# Kusuoka Representations of Coherent Risk Measures in General Probability Spaces

Nilay Noyan and Gábor Rudolf

Manufacturing Systems/Industrial Engineering Program, Sabancı University, 34956 Istanbul, Turkey,

nnoyan@sabanciuniv.edu and grudolf@sabanciuniv.edu

Abstract: Kusuoka representations provide an important and useful characterization of law invariant coherent risk measures in atomless probability spaces. However, the applicability of these results is limited by the fact that such representations do not always exist in probability spaces with atoms, such as finite probability spaces. We introduce the class of functionally coherent risk measures, which allow us to use Kusuoka representations in any probability space. We show that this class contains every law invariant risk measure that can be coherently extended to a family containing all finite discrete distributions. Thus, it is possible to preserve the desirable properties of law invariant coherent risk measures on atomless spaces without sacrificing generality. We also specialize our results to risk measures on finite probability spaces, and prove a denseness result about the family of risk measures with finite Kusuoka representations.

*Keywords*: Kusuoka representation; coherent risk measures; spectral risk measures; law invariance; comonotonicity

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**1. Introduction** Risk measures are functionals that represent the risk associated with a random variable by a scalar value. Desirable properties of risk measures, such as *coherence*, *comonotone additivity*, and *law invariance* have been axiomatized starting with the work of Artzner et al. (1999).

The seminal paper by Kusuoka (2001), in conjunction with the additional technical details provided in Jouini et al. (2006), establishes two important representation theorems for risk measures on bounded variables in atomless probability spaces. One theorem states that the class of law invariant coherent and comonotone additive risk measures coincides with the class of spectral risk measures, and thus can be represented as a mixture of conditional value-at-risk (CVaR) measures. The other theorem states that all law invariant coherent risk measures can be represented as the infimum of a family of spectral measures. These results were later extended to $\mathcal{L}^p$ spaces of random variables on atomless spaces for any $p \in [1, \infty]$, see e.g. Pflug and Römisch (2007) and Shapiro et al. (2009).

Despite the significant recent interest in these representations (see, e.g., Dentcheva et al., 2010; Pichler and Shapiro, 2012; Shapiro, 2013), to the best of our knowledge only one study (Leitner, 2005) focuses on characterizations for general (i.e., not necessarily atomless) probability spaces. The chief difficulty in extending Kusuoka's theorems lies in the fact that in the presence of atoms there exist certain "pathological" law invariant coherent risk measures for which representations of the desired form are not possible. To exclude these ill-behaved risk measures, in this paper we introduce the class of *functionally coherent* risk measures, and show that this class always coincides with the class of risk measures which admit Kusuoka representations. We also provide an analogous result that shows the equivalence between the class of functionally coherent and comonotone additive risk measures, and the class of spectral risk measures. In addition, we show that the assumption of functional coherence is not overly restrictive, as every law invariant coherent risk measure that can be evaluated on all finite discrete distributions is functionally coherent. A distinguishing feature of our approach is to view law invariant risk measures as mappings from the space of cumulative distribution functions (a similar approach is briefly used in Pichler and Shapiro, 2012).

Finite probability spaces are often important in practical contexts, in particular when using a scenario-based approach. For such spaces we prove that the mixtures of CVaR measures in Kusuoka representations can be replaced by finite convex combinations. In the equal probability case these combinations take a special form, where the number of terms is bounded from above by the size of the sample space. We also examine the class of risk measures where the infimum in the Kusuoka representation can be replaced by a finite minimum, and show that, over bounded families of random variables, they are dense among functionally coherent risk measures. These finitely representable risk measures are computationally tractable, which makes them highly useful in applications. For instance, Noyan and Rudolf (2013) develop a finite convergent solution method for optimization problems with constraints that are based on such risk measures.

The rest of the paper is structured as follows. In Section 2 we establish necessary notation and recall some basic definitions. In Section 3 we review existing Kusuoka representations for various classes of coherent risk measures, and provide a result for the case of finite probability spaces where every elementary event has the same probability. We introduce the class of functionally coherent risk measures in Section 4, show that they have Kusuoka representations, and then specialize these results for finite probability spaces. In Section 5 we describe a general framework for coherently extending mappings on families of distribution functions, then use this framework to argue the generality of the class of functionally coherent risk measures. Section 6 contains our concluding remarks.

**2. Notation and basic concepts** Let us begin by establishing some notation. Throughout this paper, let $\mathcal{S} = (\Omega_{\mathcal{S}}, \mathcal{A}_{\mathcal{S}}, \Pi_{\mathcal{S}})$ denote a standard atomless probability space (for instance, the interval $[0,1]$ equipped with the Borel sigma algebra and the Lebesgue measure). Let us denote the set of all cumulative distribution functions (CDFs) by

$$\mathfrak{F} = \left\{ F : \mathbb{R} \to [0,1] \,\middle|\, \lim_{v\to-\infty} F(v) = 0,\ \lim_{v\to\infty} F(v) = 1,\ F \text{ is right-continuous and non-decreasing} \right\}.$$

Note that, as standard uniform random variables exist on any atomless space (see, e.g., Föllmer and Schied, 2004), using probability integral transforms we obtain $\mathfrak{F} = \{F_V \,|\, V : \Omega \to \mathbb{R} \text{ is measurable}\}$. Let us denote the set of $p$-integrable random variables on $\mathcal{S}$ for some $p \in [1,\infty]$ by $\mathcal{L}^p(\mathcal{S})$, and the corresponding set of CDFs by $\mathfrak{F}^p = \{F_V \,|\, V \in \mathcal{L}^p(\mathcal{S})\}$. Note that $\mathfrak{F}^p$ does not depend on the choice of $\mathcal{S}$.

We say that random variables $V_1$, $V_2$ are equal in distribution if we have $F_{V_1} = F_{V_2}$, and denote this fact by $V_1 \sim V_2$. The relation $V_1 \leq V_2$ denotes inequality everywhere, i.e., that $V_1(\omega) \leq V_2(\omega)$ holds for all $\omega \in \Omega$.

The extended real line is denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For a value $x \in \mathbb{R}$ we denote the Dirac delta measure concentrated on $x$ by $\delta_x$.

Before proceeding to give our basic definitions, we need to make a note of some conventions used throughout this paper. We consider larger values of random variables to be preferable, while higher values of risk measures indicate less risky random outcomes. In this context, risk measures are often referred to as *acceptability functionals*. In the literature the opposite conventions are also widespread, especially when dealing with loss functions. When citing such sources, the definitions and formulas are altered to reflect this difference.

Let us now recall some well-known definitions, starting with two important risk measures that serve as the basis of the representations we consider in the following sections.

- Let $V$ be a random variable with a CDF denoted by $F_V$. The *value-at-risk* (VaR) at confidence level $\alpha \in (0,1]$ is defined as
$$\mathrm{VaR}_\alpha(V) = \inf\{\eta \ : \ F_V(\eta) \geq \alpha\}. \tag{1}$$
- The *conditional value-at-risk* at confidence level $\alpha$ is given by
$$\mathrm{CVaR}_\alpha(V) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\gamma(V)\,\mathrm{d}\gamma. \tag{2}$$
CVaR is also known in the literature as *average value-at-risk* and *tail value-at-risk*.
- We formally define $\mathrm{VaR}_0(V) = \mathrm{CVaR}_0(V) = \operatorname{ess\,inf}(V)$.

