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An interior proximal point method with φ-divergence for Equilibrium Problems

Paulo R. Oliveira · Paulo S. M. Santos · Afonso N. Silva · Arnaldo S. Brito

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Abstract In this paper, we consider the problem of general equilibrium in a finite-dimensional space on a closed convex set. For solving this problem, we developed an interior proximal point algorithm with φ-divergence. Under reasonable assumptions, we prove that the sequence generated by the algorithm converges to a solution of the Equilibrium Problem, when the regularization parameters are bounded.

Keywords Equilibrium problem · interior proximal point method · φ-divergence proximal distance.

1 Introduction

Let E be a Euclidean space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Take a nonempty closed and convex set $C \subset E$ and $f : C \times C \to \mathbb{R}$ an equilibrium bifunction, i.e., $f(x, x) = 0$ for every $x \in C$.

The equilibrium problem (in short, $EP(f, C)$) consists of:

$$EP(f,C) \quad \begin{cases} \text{Find } x^* \in \ C \ \text{ such that} \\ f(x^*, y) \geq 0 \quad \forall \ \ y \ \in C. \end{cases} \tag{1}$$

The set of solutions of EP(f, C) will be denoted by S(f, C).

This problem is very general in the sense that it includes, among its particular cases, the optimization problem, the variational inequality problem, the Nash equilibrium problem in cooperative games, the fixed point problem, the nonlinear complementary problem, and other applications, see for example [4,9] and their references. The interest of this problem is that it unifies all these particular problems in a convenient way. Moreover, many methods devoted to solving one of these problems can be extended, with suitable modifications, to solving the general equilibrium problem.

Existence results for solutions to problems of equilibrium have been extensively studied, as it can be seen in [4, 10, 13].

P.R. Oliveira
This author was supported in part by CNPq. COPPE/Sistemas-Universidade Federal do Rio de Janeiro, Rio de Janeiro, BR E-mail: poliveir@cos.ufrj.br

P.S.M. Santos
Federal University of Piaui, Piaui, BR E-mail: psergio@ufpi.edu.br

A.N. Silva
State University of Piaui and Instituto Dom Barreto, Piaui, BR E-mail: afonsonorberto@gmail.com

A.S. Brito
Corresponding author. State University of Piaui, Piaui, BR E-mail: bsarnaldo@gmail.com

In recent years, methods to solve the Equilibrium Problem has been much studied. One approach commonly used is the Proximal Point Method. This method was introduced by Martinet [15] to variational inequalities and was extended by Moudafi [17] to solve monotone equilibrium problems. Konnov [12] used the proximal point method for solving Problem (1) with f being a weakly monotone equilibrium bifunction.

Other methods have been developed to solve the equilibrium problem, such as extragradient method [19], projected subgradient [20] and gap function [16].

In [9, 12] the authors propose an algorithm to solve $EP(f, C)$, that generates a sequence $\left\{x^k\right\} \subset C$, where given $x^0 \in C$, the point x^{k+1} is obtained as a solution to the following regularized problem

$$f_k(x, y) = f(x, y) + \lambda_k \langle x - x^k, y - x\rangle, \tag{2}$$

with $\lambda_k \subset (\theta, \overline{\lambda})$; $\theta \geq 0$; $\overline{\lambda} > 0$.

Some researchers have considered the possibility of replacing the Euclidean distance in (2) by other types of distances such as Bregman distances, see [14] for finite-dimensional case and see [5] for a method in Banach spaces setting.

In [18] an interior proximal extragradient method is discussed for equilibrium problems where the feasible set is a polyhedron and quadratic term is replaced by logarithmic quadratic distance function, i.e.,

$$\min_{y \in \mathbb{R}^n} \left\{ f(x^k, y) + \lambda_k D_\varphi(y, x^k) \right\},$$

where f is continuous on $C \times C$, satisfying

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|y - x\|^2 - c_2\|z - y\|^2, \forall\, x, y, z \in C, \tag{3}$$

with $c_1 > 0$ and $c_2 > 0$.

