Exploring the Modeling Capacity of Two-stage Robust Optimization – Two Variants of Robust Unit Commitment Model

Yu An and Bo Zeng

Abstract—**To handle significant variability in loads, renewable energy generation, as well as various contingencies, two-stage robust optimization method has been adopted to construct unit commitment models and to ensure reliable solutions. In this paper, we further explore and extend the modeling capacity of two-stage robust optimization and present two new robust unit commitment variants, the *expanded robust unit commitment* and *the risk constrained robust unit commitment* models. We derive some structural properties, show the connection to the scenario based stochastic unit commitment model, and present a customized column-and-constraint generation method. Numerical experiments on those models are performed using a practical data set, which illustrate their modeling strength, economic outcomes and the algorithm performance in solving those models.**

Index Terms—**robust optimization, unit commitment problem, uncertainty, column-and-constraint generation method**

I. Introduction

RECENTLY, robust optimization (RO) techniques [1, 2, 3, 4, 5], especially two-stage robust optimization method [6], have attracted many researchers' attentions and been utilized to solve practical system design and operation problems. Different from classical stochastic programming models, an RO formulation has two features: (i) instead of assuming any probabilistic information on random factors, it assumes uncertain sets to capture randomness. (ii) instead of seeking for solutions with the optimal *expected* value, it derives solutions with the best performance with respect to the worst case situations in the uncertainty sets. So, the RO model is less demanding on data analysis for capturing randomness and its solution is more reliable towards uncertainty. Therefore, robust optimization approaches are often adopted to address real problems, e.g., the operational problems in power industry, where it is challenging to construct a stochastic model to capture randomness or the system reliability is of more critical concern.

Nevertheless, because a solution of the regular (single-stage) RO must hedge against any possible realization within the uncertainty set, it tends to be overly conservative and may not be cost-effective. To address such issue, two-stage (and multi-stage) robust optimization model [6] has been introduced to support decision making where decisions are partitioned into two stages, i.e., before and after uncertainty is disclosed. The first stage decisions still need to be made with respect to any realization in the uncertainty set while the second stage decisions can be made after the first stage decisions are determined and the uncertainty is revealed, which essentially enables the decision maker a recourse opportunity. Hence, comparing with that of the regular (single-stage) RO, a solution to two-stage RO is less conservative and more cost-effective. Note that this decision making structure nicely matches that of the day-ahead unit commitment (UC) problem, which makes use of dispatch as the recourse tool but must handle significant variability in loads, renewable energy generation, as well as various contingencies. So, over the last few years, several two-stage robust unit commitment formulations have been developed and implemented to ensure reliable power generation and dispatch, see [7, 8, 9, 10, 11, 12, 13].

As a new optimization scheme, we note that two key concepts of two-stage RO, i.e., the uncertainty set and the consideration of the worst case performance with recourse opportunities, jointly provide a very flexible mechanism that can actually be used to satisfy more complicated modeling needs. For example, on the one hand, when a single uncertainty set maybe too rough to describe the random factor, we can utilize different uncertainty sets to jointly define it. On the other hand, hard constraints on the worst case performance can be included to control the overall risks. As a result, the standard two-stage RO can be extended to capture different random situations, diverse data availabilities and qualities, and to meet various requirements. We mention that, although new models may be more involved than the typical robust UC models in existing literatures, they generally can be solved efficiently by the recent *column-and-constraint generation* computing method with minor customizations [14].

Under this direction, we present two robust unit commitment variants in this paper to demonstrate the advanced modeling capabilities of two-stage RO in solving practical problems. The first one is the *expanded robust unit commitment model* that considers the weighted summation of performances over multiple uncertainty sets. It actually generalizes the the scenario based stochastic unit commitment model and can yield solutions that are less conservative than those from a basic robust unit commitment model. The second one is the *risk constrained robust unit commitment model* that derives solutions subject to constraints on the worst case performances in those uncertainty sets. Therefore, any feasible solution, if it exists, will have a guaranteed performance under those uncertainty sets.

The paper is organized as follows. In Section II, we first

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provide the *expanded robust UC model* and discuss its properties and connection with the scenario based stochastic UC model. Then, we introduce *risk constrained robust UC model* as well as present a customized column-and-constraint solution method. In Section III, we perform a set of computational study of these two models and report numerical results. Section IV concludes the paper and discusses future research directions.