$\mathrm{VaR}_\alpha(V)$ is also known as the $\alpha$-quantile of the random variable $V$. Accordingly, the first quantile function $F_V^{-1} : (0,1] \to \mathbb{R}$ of $V$ is defined as $F_V^{-1}(\alpha) = \mathrm{VaR}_\alpha(V)$. We recall the following well-known facts:

- The mapping $F_V \mapsto F_V^{-1}$ is order reversing, i.e., the relation $F_{V_1} \leq F_{V_2}$ implies $F_{V_1}^{-1} \geq F_{V_2}^{-1}$.
- *Probability integral transform*: For any random variable $V$ and standard uniform random variable $U$ we have $V \sim F_V^{-1}(U)$.

The *generalized Lorenz curve* $G_V : (0,1] \to \mathbb{R}$ of a random variable $V$ is defined as $G_V(\alpha) = \alpha\,\mathrm{CVaR}_\alpha(V)$. Note that according to (2) the generalized Lorenz curve is continuous, since it is the integral of the first quantile function. The next lemma will be critical in establishing certain finiteness results.

Lemma 2.1 *Given a probability space $(\Omega, \mathcal{A}, \Pi)$, let $\mathcal{K} = \{\Pi(S) \ : \ S \in \mathcal{A},\ \Pi(S) > 0\}$ denote the set of all non-zero probabilities of events, and consider an interval $I \subset [0,1]$ such that $(\operatorname{int} I) \cap \mathcal{K} = \emptyset$, where $\operatorname{int} I$ denotes the interior of $I$.*

*(i) The quantile function $F_V^{-1}$ is constant on the interval $I$.*

(*ii*) *The generalized Lorenz curve $G_V$ is affine on the interval $I$. Furthermore, if the left endpoint of $I$ is 0, then $G_V$ is linear on $I$.*

Proof. Consider a value $\alpha \in I$. As the level sets of the random variable $V$ are measurable, both of the values

$$\alpha_- = \Pi\left(V < \mathrm{VaR}_\alpha(V)\right) \qquad \text{and} \qquad \alpha_+ = \Pi\left(V \leq \mathrm{VaR}_\alpha(V)\right)$$

are in $\in \mathcal{K} \cup \{0\}$. By the definition of VaR we have $\alpha_- \leq \alpha \leq \alpha_+$, implying $I \subset [\alpha_-, \alpha_+]$. Since $\mathrm{VaR}_\gamma(V) = \mathrm{VaR}_\alpha(V)$ holds for any $\gamma \in (\alpha_-, \alpha_+]$, part (i) immediately follows. Part (ii) is now a direct consequence of (2). □

**3. Coherent risk measures and Kusuoka representations** In this section we first define some important classes of risk measures, and then discuss representations of such measures based on the seminal results of Kusuoka (2001).

For $p \in [1, \infty]$, let $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{A}, \Pi)$ denote the family of $p$-integrable random variables on some probability space $(\Omega, \mathcal{A}, \Pi)$. We note that for finite probability spaces $\mathcal{L}^p$ coincides with the set $\mathcal{V} = \mathcal{V}(\Omega, \mathcal{A}, \Pi)$ of *all* random variables. Following the influential work of Artzner et al. (1999), we say that a mapping $\rho : \mathcal{L}^p \to \overline{\mathbb{R}}$ is a *coherent risk measure* if it has the following properties (for all $V, V_1, V_2 \in \mathcal{L}^p$):

- *Normalized*: $\rho(0) = 0$.
- *Monotone*: $V_1 \leq V_2 \;\Rightarrow\; \rho(V_1) \leq \rho(V_2)$.
- *Superadditive*: $\rho(V_1 + V_2) \geq \rho(V_1) + \rho(V_2)$.
- *Positive homogeneous*: $\rho(\lambda V) = \lambda \rho(V)$ for all $\lambda > 0$.
- *Translation invariant*: $\rho(V + \lambda) = \rho(V) + \lambda$.

A pair $V_1$, $V_2$ of random variables is said to be *comonotone*, if the following condition holds.

$$\left(V_1(\omega_1) - V_1(\omega_2)\right)\left(V_2(\omega_1) - V_2(\omega_2)\right) \geq 0 \qquad \Pi(\mathrm{d}\omega_1) \otimes \Pi(\mathrm{d}\omega_2)\text{-almost surely.} \tag{3}$$

It is easy to verify that if $V_1$ and $V_2$ are comonote then they are *quantile additive* (see, e.g., McNeil et al., 2005):

$$F_{V_1}^{(-1)}(\alpha) + F_{V_2}^{(-1)}(\alpha) = F_{V_1+V_2}^{(-1)}(\alpha) \quad \text{for all } \alpha \in (0, 1]. \tag{4}$$

If the probability space is atomless, (3) holds if and only if there exists a standard uniform random variable $U$ such that we have $(V_1, V_2) \sim \left(F_{V_1}^{(-1)}(U), F_{V_2}^{(-1)}(U)\right)$. We say that a risk measure $\rho$ is *comonotone additive* if it has the following property.

- *comonotone additive*: $V_1$, $V_2$ comonotone $\Rightarrow\;\rho(V_1 + V_2) = \rho(V_1) + \rho(V_2)$.

Let us denote the family of CDFs for $p$-integrable random variables by $\mathcal{F}^p = \mathcal{F}^p(\Omega, \mathcal{A}, \Pi) = \{F_V \,:\, V \in \mathcal{L}^p(\Omega, \mathcal{A}, \Pi)\}$, and again note that for a finite probability space $\mathcal{F}^p(\Omega, \mathcal{A}, \Pi)$ coincides with the family $\mathcal{F}(\Omega, \mathcal{A}, \Pi)$ of the CDFs of all random variables on $(\Omega, \mathcal{A}, \Pi)$. We say that a mapping $\rho : \mathcal{L}^p \to \overline{\mathbb{R}}$

is *law invariant* if the value $\rho(V)$ depends only on the distribution of the random variable $V$, i.e., if there exists a mapping $\varphi_\rho : \mathcal{F}^p \to \overline{\mathbb{R}}$ such that $\rho(V) = \varphi_\rho(F_V)$ holds for all $V \in \mathcal{L}^p$. Note that in this case $\varphi_\rho$ is uniquely determined by $\rho$.

It is well known (Pflug, 2000) that CVaR is a law invariant coherent risk measure. In the next subsection we present several results indicating that CVaR can be viewed as a basic building block of other law invariant coherent risk measures.

**3.1 Kusuoka representations** In this section we provide an overview of existing results in the literature that establish the existence of Kusuoka representations for coherent risk measures under various conditions. We begin with Kusuoka's original theorem; the result for $\mathcal{L}^\infty$ is due to Kusuoka (2001) and Jouini et al. (2006), while the proof for other $\mathcal{L}^p$-spaces can be found in Pflug and Römisch (2007) and Shapiro et al. (2009).

