

Facially exposed cones are not always nice

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Abstract

We address the conjecture proposed by Gábor Pataki that every facially exposed cone is nice. We show that the conjecture is true in the three-dimensional case, however, there exists a four-dimensional counterexample of a cone that is facially exposed but is not nice.

1 Introduction

Nice cones provide a simple characterisation of the closedness of a linear image [4], they also feature in the study of lifting of convex sets [2], and in the analysis of the facial reduction algorithm [1]. Facial exposedness is another classical notion, and is important in the analysis of optimization problems (e.g. see [8],[10]). It is known that homogeneous [9] and projectively exposed cones [7] are facially exposed. The standard cones used in optimization (nonnegative orthant, second-order cone and the cone of positively semidefinite matrices) are both facially exposed and nice. We refer the reader to [5] and [6] for the detailed overview of the relevance of these properties for convex optimization.

A better understanding of the relation of facial exposedness and niceness may give a valuable insight into the fundamental properties of convex programs, performance of numerical algorithms and help simplify the verification of relevant conditions. In [5] it is proved that every nice cone is facially exposed, and it is conjectured that the reverse implication is true. The goal of this work is to demonstrate that while the conjecture is true in up to three dimensions, it fails in general: we show that there exists a four-dimensional convex closed cone that is facially exposed but is not nice.

Our paper is organised as follows: in Section 2 we remind some essential definitions and prove that all three-dimensional facially exposed cones are nice. In Section 3 we obtain some general results, which we use in Section 4 to prove that there exists a four-dimensional closed convex cone that is facially exposed but is not nice. Throughout the paper, by \mathbb{R}^n we denote the n -dimensional Euclidean space; for $x, y \in \mathbb{R}^n$, we let $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$, and $\|x\| := \sqrt{\langle x, x \rangle}$ denote the scalar product and Euclidean norm. By \mathcal{S}_{n-1} we denote the unit sphere in the relevant n -dimensional space. For a set $C \subset \mathbb{R}^n$ by $\text{aff } C$, $\text{co } C$ and $\text{cone } C$ we denote, respectively, its affine, convex and conic hulls.

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2 Preliminaries and the three-dimensional case

A nonempty convex subset F of a convex closed set $C \subset \mathbb{R}^n$ is called a *face* of C if $\alpha x + (1 - \alpha)y \in F$ with $x, y \in C$ and $\alpha \in (0, 1)$ imply $x, y \in F$. We use the standard notation $F \trianglelefteq C$ to denote that F is a face of C . Observe that C is its own face. We say that $F \trianglelefteq C$ is *proper* if $F \neq C$.

When $K \subset \mathbb{R}^n$ is a closed convex cone, a face can be defined equivalently as a subset F of K such that $x + y \in F$ with $x, y \in K$ imply $x, y \in F$.

A face F of a closed convex set $C \subset \mathbb{R}^n$ is called *exposed* if it can be represented as the intersection of C with a supporting hyperplane, i.e. there exist $y \in \mathbb{R}^n$ and $d \in \mathbb{R}$ such that for all $x \in C$

$$\langle y, x \rangle \leq d \quad \forall x \in C; \quad \langle y, x \rangle = d \quad \text{iff} \quad x \in F. \quad (1)$$

We say that a pair $(y, d) \in \mathbb{R}^{n+1}$ exposes $F \trianglelefteq C$ if (y, d) satisfy (1). A set is *facially exposed* if all of its faces are exposed.

Observe that for every hyperplane $H = \{x \in \mathbb{R}^n \mid \langle x, y \rangle = d\}$ supporting a closed convex set $C \subset \mathbb{R}^n$ the set $C \cap H$ is a face of C ; moreover, C is always an exposed face of itself (letting $(y, d) = (0_n, 0)$). It is not difficult to observe that for a cone K and any pair (y, d) exposing a face $F \trianglelefteq K$ we have $d = 0$.

Let C be a convex set in \mathbb{R}^n . The *polar* of C is the set

$$C^\circ = \{y \in \mathbb{R}^n \mid \sup_{x \in C} \langle x, y \rangle \leq 1\}.$$

When $K \subset \mathbb{R}^n$ is a cone, the polar set coincides with the *polar cone* of K :

$$K^\circ = \{y \in \mathbb{R}^n \mid \sup_{x \in K} \langle x, y \rangle \leq 0\}.$$

The *dual cone* K^* of K is $K^* = -K^\circ$. Observe that K° and K^* are always closed.

A cone K is called *nice* if for every face F of K the set $K^* + F^\perp$ is closed, where F^\perp is the orthogonal complement to $\text{span } F$.

We will also use the notion of *recession cone*

$$\text{rec } C := \{d \in \mathbb{R}^n \mid \exists x \in C \ x + td \in C \quad \forall t \geq 0\}.$$

To prove that every facially exposed set in the three-dimensional space is nice, we will need the following two trivial statements

Lemma 1. *Let $C \subset \mathbb{R}^n$, and let L be a linear subspace of \mathbb{R}^n . Then*

$$C + L^\perp = \Pi_L C + L^\perp,$$

where by Π_L we denote the orthogonal projection onto the linear subspace L .