II. Two Robust Unit Commitment Variants

A. The Expanded Robust Unit Commitment Model

The typical robust unit commitment model is formulated as the following. We provide a matrix format for a compact exposition and the detailed formulations can be found in [8, 9, 10].

$$\min_{\mathbf{y},\mathbf{z}}(\mathbf{az}+\mathbf{ry})+\max_{\mathbf{v}\in\mathbb{V}}\min_{\mathbf{x},\mathbf{s}\in\Omega(\mathbf{y},\mathbf{z},\mathbf{v})}(\mathbf{gx}+\mathbf{qs}) \quad (1)$$

$$\text{s.t.} \quad \mathbf{Dy}+\mathbf{Fz}\geq\mathbf{f};\ \mathbf{y},\mathbf{z}\ \textit{binary}, \quad (2)$$

$$\Omega(\mathbf{y},\mathbf{z},\mathbf{v})=\{(\mathbf{x},\mathbf{s}):\mathbf{Ex}\leq\mathbf{e},$$
$$\mathbf{Ax}\leq\mathbf{L}-\mathbf{Gy}-\mathbf{Pz}-\mathbf{Rv},\mathbf{Ix}+\mathbf{Hs}=\mathbf{d}-\mathbf{Tv}\} \quad (3)$$

where $\mathbf{y},\mathbf{z}$ are the first stage commitment decisions that take binary values, $(\mathbf{x},\mathbf{s})$ are the second stage (aka. recourse) economic dispatch and market buy/sell decisions that are continuous, $\mathbf{v}$ represents some uncertain factor, e.g., the renewable energy generation, load or contingencies, whose randomness is captured by the uncertainty set $\mathbb{V}$. Note that, because of the two-stage decision making nature, the essential solution to the above robust unit commitment problem is the first stage start-up/shut-down $\mathbf{z}$ and on-off decisions $\mathbf{y}$ while the second stage decisions are made with perfect information on $\mathbf{v}$. Hence the set $\mathbb{V}$ plays a critical role in determining the quality of the first stage decisions.

Lemma 1. *Consider two uncertainty sets $\mathbb{V}_1$ and $\mathbb{V}_2$ such that $\mathbb{V}_1\subseteq\mathbb{V}_2$ and denote their corresponding optimal values of (1-3) by $\theta(\mathbb{V}_1)$ and $\theta(\mathbb{V}_2)$. We have $\theta(\mathbb{V}_1)\leq\theta(\mathbb{V}_2)$, i.e., θ is non-decreasing in $\mathbb{V}$ (in the terms of set inclusion).*

Note that this result can be easily proven by the fact that any first stage solution derived with respect to $\mathbb{V}_1$ will incur a higher or equal recourse cost in the worst case situations of $\mathbb{V}_2$. Hence, it would be ideal to adopt a tight $\mathbb{V}$ to reduce the objective function value. However, if the scope of $\mathbb{V}$ is small, it cannot sufficiently depicts the uncertain factor. For example, we consider random loads as $\mathbb{V}$ and use historical data for estimation. As demonstrated in Figure 1, which presents 7 consecutive days' loads of a city in Florida, the set defined by the average curve $\pm$ σ (standard deviation) clearly cannot ensure a coverage on all load possibilities, which may cause us to take a risky UC solution that is infeasible to meet load or with a high recourse cost. Also, if the scope of $\mathbb{V}$ is large, we may take a solution that is costly and over protective. As in Figure 2, the set defined by the average curve $\pm$ 3σ might overstate that randomness, which causes to run units more than necessary.

To balance the risk and cost, one way is to construct a sophisticated uncertainty set to capture the randomness, which could be technically challenging and make the robust UC computationally demanding [14]. We believe another improved strategy is to expand the uncertainty description by using multiple sets, along with their respective recourse problems. Then, we can integrate their impacts under the same umbrella by assigning different weights to the worst case performances of those sets. Specifically, let $\mathbb{V}_k$, $k=1,\ldots,K$ denote K uncertainty sets and ρ_k be their weight coefficients that are normalized for the totality being one. The *expanded robust unit commitment model* is formulated as

