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PRACTICAL MULTI-OBJECTIVE PROGRAMMING

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Abstract

This paper is on practical solutions to the multi-objective optimization problem; it advocates for single-point solutions either of the Nash equilibrium or the Tchebycheff compromise type, depending on whether one can reasonably ascribe competition or cooperation to the problem at hand. A transform method that greatly simplifies implementation of the compromise solution is presented and shown to be effective. The exposition is largely couched in game-theoretic terms, and with reference to an evolutionary multi-objective solver called GENO. The paper includes six numerical examples that illustrate the ideas and issues discussed.

Key Words: Multi-objective Programming, Game Theory, Compromise Solution, Nash Equilibria, Evolutionary Algorithms, Linear and Nonlinear Simultaneous Equations.

1 Introduction

Real-world optimization problems are often characterised not by a single objective, but by a set of criteria against which candidate solutions must be assessed. Such problems arise in various contexts and are synonymously called *multi-objective*, *multi-agent* or *multi-criterion* optimization problems. Unfortunately, there is no unique definition of what constitutes an optimal solution to a multi-objective problem, principally because various components of the vector-valued criterion function can (and often do) evaluate candidate solutions in disparate ways: taken pair-wise on any region of the search space, some criteria may "compete" in the sense that an improvement with respect to one degrades the quality of the solution as assessed the other; others may "collude" in the sense that an improvement in one entails the same in the other; and others may be totally independent. Thus, whereas 'optimize' is a well understood command in the uni-objective context meaning 'compute the extrema of a criterion function', the same cannot be said of the multi-objective problem. It is not surprising therefore that there exist different types of optima for such problems, and furthermore, some of the notions on what constitutes 'an optimum' yield non-unique solutions.

A common approach to the quandary posed by the non-uniqueness problem is to introduce a *model-user* or *decision-maker*—i.e. the person for whom the computed solutions are intended—and require such a person to articulate preferences at some stage during the solution process. This user-mediated approach thus comprises two discernible components—a *search procedure* and a *decision-making* stage. And depending on how the two components are combined, user-mediated methods may be classified into one of three categories:

1. *A priori Articulation of Preferences* [The Decide-then-Search Strategy]: The model-user expresses his preferences in terms of an aggregate utility function *prior to optimization*.
2. *A posteriori Articulation of Preferences* [The Search-then-Decide Strategy]: The model-user is first presented with a set of 'efficient solutions'; the model-user then proceeds to select *a* solution from the given set.
3. *Progressive Articulation of Preferences* [The Decide-and-Search Strategy]: Preference articulation by the decision maker and solution generation proceed in parallel at inter-leafed steps.

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Multi-objective solution methods may also be sorted according to how the vector-valued objective function is treated in the process, and a common dichotomy in this regard is that between *scalarizing* and *non-scalarizing* techniques: the latter deal with the criterion vector "as is"; the former introduce a proxy *scalar* objective function. This dichotomy in conjunction with the user-mediation categories described above leads to the following classification of currently available solution algorithms.[1]

Table 1: A Classification of Multi-Objective Solution Algorithms

		Scalarization Methods	Non-scalarization Methods
User-mediated Methods	Prior	Goal Programming; Goal Attainment Method; Weighted Global Criterion Methods; Weighted Metric Methods; Lexicographic Method; ε-Constraints Method; Compromise Programming;	Evolutionary Methods such as MOGA, NSGA-II, SPEAS, PAES, PESA—for summary descriptions, see [15]; Schäffler's Stochastic Method; Timmel's Population-based Method; GENO
	Progressive	Geoffrion-Dyer-Feinberg Method; Tchebycheff Method; Achievement Scalarization Method; Light Beam Search;	Generalized Data Envelopment Analysis; Goal Programming; ISMAUT; PROMETHEE—for summary descriptions, see [8];
	Post	Normal Boundary Intersection Method; Normal Constraint Method; Physical Programming	Evolutionary Methods such as MOGA, NSGA-II, SPEAS, PAES, PESA—for summary descriptions, see [15]; GENO
User-independent Methods		Global Criterion Methods; Compromise Programming; Nash Arbitration Schemes; Objective Product Method; Rao's Method	TOPSIS; VIKOR—for summary descriptions, see [31]; GENO

The purpose of this paper is to advocate for two particular solution concepts for the multi-objective problem namely, the *Nash equilibrium* and the *compromise* solution. The advocacy is on the grounds of *practicality*, by which is meant 'easy to implement'. And in the quest for practicality, a transform method is presented and shown to greatly simplify the computation of compromise solutions.

The exposition occasionally mentions a commercial solver called GENO whose technical description beyond this footnote[2] is not necessary for the arguments presented herein and is therefore excluded; otherwise, the paper is organised as follows: the generic model addressed by GENO is presented in §**2** in order to introduce notation and provide context for what is to follow; §**3** presents a game theory perspective of the solutions to the model in §**2**; some practical computational issues that prompted this paper are discussed in §**4**; §**5** presents a formal exposition of the compromise solution; §**6** presents the generalized loss transform; the incorporation of model-user preferences is briefly explained in §**7**; §**8** presents five numerical examples that serve to illustrate the ideas presented herein; §**9** summarises and concludes the presentation; and last but not least, the legal framework governing this publication is set forth in §**10**.

[1] The table entries are mostly based on [**26**].

[2] GENO is an acronym for ***G***eneral ***E***volutionary ***N***umerical ***O***ptimizer. GENO is a real-coded evolutionary algorithm that can be used to solve uni- or multi-objective optimization problems; the problems presented may be static or dynamic in character; they may be unconstrained or constrained by functional equality or inequality constraints, coupled with set constraints on the variables; the variables themselves may assume real or discrete values in any combination. For a more detailed description and performance evaluation of GENO, see [**36**].

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2 The Generic Model

The generic multi-objective optimization model under consideration may formally be described using the state-variable notation of *optimal control theory* as follows: let the prime [′] denote 'vector transposition'; let $\mathbf{x}_k$ and $\mathbf{u}_k$ be the state and control vectors at stage k respectively; and let $\mathbf{J} = (f_1, f_2, \ldots, f_p)'$ be a vector of performance criteria; then the problem to solve is a p-objective, constrained mathematical program defined on a discrete domain—say, the time-sequence $\mathbf{T} = \{0,1,2,..,T\}$—that may be stated thus:

MP1: The Generic Multi-Objective Mathematical Program

$$\underset{\bar{\mathbf{u}}_T}{\text{Opt}} \{ f_1(\bar{\mathbf{x}}_T, \bar{\mathbf{u}}_T), f_2(\bar{\mathbf{x}}_T, \bar{\mathbf{u}}_T), \cdots, f_p(\bar{\mathbf{x}}_T, \bar{\mathbf{u}}_T) \}$$

Subject to:

$$x^i_{k+1} = A^i_k(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_{k-1}) + B^i_k(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_{k-1}) \cdot u^i_k$$

$$C^j_k(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \le 0$$

$$i \in \mathbf{N} = \{1,2,..,n\}; \quad j \in \mathbf{M} = \{1,2,..,m\}; \quad k \in \mathbf{T} = \{0,1,2,..,T\}$$

Where:

$\bar{\mathbf{x}}_k$, $\bar{\mathbf{u}}_k$ are sequences (up to time k) of the state and control vectors respectively;

$\mathbf{x}_0$ is the initial state vector—a known *fixed* constant;

$\mathbf{x}_{T+1}$ is the final state vector, which may or may not have known fixed elements;

$$\bar{\mathbf{u}}_k = [\mathbf{u}_0 \,|\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_k] \in \bar{\mathbf{U}}_k$$

$$\bar{\mathbf{x}}_k = [\mathbf{x}_0 \,|\, \mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\, \cdots \,|\, \mathbf{x}_k] \in \bar{\mathbf{X}}_k$$

$$\mathbf{u}_k = (u^1_k, u^2_k, .., u^n_k)' \in \mathbf{U}^n_k \subset \mathbf{R}^n$$

$$\mathbf{x}_k = (x^1_k, x^2_k, .., x^n_k)' \in \mathbf{X}^n_k \subset \mathbf{R}^n$$

$$\bar{\mathbf{U}}_k \equiv \left\{ \bar{\mathbf{u}}_k \,|\, \left(\bar{\mathbf{x}}_k \in \bar{\mathbf{X}}_k\right) \wedge \left(C^j_k\left(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k\right) \le 0\right) \right\}$$

$$\bar{\mathbf{X}}_k \equiv \left\{ \bar{\mathbf{x}}_k \,|\, \left(\bar{\mathbf{u}}_k \in \bar{\mathbf{U}}_k\right) \wedge \left(C^j_k\left(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k\right) \le 0\right) \right\}$$

The set operator 'Opt' is a command that means 'optimize the operands': it may be "distributed" onto the individual elements of the criterion set { $f_1, f_2, \ldots, f_p$ }, after which it assumes the usual meaning associated uni-objective optimization, i.e. *minimize* or *maximize* depending on the problem specification. Note that MP1 is very general in scope: not only does the formulation accommodate *any* discrete domain—i.e. the set $\mathbf{T}$ need not be 'time'—it also subsumes, as special cases, other categories such as uni-objective static optimization problems where $P = 1$ and $T = 0$, or uni-objective dynamic optimization problems where $P = 1$ and $T > 0$; further details of the model may be found in [**36**].

GENO is an evolutionary algorithm that is specifically designed for MP1; it may be pre-set for different types of solutions; in the multi-objective case, the solution sought must be decided on the basis of whether one can reasonably ascribe 'competition' or 'cooperation' in the problem at hand—these concepts originate from *Game Theory*, a functional summary of which is presented in the next section.

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3 The Game Theory Perspective

Game Theory offers a useful perspective for solving the generic optimization model in §**2** because, as a mathematical abstraction, MP_1 is identical to multi-agent models that Başar & Olsder [**3**, Chap. 5] classify as 'nonlinear dynamic games in extensive form'.[3] In a game-theoretic view of MP_1, one assumes that each component of $\mathbf{J}$ is a pay-off accruing to a fictitious player who controls a subset of the state vector by taking a set of decisions (control actions) over the optimization period; such a sequence of controls is termed a *strategy*; and a player may use a *pure strategy*, i.e. a definite course of action, or a *mixed strategy*, i.e. a probabilistic combination of his pure strategies. Game-theoretic notions of what constitutes a solution in multi-agent problems are invariably variations on two major themes: that of *equilibrium* and that of *dominance*. In any a given situation, the type of solution deemed reasonable is ultimately based on the knowledge and behaviour ascribed to the players; a discussion of the possible scenarios may be found in the seminal text of Von Neumann & Morgenstern [**45**], the very lucid book by Luce & Raiffa [**20**], or the survey paper by Binmore & Dasgupta [**6**]. But for our purposes, the following summary descriptions will suffice.