THEOREM 3.1 *Consider the atomless space $\mathcal{S}$ and a value $p \in [1, \infty]$.*

*(i) A mapping $\rho : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ is a law invariant coherent risk measure if and only if there exists a family $\mathcal{M} \subset \mathcal{P}([0,1])$ of probability measures on the interval $[0,1]$ such that*

$$\rho(V) = \inf_{\mu \in \mathcal{M}} \int_{[0,1]} \mathrm{CVaR}_\alpha(V)\, \mu(\mathrm{d}\alpha) \quad \textit{for all} \quad V \in \mathcal{L}^p. \tag{5}$$

*(ii) A mapping $\rho : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ is a law invariant coherent and comonotone additive risk measure if and only if there exists a probability measure $\mu \in \mathcal{P}([0,1])$ on the interval $[0,1]$ such that*

$$\rho(V) = \int_{[0,1]} \mathrm{CVaR}_\alpha(V)\, \mu(\mathrm{d}\alpha) \quad \textit{for all} \quad V \in \mathcal{L}^p. \tag{6}$$

Mappings with a representation of the form (6) are also known as *spectral risk measures*, and can be expressed as

$$\rho(V) = \int_{[0,1]} F_V^{(-1)}(\alpha)\, \nu(\mathrm{d}\alpha) \quad \text{for all } V \in \mathcal{L}^p,$$

where the measure $\nu \in \mathcal{P}([0,1])$ is defined as

$$\nu(A) = \lambda \delta_0(A) + (1-\lambda) \int_A \phi(\alpha)\, \mathrm{d}\alpha \tag{7}$$

for some $\lambda \in [0,1]$ and some non-increasing function $\phi : [0,1] \to \mathbb{R}_+$ satisfying $\|\phi\|_1 = 1$. We mention that in finite probability spaces we have $F_V^{(-1)}\Big|_{[0,\epsilon]} \equiv F_V^{(-1)}(\epsilon)$ for $\epsilon = \min\{\Pi(\omega)\,|\,\omega \in \Omega,\ \Pi(\omega) > 0\} > 0$. Accordingly, in this case we can assume without loss of generality that $\lambda = 0$ in the formula (7). We remark that the representation (6) is sometimes called a *Choquet representation* (Pflug and Römisch, 2007), while risk measures with such representations are also known as *distortion risk measures* (Bertsimas and Brown, 2009).

Theorem 3.1 provides representations in atomless spaces. For general probability spaces Leitner (2005) shows that Kusuoka representations of the form (5) exist for coherent risk measures with the Fatou property that are consistent with second-order stochastic dominance. We mention that the consistency condition implies law invariance, but the reverse implication does not hold in general (for instance, the law invariant coherent risk measure $\hat{\rho}$ in Example 4.1 is not consistent with second-order dominance).

The proof of Theorem 3.1 can naturally be adapted to finite probability spaces with equally likely elementary events. This was pointed out (without a formal proof) for representations of the form (5) in Pichler and Shapiro (2012), while Bertsimas and Brown (2009) uses a different approach to prove the existence of representations of the form (6). For the sake of completeness we state these results in the next subsection, and fill a small gap in the literature by providing a proof for the first type of representation.

**3.2 Finite spaces with equal probabilities** Let us establish some preliminaries, starting with the well-known *risk envelope representation* for coherent risk measures. The proof of the next theorem is due to Artzner et al. (1999); we mention that analogous results exist for more general probability spaces (see, e.g., Pflug and Römisch, 2007).

Theorem 3.2 *Let $(\Omega, 2^\Omega, \Pi)$ be a finite probability space. For every coherent risk measure $\rho : \mathcal{V}(\Omega, 2^\Omega, \Pi) \to \mathbb{R}$ there exists a* risk envelope $\mathcal{Q} \subset \{Q \in \mathcal{V} \ : \ Q \geq 0, \ \mathbb{E}(Q) = 1\}$ *such that*

$$\rho(V) = \inf_{Q \in \mathcal{Q}} \mathbb{E}(QV) \quad \textit{holds for all } V \in \mathcal{V}. \tag{8}$$

The next lemma is an easy consequence of the Hardy-Littlewood-Polya inequality. We remark that this lemma can be utilized to obtain an analogous statement for atomless spaces (Kusuoka, 2001; Shapiro et al., 2009), which provides a key step in the proof of Kusuoka's theorem.

Lemma 3.1 *Consider two random variables $Q$, $V$ on a finite probability space $(\Omega, 2^\Omega, \Pi)$, where $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\Pi(\{\omega_i\}) = \frac{1}{n}$. Let $q_1 \leq \cdots \leq q_n$ be an ordering of the realizations $Q(\omega_1), \ldots, Q(\omega_n)$, and similarly let $v_1 \leq \cdots \leq v_n$ be an ordering of $V(\omega_1), \ldots, V(\omega_n)$. Then, we have*

$$\inf_{\hat{V} \sim V, \hat{V} \in \mathcal{V}} \mathbb{E}(Q\hat{V}) = \frac{1}{n} \sum_{i=1}^{n} q_{n-i+1} v_i. \tag{9}$$

We are now ready to state the main result of this subsection. Our proof of the first part of the following theorem closely follows the one for the atomless case (Shapiro et al., 2009, Theorem 6.24). The second part is due to Bertsimas and Brown (2009). Alternatively, it is possible to prove part (ii) from part (i) by again adapting the proof for the atomless case (Kusuoka, 2001, Theorem 7).

Theorem 3.3 *Consider a finite probability space $(\Omega, 2^\Omega, \Pi)$, where $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\Pi(\{\omega_i\}) = \frac{1}{n}$.*

*(i) A mapping $\rho : \mathcal{V} \to \overline{R}$ is a law invariant coherent risk measure if and only if it has a Kusuoka representation of the form*

$$\rho(V) = \inf_{\boldsymbol{\mu} \in \mathcal{M}} \sum_{i=1}^{n} \mu_i \, \mathrm{CVaR}_{\frac{i}{n}}(V), \qquad V \in \mathcal{V}, \tag{10}$$

*for some family of vectors $\mathcal{M} \subset \{\boldsymbol{\mu} \in [0,1]^n \ : \ \|\boldsymbol{\mu}\|_1 = 1\}$.*

*(ii) A mapping $\rho : \mathcal{V} \to \overline{R}$ is a law invariant coherent and comonotone additive risk measure if and only if it has a Kusuoka representation of the form*

$$\rho(V) = \sum_{i=1}^{n} \mu_i \operatorname{CVaR}_{\frac{i}{n}}(V), \qquad V \in \mathcal{V},$$

*for some vector $\boldsymbol{\mu} \in [0,1]^n$ which satisfies $\|\boldsymbol{\mu}\|_1 = 1$.*

Proof. To show the non-trivial direction in part (i), we first note that according to Theorem 3.2 the risk measure $\rho$ has a risk envelope representation of the form (8). Since $\rho$ is law invariant, for any random variable $V$ we then have

$$\rho(V) = \inf_{\hat{V} \sim V, \hat{V} \in \mathcal{V}} \rho(\hat{V}) = \inf_{\hat{V} \sim V, \hat{V} \in \mathcal{V}} \inf_{Q \in \mathcal{Q}} \mathbb{E}(Q\hat{V}).$$