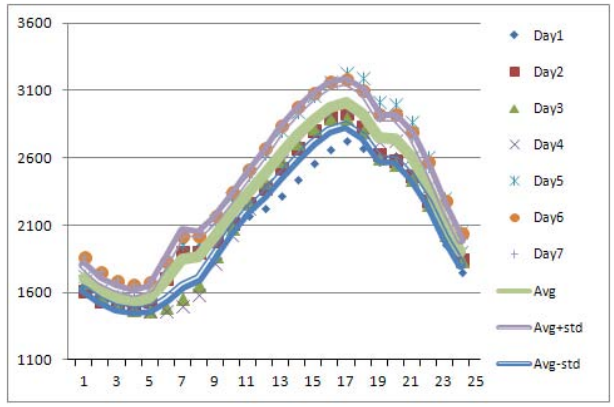

Fig. 1. 24-hour Load Distribution and Single STD Description

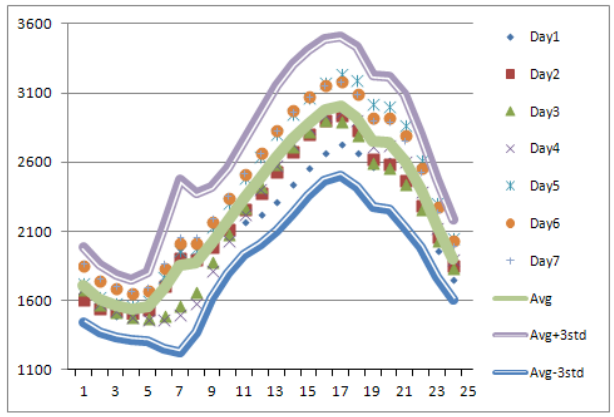

Fig. 2. 24-hour Load Distribution and Three STD Description

$$\min_{\mathbf{y},\mathbf{z}}(\mathbf{az}+\mathbf{ry})+\sum_k\rho_k\Big(\max_{\mathbf{v}\in\mathbb{V}_k}\min_{\mathbf{x},\mathbf{s}\in\Omega(\mathbf{y},\mathbf{z},\mathbf{v})}(\mathbf{gx}+\mathbf{qs})\Big) \quad (4)$$

$$\text{s.t.} \quad \mathbf{Dy}+\mathbf{Fz}\geq\mathbf{f};\ \mathbf{y},\mathbf{z}\ \textit{binary}, \quad (5)$$

$$\Omega(\mathbf{y},\mathbf{z},\mathbf{v})=\{(\mathbf{x},\mathbf{s}):\mathbf{Ex}\leq\mathbf{e},$$
$$\mathbf{Ax}\leq\mathbf{L}-\mathbf{Gy}-\mathbf{Pz}-\mathbf{Rv},\mathbf{Ix}+\mathbf{Hs}=\mathbf{d}-\mathbf{Tv}\} \quad (6)$$

The following result can be obtained easily based on Lemma 1.

Proposition 2. *Consider two uncertainty sets $\mathbb{V}_1$ and $\mathbb{V}_2$ such that $\mathbb{V}_1\subseteq\mathbb{V}_2$, and two coefficients $\rho_1\geq 0$ and $\rho_2\geq 0$ such that $\rho_1+\rho_2=1$. Denote the corresponding optimal value of (4-6) by $\Theta((\mathbb{V}_1,\mathbb{V}_2),(\rho_1,\rho_2))$. We have*

$$\theta(\mathbb{V}_1)\leq\Theta((\mathbb{V}_1,\mathbb{V}_2),(\rho_1,\rho_2))\leq\theta(\mathbb{V}_2).$$

The equality is achieved by setting $\rho_1=1$ or 0 respectively.

So, by taking non-trivial ρ_1 and ρ_2, the expanded robust UC model yields solutions that are less conservative than those derived under $\mathbb{V}_2$ exclusively while are more reliable than those derived under $\mathbb{V}_1$ exclusively.

We further point out that weight coefficients reflect decision maker's conservative/protective level and his understanding on the likelihoods of those sets. For example, $\mathbb{V}_1$ and $\mathbb{V}_2$ are defined by the average curve $\pm\ \sigma$ and 3σ respectively, to describe the overall load uncertainty. Based on data in Figures 1-2, although rigorous statistical analysis might not be obtainable, we are confident to conclude that the worst case situations of $\mathbb{V}_1$ are much more likely than those of $\mathbb{V}_2$. So, we can set ρ_1 to a value larger than ρ_2 to show our confidence. Clearly, assigning weights to multiple uncertainty sets provides us a new modeling function to take advantage of inexact but reliable information from practical data. Note that such a function is not available if any single uncertainty set is adopted.

The result in Proposition 2 can be generalized for more general expanded robust UC models.

Corollary 1. *Consider a collection of uncertainty sets* $\vec{\mathbb{V}} = \{\mathbb{V}_1, \ldots, \mathbb{V}_K\}$ *and a set of coefficients* $\vec{\rho} = \{\rho_1, \ldots, \rho_K\}$ *such that* $\sum_k \rho_k = 1$. *Denote the corresponding optimal value of (4-6) by* $\Theta(\vec{\mathbb{V}}, \vec{\rho})$. *We have*

$$\min_k \theta(\mathbb{V}_k) \leq \Theta(\vec{\mathbb{V}}, \vec{\rho}) \leq \max_k \theta(\mathbb{V}_k).$$

In fact, for a special case, this expanded robust UC reduces to the classical scenario based stochastic programming UC model.