I. Non-Cooperative Games

In a non-cooperative game, one supposes that individual players rationally work out their best strategies, and play accordingly. Given that each player can reason strategically, one can imagine that the game would "gravitate" towards a point which has some self-reinforcing property, i.e. an equilibrium point. The idea of an equilibrium solution is ultimately premised on the *individual rationality* postulate—the notion that individuals act rationally and in their own self-interest. In essence, the individual rationality postulate asserts that, given a situation in which various agents compete to optimize their individual objectives, the joint solution is an equilibrium point that provides, for each individual, a disincentive to depart from—this is the point at which each player has selected their best response to the other players' strategies, and no player may unilaterally improve his or her payoff by adopting a different strategy from that which obtains at the equilibrium point. The equilibrium solution concept may be traced back to the French mathematician Antoine Augustin Cournot (1801 – 1877), but it was John F. Nash [**28**] who formally defined it and proved that every game has at least one equilibrium solution if the players use mixed strategies. The solution is therefore alternately known as the *Cournot-Nash equilibrium*, or the *Nash equilibrium*. Formally, for a game in normal form, the Nash solution may be stated thus [**3**, p.163]:

- DEFINITION 1 [Nash Equilibrium]: Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p) \in \mathbf{U}$ denote the composite decision vector in a p-agent optimization problem in normal form and in which each decision agent wishes to maximize their respective payoffs $f_i(\mathbf{u}, \mathbf{x}_0)$; then the Nash equilibrium solution is a vector $\mathbf{u}^*$ that satisfies the following inequalities:

  $$\forall\ \mathbf{u} \in \mathbf{U} \text{ and } \forall\ i = 1, 2, \ldots, p: \qquad f_i(\mathbf{u}^*, \mathbf{x}_0) \geq f_i(\mathbf{u}, \mathbf{x}_0)$$

  REMARKS 1: A casual inspection of the inequalities above should confirm that the Nash equilibrium solution "degenerates" to the normal notion of a uni-objective *global* optimum when $p = 1$. Thus, the inequalities of the Nash equilibrium concept are statements on *global* optima, and so an immediate consequence of DEFINITION 1 is the following theorem.

[3] The more familiar 'normal form' equivalent of a dynamic game is easily derived from the extensive form by eliminating the dynamic constraints via a successive backward substitution process [**3**, p.221].

- THEOREM 1: If any of the criterion functions $f_i(\mathbf{u}, \mathbf{x}_0)$ in (2) is multi-modal in $\mathbf{u}$, then the Nash equilibrium solution occurs at the *global* maximizer of $f_i(\mathbf{u}, \mathbf{x}_0)$ on $\mathbf{U}$ for all i.

- PROOF: Clearly, THEOREM 1 must hold because it is only when each agent is already at their respective global solution that opportunities to "unilaterally improve his or her payoff by adopting a different strategy from that which obtains at the equilibrium point" (*supra*, p.4), are no longer available.

- COROLLARY 1: The following statement is self-evident: algorithms that rely on DEFINITION 1 but only guarantee convergence to *local* optima are unlikely to converge to a Nash equilibrium point ∎

II. Cooperative Games

In a co-operative game on the other hand, it is posited that players form sub-groups or coalitions; and these coalitions *collectively* work out their best strategies for the game and play accordingly; they share out the resulting 'joint utilities' in accordance with a previously agreed contract. There are various notions of what constitutes a solution in such a scenario. The original solution proffered by the founders of game theory is set-valued and called the *negotiation set*; it is based on an order relation for vectors called *dominance* as well as the reasonable idea that the negotiated solution has to be at least better than what each player would attain by acting alone; they posited that,

> ". . . the players [would] act jointly to discard all jointly dominated payoff pairs and all non-dominated payoffs which fail to give each of them at least the amount he could be sure of without co-operating." [Paraphrased from **20**, p.118]

But for practical purposes, this multi-agent solution is usually defined ignoring the latter proviso, i.e. purely on the dominance criterion. Matthias Erhgott [**10**] rightly credits Francis Ysidro Edgeworth (1845-1926) as originator of this solution concept, but it is more commonly attributed to Vilfredo Pareto (1848-1923), hence the descriptor 'Pareto-optimality'. Formally, Pareto-optimality may be defined thus:

- DEFINITION 2A [Pareto Dominance]: A succinct definition of the dominance concept is the following: A real-valued vector $\mathbf{a}$ is said to dominate another real-valued vector $\mathbf{b}$ in the 'greater-than' sense if the difference vector $\mathbf{d} = \mathbf{a} - \mathbf{b}$ only has non-negative elements of which at least one is strictly positive; and the applicable difference vector for dominance of $\mathbf{a}$ over $\mathbf{b}$ in the 'less-than' sense is $\mathbf{d} = \mathbf{b} - \mathbf{a}$.

- DEFINITION 2B [Pareto-optimality]: Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p) \in \mathbf{U}$ denote the composite decision vector in a p-agent optimization problem in normal form and in which each agent wishes to maximize their respective payoffs $f_i(\mathbf{u}, \mathbf{x}_0)$; let 'vec$\{e_i\}$' denote a vector whose i-th element is e_i; then a Pareto-optimal solution is a decision vector $\mathbf{u}^*$ whose outcome vector vec$\{ f_i(\mathbf{u}^*, \mathbf{x}_0)\}$ is not dominated by other outcome vector vec$\{ f_i(\mathbf{u}, \mathbf{x}_0)\}$. Alternatively, a solution $\mathbf{u}^*$ is said to be Pareto-optimal if, for any other vector $\mathbf{u}$, the following statement is true:

$$\forall\ \mathbf{u} \in \mathbf{U} \text{ and } \forall\ i = 1, 2, \ldots, p: \qquad f_i(\mathbf{u}^*, \mathbf{x}_0) \le f_i(\mathbf{u}, \mathbf{x}_0) \quad \Rightarrow \quad f_i(\mathbf{u}^*, \mathbf{x}_0) \equiv f_i(\mathbf{u}, \mathbf{x}_0)$$

REMARKS 2: The Pareto-optimal solution has the property that, by moving to the negotiation set or a Pareto-optimal solution, the outcome accruing to at least one player improves without degrading that of any other player in the coalition; it formalises what is called *group rationality* since it would be irrational for the coalition as a whole to settle for a solution that is *not* Pareto-optimal.

The set of all non-dominated points in outcome space is called the *Pareto-efficient frontier*, the *efficient frontier*, *Pareto frontier* or simply the *Pareto set*; the negotiation set is always a subset of the Pareto frontier; and the corresponding pre-image points in the decision or control space is called the *efficient set*.

This multiplicity of solutions has been a source of dissatisfaction with the co-operative game model in applied work. Consequently, several procedures whose purpose is to delineate a single point within the solution set have been suggested—most of these fall into two categories:

1. *Axiomatic Approaches*, where a set of axioms which the solution can reasonably be expected to satisfy are first postulated and then a function which satisfies the axioms is defined such that the solution set is mapped into a single payoff point;
2. *Strategic Approaches*, in which a dynamic bargaining process is explicitly described; one makes the players' steps of negotiation in the co-operative game become moves in a non-cooperative model and the latter is then solved using the Nash equilibrium concept.

A substantive discussion of these methods is beyond the scope of this article. It suffices to mention that most methods tend to be of the axiomatic type; but either type are rather difficult to implement in practice, and perhaps because of this, there has been a growing trend towards 'marrying' the two approaches in a research agenda called the 'Nash Program' [**6**]. The idea is to formulate a non-cooperative game whose unique Nash equilibrium coincides with the single-point solution characterised by a set of axioms. For bargaining games, the most common single-point solutions are those of Nash [**29**], and Kalai & Smorodinsky [**18**]; see also [**17**].

III. The Primary Consideration: Is it Cooperation or Competition?

A basic question that has to be addressed in a game-theoretic solution of MP_1 is whether one should adopt the cooperative or competitive concept. This question does not always arise, but when it does, the choice should be based on a sound argument on whether one should ascribe cooperation or competition onto the phenomena being modelled; but, as the following examples illustrate, such an argument may not always be obvious.

Illustration I. It is shown elsewhere that nonlinear equation systems may be solved by multi-objective programming methods [**41**].[4] To illustrate, let $\mathbf{C}(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}^m$ be a vector-valued mapping whose components are functions denoted by $c_i(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$, $i \in \{1, 2, \ldots, m\}$, some or all of which may or may not be nonlinear; define the criterion vector as $\mathbf{J} = (|c_1|, |c_2|, \ldots, |c_m|)'$; then the m-system of equations $\mathbf{C}(\mathbf{x}) = \mathbf{0}$ may be solved by "embedding" it into the following multi-objective optimization problem in which the operator 'Opt'—when "distributed" over the individual elements of the criterion set $\{|c_1|, |c_2|, \ldots, |c_m|\}$—denotes *minimisation*:

MP_{em1}:

$$\operatorname*{Opt}_{\mathbf{x}} \left\{ |c_1(\mathbf{x})|, |c_2(\mathbf{x})|, \cdots, |c_m(\mathbf{x})| \right\}$$

$$\text{Subject to: } c_i(\mathbf{x}) = 0; \quad x_i \in [\text{LB, UB}], \ \forall i$$

Remarks. The solution to the equation system $\mathbf{C}(\mathbf{x}) = \mathbf{0}$ occurs when each criterion function $|c_i(x)|$ attains its minimal value of zero; thus the optimal vector $\mathbf{x}^*$ has to be a zero simultaneously for each and every element of the criterion set $\{|c_1|, |c_2|, \ldots, |c_m|\}$; in other words, the following logical statement must hold:

$$\mathbf{x}^* \;\Rightarrow\; \{(c_1(\mathbf{x}^*) = 0) \wedge (c_2(\mathbf{x}^*) = 0) \wedge \ldots \wedge (c_m(\mathbf{x}^*) = 0)\}$$

Of course there is nothing stopping one from solving MP_{em1} for the Nash equilibrium, but numerical experience suggests that the efficiency of this approach is rather erratic; the individual rationality postulate upon which the Nash algorithm is built does not result in consistent progress towards the conjunction above in all cases. But the group rationality postulate on the other hand has an in-build bias towards the conjunction—a new estimate of the solution is only accepted if there is an improvement towards the conjunction on at least one $|c_i(\mathbf{x})|$. Thus, MP_{em1} is best treated as a cooperative game, and numerical examples in [**36**, **41**] do confirm this assertion.