Lemma 2. *Let $L \subset \mathbb{R}^n$ be a linear subspace, and assume $C \subset L$ is closed. Then the set $C + L^\perp$ is closed.*

Theorem 1. *A convex closed facially exposed cone $K \subset \mathbb{R}^3$ is nice.*

Proof. Let $K \subset \mathbb{R}^3$ be a facially exposed convex closed cone, and let $F \trianglelefteq K$. Observe that if $\text{span } F = \mathbb{R}^3$, we have $F^\perp = \{0\}$, and the set $K^* + F^\perp = K^*$ is closed since the dual cone is closed. Likewise, if $\text{span } F = \{0\}$, then $F^\perp = \mathbb{R}^3$ and $K^* + F^\perp = \mathbb{R}^3$ is closed. In the case when F is one-dimensional, i.e. $\text{span } F = \text{span}\{l\}$ for some $l \neq 0$, observe that $\Pi_{\text{span}\{l\}}K^*$ is a one-dimensional cone that contains zero, which is always closed. Hence, using Lemmas 1 and 2 the set

$$K^* + F^\perp = \Pi_{\text{span}\{l\}}K^* + F^\perp$$

is closed. It remains to consider the case when F is two-dimensional, i.e. $F = \text{cone}\{p_1, p_2\}$, where $p_1, p_2 \in \mathcal{S}_2$ are non-collinear. Observe that $E_i := \text{cone}\{p_i\}$ is a face of K for each $i \in \{1, 2\}$ (see Fig. 1). Since K is facially exposed, there exist $h_1, h_2 \in K^*$ such that

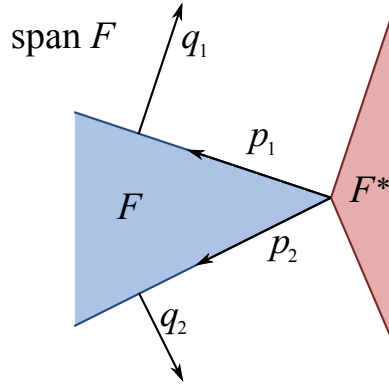


Figure 1: Illustration to the proof of Theorem 1.

$(h_i, 0)$ exposes E_i , $i \in \{1, 2\}$. Observe that $h_i \notin F^\perp$ (otherwise $(h_i, 0)$ would expose the whole face F). Let $q_i := \Pi_{\text{span } F} h_i$, and observe that $q_i \neq 0$ since $h_i \notin F^\perp$. We have

$$\langle q_i, p_i \rangle = \langle h_i, p_i \rangle = 0 \quad \forall i \in \{1, 2\}; \quad \langle q_i, p_j \rangle = \langle h_i, p_j \rangle > 0 \quad \forall i \neq j.$$

This yields (see Fig. 1)

$$F^* = \text{cone}\{q_1, q_2\} + F^\perp. \quad (2)$$

Observing that $\Pi_{\text{span } F} \text{cone}\{h_1, h_2\} = \text{cone}\{q_1, q_2\}$, from Lemma 1 we have

$$\text{cone}\{q_1, q_2\} + F^\perp = \text{cone}\{h_1, h_2\} + F^\perp. \quad (3)$$

Finally, (2) and (3) yield $F^* = \text{cone}\{h_1, h_2\} + F^\perp \subset K^* + F^\perp$. Since $F^* = \overline{(K^* + F^\perp)}$ (see [5, Remark 1]) this yields $K^* + F^\perp = F^*$, hence, $K^* + F^\perp$ is closed. By the arbitrariness of F it follows that the cone K is nice. \square

3 Technical results

We next prove two fairly trivial results that establish relations between the faces of a closed convex set $C \subset \mathbb{R}^n$ and its homogenization $K = \text{cone}(\{1\} \times C) \subset \mathbb{R}^{n+1}$.

Proposition 1. *Let $C \subset \mathbb{R}^n$ be a nonempty compact convex set, and let*

$$K = \text{cone}(\{1\} \times C) \subset \mathbb{R}^{n+1}.$$

Then K is a closed convex cone and the only faces of K are $\{0_{n+1}\}$ and the sets

$$F_K = \text{cone}\{\{1\} \times F_C\}, \quad F_C \trianglelefteq C.$$

Proof. Observe that K is a convex cone by definition. From [3, Lemma 5.41] we have $\overline{K} = K(C) \cup \{(0, d) \mid d \in \text{rec } C\}$. Observe that since C is compact, its recession cone is trivial, hence, $\overline{K} = K + \{0\} = K$, i.e. K is closed.

For every $x = (x_0, \bar{x}) \in K$ (where $x_0 \in \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$) we have $x_0 \geq 0$, and $x_0 = 0$ yields $x = 0_{n+1}$. We will use this observation several times in the proof.