Proposition 3. *Consider a case where the randomness of the uncertainty factor is completely captured by scenarios* $\hat{\mathbf{v}}_k, k = 1, \ldots, K$. *If we let* $\mathbb{V}_k$ *be a singleton* $\{\hat{\mathbf{v}}_k\}$ *and set* ρ_k *equal to the corresponding realization probability, then the expanded robust UC is equivalent to the stochastic programming UC model.*

Proof: Note that when $\mathbb{V}_k = \{\hat{\mathbf{v}}_k\}$, the $\max$ operator can be eliminated from the formulation. Hence, we have

$$\min_{\mathbf{y},\mathbf{z}} (\mathbf{az} + \mathbf{ry}) + \sum_k \rho_k \Big(\min_{\mathbf{x},\mathbf{s} \in \Omega(\mathbf{y},\mathbf{z},\hat{\mathbf{v}}_\mathbf{k})} (\mathbf{gx} + \mathbf{qs}) \Big) \quad (7)$$

$$\text{s.t.} \quad \mathbf{Dy} + \mathbf{Fz} \geq \mathbf{f};\ \mathbf{y}, \mathbf{z}\ binary, \quad (8)$$

$$\Omega(\mathbf{y},\mathbf{z},\hat{\mathbf{v}}_k) = \{(\mathbf{x},\mathbf{s}) : \mathbf{Ex} \leq \mathbf{e},$$
$$\mathbf{Ax} \leq \mathbf{L} - \mathbf{Gy} - \mathbf{Pz} - \mathbf{R}\hat{\mathbf{v}}_k, \mathbf{Ix} + \mathbf{Hs} = \mathbf{d} - \mathbf{T}\hat{\mathbf{v}}_k\} \quad (9)$$

Because economic dispatch and buy/sell decisions are made specific to individual scenarios, we can replace $\mathbf{x}, \mathbf{s}$ by introducing recourse variables $\mathbf{x}_k, \mathbf{s}_k$ for scenario $\hat{\mathbf{v}}_k$. Also, the second $\min$ operator can be removed given that it aligns with the first $\min$ operator. Hence, the overall $\min - \max - \min$ formulation can be simplified as

$$\min_{\mathbf{y},\mathbf{z}} (\mathbf{az} + \mathbf{ry}) + \sum_k \rho_k (\mathbf{gx_k} + \mathbf{qs_k})$$

$$\text{s.t.} \quad \mathbf{Dy} + \mathbf{Fz} \geq \mathbf{f}, \quad (10)$$

$$\mathbf{Ex}_k \leq \mathbf{e},\ \forall k \quad (11)$$

$$\mathbf{Ax}_k \leq \mathbf{L} - \mathbf{Gy} - \mathbf{Pz} - \mathbf{R}\hat{\mathbf{v}}_k,\ \forall k \quad (12)$$

$$\mathbf{Ix}_k + \mathbf{Hs}_k = \mathbf{d} - \mathbf{T}\hat{\mathbf{v}}_k,\ \forall k$$

$$\mathbf{y}, \mathbf{z}\ binary, \quad (13)$$

which is exactly the scenario based stochastic programming UC model. ■

Remark 1. *According to Proposition 3, we can conclude that the expanded robust UC model is a complete and flexible modeling framework to handle various randomness. Decision makers can customize their uncertainty sets and adjust weight coefficients for their conveniences, according to data availability, data quality, system requirements and their conservative/protective level. In fact, even for simple interval uncertainty sets that are easily defined by lower and upper bounds, they can be used, as building blocks, to capture or approximate any sophisticated randomness, by appropriately adjusting their weights.*

In Appendix B, we present a concrete expanded robust UC model by considering multiple load uncertainty sets. Numerical results of this model are provided in Section III.

B. The Risk Constrained Robust Unit Commitment Model

In this section, we show how to further extend our modeling capacity by introducing constraints on uncertainty sets and their worst case performances. Specifically, based on the nature of random factors and system requirements, we explicitly impose hard constraints to restrict some performance measurements in the worst case situations in the uncertainty sets. As a result, any derived solution, if exists, will guarantee its performance with respect to those uncertainty sets. Let γ_k denote our performance restriction in uncertainty sets $\mathbb{V}_k$, $k = 1, \ldots, K$. Also, let $\mathfrak{v}_0$ be the nominal situation that the decision maker would like to consider.

