_t} \Omega_Y^2 + \sum_{i=1}^{t} \Lambda_i(z), \quad \forall z \in Z.$$

By (4.22), we have

$$\Lambda_i(z) = -\frac{(1-q)\alpha_X \gamma_i}{2\eta_i} \|x_{i+1} - x_i\|^2 - \frac{(1-p)\alpha_Y \gamma_i}{2\tau_i} \|y_{i+1} - y_i\|^2 + \gamma_i \langle \Delta^i, z - z_{i+1} \rangle$$

$$= -\frac{(1-q)\alpha_X \gamma_i}{2\eta_i} \|x_{i+1} - x_i\|^2 - \frac{(1-p)\alpha_Y \gamma_i}{2\tau_i} \|y_{i+1} - y_i\|^2 + \gamma_i \langle \Delta^i, z_i - z_{i+1} \rangle + \gamma_i \langle \Delta^i, z - z_i \rangle \tag{4.29}$$

$$\leq \frac{\eta_i \gamma_i}{2(1-q)\alpha_X} \|\Delta_x^i\|_*^2 + \frac{\tau_i \gamma_i}{2(1-p)\alpha_Y} \|\Delta_y^i\|_*^2 + \gamma_i \langle \Delta^i, z - z_i \rangle,$$

where the last relation follows from Young's inequality. For all $i \geq 1$, letting $z_1^v = z_1$, and z_{i+1}^v as in (4.26),

we conclude from (4.29) and Lemma 4.5 that, $\forall z \in Z$,

$$\sum_{i=1}^{t} \Lambda_i(z) \leq \sum_{i=1}^{t} \left\{ \frac{\eta_i \gamma_i}{2(1-q)\alpha_X} \|\Delta_x^i\|_*^2 + \frac{\tau_i \gamma_i}{2(1-p)\alpha_Y} \|\Delta_y^i\|_*^2 + \gamma_i \langle \Delta^i, z_i^v - z_i \rangle + \gamma_i \langle -\Delta^i, z_i^v - z \rangle \right\}$$

$$\leq \mathcal{B}_t(z, z_{[t]}^v) + \frac{1}{2} \underbrace{\sum_{i=1}^{t} \left\{ \frac{(2-q)\eta_i \gamma_i}{(1-q)\alpha_X} \|\Delta_x^i\|_*^2 + \frac{(2-p)\tau_i \gamma_i}{(1-p)\alpha_Y} \|\Delta_y^i\|_*^2 + \gamma_i \langle \Delta^i, z_i^v - z_i \rangle \right\}}_{U_t},$$

(4.30)

where similar to (4.11) we have $\mathcal{B}_t(z, z_{[t]}^v) \leq \Omega_X^2 \gamma_t / \eta_t + \Omega_Y^2 \gamma_t / \tau_t$. Using the above inequality, (2.14), (2.15) and (4.28), we obtain

$$\beta_t \gamma_t g(z_{t+1}^{ag}) \leq \frac{2\gamma_t}{\eta_t} \Omega_X^2 + \frac{2\gamma_t}{\tau_t} \Omega_Y^2 + U_t.$$

(4.31)

Now it suffices to bound the above quantity U_t, both in expectation (part a)) and in probability (part b)).

We first show part a). Note that by our assumptions on $\mathcal{SO}$, at iteration i of Algorithm 3, the random noises Δ^i are independent of z_i and hence $\mathbb{E}[\langle \Delta^i, x - x_i \rangle] = 0$. In addition, Assumption **A1** implies that $\mathbb{E}[\|\Delta_x^i\|_*^2] \leq \sigma_{x,G}^2 + \sigma_{x,K}^2 = \sigma_x^2$ (noting that $\Delta_{x,G}^i$ and $\Delta_{x,K}^i$ are indepdent at iteration i), and $\mathbb{E}[\|\Delta_y^i\|_*^2] \leq \sigma_y^2$. Therefore,

$$\mathbb{E}[U_t] \leq \frac{1}{2} \sum_{i=1}^{t} \left\{ \frac{(2-q)\eta_i \gamma_i \sigma_x^2}{(1-q)\alpha_X} + \frac{(2-p)\tau_i \gamma_i \sigma_y^2}{(1-p)\alpha_Y} \right\}.$$

(4.32)

Taking expectation on both sides of (4.31) and using the above inequality, we obtain (3.5).