In this work, we propose an interior proximal point method for solving (1), by replacing the Euclidean distance used in (2) by the interior proximal distance given in [18], which we will be detailed in Section 2.

It is worth to point out that the main advantage of the φ-divergence proximal distance in relation to the (2); is that the term D_φ is used to force the iterates x^k to stay in the interior of the C. Such a property cannot be ensured when using the usual squared Euclidean distance $\|y - x\|^2$, see for example [2].

Our main purpose is to establish that the sequence $\left\{x^k\right\}$ generated by our algorithm is well-defined and it converges to a solution of the problem, when the parameter λ_k satisfies

$$\theta < \lambda_k \leq \bar{\lambda},$$

for some $\bar{\lambda} > 0$, moreover f does not require the condition (3) nor continuity on $C \times C$.

The paper is organized as follows. In Section 2, we recall concepts, basic results and we show an existence result for our regularized bifunction. In Section 3, we define the algorithm and we will present its convergence analysis.

2 Preliminaries

In this section, we recall definitions and known results that we present and they are important for the development of the following sections.

Definition 1 A bifunction $f : C \times C \to \mathbb{R}$ is said to be:
$(i.)$ strongly monotone on C with modulus $\beta > 0$ iff

$$f(x, y) + f(y, x) \leq -\beta\|x - y\|^2 \;\; \forall \;\; x, y \in C.$$

$(ii.)$ monotone on C iff

$$f(x,y)+f(y,x)\leq 0 \ \ \forall \ \ x,y\in C.$$

$(iii.)$ pseudomonotone on C iff

$$\forall \ \ x,y\in C: f(x,y)\geq 0 \Rightarrow f(y,x)\leq 0.$$

Clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Definition 2 A sequence $\left\{z^k\right\}\subset E$ is said to be Fejér convergent to a set $U\neq\emptyset$ with respect to the distance $\|\cdot\|_A$ iff

$$\|z^{k+1}-u\|_A\leq\|z^k-u\|_A, \ \ \forall k\geq 0, \forall u\in U.$$

The next result is important to establish the convergence of the sequence generated by Algorithm 1 studied in Section 3.

Lemma 1 *If $\left\{z^k\right\}\subset E$ is Fejér convergent to a set $U\neq\emptyset$ then $\left\{z^k\right\}$ is bounded. If a cluster point z of $\left\{z^k\right\}$ belongs to U then $\lim_{k\to\infty} z^k=z$.*

Proof See [8].

Remark 1 In [11], it has been established that if $f(.,y)$ is upper semicontinuous for all $y\in C$, $f(x,.)$ is convex for all $x\in C$ and C is compact, then $S(f,C)$ is nonempty.

Throughout the paper we assume that $C\subset E$ is unbounded.

Now we present the definition of φ-divergence and some of its basic properties used in the context of Equilibrium Problem.

In this paper, we assume that C is a polyhedral set with a nonempty interior, $\operatorname{int} C\neq\emptyset$, given by

$$C:=\left\{x\in\mathbb{R}^n \mid Ax\leq b\right\},$$

with A an $m\times n$ $(m\geq n)$ real matrix of maximal rank, and $b\in\mathbb{R}^m$.

Then the distance-like function, denoted $D_\varphi(x,y)$, is constructed from a class of functions $\varphi:\mathbb{R}\to(-\infty,+\infty]$ of the form

$$\varphi(t)=\mu h(t)+\frac{\nu}{2}(t-1)^2,$$

where $\nu>\mu>0$ and h is a closed and proper convex function satisfying the following additional properties:
$(i.)$ h is twice continuously differentiable on $\operatorname{int}(dom)h=(0,+\infty)$.
$(ii.)$ h is strictly convex on its domain.
$(iii.)$ $\lim_{t\to 0^+} h'(t)=-\infty$.
$(iv.)$ $h(1)=0$ and $h''(1)>0$, and
$(v.)$ For $t>0$, $1-t^{-1}\leq h'(t)\leq t-1$.