The *risk constrained robust unit commitment model* is formulated as

$$\min_{\mathbf{y},\mathbf{z},\mathbf{x_0},\mathbf{s_0}} \mathbf{az} + \mathbf{ry} + \mathbf{g}_0\mathbf{x}_0 + \mathbf{q}_0\mathbf{s}_0 \quad (14)$$

$$\text{s.t.} \quad \mathbf{Dy} + \mathbf{Fz} \geq \mathbf{f};\ \mathbf{y}, \mathbf{z}\ binary, \quad (15)$$

$$\mathbf{Ex}_0 \leq \mathbf{e}, \quad (16)$$

$$\mathbf{Ax}_0 \leq \mathbf{L} - \mathbf{Gy} - \mathbf{Pz} - \mathbf{Rv}, \quad (17)$$

$$\mathbf{Ix}_0 + \mathbf{Hs}_0 = \mathbf{d} - \mathbf{T}\mathfrak{v}_0, \quad (18)$$

$$\max_{\mathbf{v} \in \mathbb{V}_k} \min_{\mathbf{x},\mathbf{s} \in \Omega(\mathbf{y},\mathbf{z},\mathbf{v})} (\mathbf{g}_k\mathbf{x} + \mathbf{q}_k\mathbf{s}) \leq \gamma_k,\ \forall k \quad (19)$$

$$\Omega(\mathbf{y},\mathbf{z},\mathbf{v}) = \{(\mathbf{x},\mathbf{s}) : \mathbf{Ex} \leq \mathbf{e},$$
$$\mathbf{Ax} \leq \mathbf{L} - \mathbf{Gy} - \mathbf{Pz} - \mathbf{Rv}, \mathbf{Ix} + \mathbf{Hs} = \mathbf{d} - \mathbf{Tv}\} \quad (20)$$

Note that it is not necessary to keep g_k and q_k identical among $k = 0, \ldots, K$. So, we can consider different performance measures with respect to different $\mathbb{V}_k$. Again, we mention that the essential solution to the above robust UC model is the first stage $\mathbf{y}$ and $\mathbf{z}$ decisions. It may not be optimal to implement the solution $(\mathbf{x}_0, \mathbf{s}_0)$ because the virtual optimal one can be derived after $\mathbf{v}$ is disclosed.

Let $\vec{\gamma}$ denote $\{\gamma_1, \ldots, \gamma_K\}$ and $\Phi(\vec{\mathbb{V}}, \vec{\gamma})$ be the optimal value of (14-20) with respect to $\mathbb{V}$ and $\vec{\gamma}$. The following results can be easily proven.

Proposition 4. (i) *For a given* $\vec{\mathbb{V}}$*, the function* $\Phi(\vec{\mathbb{V}}, \vec{\gamma})$ *is non-increasing in* $\vec{\gamma}$*;* (ii) *For a given* $\vec{\gamma}$*, the function* $\Phi(\vec{\mathbb{V}}, \vec{\gamma})$ *is non-decreasing in* $\vec{\mathbb{V}}$ *(in the terms of set inclusion);* (iii) *When* $\vec{\gamma} = \vec{\infty}$*, the risk constrained UC model reduces to the standard UC model built on* $\mathfrak{v}_0$.

In Appendix C, we present a concrete risk constrained UC model by considering G-1 (i.e., one generator in forced outage) and G-2 (i.e., two generators in forced outages) contingencies as our $\mathbb{V}_1$ and $\mathbb{V}_2$. We impose upper bounds on load shedding in the worst case situations in those two uncertainty sets to control our risks. Numerical results of this model are presented in Section III.

C. A Solution Procedure: Customization of the Column-and-Constraint Generation Method

The aforementioned robust UC models can be solved by classical Benders dual methods, which have been applied to solve typical robust UC models in existing literatures [8, 9, 10, 11]. A recent *column-and-constraint generation* method has also been developed in [9, 14] that solves the typical robust UC models an order of magnitude faster than Benders dual methods. This algorithm can simply be customized to solve our new robust UC variants. Because the expanded robust UC is similar to the typical robust UC and just needs a few modifications, our illustration is within the context of the risk constrained robust UC model.

The column-and-constraint generation method is implemented in a master-subproblem framework. For a given $(\mathbf{y}^*, \mathbf{z}^*)$, we define the following subproblem

$$
\begin{aligned}
\mathbf{SP}_k: \ \mathcal{Q}_k(\mathbf{y}^*, \mathbf{z}^*) = \quad & \max_{\mathbf{v}\in\mathbb{V}_k} \min_{\mathbf{x},\mathbf{s}\in\Omega(\mathbf{y}^*,\mathbf{z}^*,\mathbf{v})} \mathbf{g}_k\mathbf{x} + \mathbf{q}_k\mathbf{s} \\
\text{s.t.} \quad & \mathbf{E}\mathbf{x} \le \mathbf{e}, \\
& \mathbf{A}\mathbf{x} \le \mathbf{L} - \mathbf{G}\mathbf{y}^* - \mathbf{P}\mathbf{z}^* - \mathbf{R}\mathbf{v}, \\
& \mathbf{I}\mathbf{x} + \mathbf{H}\mathbf{s} = \mathbf{d} - \mathbf{T}\mathbf{v}.
\end{aligned}
$$