[4] See also **Example 5** below

Illustration II. According to Chandler [7], a fundamentally important role of the headquarter (HQ) unit of a multi-business enterprise is to redistribute cash flows amongst the various strategic business units (SBUs) of the firm in an optimum manner. A non-linear dynamic optimisation model that encapsulates quantifiable aspects of this strategic function is presented in [35]. And since strategic planning is typically a 'bi-centralised' activity [1], the model is accordingly a multi-agent formulation in the MP_1 mould; the formulation integrates various theories including: (i) empirically derived methods of planning and managing the corporate portfolio; (ii) normative microeconomic notions of managerial behaviour; (iii) investor behaviour consistent with the internal market hypothesis of the HQ's resource allocation function [48]. It is tempting to reason that since the HQ and SBUs belong to the same firm, then the cooperative model is the only valid solution approach. Although normatively correct, such a view would be unnecessarily restrictive because intra-firm cooperation and competition can (and do in fact) co-exist within multi-business firms; it may be explained as follows:

> "While [multi-business firms] encourage inter-unit collaborations to realize economies of scale, they also allow inter-unit competition to achieve efficiency; subunits compete for parent resources and support, system position and market expansion; the resources for which subunits compete are diverse, ranging from technology, equipment, and key talents to capital, supplies and know-how; the competition arises because such resources are limited in quantity, and allocation and deploying them have to depend on the parent firm's global strategy" [Paraphrased from 23, p.75]

This phenomenon called 'co-opetition' which Lou defines as a "loosely coupled system in which agents maintain certain interdependence without losing their organizational separateness" [22, p.24], is common. And in regard to the strategic planning model briefly described above, it is reasonable to posit that whereas the HQ unit might endeavour to allocate investment capital for the good of the firm, SBUs will typically compete for these funds parochially. Thus, in this case, the equilibrium concept is descriptively more valid.

4 Some Methodology Issues

Our interest here however is in practical matters, particularly the computation of single-point multi-objective optimal solutions that are easy to implement. Game theory models in the non-physical sciences tend to rely on the Nash equilibrium notion and its variants:

> "[T]he Nash equilibrium is without a doubt the most ''successful''— i.e., widely used and applied — solution concept of game theory [. . .] the definition of the Nash equilibrium is in form extremely simple. Moreover, the concept is mathematically very tractable and easy to work with." [Paraphrased from 2, p.23]

On the other hand—but with the exception of control engineers (see e.g., 12 & 50) and latterly computer scientists (see e.g., 32)—researchers in the multi-objective optimization arena from the physical sciences have tended to embrace Pareto-optimality at the expense of the equilibrium notion. And it should be clear therefore that, if a multi-objective solver is to be useful in practice, then it must be capable of generating either type of solution, and preferably of the single-point variety.

By design, GENO has this dual capability: the equilibrium solution is implemented in a straight forward manner using standard algorithms—such as those suggested by Başar [4]—that are based on the 'best reply' solution characterization of John Nash [28]; and the solution multiplicity issue associated with cooperative solutions is addressed via the compromise concept that was first introduced by Salukvadze [33] and later independently presented by Yu [49] and Zeleny [51]. The Nash equilibrium solution shall not be discussed further except in the context of specific numerical examples later; the remainder of this exposition focuses on Pareto-optimal solutions.

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The multitude of methods for computing Pareto-optimal solutions have been thoroughly reviewed by, *inter alia*, Marler & Arora [**26**] and some are listed in Table 1. Most deterministic solution methods start by converting the multi-objective problem back into the familiar terrain of uni-objective optimization; the scalarization of the objective vector is usually via some 'method parameters' that may either be set to reflect model-user preferences, or be continually varied in an effort to generate the complete Pareto optimal set; the utility of a scalarization technique is evaluated based on the following questions: (i) does the method always yield efficient points? (ii) if so, can all efficient points be detected this way? (iii) does the method allow incorporation of models-user's preferences? On the other hand, evolutionary methods optimize the objective vector "as is", i.e. without scalarization; rather than test whether a computed solution is efficient, the notion of Pareto dominance is in-built in the search process itself; and the emphasis is on generating a well-sampled Pareto frontier so that the model-user may make an informed decision as part of a *search-then-decide* or a *search-and-decide* strategy.

Each solution method has its own drawbacks: (i) scalarization invariably means that the solution of the resulting uni-objective problem is dependent on the method parameters, and although there are many methods of determining such parameters, there is no guarantee that the final solution will be acceptable [**9**]; (ii) most deterministic algorithms generate only one solution at a time, and so in order to compute the Pareto set, the algorithm must be applied many times; (iii) if the number of objectives is large, i.e. greater than four, then the computational effort required to generate the efficient set is very substantial; (iv) the efficacy of many of the algorithms is dependent on the shape of the Pareto frontier; (v) in any case, generating the entire Pareto frontier is, in some sense, a waste of effort since only one point is all that is required in the final analysis.

Existing solution techniques are also rather deficient at the methodology level: (a) an implicit assumption of all user-mediated methods is that one can accurately determine the utility function that the model-user employs in articulating preferences or arbitrations—but this is an impossible task [**42**]; (b) the *decide-then-search* approach is difficult to implement when the number of objectives is large, and the *decide-and-search* strategy is equally cumbersome [**8**]; (c) the *search-then-decide* approach is doubly inadequate—quite apart from the fact that generating the entire Pareto frontier may be a waste of effort, the model-user may not be able to select a solution point that properly represents his or her preferences after all because presenting a visually discernible Pareto frontier is difficult except in the bi-objective case; (d) besides, none of the user-mediated strategies apply in some scenarios such as that in [**5**] where there cannot be a model-user present to articulate any preferences.

The problems outlined above clearly call for an automatic method for selecting a single point from the efficient set; such a method should be easy to use and be capable of admitting a few user-preferences if required; but it should also be able to generate justifiable solutions even in the absence of a model-user; the solution ideally ought to be a 'win-win' situation in which each individual objective function attains its best possible value *within* the constraints of the problem at hand. These desiderata may be fulfilled by coupling the Pareto-dominance optimality criteria to the compromise solution concept via a monotonic transform.

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5 The Compromise Solution

As mentioned previously, the compromise solution concept was first introduced in the early seventies to address the solution multiplicity issue. But the compromise concept also alleviates some implementation problems in the user-mediated approach (as shall be explained shortly). It is based on the common-sense notion that the best option is a feasible point that yields values that are closest to an ideal outcome—the *ideal* being that point at which each criterion is optimized to the fullest extent, i.e. the *global* solution. In fact, Zeleny considers the compromise notion to be an axiom of choice theory:

> "Alternatives that are closer to the ideal are preferred to those that are farther away. To be as close as possible to the perceived ideal is a rationale of human choice." [Paraphrased from **51**]

Compromise solutions—generally classified under *reference point* methods—are significantly influenced by how one quantifies the criterion 'closest to an ideal'. In Yu's study, 'closeness' is measured using distance functions based on the Minkowski L_p norms, i.e. the distance between two vectors **x** and **y** is defined as the L_p norm of the difference vector $(\mathbf{x} - \mathbf{y})$, viz.:

$$L_p(\mathbf{x},\mathbf{y}) = \left\{ \sum_{i \in I} \left(|\, x_i - y_i \,| \right)^p \right\}^{1/p} \;;\quad p \in (0, \infty] \tag{1a}$$

Voorneveld, Van Den Nouweland & McLean [**46**], amongst others, have since extended Yu's axiomatic results but focussing on the Euclidean metric, i.e., when $p = 2$; GENO uses a distance measure with $p = \infty$, which may or may not be weighted at the user's discretion. The L_∞ norm is known as the 'Supremum norm' or 'Sup-norm', and the associated metric is called the Tchebycheff metric; the μ-weighted Tchebycheff distance between two vectors **x** and **y** with weights μ_i is given by:

$$d_\infty(\mathbf{x},\mathbf{y}) = \operatorname*{Max}_{i \in I} \left\{ \mu_i \,|\, x_i - y_i \,| \right\};\; \mu_i \in (0, 1) \text{ and } \textstyle\sum \mu_i = 1 \tag{1b}$$

The rationale for choosing the weighted Tchebycheff distance is two-fold: (i) although the Euclidean metric is the natural metric in $\mathbf{R}^n$, it is sensitive to scale changes, and this could be problematic if criterion function values are subjected to a transform (as is advocated in §**6**); (ii) one can always capture any part of the Pareto frontier with a weighted Tchebycheff metric provided $\boldsymbol{\mu} > \mathbf{0}$ [**26**].

The rationale for the compromise solution is best explained in terms of the two-dimensional outcome space depicted in Figure 1 in which $f_1(\mathbf{u})$ and $f_2(\mathbf{u})$ are finite-valued criterion functions of a feasible control vector **u**; they comprise the objective or outcome vector $\mathbf{J}' \equiv \boldsymbol{\omega}' = (f_1(\mathbf{u}), f_2(\mathbf{u}))$, and the collection of all such feasible outcomes constitutes the *outcome set* $\boldsymbol{\Omega}$ which is assumed (purely for the sake of argument) to be non-empty, closed and bounded, i.e. *compact*. Associated with each outcome vector $\boldsymbol{\omega}$ are four 'boundary outcomes' $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ and $\boldsymbol{\omega}_4$; the boundary outcomes are points where lines that are parallel to the axes f_1 and f_2 intersect the boundary of the set $\boldsymbol{\Omega}$ (hereafter denoted by $\partial\boldsymbol{\Omega}$), and these lines may be defined as follows: assume $\boldsymbol{\omega}$ is fixed and let $\mathbf{v}$ be a variable vector in $\boldsymbol{\Omega}$; let the standard unit vector along the n-th dimension be denoted by $\mathbf{e}_n$, then formally, the line through $\boldsymbol{\omega}$ that is parallel to the n-th axis and lying within $\boldsymbol{\Omega}$ is the set of points:

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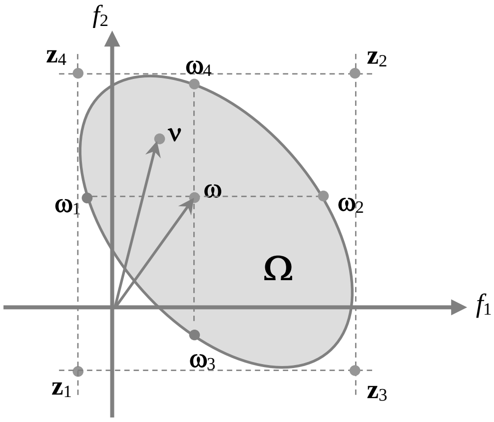
$$\mathrm{S}_{\mathrm{n}}(\boldsymbol{\omega}) = \{\forall \boldsymbol{\nu} \in \boldsymbol{\Omega} : (\boldsymbol{\omega} - \boldsymbol{\nu} = \mathrm{m}\mathbf{e}_{\mathrm{n}}) \wedge (\mathrm{m} \in \mathbf{R})\} \tag{2a}$$

Figure 1: Outcome Set of a Bi-Objective Optimization Problem



Let L_{n} and U_{n} denote the upper and lower bounds respectively for outcome f_{n} assuming all other outcomes remain constant. Then, in terms of quantities defined previously, we have that:

$$L_{\mathrm{n}}(\boldsymbol{\omega}) = \mathrm{Inf}\{f_{\mathrm{n}} : (f_{\mathrm{n}} = \boldsymbol{\nu} \cdot \mathbf{e}_{\mathrm{n}}) \wedge (\boldsymbol{\nu} \in S_{\mathrm{n}}(\boldsymbol{\omega}))\} \tag{2b}$$

$$U_{\mathrm{n}}(\boldsymbol{\omega}) = \mathrm{Sup}\{f_{\mathrm{n}} : (f_{\mathrm{n}} = \boldsymbol{\nu} \cdot \mathbf{e}_{\mathrm{n}}) \wedge (\boldsymbol{\nu} \in S_{\mathrm{n}}(\boldsymbol{\omega}))\} \tag{2c}$$

Note that under the group rationality postulate, part of the boundary set $\partial\boldsymbol{\Omega}$ constitutes *the* solution to the multi-objective problem. For example, if each player in Figure 1 seeks to maximize his or her outcome and if the components of the outcome vector $\boldsymbol{\omega}$ constitute what each player can guarantee for him or herself if there was no cooperation, then the upper-right part of the boundary curve in between $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_4$ is the negotiation set, and the upper-right part of the boundary curve in between the points of tangency for the two tangents through $\mathbf{z}_2$ is the Pareto-efficient frontier [**20**, p.118].