We present now some examples of the function h satisfying properties $(i.)-(v.)$:

$$h_1(t)=\begin{cases} t-\ln t-1, & \text{if } t>0,\\ +\infty, & \text{otherwise.}\end{cases}$$

$$h_2(t)=\begin{cases} t\ln t-t+1, & \text{if } t>0,\\ +\infty, & \text{otherwise.}\end{cases}$$

The φ corresponding to the function h_1 is called the logarithmic-quadratic function. It enjoys attractive properties for developing efficient algorithms (see [1, 2]). The function h_2 is also often used in the literature (see, for instance, [3, 7]).

Associated with φ, we consider the φ-divergence proximal distance

$$d_\varphi(u,v) := \begin{cases} \sum_{i=1}^{n} v_i^2 \varphi(u_i/v_i), & u, v \in R^n_{++}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, if a_i denotes the row i of the matrix A, for each $x \in C$ we consider $l(x) := (l_1(x), ..., l_m(x))^T$, where

$$l_i(x) := b_i - \langle a_i, x\rangle \quad \text{with} \quad i = 1, 2, ..., n.$$

For any $x, y \in \operatorname{int} C$, we define the distance-like function D_φ by

$$D_\varphi(x,y) := \begin{cases} d_\varphi(l(x), l(y)), & x, y \in \operatorname{int} C, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

As a direct consequence of (4), we obtain

$$D_\varphi(x,y) = \mu D_h(x,y) + \frac{\nu}{2} \parallel x - y \parallel_A^2 \qquad \text{for all} \quad x, y \in \operatorname{int} C.$$

The following technical lemma is important for obtaining the convergence of the sequence generated by Algorithm 1 studied in Section 3.

Lemma 2 *For all $x, y \in \operatorname{int} C$ and $z \in C$, it holds that:*
(i.) $D_\varphi(., y)$ is differentiable and strongly convex on $\operatorname{int} C$ with modulus ν, i.e.,

$$\langle \nabla_1 D_\varphi(x,p) - \nabla_1 D_\varphi(y,p), x - y\rangle \geq \nu \|x - y\|_A^2 \ \ \forall\, p \in \operatorname{int} C,$$

where $\nabla_1 D_\varphi(x,p)$ denote the gradient of $D_\varphi(.,p)$ at x,
(ii.) $D_\varphi(x,y) = 0$ if and only if $x = y$,
(iii.) $\nabla_1 D_\varphi(x,y) = 0$ if and only if $x = y$,
(iv.) $\langle \nabla_1 D_\varphi(x,y), x - z\rangle \geq (\frac{\nu+\mu}{2})(\|x - z\|_A^2 - \|y - z\|_A^2) + (\frac{\nu-\mu}{2})\|x - y\|_A^2$,
(v.) $\|x - y\|_A^2 \geq \alpha\|x - y\|^2$, where α is the minimum eigenvalue of the positive symmetric matrix $(A^T A)$.

Proof See [2, Proposition 2.1] and [6, Lemma 2.1].

The main drawback of the formulation given by Problem (1), is that in general the EP (f, C), may not have solution, and if has, it may not be unique. To avoid this situation, we replace the Problem (1) by another regularized.

Now, we define our regularization procedure for the problem EP(f, C). For this, we consider $\lambda > 0$ and $\bar{x} \in \operatorname{int} C$. To any f we will associate another bifunction $\bar{f} : C \times C \to \mathbb{R}$ which will be called a regularization of f. We define the regularized bifunction $\bar{f}$: $C \times C \to \mathbb{R}$ by:

$$\bar{f}(x,y) = f(x,y) + \lambda\langle \nabla_1 D_\varphi(x, \bar{x}), y - x\rangle,$$

where $\nabla_1 D_\varphi(x, \bar{x})$ denote the gradient of $D_\varphi(., \bar{x})$ at x. The Equilibrium Problem for the bifunction $\bar{f}$ will be denoted by EP$(\bar{f}, C)$ and will indicate their solution set by S$(\bar{f}, C)$. Note also that $\bar{f}(x,x) = 0 \ \ \forall\, x \in C$.