Changing the order of the infimums and substituting (9) we obtain

$$\rho(V) = \inf_{Q \in \mathcal{Q}} \inf_{\hat{V} \sim V, \hat{V} \in \mathcal{V}} \mathbb{E}(Q\hat{V}) = \inf_{Q \in \mathcal{Q}} \frac{1}{n} \sum_{i=1}^{n} q_{n-i+1} v_i, \tag{11}$$

where $v_1 \leq \cdots \leq v_n$ is an ordering of the realizations $V(\omega_1), \ldots, V(\omega_n)$, while for a random variable $Q \in \mathcal{Q}$ we let $q_1 \leq \cdots \leq q_n$ denote an ordering of $Q(\omega_1), \ldots, Q(\omega_n)$. Introducing the non-negative parameters $\Delta_1 = q_1$ and $\Delta_j = q_j - q_{j-1}$ for $j = 2, \ldots, n$, we have the following chain of equalities:

$$\frac{1}{n} \sum_{i=1}^{n} q_{n-i+1} v_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \Delta_j v_i = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n-j+1} \Delta_j v_i$$

$$= \frac{1}{n} \sum_{j=1}^{n} \Delta_j \sum_{i=1}^{n-j+1} \operatorname{VaR}_{\frac{i}{n}}(V) \tag{12}$$

$$= \sum_{j=1}^{n} \frac{\Delta_j (n-j+1)}{n} \operatorname{CVaR}_{\frac{n-j+1}{n}}(V). \tag{13}$$

Here equality (12) reflects the fact that $\operatorname{VaR}_{\frac{i}{n}}(V) = v_i$ for $i = 1, \ldots, n$. To verify (13), first note that by definition we have $\operatorname{VaR}_\alpha(V) = \operatorname{VaR}_{\frac{i}{n}}(V)$ for all $\alpha \in \left(\frac{i-1}{n}, \frac{i}{n}\right]$. Taking into account (2), we see that $\operatorname{CVaR}_{\frac{k}{n}}(V) = \frac{1}{k} \sum_{i=1}^{k} \operatorname{VaR}_{\frac{i}{n}}(V)$ holds (for a more detailed explanation see, e.g., Rockafellar, 2007; Noyan and Rudolf, 2013), which implies (13). Now let us introduce the notation $\mu_j^Q = \frac{\Delta_j (n-j+1)}{n}$, and observe that

$$\sum_{j=1}^{n} \mu_j^Q = \sum_{j=1}^{n} \frac{\Delta_j (n-j+1)}{n} = \sum_{i=1}^{n} \frac{q_i}{n} = \mathbb{E}(Q) = 1.$$

Then, the desired representation (10) holds with the choice of $\mathcal{M} = \{(\mu_1^Q, \ldots, \mu_n^Q) \ : \ Q \in \mathcal{Q}\}$. □

We remark that, in connection with the aforementioned work of Leitner (2005), part (i) of the above theorem implies that in the equal probability case all law invariant coherent risk measures are consistent with second-order stochastic dominance.

**4. Functionally coherent risk measures** The results presented in the previous section guarantee the existence of Kusuoka representations under a variety of circumstances. However, such representations do not always exist in probability spaces that have atoms, as the next example (originally due to Pflug and Römisch, 2007) shows.

Example 4.1 *Let $\Omega = \{\omega_1, \omega_2\}$ with $\frac{1}{2} < \Pi(\omega_1) < 1$. It is easy to verify that the risk measure $\hat{\rho}$ defined by $\hat{\rho}(V) = V(\omega_1)$ is coherent (and even comonotone additive). In addition, the equality $\hat{\rho}(V) = \sup\left\{v \ : \ F_V(v) < \frac{1}{2}\right\}$ shows that $\hat{\rho}$ is law invariant. However, $\hat{\rho}$ has no Kusuoka representation of the form (5). We provide a simple proof of this claim, as an alternative to the one given in Pflug and Römisch (2007).*

Proof. We first note that $\mathrm{CVaR}_\alpha(V) \leq \mathrm{CVaR}_1(V) = \mathbb{E}(V)$ holds for any random variable $V$ and confidence level $\alpha \in [0,1]$. It follows that we have $\rho(V) \leq \mathbb{E}(V)$ for any risk measure $\rho$ with a Kusuoka representation of form (5). On the other hand, for the random variable $V$ given by $V(\omega_1) = 1$, $V(\omega_2) = 0$ we have the inequality $\hat{\rho}(V) = V(\omega_1) = 1 > \Pi(\omega_1) = \mathbb{E}(V)$, which implies our claim. □

Let us recall that for a law invariant risk measure $\rho : \mathcal{L}^p \to \overline{\mathbb{R}}$ there exists a corresponding mapping $\varphi_\rho : \mathcal{F}^p \to \overline{\mathbb{R}}$ which satisfies $\rho(V) = \varphi_\rho(F_V)$ for all $V \in \mathcal{L}^p$. The risk measure $\hat{\rho}$ in the above example is "pathological" in the sense that its corresponding mapping $\varphi_{\hat{\rho}}$ cannot be coherently extended to CDFs from other probability spaces. We now formalize this intuitive notion.

Definition 4.1 *Given a not necessarily atomless probability space $(\Omega, \mathcal{A}, \Pi)$ and a value $p \in [1,\infty]$, a law invariant mapping $\rho : \mathcal{L}^p(\Omega, \mathcal{A}, \Pi) \to \overline{\mathbb{R}}$ is called a* functionally coherent risk measure *if there exists a law invariant coherent risk measure $\bar{\rho} : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ such that $\varphi_\rho$ is a restriction of $\varphi_{\bar{\rho}}$, i.e., we have $\varphi_\rho = \varphi_{\bar{\rho}}\big|_{\mathcal{F}^p(\Omega,\mathcal{A},\Pi)}$.*

*If there exists some $\bar{\rho} : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ that, in addition to the above properties, is also comonotone additive, then we say that $\rho$ is a* functionally coherent and comonotone additive risk measure.

We mention that the above notion is essentially equivalent to the concept of *regular coherent risk measures* discussed in Pichler and Shapiro (2012). It is easy to see that in atomless probability spaces this class coincides with the class of law invariant coherent risk measures. However, this is not always the case in general probability spaces. The next proposition shows that functionally coherent risk measures allow us to use Kusuoka representations even in probability spaces that are not atomless.

Proposition 4.1 *Consider a (not necessarily atomless) probability space $(\Omega, \mathcal{A}, \Pi)$, and a value $p \in [1,\infty]$.*

- *(i) A mapping $\rho : \mathcal{L}^p \to \overline{\mathbb{R}}$ is a functionally coherent risk measure if and only if it has a Kusuoka representation of the form (5) for some family $\mathcal{M} \subset \mathcal{P}([0,1])$.*
- *(ii) A mapping $\rho : \mathcal{L}^p \to \overline{\mathbb{R}}$ is a functionally coherent and comonotone additive risk measure if and only if it has a Kusuoka representation of the form (6) for some measure $\mu \in \mathcal{P}([0,1])$.*
- *(iii) The equivalences established in parts (i) and (ii) still hold if we restrict ourselves to measures supported on the topological closure $\bar{\mathcal{K}} = \mathrm{cl}\,\{\Pi(S) \ : \ S \in \mathcal{A},\ \Pi(S) > 0\}$ of the set $\mathcal{K}$ introduced in Lemma 2.1.*

Proof. If $\rho$ is functionally coherent, then there exists a law invariant coherent risk measure $\bar{\rho}$ on an atomless space such that $\varphi_\rho = \varphi_{\bar{\rho}}\big|_{\mathcal{F}^p(\Omega,\mathcal{A},\Pi)}$ holds. By Theorem 3.1 the risk measure $\bar{\rho}$ has a Kusuoka representation of the form (5), which trivially also provides a representation for $\rho$. On the other hand, if $\rho$ has a Kusuoka representation, then the same representation defines a law invariant mapping $\bar{\rho} : \mathcal{L}^p(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\Pi}) \to \overline{\mathbb{R}}$ on any atomless probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\Pi})$. By Theorem 3.1 the mapping $\bar{\rho}$ is a coherent risk measure, which proves part (i). The proof of part (ii) is essentially identical.