It is easy to see that the vertices of the smallest rectangle enclosing $\boldsymbol{\Omega}$ comprise the 'utopia set' which we here define as 'the collection of all the possible utopia outcomes pertaining to the problem at hand'. For any given vertex vector $\mathbf{z}_{\mathrm{n}}$, each of its dimensions represents the best possible outcome, i.e. the *global* solution, that could be attained by maximizing or minimizing a particular criterion independently. However, only one vertex would be relevant in any given scenario—*relevance* being completely determined by the 'max' and 'min' operations that the problem specification entails; such a vertex shall hereafter be called the 'ideal point'. For example, $\mathbf{z}_1$ is the ideal outcome when both agents seek to *minimize* their respective objective functions; whereas $\mathbf{z}_4$ is for the case where Agent 1 is a *minimizer* and Agent 2 is a *maximizer*.

The two-dimensional scenario described above easily extends to $\mathbf{R}^{\mathrm{n}}$; and granted that, one may define the compromise solution in the *n*-dimensional case in two stages as follows:

- DEFINITION 3 [Ideal Point]: Let $\mathbf{\Omega} \subset \mathbf{R}^n$ denote an outcome set and assume $\mathbf{\Omega}$ is compact; let L_j and U_j denote the lower and upper bounds respectively as defined by equations (2a) and (2b); let $\mathbf{z}_i$, denote the *ideal point* of the problem at hand, then the coordinates of $\mathbf{z}_i$ are given by scalars z_{ij} defined as:

$$z_{ij} = \begin{cases} \operatorname*{Sup}_{\omega \in \Omega} \{U_j(\boldsymbol{\omega})\}, & \text{if the } j\text{th criterion requires maximizing} \\ \operatorname*{Inf}_{\omega \in \Omega} \{L_j(\boldsymbol{\omega})\}, & \text{if the } j\text{th criterion requires minimizing} \end{cases} \tag{3}$$

REMARKS 3: Given n and the outcome set $\mathbf{\Omega}$, the collection of utopia outcomes has cardinality 2^n; thus, as the number of objectives n increases, the number of utopia points grows rapidly. But, as mentioned previously, only one of these called the ideal outcome would be relevant. Ideal outcomes are normally not jointly attainable and this necessitates a compromise of some sort which is the basis for the following definition.

- DEFINITION 4 [TC Solution I]: Let the point $\mathbf{z}_i$ be the *ideal point* delineated by the 'max' and 'min' commands in a given multi-objective optimization problem; then the compromise solution is a member of those feasible controls whose outcomes are closest to the ideal outcome as measured by some distance metric in outcome space; thus, in terms of the Tchebycheff metric, the compromise solution is a feasible control vector $\mathbf{u}^*$ whose corresponding outcome vector $\boldsymbol{\omega}^*$ belongs to a set of outcomes $\xi \subset \partial\mathbf{\Omega}$ that is defined as:

$$\xi(\mathbf{z}_i) = \{\boldsymbol{\omega} \in \partial\mathbf{\Omega} : \boldsymbol{\omega} = \arg\min \| \boldsymbol{\omega} - \mathbf{z}_i \|_\infty \} \tag{4}$$

REMARKS 4: Note that the definition of $\xi(\mathbf{z}_i)$ entails two processes: (i) the obvious minimization process denoted by the 'argmin' operator; (ii) the less obvious search process that is supposed to delineate the boundary set $\partial\mathbf{\Omega}$. In the GENO scheme, the latter is approximated by evolutionary mechanisms using the Pareto-dominance criterion, and the former is a straight forward implementation of the Tchebycheff metric.

The rationale underlying this solution concept may be explained as follows: (i) there is no question that, if it were achievable, the ideal outcome vector $\mathbf{z}_i$ would constitute *the* optimal solution to the multi-objective optimization problem under consideration; (ii) but since this is usually not the case, one has to compromise downwards from the ideal outcome $\mathbf{z}_i$ to a less-than-ideal outcome $\boldsymbol{\omega}^*$ that corresponds to a feasible control $\mathbf{u}^*$ and obviously, the degree of the said "downward compromise" along each dimension, i.e. the quantity of payoff that the each coalition member must be willing to forgo—as a "price" for cooperating, if you will—has to be minimal, hence the distance-minimizing operation in the definition of the solution set ξ, and the stipulation that $\boldsymbol{\omega}^* \in \xi$.

But for one to implement TC SOLUTION I, one must first evaluate (3) and this can be a serious huddle in practice. If it is possible to reformulate the criterion functions f_i such that $\mathbf{z}_i \equiv \mathbf{0}$, then that is what should be done; but failing that, then a technique involving a monotonic transform may be used instead—this is explained next.

6 The Generalized Loss Transform

At first glance, the Tchebycheff compromise solution as described thus far may seem rather impractical on several counts: (i) the method is based on the assumption that $\mathbf{\Omega}$ is a compact set of some finite but unknown extent; (ii) the solution itself is a function of the utopia set, and as such it would seem as though it suffers a certain "curse of dimensionality" in the sense that one must first determine which one of the 2^n utopia points is relevant in any given application; (iii) but even if the general character of the ideal point $\mathbf{z}_i$ is known before hand, there still remains the problem of evaluating equation (3) in order to *locate* the point $\mathbf{z}_i$ in $\mathbf{R}^n$ —a process that assumes one can compute the *global* optimum of each criterion function—and then performing further calculations to delineate the solution set ξ; (iv) and even granted all the above, there still remains the problem of selecting $\boldsymbol{\omega}^* \in \xi$ via some unspecified procedure.

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But the real situation is not as hopeless as the foregoing suggests: for a start, the "curse of dimensionality" issue actually does not arise because the same "max and min operations" argument used to determine the relevant utopia point in the $\mathbf{R}^2$ case of Figure 1 also applies to the general $\mathbf{R}^n$ space *provided* the extent of $\boldsymbol{\Omega}$ is *finite* and *known*—and although this proviso would initially be unsatisfied, it may be resolved by recourse to a monotonic transform called the generalized loss (or *g*-loss) transform applied to the outcome space; secondly, the same *g*-loss transform also resolves the location problem alluded to above, as will be explained shortly; lastly, the problem of selecting $\boldsymbol{\omega}^*$ from ξ is also not really an issue because it is usually correct to assume that ξ is a singleton since it is the tangent point to a convex set, namely the *n*-dimensional ball centred at the ideal outcome vector $\mathbf{z}_i$, and the curvature of the Pareto frontier is unlikely to exactly coincide with the *n*-ball at more than one point. The *g*-loss transform mentioned above is defined as follows.

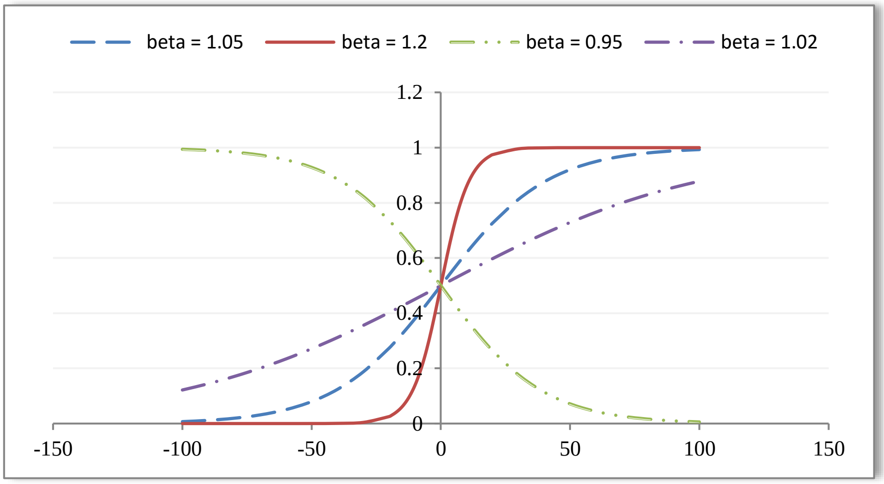
- DEFINITION 5A [Generalized Loss Transform]: Let f_i be finite and let it denote the *i*-th element of the vector-valued objective function $\mathbf{J} \in \boldsymbol{\Omega} \subset \mathbf{R}^n$. The generalized loss (or *g*-loss) transform is a monotonic function Γ_i parameterised by β_i, that is defined as follows:

$$\Gamma_i : \mathbf{R} \mapsto (0,1) \tag{5a}$$

$$\Gamma_i(f_i;\beta_i) = \frac{\beta_i^{f_i}}{1+\beta_i^{f_i}} \tag{5b}$$

REMARKS 5A: The transform Γ_i is a sigmoid function whose range is the open interval (0, 1). If all elements of the vector $\mathbf{J}$ are transformed by their respective Γ_i's over the outcome set $\boldsymbol{\Omega}$, then the extent and location of the image set denoted by $\boldsymbol{\Lambda}$ would be known completely, namely the *n*-dimensional unit box in $\mathbf{R}^n$ (see DEFINITION 6 below); furthermore, because each Γ_i is monotonic, the boundary $\partial\boldsymbol{\Omega}$ maps to the boundary $\partial\boldsymbol{\Lambda}$.

Figure 2: Graphs of the g-loss transform of DEFINITION 5A for various values of the β parameter



The graph of Γ_i as defined in equation (5b) is the constant 0.5 when $\beta_i = 1$; it is 'forward S-shaped' for $\beta_i > 1$, and 'reverse S-shaped' for $\beta_i \in (0, 1)$, as Figure 2 above illustrates. The steepness of the gradient at the point of inflection is determined by the value of β_i and one may use different values for each criterion f_i; if the expected range of f_i is large, then a β_i value that is "slightly greater than 1" is required in order to have an appreciable non-zero gradient at the "head" and "tail" of the graph that ensures discrimination amongst large values of f_i; but if values of f_i are expected to cluster around zero, then a fairly large value of β may be more appropriate.