We present now our basic assumptions for the bifunction $f : C \times C \to \mathbb{R}$:

A1 $f(., y) : C \to \mathbb{R}$ is upper semicontinuous for all $y \in C$.
A2 $f(x, .) : C \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in C$.
A3 Whenever $f(x,y) \geq 0$ with $x, y \in C$, it holds that $f(y,x) \leq 0$.

A4 There exist $\theta > 0$ such that

$$f(x, y) + f(y, x) \leq \theta \langle \nabla_1 D_\varphi(x, p) - \nabla_1 D_\varphi(y, p), x - y \rangle \quad \forall\, x, y \in \operatorname{int} C$$

and $p \in \operatorname{int} C$.

A5 For any sequence $\{x^n\} \subset C$ with $\lim_{n\to\infty} \|x^n\| = +\infty$, there exists $u \in C$ and $n_0 \in \mathbb{N}$ such that $f(x^n, u) \leq 0 \ \ \forall\, n \geq n_0$.

Remark 2 Assumption *A*4 is weaker than the monotonicity of the f. In [9], this condition is denoted θ-undermonotonicity by using Euclidean distance and in [5] it is assumed by considering Bregman distances.

The following result existence is important to ensure that the EP$(\bar{f}, C)$ has a solution. First, we recall an important result.

Theorem 1 *Assume that f satisfies A*1*-A*3 *and A*5*. Then $S(f, C)$ is nonempty.*

Proof See [10, Theorem 4.3].

Theorem 2 *Assume that f satisfies A*1*-A*2 *and A*4*. Fix $\bar{x} \in \operatorname{int} C$ and $\lambda > \theta$. Then $EP(\bar{f}, C)$ has a unique solution.*

Proof Firstly, we prove the existence. It is easy to see that the regularized bifunction $\bar{f}$ satisfies assumptions *A*1 and *A*2. We now show that $\bar{f}$ satisfies *A*3. For this, note that:

$$\bar{f}(x, y) = f(x, y) + \lambda \langle \nabla_1 D_\varphi(x, \bar{x}), y - x \rangle. \tag{5}$$

$$\bar{f}(y, x) = f(y, x) + \lambda \langle \nabla_1 D_\varphi(y, \bar{x}), x - y \rangle. \tag{6}$$

Adding (5) with (6) and using *A*4, we get:

$$\bar{f}(x, y) + \bar{f}(y, x) \leq (\theta - \lambda) \langle \nabla_1 D_\varphi(x, \bar{x}) - \nabla_1 D_\varphi(y, \bar{x}), x - y \rangle. \tag{7}$$

Using the fact that $\lambda > \theta$ and from Lemma 2-*i*, it follows from (7) that

$$\bar{f}(x, y) + \bar{f}(y, x) \leq -\beta \|x - y\|_A^2, \tag{8}$$

where $\beta = \lambda\nu - \theta\nu$. This shows that $\bar{f}$ is strongly monotone with modulus $\beta > 0$, in particular, $\bar{f}$ is pseudomonotone and therefore satisfies *A*3. To apply Theorem 1 we must show that $\bar{f}$ satisfies assumption *A*5. For this, take $u \in \operatorname{int} C$ and consider a sequence $\{x^n\} \subset \operatorname{int} C$ with $\lim_{n\to\infty} \|x^n\| = +\infty$. Note that:

$$\bar{f}(x^n, u) = f(x^n, u) + \lambda \langle \nabla_1 D_\varphi(x^n, \bar{x}), u - x^n \rangle. \tag{9}$$

It follows from (9) that:

$$\begin{aligned} \bar{f}(x^n, u) &= f(x^n, u) \\ &\quad + \lambda \langle \nabla_1 D_\varphi(x^n, \bar{x}) - \nabla_1 D_\varphi(u, \bar{x}), u - x^n \rangle \\ &\quad + \lambda \langle \nabla_1 D_\varphi(u, \bar{x}), u - x^n \rangle. \end{aligned} \tag{10}$$