Although $\mathbf{SP}_k$ is a bi-level program, because the inner problem is a linear program, it can be converted into a mixed integer program by using classical Karush-Kuhn-Tucker (KKT) conditions and big-M linearization method [14]. To avoid distraction, we simply assume an oracle can solve it or identify some $\mathbf{v}^* \in \mathbb{V}_k$ for which the inner problem is infeasible (and $\mathcal{Q}_k(\mathbf{y}^*, \mathbf{z}^*)$ is conventionally set to $+\infty$). Next, we give the algorithm details to solve the risk constrained UC model.

1) Set $i = 0$, VIOLATION = VIOL$_k$= FALSE, and $L_k = 0$ for $k = 1, \ldots, K$.
2) Solve the following mixed integer master problem

$$
\begin{aligned}
\mathbf{MP}: \quad & \min_{\mathbf{y},\mathbf{z},\mathbf{x}_0,\mathbf{s}_0} \mathbf{a}\mathbf{z} + \mathbf{r}\mathbf{y} + \mathbf{g}_0\mathbf{x}_0 + \mathbf{q}_0\mathbf{s}_0 \\
\text{s.t.} \quad & \mathbf{D}\mathbf{y} + \mathbf{F}\mathbf{z} \ge \mathbf{f}; \ \mathbf{y}, \mathbf{z} \ binary, \\
& \mathbf{E}\mathbf{x}_0 \le \mathbf{e}, \\
& \mathbf{A}\mathbf{x}_0 \le \mathbf{L} - \mathbf{G}\mathbf{y} - \mathbf{P}\mathbf{z} - \mathbf{R}\mathfrak{v}_0, \\
& \mathbf{I}\mathbf{x}_0 + \mathbf{H}\mathbf{s}_0 = \mathbf{d} - \mathbf{T}\mathfrak{v}_0, \\
& \mathbf{g}_k\mathbf{x}_{k,l} + \mathbf{q}_k\mathbf{s}_{k,l} \le \gamma_k, \ \forall k, l \le L_k \\
& \mathbf{E}\mathbf{x}_{k,l} \le \mathbf{e}, \ \forall k, l \le L_k \\
& \mathbf{A}\mathbf{x}_{k,l} \le \mathbf{L} - \mathbf{G}\mathbf{y} - \mathbf{P}\mathbf{z} - \mathbf{R}\mathbf{v}_{k,l}, \ \forall k, l \le L_k \\
& \mathbf{I}\mathbf{x}_{k,l} + \mathbf{H}\mathbf{s}_{k,l} = \mathbf{d} - \mathbf{T}\mathbf{v}_{k,l}, \ \forall k, l \le L_k
\end{aligned}
$$

   Derive an optimal solution $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{x}_0^*, \mathbf{s}_0^*, \mathbf{x}_{1,1}^* \ldots, \mathbf{x}_{K,L_K}^*, \mathbf{s}_{1,1}^*, \ldots, \mathbf{s}_{K,L_K}^*)$.
3) With given $(\mathbf{y}^*, \mathbf{z}^*)$, for $k = 1, \ldots, K$, do
   a) call the oracle to solve $\mathbf{SP}_k$;
   b) if $\mathcal{Q}_k(\mathbf{y}^*, \mathbf{z}^*) > \gamma_k$, set VIOLATION = VIOL$_k$ = TRUE, use $\mathbf{v}_{k,L_k+1}$ to record the optimal $\mathbf{v}^*$ and update $L_k = L_k + 1$.
4) If VIOLATION = FALSE, return $(\mathbf{y}^*, \mathbf{z}^*)$ and terminate. Otherwise, set VIOLATION = FALSE and for $k = 1, \ldots, K$, do
   a) If VIOL$_k$ = TRUE, create variables $(\mathbf{x}_{k,L_k}, \mathbf{s}_{k,L_k})$ and add the following constraints

$$
\begin{aligned}
& \mathbf{g}_k\mathbf{x}_{k,L_k} + \mathbf{q}_k\mathbf{s}_{k,L_k} \le \gamma_k, \\
& \mathbf{E}\mathbf{x}_{k,L_k} \le \mathbf{e}, \\
& \mathbf{A}\mathbf{x}_{k,L_k} \le \mathbf{L} - \mathbf{G}\mathbf{y} - \mathbf{P}\mathbf{z} - \mathbf{R}\mathbf{v}_{k,l}, \\
& \mathbf{I}\mathbf{x}_{k,L_k} + \mathbf{H}\mathbf{s}_{k,L_k} = \mathbf{d} - \mathbf{T}\mathbf{v}_{k,L_k},
\end{aligned}
$$

   to $\mathbf{MP}$.
   b) Set VIOL$_k$= FALSE,
5) Update $i = i + 1$ and go to Step 2). ∎