Let $F_C \trianglelefteq C$ and $F_K := \text{cone}\{\{1\} \times F_C\}$. We show that $F_K \trianglelefteq K$. Pick a $z = (z_0, \bar{z}) \in F_K$, and let $x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in K$ be such that $z = x + y$. If $x_0 = 0$, we have $x = 0$, hence, $z = y \in F_K$. Similarly, $y_0 = 0$ yields $z = x \in F_K$. Assume that both x_0 and y_0 are not zero. Then $x_0, y_0, z_0 > 0$. Let

$$\bar{z}' = \frac{1}{z_0} \bar{z}, \quad \bar{x}' = \frac{1}{x_0} \bar{x}, \quad \bar{y}' = \frac{1}{y_0} \bar{y}.$$

Observe that $\bar{x}', \bar{y}' \in C$ by the definition of K ; moreover, $\bar{z}' \in F_C$ and

$$\bar{z}' = \frac{x_0}{z_0} \bar{x}' + \frac{y_0}{z_0} \bar{y}', \quad \frac{x_0}{z_0} + \frac{y_0}{z_0} = 1, \quad \frac{x_0}{z_0}, \frac{y_0}{z_0} > 0,$$

hence, $\bar{z}' \in (\bar{x}', \bar{y}')$. Since $\bar{z}' \in F_C$, and $F_C \trianglelefteq C$, this yields $\bar{x}', \bar{y}' \in F_C$, hence, $x, y \in F_K$, and by the arbitrariness of x, y, z the set F_K is a face of K . It is not difficult to observe that $\{0_{n+1}\}$ is a face of K as $\{x = (x_0, \bar{x}) \mid x_0 = 0\} \cap K = \{0_{n+1}\}$.

Now assume F_K is a face of K . Let

$$F_C := \{\bar{x} \in \mathbb{R}^n \mid \exists \alpha > 0 : \alpha(1, \bar{x}) \in F_K\}.$$

In the case when $F_C = \emptyset$, for every $x \in F_K$ we have $x_0 = 0$, hence, $x = 0$, and therefore $F_K = \{0_{n+1}\}$. Consider the case when $F_C \neq \emptyset$. Then $F_K = \text{cone}\{\{1\} \times F_C\}$, and it remains to show that F_C is a face of C . Pick any point $\bar{z}' \in F_C$, and let $\bar{z}' = \alpha \bar{x}' + (1 - \alpha) \bar{y}'$, where $\bar{x}', \bar{y}' \in C$ and $\alpha \in (0, 1)$. Since $\bar{z}' \in F_C$, the point $z = (1, \bar{z}') \in F_K$. Let $x = \alpha(1, \bar{x}')$, $y = (1 - \alpha)(1, \bar{y}')$. Observe that $z = x + y$. Since F_K is a face, $x, y \in F_K$, therefore, $\bar{x}', \bar{y}' \in F_C$, and hence F_C is a face of C . \square

Proposition 2. *Let C be a compact convex set in \mathbb{R}^n . If C is facially exposed, then so is $K = \text{cone}\{\{1\} \times C\} \subset \mathbb{R}^{n+1}$.*

Proof. First observe that $\{0_{n+1}\}$ is an exposed face of K , since $\{x = (x_0, \bar{x}) \mid x_0 = 0\} \cap K = \{0_{n+1}\}$. By Proposition 1 the only remaining faces of K are

$$F_K = \text{cone}\{\{1\} \times F_C\}, \quad F_C \trianglelefteq C.$$

Assume that $F_C \trianglelefteq C$. Since F_C is exposed, there exist $\bar{y} \in \mathbb{R}^n$ and $d \in \mathbb{R}$ such that

$$\langle \bar{x}, \bar{y} \rangle < d \quad \forall \bar{x} \in C \setminus F_C; \quad \langle \bar{x}, \bar{y} \rangle = d \quad \forall \bar{x} \in F_C. \quad (4)$$

Let $y := (-d, \bar{y}) \in \mathbb{R}^{n+1}$. Pick any $x = (x_0, \bar{x}) \in F_K$. If $x_0 = 0$, we have $\langle x, y \rangle = 0$. If $x_0 > 0$, observe that $\bar{x}' = \frac{1}{x_0} \bar{x} \in F_C$, hence, we have

$$\langle x, y \rangle = x_0 (-d + \langle \bar{y}, \bar{x}' \rangle) = x_0(-d + d) = 0.$$

Now suppose $x \in K \setminus F_K$. Then $x_0 \neq 0$ and $\bar{x}' := \frac{1}{x_0} \bar{x} \notin F_C$. We have from (4)

$$\langle x, y \rangle = x_0 (-d + \langle \bar{y}, \bar{x}' \rangle) < x_0(-d + d) = 0.$$

We have therefore shown that the pair $((-d, y), 0)$ exposes $F_K \trianglelefteq K$. \square

The following statement relates the polar sets of $C \subset \mathbb{R}^n$ and $K = \text{cone}(\{1\} \times C) \subset \mathbb{R}^{n+1}$.