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An alternative definition of the *g*-loss transform which is also accommodated by GENO is the following:

- DEFINITION 5B [Generalized Loss Transform]: Let f_i be finite and let it denote the *i*-th element of the vector-valued objective function $\mathbf{J} \in \mathbf{\Omega} \subset \mathbf{R}^n$. The generalized loss transform is a monotonic function Γ_i parameterised by β_i, that is defined as follows:

$$\Gamma_i : \mathbf{R} \mapsto (0, 1) \tag{6a}$$

$$\Gamma_i(f_i; \beta_i) = \frac{1}{1 + \beta_i^{f_i}} \tag{6b}$$

REMARKS 5B: The *g*-loss transform in equation (6b) is still a sigmoid function with range (0, 1). But now however, the graph is 'reverse S-shaped' for $\beta_i > 1$, and 'forward S-shaped' for $\beta_i \in (0, 1)$. GENO is designed for the "forward" *g*-loss transform only—the algorithm assumes DEFINITION 5A when the restriction $\beta_i > 1$ applies, or it assumes DEFINITION 5B when the set constraint $\beta_i \in (0, 1)$ is true. The user is free to choose either definition by merely specifying an appropriate value for the parameter β_i. If a problem requires different values of β_i to be used, it is highly recommended that they should pertain to the same definition.

- DEFINITION 6 [Aspiration Point]: Assume all outcome vectors $\boldsymbol{\omega}_i$ in $\mathbf{\Omega}$ are finite and let a *g*-loss transform be applied to each $\boldsymbol{\omega}_i$; such a mapping converts the outcome set $\mathbf{\Omega}$ — which, in general, is of unknown extent and location in $\mathbf{R}^n$ — into a set $\mathbf{\Lambda}$ that is totally enclosed by the unit box of $\mathbf{R}^n$ defined by:

$$\mathbf{I} = \left\{ \upsilon_i = (\upsilon_{i1}, \upsilon_{i2}, \cdots, \upsilon_{in})' \in \mathbf{R}^n \mid \left\{ (\upsilon_{ii} = 0) \vee (\upsilon_{ii} = 1) \right\} \wedge (0 \le \upsilon_{ij} \le 1), \forall j \ne i \right\} \tag{7}$$

The 2^n vertices of the unit box $\mathbf{I}$ are a collection of binary vectors $\{\boldsymbol{\theta}_i\}$ whose components are suprema and infima of the transformed outcome set $\mathbf{\Lambda}$; the collection is only an approximation of the real utopia set pertaining to $\mathbf{\Lambda}$ because, unlike the real utopia set, none of the components of the vectors $\boldsymbol{\theta}_i$ are attainable along any dimension of $\mathbf{R}^n$. So the term 'displaced utopia set' shall be used to denote the collection $\{\boldsymbol{\theta}_i\}$ and, following Wiersbiki [**47**], the term 'aspiration point' shall denote the equivalent to the ideal point of DEFINITION 3. Let $\boldsymbol{\theta}_a$ denote the *aspiration point* after applying the "forward" *g*-loss transform; then the coordinates of $\boldsymbol{\theta}_a$ are given by binary scalars θ_{aj} where $j \in \{1, 2, \ldots, n\}$, and:

$$\theta_{aj} = \begin{cases} 1, & \text{if the } j\text{th criterion requires maximizing} \\ 0, & \text{if the } j\text{th criterion requires minimizing} \end{cases} \tag{8}$$

REMARKS 6: In any given optimization problem, the aspiration point is a vertex $\boldsymbol{\theta}_a$ from the collection of 2^n vertices of $\mathbf{I}$ that is uniquely defined as a binary number by the '0' and '1' entries made according to equation (8) whose pattern is determined by the problem specification itself. Note that each vertex dominates the corresponding ideal point in $\mathbf{\Lambda}$ and so the approximation is consistent with Zeleny's notion of the 'displaced ideal' [**52**] whose essence is that

> ". . . we should select goals that are sufficiently distant from the set of attainable outcomes, and we can prove that norm minimization will result only in efficient outcomes, no matter what norm we use or what properties has the set of attainable outcomes." [Paraphrased from **47**, p.3]

The approximation may be made as accurate as one wishes by choosing an appropriate value of β. And as with assumes DEFINITION 5A, one may use of a different value of β_i for each criterion function f_i. But in this case, a β_i value that is very close to 1 is required in order to have an appreciable non-zero gradient at the "head" and "tail" of the graph that ensures discrimination amongst large values of f_i; and if values of f_i are expected to cluster around zero, then a value of β that is closer to zero is more appropriate.

In view of the foregoing, it should now be apparent that a very practical single-point solution to the multi-objective optimization problem is the Tchebycheff compromise solution defined as follows:

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- DEFINITION 7 [TC Solution II]: Let the binary vector $\boldsymbol{\theta}_a$ denote the *ideal* aspiration point following the application of the "forward" *g*-loss transform Γ to the outcome set $\boldsymbol{\Omega}$; let $\boldsymbol{\pi}_i \in \boldsymbol{\Lambda}$ be the image of the feasible outcome $\boldsymbol{\omega}_i$ under Γ; then the TC solution is the set $\zeta(\boldsymbol{\theta}_a) \subset \partial\boldsymbol{\Lambda}$ defined as follows:

$$\zeta(\boldsymbol{\theta}_a) = \{\boldsymbol{\pi} \in \partial\Lambda : \boldsymbol{\pi} = \arg\min \| \boldsymbol{\pi} - \boldsymbol{\theta}_a \|_\infty \} \tag{9}$$

REMARKS 7: Note that unlike the previous Tchebycheff compromise solution $\xi(\mathbf{z}_i)$ of DEFINITION 4 which, in most cases, requires auxiliary computations to locate the ideal point $\mathbf{z}_i$ in $\mathbf{R}^n$, the applicable reference point in the case of $\zeta(\boldsymbol{\theta}_a)$—i.e. the binary vector $\boldsymbol{\theta}_a$—is known immediately. But just like $\xi(\mathbf{z}_i)$, the definition of $\zeta(\boldsymbol{\theta}_a)$ entails two search processes that are implemented in the same manner in the GENO scheme: (i) the obvious minimization process denoted by the 'argmin' operator; (ii) the implicit search process that delineates the efficient set $\partial\boldsymbol{\Lambda}$. The solution computed via TC SOLUTION II may not always coincide with that based on TC SOLUTION I of DEFINITION 4; though the rationale for latter is logically more compelling (because components of the ideal outcome are actually achievable in the absence of cooperation), this should be traded-off on a case by case basis against the ease of implementing TC SOLUTION II which does not require auxiliary computations to determine the reference point.

7 Incorporating User Preferences

Although the TC solution has been defined without assuming the presence of a model-user, both TC SOLUTION I and TC SOLUTION II are capable of admitting user preferences expressed as inequality restrictions on components of the vector objective function. For example, assume the *i*-th criterion is required to be maximized and suppose the model-user is only interested in values of f_i greater than some constant f_0; this is easily implemented in the GENO framework by including the following lines of code (in GAUSS syntax) [**36**]:

```
fi = ...;
fv = fi - f0;
retp(fv);
```

This code fragment effectively translates the outcome set $\boldsymbol{\Omega}$ by f_0 units to the left—or equivalently the origin of the outcome space by f_0 units to the right—along the *i*-th dimension of the outcome space.

8 Numerical Examples

The following examples showcase GENO's efficacy at generating single-point solutions to the multi-objective optimization problem that have been discussed. All the examples are artificial constructs in which the primary question of whether to adopt the equilibrium or the compromise solution concept has already been decided: Example 1 and Example 2 are competitive models that are solved for the Nash equilibrium solution—the first example also serves as a "visual-numerical proof" of THEOREM 1; Example 3 is a cooperative model that is solved for the compromise solution with the ideal point as the reference outcome; Example 4 is essentially a repeat of Example 3 but this time using the *g*-loss transform and with the aspiration point as the reference outcome; Example 5 illustrates how systems of equations may be solved via multi-objective programming—this is one instance in which the TC SOLUTION I of DEFINITION 4 is naturally applicable; finally, Example 6 shows how a simple reformulation allows one to exploit the simpler TC SOLUTION I as advocated in REMARKS 4. In all cases, GENO's solution is compared to one determined by an alternative method, or by others.

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Example 1: $\underset{x_i}{\text{Max}}\left\{ f_i(\mathbf{x}) = e^{-0.1x_i}\text{Cos}(2\pi x_i/5) - 0.04\sum_{j=1}^{5} x_j \right\};\quad i = \{1, 2, ..., 5\}$

Subject to: $x_1 \in [-18,\ 200]$

$x_2 \in [-13,\ 200]$

$x_3 \in [-8,\ 200]$

$x_4 \in [-3,\ 200]$

$x_5 \in [2,\ 200]$

I. GENO Output

Generation	x_1	x_2	x_3	x_4	x_5
0	8.000000	-4.000000	6.000000	1.000000	17.000000
10	-15.062500	-10.062500	-5.093750	-0.093750	4.906250
20	-15.068787	-10.072418	-5.078400	-0.088226	4.895599
30	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
40	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
50	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
60	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
70	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
80	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
90	-15.068788	-10.072415	-5.078389	-0.088225	4.895590
100	-15.068788	-10.072415	-5.078389	-0.088225	4.895590

Nash Equilibrium Solution Vector: $\mathbf{x}^*$ = (- 15.068788, - 10.072415, - 5.078389, - 0.088225, 4.895590)^T

II. General Remarks

Nash Equilibrium Solution: The Optimality Equations Approach. To afford comparison with the GENO solution, an alternative method of computing the Nash equilibrium solution via *Mathematica's* `FindRoot` equation solver of was employed; the Nash equilibrium for this example is easily determined as follows.

Since the solution is, by design, in the interior of the search space, it satisfies the first-order optimality conditions:

$$\frac{\partial f_i(\mathbf{x}^*)}{\partial x_i} = -e^{-0.1x_i^*}\left(0.1Cos(2\pi x_i^*/5) + \frac{2\pi}{5}Sin(2\pi x_i^*/5)\right) - 0.04 = 0\,,\quad \forall i \in \{1,2,3,4,5\} \tag{10}$$

Each equation in (10) has several roots on its respective domain, and if an algorithm is to converge to the Nash equilibrium, it needs to locate the roots that correspond to the *global* maxima of the criterion functions f_i since it is only at the global solution **x*** (say) that there exists—as per definition of the Nash equilibrium— "a disincentive to depart from" **x***. The presence of a decaying exponential term in equation (10) implies that the global maximum for each individual criterion is to be found in the left-side region of the respective search space; to compute a precise location for the optimal solution to f_1 for example, one writes:[5]

```
f(x_):= - Exp[-0.1*x]*(0.1*Cos[(2*Pi*x)/5]+(2*Pi/5)*Sin[(2*Pi*x)/5]) - 0.04;
FindRoot[f(x) == 0.0, {x, -15}[6]];
```

[5] This statement was valid in *Mathematica* 2.2—the syntax may be different for later versions of *Mathematica*.