Given that the second stage recourse problems are linear programs, the convergence and the complexity results follow from the complexity analysis of the column-and-constraint generation method presented in [14]. Let p_k be the number of extreme points of $\mathbb{V}_k$ if it is a polytope (e.g., the uncertainty sets for random loads) or the set cardinality if it is a finite discrete set (e.g., the G-k contingency set). We have

Proposition 5. *The column-and-constraint generation method either terminates with an optimal solution or reports infeasibility of the risk constrained robust UC model (the expanded robust UC model, respectively) in* $O(\prod_{k=1}^{K} p_k)$ *iterations.*

Actually, the computational performance of this method on solving practical problems is drastically better than the theoretical result, which can be seen in the next section.

III. Numerical Examples

In this section, we numerically investigate our proposed robust UC models for illustration, based on a dataset with loads of 7 consecutive days and 11 generators from a utility company in Florida (see specifications provided in [15]). The

TABLE I
RESULTS FOR TWO UNCERTAINTY SETS WITH $\rho_1 = 0.86$ AND $\rho_2 = 0.14$

Case	Γ_1, Γ_2	iter.	time(s)	Obj.	Cost in $\mathbb{V}_1$	Cost in $\mathbb{V}_2$
1	12,12	2	0.869	973087.4	951082.07	1035048.71
2	12,10	2	0.853	969668.16	951082.07	1010625.6
3	12,8	2	0.837	966147.13	951082.07	985475.38
4	12,6	2	0.869	962286.85	951082.07	957901.93
5	12,4	2	0.837	958204.08	951082.07	928739.26
6	12,2	2	0.869	954041.31	951082.07	899005.23
7	6,6	2	0.79	929118.42	913618.57	958974.63
8	6,4	2	0.806	925028.02	913618.57	929757.5
9	6,2	2	0.885	920861.62	913618.57	899997.51

TABLE II
RESULTS FOR TWO UNCERTAINTY SETS WITH $\rho_1 = 0.6$ AND $\rho_2 = 0.4$

Case	Γ_1, Γ_2	iter.	time(s)	Obj.	Cost in $\mathbb{V}_1$	Cost in $\mathbb{V}_2$
1	12,12	2	0.79	994916.84	951156.8	1035056.91
2	12,10	2	0.917	985147.59	951156.8	1010633.8
3	12,8	2	0.932	975089.39	951082.07	985475.38
4	12,6	2	0.917	964060.01	951082.07	957901.93
5	12,4	2	0.933	952394.95	951082.07	928739.26
6	12,2	2	0.901	940490.81	952236.35	899997.51
7	6,6	2	0.837	940911	913618.57	958974.63
8	6,4	2	0.885	929224.14	913618.57	929757.5
9	6,2	2	0.821	917320.15	913618.57	899997.51

column-and-constraint generation method is implemented in C++ and tested on a Dell Optiplex 760 desktop computer (Intel Core 2 Duo CPU, 3.0GHz, 3.25GB of RAM) in Windows XP environment. CPLEX 12.5 is adopted as the mixed integer programming solver and its optimality tolerance is set to 10^{-4}.

In the computation of the expanded robust UC model, we consider the formulation in Appendix B with two uncertainty sets. They are in the form of (26) constructed by the approach presented in [8]. In (26), $\bar{v}_t$ is set to the average load in time t, and $\hat{v}_t$ is set to 1.5σ and 3σ in $\mathbb{V}_1$ and $\mathbb{V}_2$ respectively, noting that taking values beyond $\pm 3\sigma$ is very unlikely ($< .27\%$) if loads follow normal distributions.

Parameters Γ_1 and Γ_2 are chosen such that $\Gamma_2 \leq \Gamma_1$, knowing that random loads are less likely to reach lower/upper bounds if the uncertain interval is larger. We consider two sets of weight coefficients, i.e., $\rho_1 = 0.86$ and $\rho_2 = 0.14$, and $\rho_1 = 0.6$ and $\rho_2 = 0.4$, in our computation. The first set is selected according to the fact that loads will fall within $\pm 1.5\sigma$ range with probability 0.86 under the normal distribution. The second set is simply selected to emphasize the importance of $\mathbb{V}_2$, which might lead to a more conservative solution. All numerical results are presented in Tables I and II.