Proposition 3. *Let $C \subset \mathbb{R}^n$ be a compact convex set such that $0_n \in \text{int } C$, and let $K = \text{cone}(\{1\} \times C)$. Then*

$$K^\circ = \text{cone}\{\{-1\} \times C^\circ\}.$$

Proof. From the definition of a polar cone

$$\begin{aligned} K^\circ &= \{(y_0, \bar{y}) \mid \sup_{\alpha \geq 0, \bar{x} \in C} \alpha(y_0 + \langle \bar{x}, \bar{y} \rangle) \leq 0\} \\ &= \{(y_0, \bar{y}) \mid \sup_{\alpha > 0, \bar{x} \in C} \alpha(y_0 + \langle \bar{x}, \bar{y} \rangle) \leq 0\} \\ &= \{(y_0, \bar{y}) \mid \sup_{\bar{x} \in C} (y_0 + \langle \bar{x}, \bar{y} \rangle) \leq 0\} \\ &= \{(y_0, \bar{y}) \mid \sup_{\bar{x} \in C} \langle \bar{x}, \bar{y} \rangle \leq -y_0\}. \end{aligned}$$

Observe that since $0_n \in \text{int } C$, for every $y \in \mathbb{R}^n$ we have $\sup_{\bar{x} \in C} \langle \bar{x}, y \rangle > 0$, hence,

$$\begin{aligned} K^\circ &= \{\alpha(-1, \bar{y}), \alpha \geq 0 \mid \alpha \sup_{\bar{x} \in C} \langle \bar{x}, \bar{y} \rangle \leq \alpha\} \\ &= \{0_{n+1}\} \cup \{\alpha(-1, \bar{y}), \alpha > 0 \mid \sup_{\bar{x} \in C} \langle \bar{x}, \bar{y} \rangle \leq 1\} \\ &= \{0_{n+1}\} \cup \{\alpha(-1, \bar{y}), \alpha > 0 \mid \bar{y} \in C^\circ\} \\ &= \text{cone}\{\{-1\} \times C^\circ\}. \end{aligned}$$

\square

4 Four-dimensional Counterexample

The goal of this section is to prove the following result

Theorem 2. *There exists a facially exposed closed convex cone $K \subset \mathbb{R}^4$ such that K is not nice*

We first describe the construction of our counterexample. Let the three-dimensional curves $\gamma_i : [0, T] \rightarrow \mathbb{R}^3$, with $T = \pi/4$, $i \in I := \{1, 2, 3, 4\}$ be defined as follows:

$$\begin{aligned} \gamma_1(t) &:= (0, -\sin t, \cos t - 1); & \gamma_2(t) &:= (0, \cos t - 1, -\sin t); \\ \gamma_3(t) &:= (-\sin t, 1 - \cos t, 0); & \gamma_4(t) &:= (\cos t - 1, \sin t, 0). \end{aligned} \quad (5)$$

For convenience, we use γ_i to denote the whole segment $\gamma_i([0, T])$. Let $C := \text{co}_{i \in I} \{\gamma_i\}$.

The three-dimensional set C is shown in Fig. 2, and in Fig. 3 we give some geometric details related to the construction of C . To finish the construction of the counterexample,

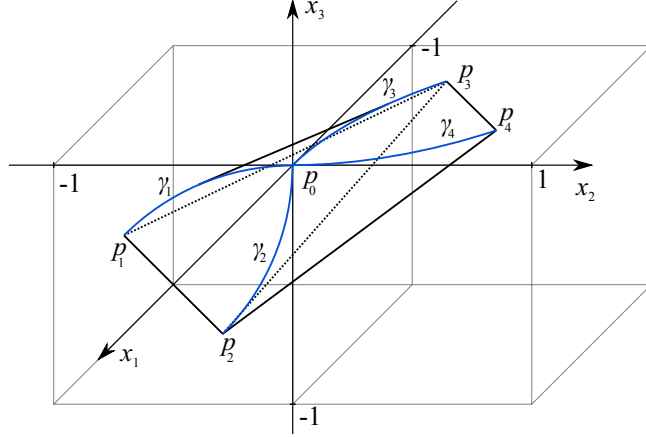


Figure 2: The three-dimensional set $C = \text{co}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.

let

$$C' := 2C + c \quad \text{with} \quad c = \left(\frac{1}{2}, 0, \frac{1}{2}\right) \quad \text{and} \quad K = \text{cone}(\{1\} \times C'). \quad (6)$$

The set C' is shown in Fig. 4. We prove Theorem 2 by showing that this set is a convex closed facially exposed cone which is not nice. Throughout this section, we always use the notation C , C' and K to refer to the aforementioned sets.

For the proof of Theorem 2 we need several technical statements related to the geometry of the sets C , C' and K .