It is sufficient to prove part (iii) for representations of the form (6). For a confidence level $\alpha \in [0,1]$ let us define $\alpha_- = \sup\{\gamma \in \mathcal{K} \cup \{0\} : \gamma \le \alpha\} \in \bar{\mathcal{K}} \cup \{0\}$ and $\alpha_+ = \inf\{\gamma \in \mathcal{K} : \gamma \ge \alpha\} \in \bar{\mathcal{K}}$. According to Lemma 2.1, the mapping $\gamma \mapsto \gamma \operatorname{CVaR}_\gamma$ is affine on the interval $[\alpha_-, \alpha_+]$, therefore $\operatorname{CVaR}_\alpha$ can be expressed as a convex combination $\lambda_\alpha^- \operatorname{CVaR}_{\alpha_-} + \lambda_\alpha^+ \operatorname{CVaR}_{\alpha_+}$ for some coefficients $\lambda_\alpha^-$ and $\lambda_\alpha^+$. In addition, if $\alpha_- = 0$, then the mapping $\gamma \mapsto \gamma \operatorname{CVaR}_\gamma$ is linear on $[0, \alpha_+]$, therefore we can set $\lambda_\alpha^- = 0$, $\lambda_\alpha^+ = 1$.

Let us define the probability measure $\bar{\mu}$ on the set $\bar{\mathcal{K}}$ by

$$\bar{\mu}(A) = \int\limits_{\{\alpha \in (0,1] \,:\, \alpha_- \in A\}} \lambda_\alpha^- \, \mu(\mathrm{d}\alpha) + \int\limits_{\{\alpha \in (0,1] \,:\, \alpha_+ \in A\}} \lambda_\alpha^+ \, \mu(\mathrm{d}\alpha).$$

It is easy to verify that $\int_0^1 \operatorname{CVaR}_\alpha(V)\, \mu(\mathrm{d}\alpha) = \int_{\bar{\mathcal{K}}} \operatorname{CVaR}_\alpha(V)\, \bar{\mu}(\mathrm{d}\alpha)$ holds for all $V \in \mathcal{L}^p(\Omega, \mathcal{A}, \Pi)$, which proves our claim. □

Note that, unlike in the atomless case, for probability spaces with atoms the supporting set $\bar{\mathcal{K}}$ in the above proposition can be a proper subset of the interval $[0,1]$. In particular, this is the case for finite probability spaces, which yields some useful consequences.

Corollary 4.1 *Consider a functionally coherent and comonotone additive risk measure $\rho$ on some finite probability space $(\Omega, 2^\Omega, \Pi)$, and let $n = |\Omega|$.*

*(i) The risk measure $\rho$ can be written as a convex combination of finitely many* CVaR *measures, i.e., it has a representation*

$$\rho(V) = \sum_{i=1}^{M} \mu_i \operatorname{CVaR}_{\alpha_i}(V) \quad \textit{for all} \quad V \in \mathcal{V}, \tag{14}$$

*for some $\alpha_1, \ldots, \alpha_M \in (0,1]$ and $\mu_1, \ldots, \mu_M \in \mathbb{R}_+$ satisfying $\sum_{i=1}^M \mu_i = 1$.*

*(ii) If every elementary event is equally likely, then $\rho$ can be expressed in the form*

$$\rho(V) = \sum_{i=1}^{n} \mu_i \operatorname{CVaR}_{\frac{i}{n}}(V) \quad \textit{for all} \quad V \in \mathcal{V}, \tag{15}$$

*for some $\mu_1, \ldots, \mu_n \in \mathbb{R}_+$ satisfying $\sum_{i=1}^n \mu_i = 1$.*

Proof. We trivially have $|\mathcal{K}| < 2^n < \infty$, which also implies $\bar{\mathcal{K}} = \mathcal{K}$. Furthermore, in the equal probability case $\mathcal{K} = \left\{\frac{1}{n}, \ldots, \frac{n}{n}\right\}$ holds. □

Remark 4.1 *Shapiro (2013) shows that Kusuoka representations of type (6) are unique in atomless spaces, while in finite probability spaces a single risk measure can have infinitely many different representations. Similar results for* minimal *representations of type (5) can be found in Pichler and Shapiro (2012).*

We mention that risk measures of the form (14) are also known as *mixed* CVaR (Rockafellar, 2007). Combining Corollary 4.1 and Proposition 4.1, we see that functionally coherent risk measures on finite probability spaces have Kusuoka representations of the form

$$\rho(V) = \inf_{\boldsymbol{\mu}\in\mathcal{M}} \sum_{i=1}^{M} \mu_i \operatorname{CVaR}_{\alpha_i}(V) \quad \text{for all} \quad V \in \mathcal{V}, \tag{16}$$

for some integer $M$, confidence levels $\alpha_1, \dots, \alpha_M \in (0,1]$, and a family of vectors $\mathcal{M}$ in the unit simplex $S_M = \left\{\boldsymbol{\mu} \in [0,1]^M \ : \ \|\boldsymbol{\mu}\|_1 = 1\right\}$. If a risk measure has a representation where the set $\mathcal{M}$ is finite, then we say that it is a *finitely representable coherent risk measure*. The next result shows that, over bounded families of random variables, these risk measures are dense among functionally coherent risk measures.

Proposition 4.2 *Consider a family $\bar{\mathcal{V}}$ of random variables on a finite probability space, and assume that $\bar{\mathcal{V}}$ is bounded in the $\mathcal{L}^1$-norm. Then, for any functionally coherent risk measure $\rho$ and $\epsilon > 0$ there exists a finitely representable coherent risk measure $\bar{\rho}$ such that $|\rho(V) - \bar{\rho}(V)| < \epsilon$ holds for all $V \in \bar{\mathcal{V}}$.*

Proof. The risk measure $\rho$ has a Kusuoka representation of the form (16). Let us now select an integer $k > \frac{1}{\epsilon} M \sup\left\{\|V\|_1 \ : \ V \in \bar{\mathcal{V}}\right\}$, and consider the finite set

$$S_M^{(k)} = \left\{ \boldsymbol{\mu} \in \left\{\frac{0}{k}, \frac{1}{k}, \dots, \frac{k}{k}\right\}^M \ : \ \sum_{i=1}^{M} \mu_i = 1 \right\} \subset S_M.$$