[6] The *Mathematica* solver `FindRoot` requires the specification of a starting point. A graph of equation (10) in the range [-20, 10] is shown in Figure 3a from which one can read off the approximate location of the required solution for input to `FindRoot`. The choice of -15 directs it away from the solution in between -20 and -15 since this solution is a *minimum* (see Figure 3b), yet we seek a maximum.

Thus, the Nash equilibrium solution as determined by `FindRoot` can be shown to be:

$$x_1^* = -15.0688; \quad x_2^* = -10.0724; \quad x_3^* = -5.07839; \quad x_1^* = -0.0882254; \quad x_1^* = 4.89559$$

Figure 3a: Graph of $\partial f_i(x_i) / \partial x_i$, for $x_i \in [-20, 10]$ and $x_j = 0$, for all $j \neq i$

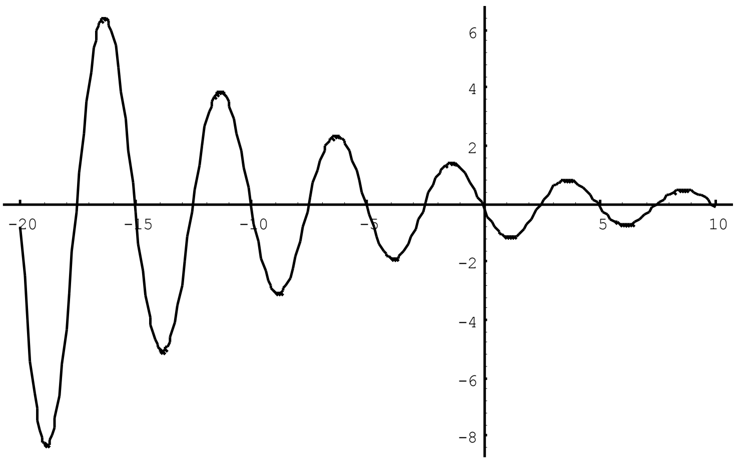
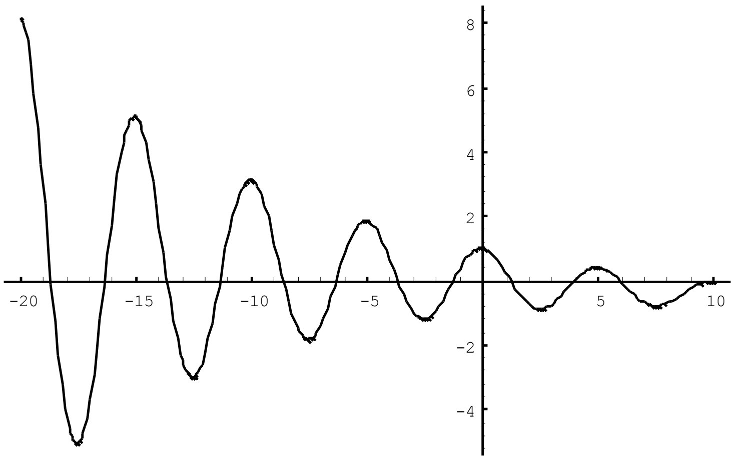


Figure 3b: Graph of $f_i(x_i)$ for $x_i \in [-20, 10]$ and $x_j = 0$, for all $j \neq i$



Concluding Remarks. As can be seen from the results presented above, GENO converges to the solution computed via `FindRoot` within 30 generations.

Example 2: $\underset{\mathbf{q}}{\text{Opt}}\{J^1, J^2, \cdots, J^5\}$

Subject to: $J^i(\mathbf{q}) = q_i p(\mathbf{q}) - f_i(q_i)$; $f_i(q_i) = c_i q_i + q_i^{(1+\alpha_i)}(1+\alpha_i)^{-1} K_i^{-\alpha_i}$; $i = \{1,2,\ldots,5\}$

Where: $p(\mathbf{q}) \equiv 5000^{\beta} \mathbf{Q}^{-\beta}$; $\mathbf{Q} \equiv \sum_{i=1}^{n} q_i$; $q_i \in [0, \infty)$

Model Parameters:

Firm	c_i	K_i	α_i	β_i
1	10	5	1/1.2	1/1.1
2	8	5	1/1.1	1/1.1
3	6	5	1/1.0	1/1.1
4	4	5	1/0.9	1/1.1
5	2	5	1/0.8	1/1.1

I. GENO Output

Generation	q_1	q_2	q_3	q_4	q_5
0	50.000000	43.000000	43.000000	34.000000	43.000000
10	36.875000	41.750000	43.625000	42.750000	39.250000
20	36.932678	41.818054	43.706482	42.659363	39.178894
30	36.932511	41.818142	43.706578	42.659240	39.178952
60	36.932511	41.818142	43.706579	42.659240	39.178953
80	36.932511	41.818142	43.706579	42.659240	39.178953
100	36.932511	41.818142	43.706579	42.659240	39.178953

Nash Equilibrium Solution Vector: $\mathbf{q}^* = (36.932511, 41.818141, 43.706579, 42.659240, 39.178952)^T$

II. General Remarks

This model of market competition in which the individual criteria are to be maximized, was originally formulated by Murphy, Sherali & Soyster [**27**]; it has since been tackled using various methods by Harker [**14**], Jörnsten [**16**], and Kolstad & Mathiesen [**19**]. To provide a benchmark against which the various algorithms may be assessed, a solution that is accurate to 18 decimal places was found by solving the Karush-Kuhn-Tucker conditions using and *Mathematica's* `FindRoot` equation solver. The table below clearly shows the solution by GENO to be the most accurate, and this may be partly due to that fact that, unlike the other methods, GENO has the capacity to converge onto global optima even in the absence of a fortuitous starting point.

q_i	Benchmark	Murphy, *et al.*	Harker	Jörnsten	Kolstad, *et al.*	GENO
1	36.932510815735757481	36.9120	36.93180	36.9300	36.9350	36.932511
2	41.818141660437635128	41.8200	41.81755	41.8200	41.8182	41.818142
3	43.706578522274216542	43.7050	43.7060	43.7100	43.7066	43.706579
4	42.659239743305114839	42.6650	42.6588	42.6600	42.6593	42.659240
5	39.178952516625022418	39.1820	39.1786	39.1800	39.1790	39.178953

Example 3: $\underset{\mathbf{x}}{\text{Opt}} \{ f_1(\mathbf{x}), f_2(\mathbf{x}) \}$

Subject to: $x_i \in [-2, 2], \quad \forall i \in \{1, 2, \ldots, 8\}$

Where: $$f_1(\mathbf{x}) \equiv 1 - \exp\left\{ -\sum_{i=1}^{n} \left(x_i - 1/\sqrt{n} \right)^2 \right\}$$

$$f_2(\mathbf{x}) \equiv 1 - \exp\left\{ -\sum_{i=1}^{n} \left(x_i + 1/\sqrt{n} \right)^2 \right\}$$

I. GENO Output

Generation	*Objective [1]*	*Objective [2]*
0	0.965363	0.939017
10	0.632149	0.632149
20	0.632121	0.632121
30	0.632121	0.632121
40	0.632121	0.632121
60	0.632121	0.632121
80	0.632121	0.632121
100	0.632121	0.632121

TC Solution Vector: $\mathbf{x} = (0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000)^T$

TC Solution Outcome: $\mathbf{J} = (0.632121,\ 0.632121)^T$

II. General Remarks

Preamble. The structure of this example from [**11**] is simple enough to allow an analytical determination of the Tchebycheff compromise (TC) Solution against which GENO's performance may be measured. The 'Opt' operator denotes minimization on both dimensions of the objective vector.

Tchebycheff Compromise Solution: An Analytical Approach. In order to compute the TC solution, one needs to know the location of the reference point in the space of outcomes. And because the ideal outcome is easily determined in this case, there is no need to evoke the *g*-loss transform—all evaluations pertain to an *n*-dimensional real space that contains the outcome set $\mathbf{\Omega}$, and these are as follows.

The Pareto frontier is the set of all vectors **x** that satisfy the following relation:

$$(x_1 = x_2 = \ldots = x_n) \wedge \left(-1/\sqrt{n} \le x_1 \le 1/\sqrt{n} \right) \tag{11}$$

It is easy to show that the minimum of f_1 is 0, and this is located (for all *n*) at:

$$(x_1, \ldots, x_n) = \left(1/\sqrt{n}, \cdots, 1/\sqrt{n} \right) \tag{12a}$$

Similarly, the minimum of f_2 is 0 and this is located (for all *n*) at

$$(x_1, \ldots, x_n) = \left(-1/\sqrt{n}, \cdots, -1/\sqrt{n} \right) \tag{12b}$$

The ideal solution in the space of outcomes is therefore the point $(f_1, f_2) = (0, 0)$, and clearly, this is not jointly attainable. To ascertain the point on the Pareto frontier that is closest to the ideal, one proceeds as follows.