For convenience, denote the endpoints of the curves in (5) as follows.

$$\begin{aligned} p_0 &:= 0_3 = \gamma_1(0) = \gamma_2(0) = \gamma_3(0) = \gamma_4(0); \\ p_1 &:= \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - 1\right) = \gamma_1(T); & p_2 &:= \left(0, \frac{1}{\sqrt{2}} - 1, -\frac{1}{\sqrt{2}}\right) = \gamma_2(T); \\ p_3 &:= \left(-\frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}, 0\right) = \gamma_3(T); & p_4 &:= \left(\frac{1}{\sqrt{2}} - 1, \frac{1}{\sqrt{2}}, 0\right) = \gamma_4(T). \end{aligned} \quad (7)$$

Throughout this section, we also utilize the following notation.

$$t_\theta := \arccos\left(\frac{\sin \theta}{1 + \sin \theta - \cos \theta}\right); \quad (8)$$

$$y_\theta := (-\sin t_\theta \sin \theta, -\cos t_\theta \sin \theta, \cos t_\theta \cos \theta); \quad (8)$$

$$d_\theta := \cos t_\theta(1 - \cos \theta) = \sin \theta(1 - \cos t_\theta). \quad (9)$$

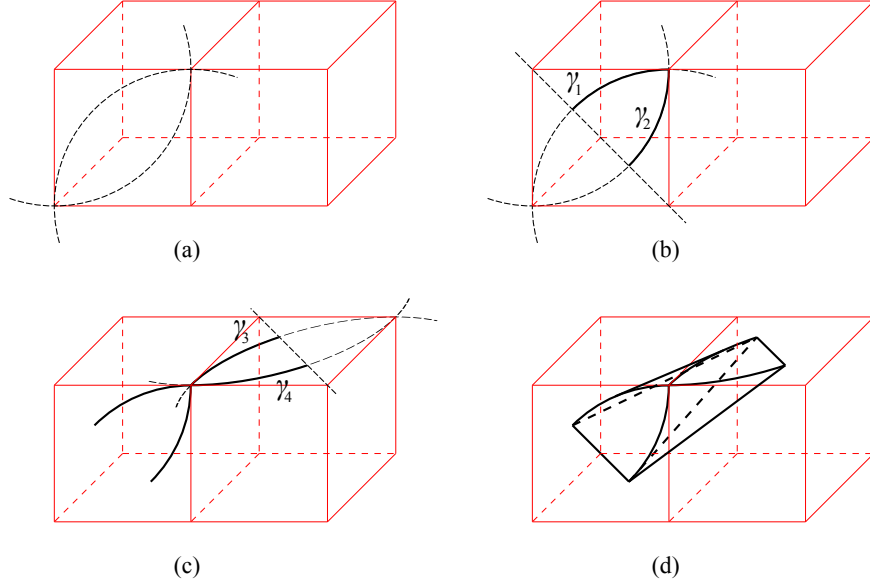


Figure 3: The geometric construction of the set C : (a) consider two adjacent cubes as shown; on the frontal surface of the left cube draw two circles with centres at the cube's top-left and bottom-right corners with the radius equal to the edge; (b) intersect the circles with the diagonal through their centres; γ_1 and γ_2 are the segments of the circles bounded by their intersections with the diagonal and the top-right corner of the face; (c) repeat the same process on the top face of the right cube to obtain the curves γ_3 and γ_4 ; (d) construct the convex hull.

Proposition 4. *Let*

$$\varphi(\theta) := \frac{\sin \theta}{1 + \sin \theta - \cos \theta}, \quad \theta \in (0, T].$$

The function $\varphi : (0, T] \rightarrow \mathbb{R}$ is strictly decreasing; moreover,

$$\lim_{\theta \downarrow 0} \varphi(\theta) = 1; \quad \varphi(T) = \frac{1}{\sqrt{2}}.$$

Hence, the mapping $t_\theta : (0, T] \rightarrow (0, T]$, $t_\theta = \arccos \varphi(\theta)$ is a bijection.

Proof. Observe that

$$\varphi'(\theta) = \frac{\cos \theta - 1}{(1 + \sin \theta - \cos \theta)^2} < 0 \quad \forall \theta \in (0, T],$$

hence, φ is strictly decreasing on $(0, T]$; multiplying the numerator and denominator of φ by $(1 + \sin \theta + \cos \theta)$, we have

$$\varphi(\theta) = \frac{1 + \cos \theta + \sin \theta}{2(1 + \sin \theta)} \xrightarrow{\theta \downarrow 0} 1 = \cos(0); \quad \varphi(T) = \frac{1}{\sqrt{2}} = \cos T. \quad (10)$$

It is evident from the strict monotonicity of φ and (10) that t_θ is bijective. \square

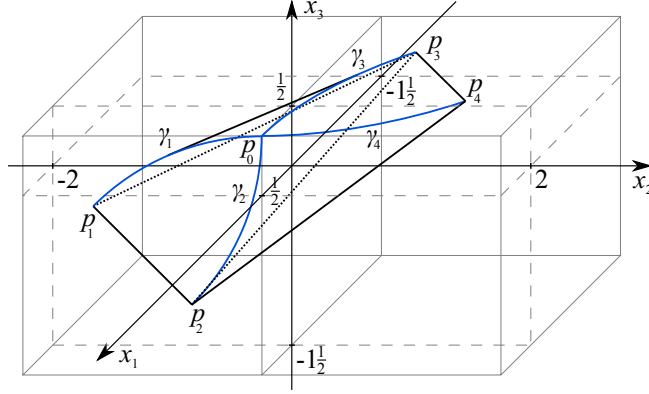


Figure 4: The three-dimensional set $C' = 2C + \{c\}$.