For every vector $\boldsymbol{\mu} \in S_M$ there exists a *controlled rounding* $\bar{\boldsymbol{\mu}} \in S_M^{(k)}$ such that $|\mu_i - \bar{\mu}_i| < \frac{1}{k}$ holds for all $i = 1, \dots, M$ (see, e.g., Bacharach, 1966). Then, using the fact that $|\operatorname{CVaR}_\alpha(V)| \le \|V\|_1$ holds for all $V \in \mathcal{V}$, $\alpha \in (0,1]$, we have

$$\left| \sum_{i=1}^{M} \mu_i \operatorname{CVaR}_{\alpha_i}(V) - \sum_{i=1}^{M} \bar{\mu}_i \operatorname{CVaR}_{\alpha_i}(V) \right| \le M \frac{1}{k} \sup\left\{\|V\|_1 \ : \ V \in \bar{\mathcal{V}}\right\} < \epsilon \quad \text{for every} \quad \boldsymbol{\mu} \in S_M, \ V \in \bar{\mathcal{V}}. \tag{17}$$

Let us introduce the finite set $\bar{\mathcal{M}} = \{\bar{\boldsymbol{\mu}} \ : \ \boldsymbol{\mu} \in \mathcal{M}\} \subset S_M^{(k)}$ and the finitely representable coherent risk measure

$$\bar{\rho}(V) = \inf_{\boldsymbol{\mu}\in\bar{\mathcal{M}}} \sum_{i=1}^{M} \mu_i \operatorname{CVaR}_{\alpha_i}(V) = \inf_{\boldsymbol{\mu}\in\mathcal{M}} \sum_{i=1}^{M} \bar{\mu}_i \operatorname{CVaR}_{\alpha_i}(V), \quad V \in \mathcal{V}.$$

Then, recalling (16) and (17), we have

$$|\rho(V) - \bar{\rho}(V)| = \left| \inf_{\boldsymbol{\mu}\in\mathcal{M}} \sum_{i=1}^{M} \mu_i \operatorname{CVaR}_{\alpha_i}(V) - \inf_{\boldsymbol{\mu}\in\mathcal{M}} \sum_{i=1}^{M} \bar{\mu}_i \operatorname{CVaR}_{\alpha_i}(V) \right| < \epsilon \quad \text{for every} \quad V \in \bar{\mathcal{V}},$$

which proves our claim. □

**5. Coherent extensions** In the previous section we utilized the notion that a law invariant risk measure corresponds to an operator that acts on a family of CDFs. We now examine what it means for such operators to be coherent, and obtain sufficient conditions under which they can be extended in a coherent fashion to a wider family of CDFs. The latter question is of particular importance because Theorem 3.1 implies that Kusuoka representations are available for coherent operators defined on the family $\mathfrak{F}^p$ for any $p \in [1, \infty]$.

Consider a family $\mathcal{F} \subset \mathfrak{F}$ of CDFs, and a mapping $\varphi : \mathcal{F} \to \overline{\mathbb{R}}$. We say that $\varphi$ is a *coherent mapping* if the following conditions hold for any random variables $V$, $V_1$, $V_2$ on $\mathcal{S}$.

- *Normalized*: $F_{\mathbb{0}} \in \mathcal{F} \implies \varphi(F_{\mathbb{0}}) = 0$.
- *Monotone*: $F_{V_1}, F_{V_2} \in \mathcal{F},\ F_{V_1} \geq F_{V_2} \implies \varphi(F_{V_1}) \leq \varphi(F_{V_2})$.
- *Superadditive*: $F_{V_1}, F_{V_2}, F_{V_1+V_2} \in \mathcal{F} \implies \varphi(F_{V_1+V_2}) \geq \varphi(F_{V_1}) + \varphi(F_{V_2})$.
- *Positive homogeneous*: $\alpha > 0,\ F_V, F_{\alpha V} \in \mathcal{F} \implies \varphi(F_{\alpha V}) = \alpha\varphi(F_V)$.
- *Translation invariant*: $\lambda \in \mathbb{R},\ F_V, F_{V+\lambda} \in \mathcal{F} \implies \varphi(F_{V+\lambda}) = \varphi(F_V) + \lambda$.

If, in addition to the above properties, the condition

- *Comonotone additive*: $V_1$, $V_2$ comonotone, $F_{V_1}, F_{V_2}, F_{V_1+V_2} \in \mathcal{F} \implies \varphi(F_{V_1+V_2}) = \varphi(F_{V_1}) + \varphi(F_{V_2})$

also holds, then $\varphi$ is a *coherent and comonotone additive mapping.*

Definition 5.1 *We say that a family $\mathcal{F} \subset \mathfrak{F}$ is* closed *if for any random variables $V$, $V_1$, $V_2$ on $\mathcal{S}$ we have the following.*

- $F_{\mathbb{0}} \in \mathcal{F}$.
- $F_V \in \mathcal{F} \implies F_{\alpha V} \in \mathcal{F}$ *for all* $\alpha \in \mathbb{R}_+$.
- $F_V \in \mathcal{F} \implies F_{\lambda+V} \in \mathcal{F}$ *for all* $\lambda \in \mathbb{R}$.
- $F_{V_1}, F_{V_2} \in \mathcal{F} \implies F_{V_1+V_2} \in \mathcal{F}$.

*Given a value $p \in [1, \infty]$ we call a family $\mathcal{F} \subset \mathfrak{F}$ $p$-*dense *if for any $F \in \mathfrak{F}^p$ there exists a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\lim_{n\to\infty} \|F^{-1} - F_n^{-1}\|_p = 0$.*

Let us now introduce some important closed families of CDFs.

Example 5.1 *For a random variable $V$ on $\mathcal{S}$ let $F_V^{-1}([0,1]) = \{F_V^{-1}(u) \,|\, u \in [0,1]\}$ denote the range of the quantile function $F_V^{-1}$. For $p \in [1, \infty]$ let $\mathfrak{F}_D^p = \{F_V : V \in \mathcal{L}^p(\mathcal{S}),\ F_V^{-1}([0,1]) \text{ is countable}\}$ denote the family of CDFs of p-integrable discrete distributions, and let $\mathfrak{F}_{FD} = \{F_V : V \in \mathcal{L}^1(\mathcal{S}),\ |F_V^{-1}([0,1])| < \infty\}$ denote the family of CDFs of finite discrete distributions. The families $\mathfrak{F}_D^p$ and $\mathfrak{F}_{FD}$ are closed and p-dense for any $p \in [1, \infty]$.*

Proof. It is easy to verify that the families in question are closed. Since $\mathfrak{F}_{FD}$ is a subset of $\mathfrak{F}^p_D$, it suffices to show that the former family is $p$-dense. To this end, let us consider an arbitrary CDF $F_V \in \mathfrak{F}^p$ for some $V \in \mathcal{L}^p(\mathcal{S})$, and define the discretized random variables $V_n = \frac{\lceil nV \rceil}{n}$. As the inequality $|V - V_n| < \frac{1}{n}$ holds everywhere, we have $\left\|F_V^{-1} - F_{V_n}^{-1}\right\|_p = \|V - V_n\|_p \leq \frac{1}{n}$ for every $n \in \mathbb{N}$. □

Theorem 5.1 *Consider a value $p \in [1, \infty]$, a closed family $\mathcal{F} \subset \mathfrak{F}^p$, and a mapping $\varphi : \mathcal{F} \to \overline{\mathbb{R}}$.*

- *(i) The mapping $\varphi$ is coherent if and only if $\varphi = \varphi_\rho\big|_{\mathcal{F}}$ holds for some risk measure $\rho : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ which has a Kusuoka representation of the form (5).*
- *(ii) If the family $\mathcal{F}$ is p-dense, then $\varphi$ is a coherent and comonotone additive mapping if and only if $\varphi = \varphi_\rho\big|_{\mathcal{F}}$ holds for some risk measure $\rho : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ which has a Kusuoka representation of the form (6).*

Proof. According to Theorem 3.1, to prove part (i) it is sufficient to show that there exists a law invariant coherent risk measure $\rho$ on the atomless space $\mathcal{S}$ such that $\varphi_\rho$ is an extension of $\varphi$.