We now show that part b) holds. Note that by our assumptions on $\mathcal{SO}$ and the definition of z_i^v, the sequences $\{\langle \Delta_{x,G}^i, x_i^v - x_i \rangle\}_{i \geq 1}$ is a martingale-difference sequence. By the well-known large-deviation theorem for matrigale-difference sequence (e.g., Lemma 2 of [22]), and the fact that

$$\mathbb{E}[\exp\{\alpha_Y \gamma_i^2 \langle \Delta_{x,G}^i, x_i^v - x_i \rangle^2 / (2\gamma_i^2 \Omega_Y^2 \sigma_{x,G}^2)\}] \leq \mathbb{E}[\exp\{\alpha_Y \|\Delta_{x,G}^i\|_*^2 \|x_i^v - x_i\|^2 / (2\Omega_Y^2 \sigma_{x,G}^2)\}]$$

$$\leq \mathbb{E}[\exp\{\|\Delta_{x,G}^i\|_*^2 V(x_i^v, x_i) / (\Omega_Y^2 \sigma_{x,G}^2)\}] \leq \mathbb{E}[\exp\{\|\Delta_{x,G}^i\|_*^2 / \sigma_{x,G}^2\}] \leq \exp\{1\},$$

we conclude that

$$\text{Prob}\left\{\sum_{i=1}^{t} \gamma_i \langle \Delta_{x,G}^i, x_i^v - x_i \rangle > \lambda \cdot \sigma_{x,G} \Omega_X \sqrt{\frac{2}{\alpha_X} \sum_{i=1}^{t} \gamma_i^2}\right\} \leq \exp\{-\lambda^2/3\}, \forall \lambda > 0.$$

By using a similar argument, we can show that, $\forall \lambda > 0$,

$$\text{Prob}\left\{\sum_{i=1}^{t} \gamma_i \langle \Delta_y^i, y_i^v - y_i \rangle > \lambda \cdot \sigma_y \Omega_Y \sqrt{\frac{2}{\alpha_Y} \sum_{i=1}^{t} \gamma_i^2}\right\} \leq \exp\{-\lambda^2/3\},$$

$$\text{Prob}\left\{\sum_{i=1}^{t} \gamma_i \langle \Delta_{x,K}^i, x - x_i \rangle > \lambda \cdot \sigma_{x,K} \Omega_X \sqrt{\frac{2}{\alpha_X} \sum_{i=1}^{t} \gamma_i^2}\right\} \leq \exp\{-\lambda^2/3\}.$$

Using the previous three inequalities and the fact that $\sigma_{x,G} + \sigma_{x,K} \leq \sqrt{2}\sigma_x$, we have, $\forall \lambda > 0$,

$$\text{Prob}\left\{\sum_{i=1}^{t} \gamma_i \langle \Delta^i, z_i^v - z_i \rangle > \lambda \left[\frac{\sqrt{2}\sigma_x \Omega_X}{\sqrt{\alpha_X}} + \frac{\sigma_y \Omega_Y}{\sqrt{\alpha_Y}}\right] \sqrt{2\sum_{i=1}^{t} \gamma_i^2}\right\} \leq$$

$$\text{Prob}\left\{\sum_{i=1}^{t} \gamma_i \langle \Delta^i, z_i^v - z_i \rangle > \lambda \left[\frac{(\sigma_{x,G} + \sigma_{x,K})\Omega_X}{\sqrt{\alpha_X}} + \frac{\sigma_y \Omega_Y}{\sqrt{\alpha_Y}}\right] \sqrt{2\sum_{i=1}^{t} \gamma_i^2}\right\} \leq 3\exp\{-\lambda^2/3\}.$$

(4.33)

Now let $S_i := (2-q)\eta_i \gamma_i / [(1-q)\alpha_X]$ and $S := \sum_{i=1}^{t} S_i$. By the convexity of exponential function, we have

$$\mathbb{E}\left[\exp\left\{\frac{1}{S} \sum_{i=1}^{t} S_i \|\Delta_{x,G}^i\|_*^2 / \sigma_{x,G}^2\right\}\right] \leq \mathbb{E}\left[\frac{1}{S} \sum_{i=1}^{t} S_i \exp\left\{\|\Delta_{x,G}^i\|_*^2 / \sigma_{x,G}^2\right\}\right] \leq \exp\{1\}.$$