The Tchebycheff distance to the ideal point is given by:

$$d_\infty(\mathbf{x}) = \mathrm{Max}\{|\, f_1(\mathbf{x}) - 0\,|,\ |\, f_2(\mathbf{x}) - 0\,|\} \tag{13a}$$

However, a direct evaluation of the minimum of $d_\infty(\mathbf{x})$ is in this case rather problematic because one cannot unambiguously solve equation (13a) for all $\mathbf{x}$ and so an alternative approach is required. Because the reference point is the origin, the metric d_∞ degenerates to the supremum norm

$$d_\infty(\mathbf{x}) \equiv \|\mathbf{J}(\mathbf{x})\|_\infty = \mathrm{Max}\{|\, f_1(\mathbf{x})\,|,\ |\, f_2(\mathbf{x})\,|\} \tag{13b}$$

Although the supremum norm $\|\cdot\|_\infty$ and the Euclidean norm $\|\cdot\|_2$ are not equal, they are each bounded by a constant multiple of the other, viz.:

$$\|\mathbf{J}(\mathbf{x})\|_\infty \le \|\mathbf{J}(\mathbf{x})\|_2 \le \sqrt{\mathrm{n}}\,\|\mathbf{J}(\mathbf{x})\|_\infty \quad \Leftrightarrow \quad \|\mathbf{J}(\mathbf{x})\|_2 \big/ \sqrt{\mathrm{n}} \le \|\mathbf{J}(\mathbf{x})\|_\infty \le \|\mathbf{J}(\mathbf{x})\|_2 \tag{14}$$

Furthermore, it can be shown that if a sequence $\{x_n\}$ is convergent under the metric $d_\infty(\mathbf{x})$ which is defined using the supremum norm, it is also convergent under the Euclidean metric $d_2(\mathbf{x})$, and the limits of convergence are the same [**43**, p.27]. Consequently, by virtue of this convergence property and the relations in (14), one may determine the optimal point $\mathbf{x}^*$ at which the Tchebycheff metric d_∞ attains its minimum value by using the Euclidean metric d_2 instead. The Euclidean distance to the reference point in the space of outcomes is given by:

$$d_2(\mathbf{x}) = \sqrt{\left(f_1^2(\mathbf{x}) + f_2^2(\mathbf{x})\right)} \tag{15}$$

The minimum distance occurs at a solution to the following system of equations:

$$\frac{\partial d_2(\mathbf{x})}{\partial \mathrm{x}_\mathrm{i}} = \frac{1}{2\sqrt{\left(f_1^2(\mathbf{x}) + f_2^2(\mathbf{x})\right)}}\left(2 f_1(\mathbf{x})\frac{\partial f_1(\mathbf{x})}{\partial \mathrm{x}_\mathrm{i}} + 2 f_2(\mathbf{x})\frac{\partial f_2(\mathbf{x})}{\partial \mathrm{x}_\mathrm{i}}\right) = 0, \quad \forall i \tag{16a}$$

where:

$$\frac{\partial f_1(\mathbf{x})}{\partial \mathrm{x}_\mathrm{i}} = 2\left(\mathrm{x}_\mathrm{i} - 1\big/\sqrt{\mathrm{n}}\right)\exp\left\{-\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathrm{x}_\mathrm{i} - 1\big/\sqrt{\mathrm{n}}\right)^2\right\}, \quad \forall i \tag{16b}$$

and:

$$\frac{\partial f_2(\mathbf{x})}{\partial \mathrm{x}_\mathrm{i}} = 2\left(\mathrm{x}_\mathrm{i} + 1\big/\sqrt{\mathrm{n}}\right)\exp\left\{-\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathrm{x}_\mathrm{i} + 1\big/\sqrt{\mathrm{n}}\right)^2\right\}, \quad \forall i \tag{16c}$$

It follows that, $\forall i$:

$$\frac{\partial d_2(\mathbf{x})}{\partial \mathrm{x}_\mathrm{i}} = 0 \;\Rightarrow\; \left(\mathrm{x}_\mathrm{i} - 1\big/\sqrt{\mathrm{n}}\right) f_1(\mathbf{x})\exp\left\{-\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathrm{x}_\mathrm{i} - 1\big/\sqrt{\mathrm{n}}\right)^2\right\} = -\left(\mathrm{x}_\mathrm{i} + 1\big/\sqrt{\mathrm{n}}\right) f_2(\mathbf{x})\exp\left\{-\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathrm{x}_\mathrm{i} + 1\big/\sqrt{\mathrm{n}}\right)^2\right\} \tag{17}$$

By inspection, one can see that the point $\mathbf{x}^* = \mathbf{0}$ is the *only point on the Pareto frontier* that solves equation (17), and the criterion function values are equal at this point, i.e.: $f_1(\mathbf{x}^*) = f_2(\mathbf{x}^*) = 0.632121$.

Concluding Remarks. The numerical and analytical results above clearly show that GENO is capable of converging to a point on the Pareto frontier that is closest to the reference point as measured by the Tchebycheff metric. In view of the aim of this paper, a re-iteration of the practicality of the solution is in order: (i) the method does not use the rather problematic scalarization technique; (ii) it does not require an explicit estimate of the Pareto frontier; (iii) it does not require a decision-maker to express preferences prior, post or during the solution process—rather, it is implicitly (and reasonably) assumed that the model-user is rational and would therefore prefer the ideal solution if this were attainable, but he accepts the TC solution as being the best that can be done; (iv) as can be seen from the results presented above, GENO's TC solution exhibits a "middling characteristic" which (intuition tells us) is a very desirable attribute for cooperative solutions.[7]

[7] The term 'middling' was brought to the multi-objective programming lexicon by Schaffer [**34**] who noted that his VEGA technique tended to produce solutions that excelled on one criterion, but performed poorly on others. He argues that what is desirable of a compromise solution is that it should have acceptable performance simultaneously on all criteria, i.e. a "middling performance".

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Example 4: $\underset{\mathbf{x}}{\text{Opt}} \{ f_1(\mathbf{x}), f_2(\mathbf{x}) \}$

Subject to: $x_i \in [-2, 2], \quad \forall i \in \{1, 2, \ldots, 8\}$

Where: $$f_1(\mathbf{x}) \equiv \exp\left\{-\sum_{i=1}^{n}\left(x_i - 1/\sqrt{n}\right)^2\right\} - 1$$

$$f_2(\mathbf{x}) \equiv \exp\left\{-\sum_{i=1}^{n}\left(x_i + 1/\sqrt{n}\right)^2\right\} - 1$$

I. GENO Output

Generation	*Objective [1]*	*Objective [2]*
0	-0.965363	-0.939017
10	-0.632149	-0.632149
20	-0.632121	-0.632121
30	-0.632121	-0.632121
40	-0.632121	-0.632121
60	-0.632121	-0.632121
80	-0.632121	-0.632121
100	-0.632121	-0.632121

TC Solution Vector: $\mathbf{x} = (0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000,\ 0.000)^T$

TC Solution Outcome: $\mathbf{J} = (-0.632121,\ -0.632121)^T$

II. General Remarks

This is essentially a repeat of Example 3, but in this case (a) both criteria are reversed in sign and are required to be maximized, (b) the *g*-loss transform is applied, and (c) the TC solution is computed with respect to the aspiration point which, according to Definition 4, is the binary vector $(\pi_1, \pi_2) = (1, 1)$. Note that the ideal point in the space of untransformed outcomes is the same as for Example 3, namely $(f_1, f_2) = (0, 0)$, and it is not jointly attainable. And although one could employ a theoretical analysis similar to that presented in the case of Example 3 to establish the compromise solution with respect to the aspiration point, in this particular case, symmetry allows one to correctly assert that the Pareto point closest to the ideal outcome is exactly the same that closest to the aspiration point. As shown previously, the point $\mathbf{x}^* = \mathbf{0}$ is the only point on the Pareto frontier that is closest to the ideal point and, by the symmetry, closest to the aspiration point as well, and the criterion function values are equal at this point are equal, namely, $f_1(\mathbf{x}^*) = f_2(\mathbf{x}^*) = –0.632121$.

Of course not all problem instances are going to be as symmetrical as this example, and so the solution computed via TC Solution II will in general not coincide with that based on TC Solution I of Definition 4. Although the rationale for latter is logically more compelling—in the sense that individual components of the ideal outcome are potentially achievable—this should be traded-off, on a case by case basis, against the ease of implementing TC Solution II: it may be recalled that in order to compute TC Solution I one needs to know the ideal point $\mathbf{z}_i$, and this often requires one to perform some auxiliary calculations with the implicit assumption that the algorithm employed in the said computations converges to the global solution—a condition that cannot be guaranteed; on the other hand, the *g*-loss approach does not require auxiliary computations to determine the reference point—one merely "reads-off" the aspiration point from the problem definition itself. And of course, all the points on 'practicality of the TC solution' that are mentioned in the concluding remarks to the previous example equally apply to the solution computed by the transform method.

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Example 5: Combustion Application [44, §6.6]

Solve the System:

$$x_2 + 2x_6 + x_9 + 2x_{10} = 10^{-5}$$

$$x_3 + x_8 = 3 \cdot 10^{-5}$$

$$x_1 + x_3 + 2x_5 + 2x_8 + x_9 + 2x_{10} = 5 \cdot 10^{-5}$$

$$x_4 + 2x_7 = 10^{-5}$$

$$0.5140437 \cdot 10^{-7} x_5 = x_1^2$$

$$0.1006932 \cdot 10^{-6} x_6 = 2x_2^2$$

$$0.7816278 \cdot 10^{-15} x_7 = x_4^2$$

$$0.1496236 \cdot 10^{-6} x_8 = x_1 x_3$$

$$0.6194411 \cdot 10^{-7} x_9 = x_1 x_2$$

$$0.2089296 \cdot 10^{-14} x_{10} = x_1 x_2^2$$

Subject to: $x_i \in [-10.0, 10.0]$, $i \in \{1, 2, \dots, 10\}$

I. Reformulation

In [41] it is formally shown that an equivalence exists between any system of linear or nonlinear equations and a particular multi-objective optimization problem—readers are referred to that paper for further details; the same paper also introduces the 'variable endogenization' technique[8] that was applied on the last six equations above leading to an 'internalization' of x_5, x_6, x_7, x_8, x_9, and x_{10}; the final equivalent multi-objective program to the equation system above is:

Given:

$$z_1 = x_1^2 \,/\, 0.5140437 \cdot 10^{-7}$$

$$z_2 = 2x_2^2 \,/\, 0.1006932 \cdot 10^{-6}$$

$$z_3 = x_4^2 \,/\, 0.7816278 \cdot 10^{-15}$$

$$z_4 = x_1 x_3 \,/\, 0.1496236 \cdot 10^{-6}$$

$$z_5 = x_1 x_2 \,/\, 0.6194411 \cdot 10^{-7}$$

$$z_6 = x_1 x_2^2 \,/\, 0.2089296 \cdot 10^{-14}$$

$$\operatorname*{Opt}_{\mathbf{x}} \left\{ |c_1(\mathbf{x})|, |c_2(\mathbf{x})|, |c_3(\mathbf{x})|, |c_4(\mathbf{x})| \right\}$$

Subject to:

$$c_1 \equiv x_2 + 2z_2 + z_5 + 2z_6 - 10^{-5} = 0$$

$$c_2 \equiv x_3 + z_4 - 3 \cdot 10^{-5} = 0$$

$$c_3 \equiv x_1 + x_3 + 2z_1 + 2z_4 + z_5 + 2z_6 - 5 \cdot 10^{-5} = 0$$

$$c_4 \equiv x_4 + 2z_3 - 10^{-5} = 0$$

$x_i \in [-10.0, 10.0]$, $i \in \{1, 2, 3, 4\}$;

$z_i \in [-10.0, 10.0]$, $i \in \{1, 2, \dots, 6\}$

[8] This technique is described more substantively in [38, 39]

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II. Solution Method

Assume each element $|c_i|$ of the criterion set $\{|c_1|, |c_2|, |c_3|, |c_4|\}$ is finite for all **x** and let S_i denote its supremum; in that case, the outcome set is given by $\boldsymbol{\Omega} = \left\{ \mathbf{C} = (c_1, \cdots, c_4)' \in \mathbf{R}^4 \;\middle|\; |c_i| \in [0, S_i) \right\}$; and if the equation system is well-posed, then there exists at least one **x*** such that $\forall i$, $|c_i(\mathbf{x}^*)| = 0$; in other words, the outcome set $\boldsymbol{\Omega}$ must have a vertex at the origin of $\mathbf{R}^4$. The Pareto frontier in this case is a singleton, namely the point **0** in of $\mathbf{R}^4$, this being *the one and only point* that is not dominated (in the 'less-than' sense) by any other in $\boldsymbol{\Omega}$. Furthermore, the Pareto frontier happens to coincide with the ideal point as per DEFINITION 3, which means that the ideal outcome *is jointly attainable and of a known location*, and so one may evoke TC SOLUTION I of DEFINITION 4.