For the family $\mathcal{Z} = \{Z \in \mathcal{L}^p(\mathcal{S}) \,|\, F_Z \in \mathcal{F}\}$ of random variables we have $\mathcal{F} = \{F_Z \,|\, Z \in \mathcal{Z}\}$. Let us now define the law invariant mapping $\rho : \mathcal{L}^p(\mathcal{S}) \to \overline{\mathbb{R}}$ by

$$\rho(V) = \sup\left\{\varphi(F_Z) \mid Z \in \mathcal{Z}, F_Z \geq F_V\right\} \quad \text{for all } V \in \mathcal{L}^p(\mathcal{S}). \tag{18}$$

We first show that $\varphi_\rho : \mathfrak{F}^p \to \overline{\mathbb{R}}$, defined by

$$\varphi_\rho(F_V) = \rho(V) \qquad \text{for all } V \in \mathcal{L}^p,$$

is an extension of $\varphi$, i.e., that $\varphi = \varphi_\rho\big|_{\mathcal{F}}$ holds. If we consider an arbitrary function $F_V \in \mathcal{F}$ for some $V \in \mathcal{Z}$, we immediately have $\varphi_\rho(F_V) = \rho(V) = \sup\{\varphi(F_Z) \,|\, Z \in \mathcal{Z}, F_Z \geq F_V\} \geq \varphi(F_V)$. On the other hand, due to the monotone property of $\varphi$, the inequality $\varphi(F_Z) \leq \varphi(F_V)$ holds for any $Z \in \mathcal{Z},\ F_Z \geq F_V$ featured in the above supremum, implying $\varphi_\rho(F_V) \leq \varphi(F_V)$.

We next prove that $\rho$ is a coherent risk measure. It is easy to verify that $\rho$ is *normalized, positive homogeneous,* and *translation invariant*. For example, the translation invariance of $\rho$ follows from the next simple chain of equalities.

$$\rho(V + \lambda) = \sup\left\{\varphi(F_Z) \mid Z \in \mathcal{Z},\, F_Z \geq F_{V+\lambda}\right\} \tag{19}$$

$$= \sup\left\{\varphi(F_{\bar{Z}+\lambda}) \;\middle|\; \bar{Z} \in \mathcal{Z},\, F_{\bar{Z}+\lambda} \geq F_{V+\lambda}\right\} \tag{20}$$

$$= \sup\left\{\varphi(F_{\bar{Z}}) + \lambda \;\middle|\; \bar{Z} \in \mathcal{Z},\, F_{\bar{Z}+\lambda} \geq F_{V+\lambda}\right\} \tag{21}$$

$$= \sup\left\{\varphi(F_{\bar{Z}}) \;\middle|\; \bar{Z} \in \mathcal{Z},\, F_{\bar{Z}} \geq F_V\right\} + \lambda \tag{22}$$

$$= \rho(V) + \lambda. \tag{23}$$

Here equalities (19) and (23) are provided by the definition of $\rho$. Equality (20) reflects the fact that, as $\mathcal{F}$ is a closed family, for $Z = \bar{Z} + \lambda$ the conditions $Z \in \mathcal{Z}$ and $\bar{Z} \in \mathcal{Z}$ are equivalent. Equality (21) follows from the translation invariance of $\varphi$, while the trivial equivalence $F_{\bar{Z}+\lambda} \geq F_{V+\lambda} \Leftrightarrow F_{\bar{Z}} \geq F_V$ implies (22). To show that $\rho$ is *monotone* it is sufficient to observe that if $V_1 \leq V_2$ holds for random variables $V_1, V_2 \in \mathcal{L}^p$, then the supremum that defines $\rho(V_1)$ in (18) is taken over a subset of the set featured in the supremum that defines $\rho(V_2)$. It remains to show that $\rho$ is *superadditive*. Let us consider two random variables $V_1, V_2 \in \mathcal{L}^p$. According to Sklar's Theorem on copulas (see, e.g., Nelsen, 1999), there exist standard uniform random variables $U_1$ and $U_2$ on $\mathcal{S}$ for which $\big(F_{V_1}^{-1}(U_1), F_{V_2}^{-1}(U_2)\big)$ has the same joint distribution as $(V_1, V_2)$. Introducing the notation $\bar{Z}_1 = F_{Z_1}^{-1}(U_1)$, $\bar{Z}_2 = F_{Z_2}^{-1}(U_2)$, we have the following chain of implications:

$$
\begin{aligned}
& Z_1, Z_2 \in \mathcal{Z}, \quad F_{Z_1} \geq F_{V_1}, \quad F_{Z_2} \geq F_{V_2} && (24)\\
\Longrightarrow \quad & Z_1, Z_2 \in \mathcal{Z}, \quad F_{Z_1}^{-1}(U_1) \leq F_{V_1}^{-1}(U_1), \quad F_{Z_2}^{-1}(U_2) \leq F_{V_2}^{-1}(U_2) && (25)\\
\Longrightarrow \quad & \bar{Z}_1, \bar{Z}_2 \in \mathcal{Z}, \quad \bar{Z}_1 \leq F_{V_1}^{-1}(U_1), \quad \bar{Z}_2 \leq F_{V_2}^{-1}(U_2) && (26)\\
\Longrightarrow \quad & \bar{Z}_1 + \bar{Z}_2 \in \mathcal{Z}, \quad \bar{Z}_1 + \bar{Z}_2 \leq F_{V_1}^{-1}(U_1) + F_{V_2}^{-1}(U_2) && (27)\\
\Longrightarrow \quad & \bar{Z}_1 + \bar{Z}_2 \in \mathcal{Z}, \quad F_{\bar{Z}_1+\bar{Z}_2} \geq F_{V_1+V_2}. && (28)
\end{aligned}
$$

Here (25) follows from the order-reversing property of the mapping $F \mapsto F^{-1}$. The definitions of $Z_1$, $Z_2$ and $\mathcal{Z}$, together with the relations $Z_1 \sim \bar{Z}_1$, $Z_2 \sim \bar{Z}_2$ immediately yield (26). To verify (27), we note that $\mathcal{Z}$ is closed under addition because $\mathcal{F}$ is a closed family. Finally, (28) holds because $\big(F_{V_1}^{-1}(U_1), F_{V_2}^{-1}(U_2)\big) \sim (V_1, V_2)$ implies $F_{V_1}^{-1}(U_1) + F_{V_2}^{-1}(U_2) \sim V_1 + V_2$. The superadditivity of $\rho$, and thus part (i), now follows from the next chain of inequalities.