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where the last inequality follows from Assumption **A2**. Therefore, by Markov's inequality, for all $\lambda > 0$,

$$\text{Prob}\left\{\sum_{i=1}^{t}\frac{(2-q)\eta_i\gamma_i}{(1-q)\alpha_X}\|\Delta_{x,G}^i\|_*^2 > (1+\lambda)\sigma_{x,G}^2\sum_{i=1}^{t}\frac{(2-q)\eta_i\gamma_i}{(1-q)\alpha_X}\right\}$$
$$= \text{Prob}\left\{\exp\left\{\frac{1}{S}\sum_{i=1}^{t}S_i\|\Delta_y^i\|^2/\sigma_y^2\right\} \geq \exp\{1+\lambda\}\right\} \leq \exp\{-\lambda\}.$$

Using an similar argument, we can show that

$$\text{Prob}\left\{\sum_{i=1}^{t}\frac{(2-q)\eta_i\gamma_i}{(1-q)\alpha_X}\|\Delta_{x,K}^i\|_*^2 > (1+\lambda)\sigma_{x,K}^2\sum_{i=1}^{t}\frac{(2-q)\eta_i\gamma_i}{(1-q)\alpha_X}\right\} \leq \exp\{-\lambda\},$$
$$\text{Prob}\left\{\sum_{i=1}^{t}\frac{(2-p)\tau_i\gamma_i}{(1-p)\alpha_Y}\|\Delta_y^i\|_*^2 > (1+\lambda)\sigma_y^2\sum_{i=1}^{t}\frac{(2-p)\tau_i\gamma_i}{(1-p)\alpha_Y}\right\} \leq \exp\{-\lambda\}.$$

Combining the previous three inequalities, we obtain

$$\text{Prob}\left\{\sum_{i=1}^{t}\frac{(2-q)\eta_i\gamma_i}{(1-q)\alpha_X}\|\Delta_x^i\|_*^2 + \sum_{i=1}^{t}\frac{(2-p)\tau_i\gamma_i}{(1-p)\alpha_Y}\|\Delta_y^i\|_*^2 > \right.$$
$$\left.(1+\lambda)\left[\sigma_x^2\sum_{i=1}^{t}\frac{(2-q)\eta_i\gamma_i}{(1-q)\alpha_X} + \sigma_y^2\sum_{i=1}^{t}\frac{(2-p)\tau_i\gamma_i}{(1-p)\alpha_Y}\right]\right\} \leq 3\exp\{-\lambda\}, \tag{4.34}$$

Our result now follows directly from (4.30), (4.31), (4.33) and (4.34). $\square$

In the remaining part of this subsection, our goal is to prove Theorem 3.3, which describes the convergence rate of Algorithm 3 when X and Y are both unbounded. Similar as proving Theorem 2.3, first we specialize the result of Lemma 4.4 under (2.16), (2.23) and (3.4). The following lemma is analogous to Lemma 4.3.

LEMMA 4.6. *Let $\hat{z} = (\hat{x}, \hat{y}) \in Z$ be a saddle point of (1.1). If $V_X(x, x_t) = \|x - x_t\|^2/2$ and $V_Y(y, y_t) = \|y - y_t\|^2/2$ in Algorithm 3, and the parameters $\beta_t, \theta_t, \eta_t$ and τ_t satisfy (2.16), (2.23) and (3.4), then*
(a).
$$\|\hat{x} - x_{t+1}\|^2 + \|\hat{x} - x_{t+1}^v\|^2 + \frac{\eta_t(1-p)}{\tau_t}\|\hat{y} - y_{t+1}\|^2 + \frac{\eta_t}{\tau_t}\|\hat{y} - y_{t+1}^v\|^2$$
$$\leq 2\|\hat{x} - x_1\|^2 + \frac{2\eta_t}{\tau_t}\|\hat{y} - y_1\|^2 + \frac{2\eta_t}{\gamma_t}U_t, \text{ for all } t \geq 1, \tag{4.35}$$

where (x_{t+1}^v, y_{t+1}^v) and U_t are defined in (4.26) and (4.30), respectively.
(b).
$$\tilde{g}(z_{t+1}^{ag}, v_{t+1}) \leq \frac{1}{\beta_t\eta_t}\|x_{t+1}^{ag} - x_1\|^2 + \frac{1}{\beta_t\tau_t}\|y_{t+1}^{ag} - y_1\|^2 + \frac{1}{\beta_t\gamma_t}U_t =: \delta_{t+1}, \text{ for all } t \geq 1, \tag{4.36}$$

where $\tilde{g}(\cdot, \cdot)$ is defined in (2.22) and
$$v_{t+1} = \left(\frac{1}{\beta_t\eta_t}(2x_1 - x_{t+1} - x_{t+1}^v), \frac{1}{\beta_t\tau_t}(2y_1 - y_{t+1} - y_{t+1}^v) + \frac{1}{\beta_t}K(x_{t+1} - x_t)\right). \tag{4.37}$$