Using *A*4, (10) results in:

$$\begin{aligned} \bar{f}(x^n, u) &\leq -f(u, x^n) + \theta \langle \nabla_1 D_\varphi(x^n, \bar{x}) - \nabla_1 D_\varphi(u, \bar{x}), x^n - u \rangle \\ &\quad + \lambda \langle \nabla_1 D_\varphi(x^n, \bar{x}) - \nabla_1 D_\varphi(u, \bar{x}), u - x^n \rangle + \lambda \langle \nabla_1 D_\varphi(u, \bar{x}), u - x^n \rangle \\ &= -f(u, x^n) + (\theta - \lambda) \langle \nabla_1 D_\varphi(x^n, \bar{x}) - \nabla_1 D_\varphi(u, \bar{x}), x^n - u \rangle \\ &\quad + \lambda \langle \nabla_1 D_\varphi(u, \bar{x}), u - x^n \rangle. \end{aligned} \tag{11}$$

Using (11), Lemma 2-i, v and the Cauchy-Schwartz inequality, we get:

$$\bar{f}(x^n, u) \leq -f(u, x^n) + (\theta - \lambda)\alpha\nu \|x^n - u\|^2 + \lambda \|\nabla_1 D_\varphi(u, \bar{x})\| \|u - x^n\|. \tag{12}$$

Take $\hat{x} \in \operatorname{int} C$. Since $f(u,.)$ is convex, by $A2$, its subdifferential at $\hat{x}$, denoted by $\partial f(u,\hat{x})$, is nonempty. So, there exists $\hat{v} \in \partial f(u,\hat{x})$. By the definition of subdifferential, we have:

$$\langle \hat{v}, x^n - \hat{x} \rangle \leq f(u,x^n) - f(u,\hat{x}). \tag{13}$$

It follows from (13) that:

$$\begin{aligned} -f(u,x^n) &\leq \langle \hat{v}, \hat{x} - x^n \rangle - f(u,\hat{x}) \\ &\leq \|\hat{v}\|\|x^n - \hat{x}\| - f(u,\hat{x}) \\ &\leq \|\hat{v}\|\|u - \hat{x}\| + \|\hat{v}\|\|x^n - u\| - f(u,\hat{x}). \end{aligned} \tag{14}$$

We use the Cauchy-Schwartz and triangular inequality in the second and third inequalities respectively. Replacing (14) in (12), we obtain:

$$\begin{aligned} \bar{f}(x^n,u) &\leq \|\hat{v}\|\|u - \hat{x}\| + \|\hat{v}\|\|x^n - u\| - f(u,\hat{x}) \\ &\quad + (\theta - \lambda)\alpha\nu\|x^n - u\|^2 + \lambda\|\nabla_1 D_\varphi(u,\bar{x})\|\|u - x^n\| \\ &= \|x^n - u\| \left[\|\hat{v}\| + (\theta - \lambda)\alpha\nu\|x^n - u\| + \|\nabla_1 D_\varphi(u,\bar{x})\|\right] \\ &\quad + \|\hat{v}\|\|u - \hat{x}\| - f(u,\hat{x}). \end{aligned} \tag{15}$$

Thus, as $\theta - \lambda < 0$ and $\lim_{n\to\infty} \|x^n\| = +\infty$; it follows from (15) that $\lim_{n\to\infty} \bar{f}(x^n,u) = -\infty$. Therefore, for n large enough, it follows that $\bar{f}(x^n,u) \leq 0$. We show thus that $\bar{f}$ satisfies $A5$. Since $\bar{f}$ satisfies all the assumptions of Theorem 1, it follows that, $S(\bar{f},C)$ is nonempty.
From (8), is straightforward that the solution set is a singleton.

3 The algorithm and its convergence analysis

We begin this section by presenting an interior proximal point algorithm with φ-divergence for equilibrium problems in Hilbert spaces. Take a sequence of regularization parameters $\{\lambda_k\} \subset (\theta, \bar{\lambda})$.