III. GENO Output

Generation	*Time*	c_1	c_2	c_3	c_4
0	0.00	4.60083184	0.79997909	2.49997057	0.69999000
10	7.07	0.00936500	0.00309500	0.00932500	0.00311500
20	7.00	0.00000084	0.00000052	0.00000117	0.00000221
30	7.22	0.00000004	0.00000000	0.00000000	0.00000004
40	7.21	0.00000000	0.00000000	0.00000000	0.00000000
60	5.57	0.00000000	0.00000000	0.00000000	0.00000000
80	7.18	0.00000000	0.00000000	0.00000000	0.00000000
100	6.99	0.00000000	0.00000000	0.00000000	0.00000000

TC Solution Vector: **x*** = $(0.00002000, 0.00001000, 0.00003000, 0.00001000)^T$

TC Endogenous Vector: **z*** = $(0.00000000, 0.00000000, 0.00000000, 0.00000000, 0.00000000, 0.00000000)^T$

TC Outcome Vector: **C**(**x***) = $(0.00000000, 0.00000000, 0.00000000, 0.00000000)^T$

Average execution time per 10 generations: 6.94 seconds

Overall execution time on 100 generations: 69.44 seconds

Approximate time to first optimum: 27.76 seconds

IV. General Remarks

Table 2 lists solutions (in outcome space) found by the evolutionary algorithm (EA-GA) in [**13**]; by the probability-driven algorithm (PDA) in [**30**]; and by GENO. Run-times for the EA-GA and PDA algorithms were 151.12 and 30 seconds respectively, whereas GENO attains a 27.76 sec 'time-to-first optimum' [**41**, p.13]; and as regards the quality of the final solution, the results displayed below "speak" for themselves.

Table 2: Comparative Solutions to Example 5

METHOD	EA-GA	PDA	GENO
f_1(x*)	0.0274133880	0.0000000000	0.0000000000
f_2(x*)	0.0841848522	0.0000000000	0.0000000000
f_3(x*)	0.1482418892	0.0000000000	0.0000000000
f_4(x*)	0.0839188566	0.0000000000	0.0000000000
f_5(x*)	-0.0030517851	-0.0000000881	0.0000000000
f_6(x*)	-0.0000109317	-0.0000000753	0.0000000000
f_7(x*)	-0.0165644486	0.0000000000	0.0000000000
f_8(x*)	0.0025184283	0.0000001896	0.0000000000
f_9(x*)	-0.0001291516	-0.0000002470	0.0000000000
f_{10}(x*)	0.0000003019	0.0000000000	0.0000000000

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Example 6: Portfolio Optimization [40]

$$\underset{\mathbf{u}}{\mathbf{Opt}} \ \{g(\mathbf{u}), h(\mathbf{u})\}$$

Subject to:
$$x_{k+1} = x_k - u_k; \qquad k \in \{1, 2, \ldots, 5\}$$
$$x_1 = 1$$
$$x_{n+1} = 0$$
$$\sum_{k=1}^{n} u_k = 1$$
$$u_k \in [0,1]$$

Where: $g(\mathbf{u}) = \langle \mathbf{r}, \mathbf{u} \rangle = \sum_{k=1}^{5} r_k u_k$ is the expected portfolio return (the 'reward' function);

$h(\mathbf{u}) = \sum_{t=1}^{5} \sum_{k=1}^{5} u_t u_k c_{tk}$ is the portfolio variance (the 'risk' function)

Tables 3A: Expected Annual Returns (%) [**21**, p.163]

Asset Name	Security 1	Security 2	Security 3	Security 4	Security 5
Mean Return	15.1	12.5	14.7	9.02	17.68

Tables 3B: Covariance Matrix [**21**, p.163]

Covariance Matrix	Security 1	Security 2	Security 3	Security 4	Security 5
Security 1	2.30	0.93	0.62	0.74	-0.23
Security 2	0.93	1.40	0.22	0.56	0.26
Security 3	0.62	0.22	1.80	0.78	-0.27
Security 4	0.74	0.56	0.78	3.40	-0.56
Security 5	-0.23	0.26	-0.27	-0.56	2.60

I. Reformulation and Solution Method

The Markowitz portfolio optimization problem [24, 25] belongs to a "genre" of models in which a trade-off between 'risk' and 'reward' is required—a natural problem-domain for the compromise solution concept. The rationale underlying *this* particular formulation is presented in full in [40]. And with the reward function defined as above, one is compelled to evoke TC Solution II of Definition 7. But, in this particular case where asset returns are positive and at least one of the decision variables u_i is required to be non-zero, the reward function $g(\mathbf{u})$ is strictly positive, and a simple reformulation of the problem significantly simplifies the solution process: if one re-defines the 'reward' function as $g(\mathbf{u}) \equiv \langle \mathbf{r}, \mathbf{u} \rangle^{-1}$—the reciprocal of the portfolio return, which in some sense, represents a "loss" that must be *minimised*—then the (unattainable) infima for both the reward and risk functions is '0', and the outcome set $\mathbf{\Omega}$ is thus the positive quadrant of $\mathbf{R}^2$ but with a hyperbolic lower boundary whose asymptotes are the abscissa and ordinate axes. Thus, one may view the origin of $\mathbf{R}^2$ as *the* aspiration (and reference) point and legitimately evoke TC Solution I of Definition 4.

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II. GENO Output

Generation Number	*Time[9] (sec)*	*Portfolio "Loss"*	*Portfolio Risk*
0	0.00	0.06452204	0.95838589
10	4.13	0.06935328	0.79053227
20	4.10	0.06935326	0.79053228
30	4.15	0.06935326	0.79053228
40	4.15	0.06935326	0.79053228
50	4.13	0.06935326	0.79053228
60	4.09	0.06935326	0.79053228
70	4.12	0.06935326	0.79053228
80	4.09	0.06935326	0.79053228
90	4.12	0.06935326	0.79053228
100	4.07	0.06935326	0.79053228

Asset Name:	SECURITY 1	SECURITY 2	SECURITY 3	SECURITY 4	SECURITY 5
Allocation:	0.088790530	0.250125980	0.282629880	0.103244140	0.275209470

Portfolio Return (%):	14.41893378
Portfolio Risk:	0.79053228
Return-Risk Ratio:	18.23952574

III. General Remarks

Luenberger [21] solves this problem by the sequential quadratic programming method using the minimum variance version of the mean-variance optimization model [40]—the quality characteristics of his solution are as shown in Table 5 under L-PORTFOLIO. GENO generates a sequence that converges to a different portfolio whose quality characteristics are those under G-PORTFOLIO.

Table 5: A Comparative Analysis for **Example 6**

	G-PORTFOLIO	L-PORTFOLIO
PORTFOLIO RETURN (%)	14.41893378	14.41178000
PORTFOLIO RISK	0.79053228	0.79053140
REWARD-RISK RATIO	18.23952574	18.23049660
COMPROMISE METRIC	0.79356862	0.79357076

These portfolios yield comparable performances. But according to the portfolio evaluation criterion explained elsewhere, [10] it is clear that the G-PORTFOLIO, which has the lower compromise metric value of 0.79356862, is the better option. Note also that portfolio G-PORTFOLIO is only marginally poorer in terms of the variance outcome than portfolio L-PORTFOLIO that was computed by Luenberger by specifically minimizing the risk function.

[9] The execution times pertain to a GAUSS version of GENO running under Windows 7 on a Desktop machine with the following hardware specs: AMD A4-5000 APU Processor, 1.5GHz, 4GB RAM. The mating population used was of size 30

[10] See §8 in [40]

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9 Summary and Conclusions

This article has presented an exposition of the multi-objective optimization problem and its solutions. Two types of solution have been identified as being practical, namely the Nash equilibrium and the Tchebycheff compromise solution: the former pertains to scenarios in which the components of the vector-valued criterion function may be said to "compete" whereas the later pertains to scenarios where there is "cooperation"; and in any given problem instance, the solution concept to be adopted must be decided on the basis of whether one can reasonably discern competition or cooperation in the problem at hand.

The Nash equilibrium solution reverts to the commonly-understood optimum when the dimension of the objective vector is one. But this should not fool one into believing that one can simply adapt any nonlinear programming algorithm into a Nash equilibrium solver because algorithms that only guarantee convergence to local optima are unlikely to converge to a Nash equilibrium point. In order to discern equilibrium points, i.e. those points in the search space at which "there is no individual incentive to depart from", it is imperative for 'Nash equilibrium solvers' to have the capacity for converging to global optima. Example 1 and 2 as well as others reported in [36] show that GENO readily converges to Nash equilibria.

Cooperative solutions to the multi-objective optimization problem are ultimately based on the notion of Pareto-dominance. But this criterion alone is usually not sufficient to delineate a unique solution point; thus, cooperative solutions are typically presented as a set known as the 'efficient' or 'Pareto' frontier that is comprised of non-dominated points in outcome space; and sometimes, a subset of the Pareto frontier called negotiation set is used instead. The quandary that the multiplicity of solution points poses has traditionally been addressed by requiring a decision-maker to articulate preferences at some stage during the solution process. And to assist the process of selecting the final solution, most research has tended to emphasize the development of algorithms that generate a well sampled Pareto frontier. But it has been argued that this approach is problematic on several counts: (i) generating the entire Pareto frontier is a waste of effort since in the end only one solution point is all that is required; (ii) the decision-maker may not be able to select a solution point that properly represents his or her preferences because presenting a visually discernible Pareto set is difficult except in the bi-objective case; (iii) eliciting preferences is only viable on small problems; (iv) and in any case, some problem settings are such as that there cannot be a model-user present to articulate any preferences.

Accordingly, this paper advocates for an automatic method for selecting a single point from the Pareto frontier based on the compromise solution concept. The generalized loss (or g-loss) transform has been proffered to address implementation issues associated with the compromise solution as originally conceived; the g-loss transform effectively "automates" the determination of the reference vector required by the compromise solution method. Empirical evidence—in the form of Example 3 and Example 4 as well as several others reported in [36]—suggests that the g-loss transform technique is effective at generating single-point cooperative solutions that are logically sound. The multi-objective solution techniques presented herein also apply to systems of (linear or nonlinear) equations [41].

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