In the next statement we list all proper faces of the set C and show that they are exposed. Note that by the *dimension* of a face we mean the smallest affine subspace spanned by the face.

Proposition 5. *The following singletons are the only zero-dimensional faces of C :*

$$F_{00} = \{p_0\}; \quad F_{0i}(t) = \{\gamma_i(t)\}, \quad t \in (0, T], \quad i \in I.$$

The only one-dimensional faces of F are the following line segments:

$$\begin{aligned} F_{11}(\theta) &= \text{co}\{\gamma_1(\theta), \gamma_3(t_\theta)\}, & \theta \in (0, T]; \\ F_{12}(\theta) &= \text{co}\{\gamma_4(\theta), \gamma_2(t_\theta)\}, \end{aligned}$$

$$F_{13} = \text{co}\{p_1, p_2\}; \quad F_{14} = \text{co}\{p_3, p_4\}; \quad F_{15} = \text{co}\{p_2, p_3\}.$$

The only two-dimensional faces of C are

$$F_{21} = \text{co}\{p_1, p_2, p_3\}, \quad F_{22} = \text{co}\{p_2, p_3, p_4\}, \quad F_{23} = \text{co}\{\gamma_1, \gamma_2\}, \quad F_{24} = \text{co}\{\gamma_3, \gamma_4\}.$$

All these faces are exposed.

Proof. It is evident from the plot in Fig. 2 that the only two-dimensional faces of F are, indeed, F_{21} , F_{22} , F_{23} and F_{24} , and that they are exposed. It is also clear that F_{1i} , $i = 3, 4, 5$ are one-dimensional exposed faces of C , and that the singletons in $\cup_{i=1}^4 \gamma_i$ are the only zero-dimensional faces of C . It remains to show that the one-dimensional sets $F_{11}(\theta)$ and $F_{12}(\theta)$ are exposed faces of C , that all zero-dimensional faces are exposed and that C does not have any other faces.

It is a technical exercise to verify that the following relations hold for each $\theta \in (0, T]$.

$$\langle \gamma_1(\theta), y_\theta \rangle = \langle \gamma_3(t_\theta), y_\theta \rangle = d_\theta; \tag{11}$$

$$\begin{aligned} \langle \gamma_1(\theta'), y_\theta \rangle &= \cos t_\theta (\cos(\theta - \theta') - \cos \theta) < d_\theta, & \forall \theta', t' \in [0, T] \\ \langle \gamma_3(t'), y_\theta \rangle &= \sin \theta (\cos(t' - t_\theta) - \cos t_\theta) < d_\theta & \text{s.t. } \theta' \neq \theta, t' \neq t_\theta. \end{aligned} \tag{12}$$

$$\begin{aligned} \langle \gamma_2(t), y_\theta \rangle &= \cos t_\theta (\sin \theta - \sin(t + \theta)) \leq 0 < d_\theta & \forall t \in [0, T], \\ \langle \gamma_4(t), y_\theta \rangle &= \sin \theta (\sin t_\theta - \sin(t + t_\theta)) \leq 0 < d_\theta & \forall t \in [0, T]. \end{aligned} \tag{13}$$

From (11)-(13) we deduce that for every $\theta \in (0, T]$ the pair (y_θ, d_θ) exposes $F_{11}(\theta)$, but no other points of C . Hence, for every $\theta \in (0, T]$ the set $F_{11}(\theta)$ is an exposed face.

Similarly, it can be shown that the sets $F_{22}(\theta)$ are exposed faces of C . In this case the normals y_θ must be replaced by

$$y'_\theta = (\cos t_\theta \cos \theta, \sin \theta \cos t_\theta, -\sin t_\theta \sin \theta).$$

We show that the zero-dimensional faces of C are exposed. For each $i \in I$ and $t \in (0, T]$ the faces $F_{01}(t)$ and $F_{04}(t)$ are exposed by $(y_i(t), d_i(t)) \in \mathbb{R}^{3+1}$, $i \in \{1, 4\}$, where

$$y_1(t) = (1, -\sin t, \cos t), \quad y_4(t) = (\cos t, \sin t, 1), \quad d_1(t) = d_4(t) = 1 - \cos t.$$

For each $\theta \in (0, T]$ (recall that by Proposition 4 the mapping $\theta \rightarrow t_\theta$ is a bijection on $[0, T]$) the faces $F_{02}(t_\theta)$ and $F_{03}(t_\theta)$ are exposed by $(y_i(\theta), d_\theta) \in \mathbb{R}^{3+1}$, $i \in \{2, 3\}$, where

$$y_2(\theta) = y'_\theta + (1, 0, 0), \quad y_3(\theta) = y_\theta + (0, 0, 1).$$

For the remaining point p_0 the supporting plane is defined by $y_0 = (1, 0, 1)$, $d_0 = 0$.