The algorithm generates a sequence $\left\{x^k\right\} \subset \mathcal{H}$ as follows, x^0 is an arbitrary point in $\operatorname{int} C$, and, given x^k, x^{k+1} is the solution of $\mathrm{EP}(f_k, C)$ with

$$f_k(x,y) = f(x,y) + \lambda_k \langle \nabla_1 D_\varphi(x,x^k), y - x \rangle. \tag{16}$$

The Equilibrium Problem for the bifunction $f_k : C \times C \to \mathbb{R}$ will be denoted by $\mathrm{EP}(f_k, C)$ and will indicate their solution set by $\mathrm{S}(f_k, C)$. Note also that $f_k(x,x) = 0 \ \ \forall\, x \in C$, i.e., f_k is an equilibrium bifunction.

Bellow we present the Algorithm and a requirement for the analysis of convergence. For this, we assume:

R1 $\theta < \lambda_k \leq \bar{\lambda}$, where $\theta \geq 0$.

Algorithm 1
1. Given $x^0 \in \operatorname{int} C$, choice $\lambda_0 > 0$, and make $k := 0$.
2. Given $x^k \in \operatorname{int} C$, find $x^{k+1} \in \operatorname{int} C$ such that

$$f(x^{k+1}, y) + \lambda_k \langle \nabla_1 D_\varphi(x^{k+1}, x^k), y - x^{k+1} \rangle \geq 0, \forall\, y \in C. \tag{17}$$

3. If $\|x^k - x^{k+1}\| = 0$, then stop, otherwise put $k = k+1$ and go to step 2.

Remark 3 When $x^{k+1} = x^k$, the inequality (17) becomes

$$f(x^{k+1}, y) \geq -\lambda_k \langle \nabla_1 D_\varphi(x^{k+1}, x^{k+1}), y - x^{k+1} \rangle \ \ \forall\, y \in C.$$

Since $\langle \nabla_1 D_\varphi(x^{k+1}, x^{k+1}), y - x^{k+1} \rangle = 0$, by Lemma 2-*iii*, it follows that

$$f(x^{k+1}, y) \geq 0 \ \ \forall\, y \in C,$$

i.e., $x^{k+1} \in S(f,C)$.

In the following we show that the sequence generated by the Algorithm **??** is bounded.

Theorem 3 *Assume that f satisfies A1-A4, R1 occurs, $\nu > \mu$ and $S(f,C) \neq \emptyset$. Then:*
(i) For all $x^* \in S(f,C)$ the sequence $\left\{\|x^k - x^*\|_A\right\}$ is convergent.
(ii) The sequence $\left\{x^k\right\}$ is Fejér convergent to a set $S(f,C)$ with respect to the distance $\|\cdot\|_A$.
(iii) $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0.$

Proof From Theorem 2 it follows that the sequence $\left\{x^k\right\}$ is well-defined. Thus there exists $x^{k+1} \in \operatorname{int} C$ such that

$$f_k(x^{k+1}, y) = f(x^{k+1}, y) + \lambda_k \langle \nabla_1 D_\varphi(x^{k+1}, x^k), y - x^{k+1} \rangle \geq 0. \tag{18}$$

Take any $x^* \in \mathrm{S}(f,C)$. By doing $y = x^*$ in (18) we obtain:

$$0 \leq f(x^{k+1}, x^*) + \lambda_k \langle \nabla_1 D_\varphi(x^{k+1}, x^k), x^* - x^{k+1} \rangle. \tag{19}$$

Note that $f(x^*, x^{k+1}) \geq 0$, because $x^* \in S(f,C)$. Since by $A3$ we know that $f(x^{k+1}, x^*) \leq 0$ then (19) result in:

$$0 \leq -\lambda_k \langle \nabla_1 D_\varphi(x^{k+1}, x^k), x^{k+1} - x^* \rangle. \tag{20}$$

Using Lemma 2-iv, with $x = x^{k+1}$, $y = x^k$ and $z = x^*$ in (20), we obtain that

$$0 \leq -\lambda_k \left(\frac{\nu+\mu}{2}\right)\left(\|x^{k+1} - x^*\|_A{}^2 - \|x^k - x^*\|_A^2\right) - \lambda_k \left(\frac{\nu-\mu}{2}\right)\|x^{k+1} - x^k\|_A^2. \tag{21}$$