Proof. Apply (3.4), (4.18) and (4.30) to (4.21) in Lemma 4.4, we get

$$\beta_t\gamma_t Q(z_{t+1}^{ag}, z) \leq \bar{\mathcal{B}}(z, z_t) + \frac{p\gamma_t}{2\tau_t}\|y - y_t\|^2 + \bar{\mathcal{B}}(z, z_t^v) + U_t,$$

where $\bar{\mathcal{B}}(\cdot, \cdot)$ is defined as

$$\bar{\mathcal{B}}(z, \tilde{z}) := \frac{\gamma_t}{2\eta_t}\|x - x_1\|^2 - \frac{\gamma_t}{2\eta_t}\|x - \tilde{x}\|^2 + \frac{\gamma_t}{2\tau_t}\|y - y_1\|^2 - \frac{\gamma_t}{2\tau_t}\|y - \tilde{y}\|^2.$$

thanks to (2.23). Now letting $z = \hat{z}$, and noting that $Q(z_{t+1}^{ag}, \hat{z}) \geq 0$, we get (4.35).
On the other hand, if we only apply (3.4) and (4.30) to (4.21) in Lemma 4.4, then we get

$$\beta_t\gamma_t Q(z_{t+1}^{ag}, z) \leq \bar{\mathcal{B}}(z, z_t) + \gamma_t\langle K(x_{t+1} - x_t), y - y_{t+1}\rangle + \bar{\mathcal{B}}(z, z_t^v) + U_t.$$

Apply (2.23) and (4.19) to $\mathcal{B}(z, z_t)$ and $\bar{\mathcal{B}}(z, z_t^v)$ in the above inequality, we get (4.36). $\square$

With the help of Lemma 4.6, we are ready to prove Theorem 3.3.

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Proof of Theorem 3.3 Let δ_{t+1} and v_{t+1} be defined in (4.36) and (4.37), respectively. Also let C and D, respectively, be defined in (3.15) and (2.27). It suffices to estimate $\mathbb{E}[\|v_{t+1}\|]$ and $\mathbb{E}[\delta_{t+1}]$. First it follows from (2.23), (3.15) and (4.32) that

$$\mathbb{E}[U_t] \leq \frac{\gamma_t}{2\eta_t} C^2. \tag{4.38}$$

Using the above inequality, (2.23), (2.27) and (4.35), we have

$$\mathbb{E}[\|\hat{x} - x_{t+1}\|^2] \leq 2D^2 + C^2 \text{ and } \mathbb{E}[\|\hat{y} - y_{t+1}\|^2] \leq (2D^2 + C^2)\frac{\tau_1}{\eta_1(1-p)},$$

which, by Jensen's inequality, then imply that

$$\mathbb{E}[\|\hat{x} - x_{t+1}\|] \leq \sqrt{2D^2 + C^2} \text{ and } \mathbb{E}[\|\hat{y} - y_{t+1}\|^2] \leq \sqrt{2D^2 + C^2}\sqrt{\frac{\tau_1}{\eta_1(1-p)}}.$$

Similarly, we can show that

$$\mathbb{E}[\|\hat{x} - x_{t+1}^v\|] \leq \sqrt{2D^2 + C^2} \text{ and } \mathbb{E}[\|\hat{y} - y_{t+1}^v\|^2] \leq \sqrt{2D^2 + C^2}\sqrt{\frac{\tau_1}{\eta_1}}.$$