It remains to show that we have listed all faces of C . From Proposition 4 the mapping $t_\theta : (0, T] \rightarrow (0, T]$ is continuous and bijective, hence, every point on the curve γ_1 is connected with a point on γ_3 by a one-dimensional face in a continuous manner, and vice versa. Hence, these faces $F_{11}(\theta)$ together with the point p_0 cover the part of the surface of C bounded by γ_1 , γ_3 and $[p_1, p_3]$. Similar argument works for the part of the surface bounded by γ_2 , γ_4 and $[p_2, p_4]$. It is evident from the plot in Fig. 4 that the rest of the surface of C is covered by the two-dimensional faces F_{2i} , $i = 1, \dots, 4$. Hence, if we have missed any faces, they must belong to either F_{2i} , $i = 1, \dots, 4$, $F_{1i}(\theta)$, $\theta \in (0, T]$ or $F_{00} = \{p_0\}$. It is evident that we have already listed all zero- and one-dimensional faces that comprise the relative boundaries of the aforementioned faces. \square

Proposition 6. *The set C' is facially exposed.*

Proof. Since C' is obtained from C via an affine transformation, and hence facial exposedness of C yields facial exposedness of C' , it is sufficient to show that C is facially exposed. Observe that any proper face of C is at most two-dimensional (a face $F \trianglelefteq C$ is proper if $F \neq C$), hence, Proposition 5 implies that all proper faces of C are exposed. Hence, C is exposed. \square

Proposition 7. *Let $F = \text{cone}(\{1\} \times (2 \text{co}\{\gamma_3, \gamma_4\} + \{c\}))$, where $c = (1/2, 0, 1/2)$. Then $F^\perp = \text{span}\{(1, 0, 0, -2)\}$.*

Proof. Observe that the points $q_i = (1, 2p_i + c) \in \mathbb{R}^4$, $i = 0, 3, 4$, where p_i 's are defined by (7), belong to F . Moreover, it is not difficult to observe that the affine hull $\text{aff}\{p_0, p_3, p_4\}$ coincides with $\text{aff}\{\gamma_3, \gamma_4\}$. Hence, $\text{span } F = \text{span}\{q_i, i \in \{0, 3, 4\}\}$. For any point $y = (y_0, \bar{y}) \in F^\perp$ we have

$$\langle y, q_i \rangle = 0, \quad i = 0, 3, 4,$$

or, equivalently,

$$\begin{bmatrix} 1 & & & \\ 1 & \frac{1}{2} - \sqrt{2} & 2 - \sqrt{2} & \frac{1}{2} \\ 1 & \sqrt{2} - \frac{3}{2} & \sqrt{2} & \frac{1}{2} \end{bmatrix} y = 0.$$

Solving this linear system, we obtain $y_1 = y_2 = 0$, $y_3 = -2y_0$ (where $(y_1, y_2, y_3) = \bar{y}$), hence, $F^\perp = \text{span}\{(1, 0, 0, -2)\}$. \square

Proposition 8. *Let*

$$\chi(\theta) = \left(-1, \frac{y_\theta}{d'_\theta}\right), \quad \theta \in (0, T), \quad (14)$$

where $d'_\theta = 2d_\theta + \frac{1}{2} \cos(t_\theta - \theta)$, and y_θ and d_θ are defined by (8). Then

$$\chi = \chi((0, T)) \subset K^\circ. \quad (15)$$

Proof. It is shown in the proof of Proposition 5 that the pairs (y_θ, d_θ) expose faces of C . It is not difficult to observe that then the pairs (y_θ, d'_θ) expose the relevant faces of C' for all $\theta \in (0, T)$. Indeed, let $y = 2x + c$, where $x \in C$. Then

$$\langle y_\theta, y \rangle = 2\langle y_\theta, x \rangle + \frac{1}{2} \cos(t_\theta - \theta),$$

which yields the desired relation.

We have by the definition of the polar set

$$\frac{y_\theta}{d'_\theta} \in (C')^\circ \quad \forall \theta \in (0, T).$$

It is not difficult to observe that $0 \in \text{int } C'$ (for example, by checking that the only solution of the system

$$\langle 2p_i + c, x \rangle \leq 0, \quad i \in \{0, \dots, 5\}$$

is $x^* = 0$, hence, zero can not be separated from C'), hence, (15) follows from Proposition 3. \square

4.1 Proof of Theorem 2

The set K defined by (6) is obviously a convex cone and is closed by Proposition 1. We first prove that K is facially exposed (Proposition 9) and then show that K is not nice (Proposition 11). These two results together yield Theorem 2.

Proposition 9. *The cone K is facially exposed.*

Proof. By Proposition 6 the set C' is facially exposed, hence, by Proposition 2 the cone $K = \text{cone}(\{1\} \times C)$ is facially exposed. \square

We need the following technical result for the proof of Proposition 11.