It follows from (21) that:

$$0 \leq \lambda_k \left(\frac{\nu+\mu}{2}\right)\|x^k - x^*\|_A^2 - \lambda_k \left(\frac{\nu+\mu}{2}\right)\|x^{k+1} - x^*\|_A^2 - \lambda_k \left(\frac{\nu-\mu}{2}\right)\|x^{k+1} - x^k\|_A^2. \tag{22}$$

As $\nu > \mu$, (22) result in

$$\|x^{k+1} - x^*\|_A^2 \leq \|x^k - x^*\|_A^2. \tag{23}$$

From (23), it follows that the sequence $\left\{\|x^k - x^*\|_A\right\}$ is monotone and bounded, so $\left\{\|x^k - x^*\|_A\right\}$ is convergent.

Note now that (23) implies $\|x^{k+1} - x^*\|_A \leq \|x^k - x^*\|_A$, by Definition 2, follows the item (ii).

Taking limits in (22) with $k \to \infty$, using (i) of this theorem, and $R1$, we get

$$\lim_{k\to\infty} \|x^{k+1} - x^k\|_A^2 = 0 \Rightarrow \lim_{k\to\infty} \|x^{k+1} - x^k\| = 0.$$

Below we show that the whole sequence $\left\{x^k\right\}$ converges to a solution of $EP(f,C)$.

Theorem 4 *Assume that f satisfies A1-A4, R1 occurs, $\nu > \mu$ and $S(f,C) \neq \emptyset$. Then the whole sequence $\{x^k\}$, generated by Algorithm **??**, converges to a solution of $EP(f,C)$.*

Proof From Lemma 1 and Theorem 3-ii we have that $\left\{x^k\right\}$ is bounded. Therefore, these sequence have cluster points, and how C is closed it follows that all cluster points of $\left\{x^k\right\}$ belong to C. Let $\bar{x}$ any cluster point of $\left\{x^k\right\}$ and $\left\{x^{kj}\right\} \subseteq \left\{x^k\right\}$ such that $x^{kj} \rightharpoonup \bar{x}$.

Using Theorem 3-i we get that

$$x^{kj+1} \rightharpoonup \bar{x}.$$

On the other hand, we know that $x^{kj+1} \in \operatorname{int} C$ and

$$f(x^{kj+1}, y) + \lambda_{kj} \langle \nabla_1 D_\varphi(x^{kj+1}, x^{kj}), y - x^{kj+1} \rangle \geq 0 \quad \forall\, y \in C. \tag{24}$$

It follows from (24) that

$$f(x^{kj+1}, y) \geq \lambda_{kj} \langle \nabla_1 D_\varphi(x^{kj+1}, x^{kj}), x^{kj+1} - y \rangle \quad \forall\, y \in C. \tag{25}$$

Using Lemma 2-*iv*, in (25) we have for all $y \in C$ and all j that

$$f(x^{kj+1}, y) \geq \lambda_{kj}(\frac{\nu + \mu}{2})(\|x^{kj+1} - y\|_A^2 - \|x^{kj} - y\|_A^2) + \lambda_{kj}(\frac{\nu - \mu}{2})\|x^{kj+1} - x^{kj}\|_A^2. \quad (26)$$

By using $A1$, $R1$ and the fact that $\lim_{j\to\infty} x^{kj} = \lim_{j\to\infty} x^{kj+1} = \bar{x}$, it follows from (26) that

$$f(\bar{x}, y) \geq \limsup_{j\to\infty} f(x^{k_j}, y) \geq 0 \ \ \forall\, y \in C.$$

Therefore $\bar{x} \in S(f, C)$. By Lemma 1 we then conclude that the sequence $\left\{x^k\right\}$ converges to a solution of $\mathrm{EP}(f, C)$.

4 Conclusions

In this work, we developed an interior proximal point algorithm with φ-divergence in finite-dimensional space to solve the Equilibrium problem $\mathrm{EP}(f, C)$. Under mild assumptions we show that the sequence generated by the algorithm converges to a solution of the problem, when the regularization parameters are bounded.

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