Therefore, by (4.37) and the above four inequalities, we have

$$\mathbb{E}[\|v_{t+1}\|]$$
$$\leq \mathbb{E}\left[\frac{1}{\beta_t \eta_t}\left(\|x_1 - x_{t+1}\| + \|x_1 - x_{t+1}^v\|\right) + \frac{1}{\beta_t \tau_t}\left(\|y_1 - y_{t+1}\| + \|y_1 - y_{t+1}^v\|\right) + \frac{L_K}{\beta_t}\|x_{t+1} - x_t\|\right]$$
$$\leq \mathbb{E}\left[\frac{1}{\beta_t \eta_t}\left(2\|\hat{x} - x_1\| + \|\hat{x} - x_{t+1}\| + \|\hat{x} - x_{t+1}^v\|\right)\right.$$
$$\left. + \frac{1}{\beta_t \tau_t}\left(2\|\hat{y} - y_1\| + \|\hat{y} - y_{t+1}\| + \|\hat{y} - y_{t+1}^v\|\right) + \frac{L_K}{\beta_t}\left(\|\hat{x} - x_{t+1}\| + \|\hat{x} - x_t\|\right)\right]$$
$$\leq \frac{2\|\hat{x} - x_1\|}{\beta_t \eta_t} + \frac{2\|\hat{y} - y_1\|}{\beta_t \tau_t} + \sqrt{2D^2 + C^2}\left[\frac{2}{\beta_t \eta_t} + \frac{1}{\beta_t \tau_t}\sqrt{\frac{\tau_1}{\eta_1}}\left(\sqrt{\frac{1}{1-p}} + 1\right) + \frac{2L_K}{\beta_t}\right],$$

thus (3.14) holds.

Now let us estimate a bound on δ_{t+1}. By (4.20), (4.32), (4.35) and (4.38), we have

$$\mathbb{E}[\delta_{t+1}] = \mathbb{E}\left[\frac{1}{\beta_t \eta_t}\|x_{t+1}^{ag} - x_1\|^2 + \frac{1}{\beta_t \tau_t}\|y_{t+1}^{ag} - y_1\|^2\right] + \frac{1}{\beta_t \gamma_t}\mathbb{E}[U_t]$$
$$\leq \mathbb{E}\left[\frac{2}{\beta_t \eta_t}\left(\|\hat{x} - x_{t+1}^{ag}\|^2 + \|\hat{x} - x_1\|^2\right) + \frac{2}{\beta_t \tau_t}\left(\|\hat{y} - y_{t+1}^{ag}\|^2 + \|\hat{y} - y_1\|^2\right)\right] + \frac{1}{2\beta_t \eta_t}C^2$$
$$= \mathbb{E}\left[\frac{1}{\beta_t \eta_t}\left(2D^2 + 2\|\hat{x} - x_{t+1}^{ag}\|^2 + \frac{2\eta_t(1-p)}{\tau_t}\|\hat{y} - y_{t+1}^{ag}\|^2 + \frac{2\eta_t p}{\tau_t}\|\hat{y} - y_{t+1}^{ag}\|^2\right)\right] + \frac{1}{2\beta_t \eta_t}C^2$$
$$\leq \frac{1}{\beta_t \eta_t}\left[2D^2 + \frac{2}{\beta_t \gamma_t}\sum_{i=1}^{t}\gamma_i\left(\mathbb{E}\left[\|\hat{x} - x_{i+1}\|^2\right] + \frac{\eta_t(1-p)}{\tau_t}\mathbb{E}\left[\|\hat{y} - y_{i+1}\|^2\right] + \frac{\eta_t p}{\tau_t}\mathbb{E}\left[\|\hat{y} - y_{i+1}\|^2\right]\right) + \frac{C^2}{2}\right]$$
$$\leq \frac{1}{\beta_t \eta_t}\left[2D^2 + \frac{2}{\beta_t \gamma_t}\sum_{i=1}^{t}\gamma_i\left(2D^2 + C^2 + \frac{\eta_t p}{\tau_t}\cdot\frac{\tau_1}{\eta_1(1-p)}(2D^2 + C^2)\right) + \frac{C^2}{2}\right] = \frac{1}{\beta_t \eta_t}\left(\frac{6-4p}{1-p}D^2 + \frac{5-3p}{2-2p}C^2\right).$$

Therefore (3.13) holds.

$\square$

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5. Conclusion. We present in this paper the APD method by incorporating a multi-step acceleration scheme into the primal-dual method in [6]. We show that this algorithm can achieve the optimal rate of convergence for solving both deterministic and stochastic SPP. In particular, the stochastic APD algorithm seems to be the first optimal algorithm for solving this important class of stochastic saddle-point problems in the literature. For both deterministic and stochastic SPP, the developed APD algorithms can deal with either bounded or unbounded feasible sets as long as a saddle point of SPP exists. In the unbounded case, the rate of convergence of the APD algorithms will depend on the distance from the initial point to the set of optimal solutions.

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