Proposition 10. *For every $\alpha \in \mathbb{R}$ there exists $t_\alpha > 0$ such that*

$$\varphi_\alpha(t) = \alpha(\cos t - 1) + \sin t > 0 \quad \forall t \in (0, t_\alpha).$$

Proof. First assume that $\alpha \leq 0$. Then $\varphi_\alpha(t) > 0$ for all $t \in (0, \frac{\pi}{2})$, and we let $t_\alpha := \frac{\pi}{2}$. Consider the case when $\alpha > 0$. Observe that for all $t > 0$

$$\cos t > 1 - \frac{t^2}{2}, \quad \sin t > t - \frac{t^3}{6},$$

therefore, for $t > 0$ we have

$$\alpha(\cos t - 1) + \sin t > \alpha \left(1 - \frac{t^2}{2} - 1\right) + t - \frac{t^3}{6} = t \left(1 - \frac{t}{2}\alpha - \frac{t^2}{6}\right).$$

Choose $t_\alpha > 0$ such that

$$\frac{t}{2}\alpha + \frac{t^2}{6} < 1 \quad \forall t \in (0, t_\alpha),$$

hence, $\varphi_\alpha(t) > 0$ for all $t \in (0, t_\alpha)$. □

Proposition 11. *The cone K is not nice.*

Proof. Observe that there is a one-to-one correspondence between the faces of C and C' , namely, for every $E_0 \trianglelefteq C$ the set $2E_0 + \{c\}$ is a face of C' and vice versa. In particular, the face $F_{24} = \text{co}\{\gamma_3, \gamma_4\}$ of C (see Proposition 5) corresponds to $E := 2F_{24} + \{c\}$, hence, $E \trianglelefteq C'$. Proposition 1 yields $F = \text{cone}(\{1\} \times E) \trianglelefteq K$. By Proposition 7

$$F^\perp = \text{span}\{u\}, \quad u = (1, 0, 0, -2).$$

We show that $q := (-1, 0, -1, 2) \in \overline{(K^\circ + F^\perp)} \setminus (K^\circ + F^\perp)$, hence the set $K^\circ + F^\perp = -(K^* + F^\perp)$ is not closed, and K is not nice.

We first show that $q \notin (K^\circ + F^\perp)$. Assume the contrary. Then $q = p + \lambda u$ for some $p \in K^\circ, \lambda \in \mathbb{R}$. Let

$$\gamma'(t) := (1, 2\gamma_1(t) + c), \quad t \in [0, T].$$

Since $\gamma' = \gamma'([0, T]) \subset (\{1\} \times C') \subset K$, and $p \in K^\circ$,

$$\psi(t) := \langle \gamma'(t), p \rangle \leq 0 \quad \forall t \in [0, T]. \tag{16}$$

Observe that

$$\psi(t) = \langle \gamma'(t), p \rangle = \langle \gamma'(t), q - \lambda u \rangle = 2(2(\lambda + 1)(\cos t - 1) + \sin t).$$

Let $\alpha := 2(\lambda + 1)$ in Proposition 10. Then there exists $t_\alpha > 0$ such that $\psi(t) = 2\varphi_\alpha(t) > 0$ for all $t \in (0, t_\alpha)$. This contradicts (16), hence, our assumption is wrong and $q \notin K^\circ + F^\perp$.

It remains to show that

$$q \in \overline{K^\circ + F^\perp}. \tag{17}$$

Let $\chi = \chi((0, T))$ be defined by (14). By Proposition 8 we have $\chi \subset K^\circ$. Therefore, for all $\theta \in (0, T)$, $\alpha \geq 0$ and $\lambda \in \mathbb{R}$

$$q(\theta, \alpha, \lambda) := \alpha\chi(\theta) + \lambda u \in K^\circ + F^\perp.$$

Let

$$\alpha_\theta := \frac{d'_\theta}{\cos t_\theta \sin \theta}, \quad \lambda_\theta := \frac{\cos \theta}{2 \sin \theta} - 1,$$

where d'_θ is as in Proposition 8. Then

$$q_\theta := q(\theta, \alpha_\theta, \lambda_\theta) = \left(\lambda_\theta - \alpha_\theta, -\frac{\sin t_\theta}{\cos t_\theta}, -1, 2 \right).$$

Observe that

$$\lambda_\theta - \alpha_\theta = \frac{\cos \theta}{2 \sin \theta} - \frac{d'_\theta}{\cos t_\theta \sin \theta} - 1 = \frac{\sin t_\theta}{\cos t_\theta} + 2 \frac{\cos \theta - 1}{\sin \theta} - 1.$$

Notice that

$$\frac{\cos \theta - 1}{\sin \theta} = \frac{\cos^2 \theta - 1}{\sin \theta (\cos \theta + 1)} = \frac{\sin \theta}{\cos \theta + 1} \xrightarrow{\theta \downarrow 0} 0; \quad \lim_{\theta \downarrow 0} t_\theta = 0,$$

where the last relation is shown in Proposition 4. Hence,

$$\lim_{\theta \downarrow 0} \frac{\sin t_\theta}{\cos t_\theta} = 0; \quad \lim_{\theta \downarrow 0} (\lambda_\theta - \alpha_\theta) = -1.$$

Therefore,

$$\lim_{\theta \downarrow 0} q_\theta = (-1, 0, -1, 2) = q.$$

Since $q_\theta \in K^\circ + F^\perp$ for all $\theta \in (0, T)$, this yields (17). □

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