$$
\begin{aligned}
\rho(V_1 + V_2) &= \sup\{\varphi(F_Z) \mid Z \in \mathcal{Z}, F_Z \geq F_{V_1+V_2}\} && (29)\\
&\geq \sup\big\{\varphi(F_{\bar{Z}_1+\bar{Z}_2}) \;\big|\; Z_1, Z_2 \in \mathcal{Z},\, F_{Z_1} \geq F_{V_1}, F_{Z_2} \geq F_{V_2}\big\} && (30)\\
&\geq \sup\big\{\varphi(F_{\bar{Z}_1}) + \varphi(F_{\bar{Z}_2}) \;\big|\; Z_1, Z_2 \in \mathcal{Z},\, F_{Z_1} \geq F_{V_1}, F_{Z_2} \geq F_{V_2}\big\} && (31)\\
&= \sup\{\varphi(F_{Z_1}) \mid Z_1 \in \mathcal{Z},\, F_{Z_1} \geq F_{V_1}\} + \sup\{\varphi(F_{Z_2}) \mid Z_2 \in \mathcal{Z},\, F_{Z_2} \geq F_{V_2}\} && (32)\\
&= \rho(V_1) + \rho(V_2) && (33)
\end{aligned}
$$

Here (29) and (33) are again provided by the definition of $\rho$. The inequality (30) follows from the implication (24)-(28), while (31) is a consequence of the superadditivity of $\varphi$. Finally, the relations $Z_1 \sim \bar{Z}_1$, $Z_2 \sim \bar{Z}_2$ imply (32).

To prove the non-trivial implication in part (ii), assume that $\varphi$ is a coherent and comonotone additive mapping. Let us again define $\rho$ as in (18), and note that according to part (i) it is a law invariant coherent risk measure. It only remains to show that $\rho$ is comonotone additive. Let us consider two comonotone random variables $V_1, V_2 \in \mathcal{L}^p(\mathcal{S})$, and recall that there exists a standard uniform random variable $U$ such that $(V_1, V_2) \sim \Big(F_{V_1}^{(-1)}(U), F_{V_2}^{(-1)}(U)\Big)$ holds. Since the family $\mathcal{F}$ is $p$-dense, there exist sequences $\{F_{i,n}\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that we have $\lim\limits_{n\to\infty} \|F_{V_i}^{-1} - F_{i,n}^{-1}\|_p = 0$ for $i \in \{1, 2\}$. Introducing the notation $Z_{i,n} = F_{i,n}^{-1}(U)$, our statement follows from the next chain of equalities.

$$
\rho(V_1) + \rho(V_2) = \rho(F_{V_1}^{(-1)}(U)) + \rho(F_{V_2}^{(-1)}(U)) \tag{34}
$$

$$= \lim_{n\to\infty} \rho(Z_{1,n}) + \rho(Z_{2,n}) \tag{35}$$

$$= \lim_{n\to\infty} \varphi(F_{Z_{1,n}}) + \varphi(F_{Z_{2,n}}) \tag{36}$$

$$= \lim_{n\to\infty} \varphi(F_{Z_{1,n}+Z_{2,n}}) \tag{37}$$

$$= \lim_{n\to\infty} \rho(Z_{1,n} + Z_{2,n}) \tag{38}$$

$$= \rho(F_{V_1}^{(-1)}(U) + F_{V_2}^{(-1)}(U)) \tag{39}$$

$$= \rho(F_{V_1+V_2}^{(-1)}(U)) \tag{40}$$

$$= \rho(V_1 + V_2) \tag{41}$$

Equalities (34) and (41) hold due to the law invariance of $\rho$. We note that, as $\rho$ is a coherent risk measure, it is continuous in the $\mathcal{L}^p$-norm (Ruszczyński and Shapiro, 2006). Then, if $p \in [1, \infty)$, for $i \in \{1, 2\}$ we have

$$\lim_{n\to\infty} \|F_{V_i}^{-1}(U) - Z_{i,n}\|_p = \lim_{n\to\infty} \|F_{V_i}^{-1}(U) - F_{i,n}^{-1}(U)\|_p = \lim_{n\to\infty} \left(\int_0^1 \left|F_{V_i}^{-1}(u) - F_{i,n}^{-1}(u)\right|^p \mathrm{d}u\right)^{\frac{1}{p}} \tag{42}$$

$$= \lim_{n\to\infty} \|F_{V_i}^{-1} - F_{i,n}^{-1}\|_p = 0. \tag{43}$$

Similarly, if $p = \infty$, we have

$$\lim_{n\to\infty} \|F_{V_i}^{-1}(U) - Z_{i,n}\|_\infty = \lim_{n\to\infty} \|F_{V_i}^{-1} - F_{i,n}^{-1}\|_\infty = 0. \tag{44}$$

Equality (35) immediately follows. By the definition of $\rho$ we obtain (36) and (38), while (37) follows from the comonotone additivity of $\varphi$. Similarly to (42)-(44), for any $p \in [1, \infty]$ we have

$$\lim_{n\to\infty} \left\|F_{V_1}^{-1}(U) + F_{V_1}^{-1}(U) - (Z_{1,n} + Z_{2,n})\right\|_p = 0,$$

therefore equality (39) again follows from the fact that $\rho$ is continuous in the $p$-norm. Finally, recalling (4), the quantile additivity of comonotone random variables yields (40). □

Keeping in mind Example 5.1, the next result is an easy consequence of the above theorem.

Corollary 5.1 *If a law invariant risk measure $\rho$ can be coherently extended to a family containing all finite discrete distributions, then $\rho$ is functionally coherent. More precisely, consider a mapping $\rho : \mathcal{V}(\Omega, 2^\Omega, \Pi) \to \overline{R}$ on a finite probability space. If for some $p \in [1, \infty]$ there exists a coherent mapping $\varphi : \mathfrak{F}_{FD} \to \overline{R}$ such that $\varphi\big|_{\mathcal{F}(\Omega,2^\Omega,\Pi)} = \varphi_\rho$ holds, then $\rho$ is a functionally coherent risk measure. It follows that $\rho$ has a Kusuoka representation of the form (5). If, in addition, $\varphi$ is also comonotone additive, then $\rho$ is a functionally coherent and comonotone additive risk measure, and thus has a Kusuoka representation of the form (6).*

**6. Conclusion and future research** We have seen that in atomless probability spaces, as well as in finite spaces where each elementary event has the same probability, all law invariant coherent risk measures have Kusuoka representations of the form (5). Similarly, in such spaces all law invariant coherent and comonotone additive risk measures have representations of the form (6). An interesting question for future research is whether there exist other probability spaces with similar properties.

Another open problem is to find a method to determine whether a given law invariant coherent risk measure on a non-atomless space has a Kusuoka representation. We note that the characterization by Leitner (2005) provides a partial answer for risk measures with the Fatou property. Since law invariant risk measures can be viewed as operators on CDFs, a natural generalization of the previous problem is to characterize coherent mappings from arbitrary families of CDFs which admit Kusuoka representations. The sufficient condition established in Theorem 5.1 can serve as a starting point in studying this problem.

Finally we mention that from a practical point of view random variables with finitely many realizations are often of interest. We have shown that any law invariant coherent risk measure that is defined on all such variables has a Kusuoka representation. Risk measures with finite representations are of particular interest due to the fact that they are computationally tractable. According to our denseness results, the class of finitely representable risk measures is sufficiently rich to allow an arbitrarily close approximation of any risk measure that admits a Kusuoka representation.

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