We note in Table I that the (worst case) cost in $\mathbb{V}_1$ does not change, in spite of the change of Γ_2 in $\mathbb{V}_2$, which indicates that the first stage commitment solution remains the same. Such an observation can be explained by the fact that ρ_2 is small and the uncertainty set $\mathbb{V}_1$ dominates the final solution. Nevertheless, more interesting interactions between $\mathbb{V}_1$ and $\mathbb{V}_2$ can be seen in Table II where weight coefficient ρ_2 is larger. Specifically, when $\Gamma_1 = 12$ and Γ_2 changes from 12 to 2, we note that the (worst case) cost in $\mathbb{V}_1$ starts with 951156.8, then deceases to 951082.07 and finally increases to 952236.35. Such a behavior is basically due to the large value of ρ_2 that enables $\mathbb{V}_2$ to heavily affect the commitment solution. When Γ_2 is large, more generators are committed that may be unnecessary for any realization in $\mathbb{V}_1$. So, the (worst case) cost in $\mathbb{V}_1$ has to include cost associated with their minimum generation levels. When Γ_2 is moderate, less generators are committed but they are sufficient to handle any possibility inside $\mathbb{V}_1$. Then, the (worst case) cost in $\mathbb{V}_1$ decreases. Finally, when Γ_2 is small, much less generators are committed, which in turn causes expensive market purchases to deal with the worst cases of $\mathbb{V}_1$. As a result, the (worst case) cost in $\mathbb{V}_1$ increases.

In the computation of the risk constrained robust UC model, we consider the formulation in Appendix C with load shedding restrictions on G-1/G-2 contingencies. Numerical results, along with the algorithm performance on load shedding (LS) over iterations, for different γ_1 and γ_2 are presented in Table III.

Note that with γ_1 and γ_2 become more restrictive in G-1 and G-2 contingencies, solutions with higher costs and more units turned on are derived. When constraints with $\gamma_1 = 200$ and $\gamma_2 = 2000$ are imposed, the model actually becomes infeasible. Hence, we can conclude that if the reliability standard with $\gamma_1 = 200$ and $\gamma_2 = 2000$ is required, the system needs to obtain and operate extra generators. Therefore, this model can also be treated as a decision support tool for system expansion under reliability consideration.

TABLE III
RESULTS OF G-1/G-2 RISK CONSTRAINED MODEL

Case	Case 1		Case 2		Case 3	
Obj.	885890.90		886219.02		NA	
LS$\backslash\gamma_k$	300	3000	250	2500	200	2000
iter.1	4793.44	10812.8	4793.44	10812.8	4793.44	10812.8
iter.2	4793.44	10812.8	4793.44	10812.8	4793.44	10812.8
iter.3	821.99	4793.44	1729.08	4793.44	650.99	4793.44
iter.4	325.8	3197.15	420.46	2720.74	INF	INF
iter.5	268.8	2912.15	225.26	2499.03		
time(s)	70.203		44.312		10.35	

IV. CONCLUSION

In this paper, we explore and extend the modeling capacity of two-stage robust optimization method. We demonstrate the improved capability by presenting two new robust unit commitment models, i.e., the expanded robust unit commitment and the risk constrained robust unit commitment models. We derive some structural properties, show that the first model generalizes the scenario based stochastic unit commitment model, and present a customized column-and-constraint generation method. We then perform a set of numerical experiments on those models to illustrate their modeling strength, economic outcomes with respect to different uncertainty sets and the algorithm performance in solving those models.

Although those unit commitment models improve our ability to capture uncertainties and handle risks in power systems, we mention that the presented research is a basic work in exploring robust optimization, a new optimization paradigm that may have many powerful modeling and solution features. For example, a natural extension is to adopt mixed integer recourse programs [16] that can model quick-start generators and transmission line switching in the second stage [17]. Also, the concepts of modeling presented in this paper can be applied into other robust optimization applications, e.g., [18, 19], to address practical needs.

APPENDIX A
NOMENCLATURE

Indices and Sets

i	Generator, $i = 0, 1, ..., I-1$
t	Planning period, $t = 0, 1, ..., T-1$
$\mathbb{V}_k$	The k-th uncertainty set

Parameters

a_i	Start up cost of unit i
r_i	Running cost of unit i
c_i	Fuel cost of unit i
q_t^+	Purchase price at time t in power market
q_t^-	Sale price at time t in power market
v_t	Load (or generator forced outage) at time t
$\bar{v}_t$	Nominal value of load at time t
$\hat{v}_t$	Bound on load deviation from the nominal value at time t
Γ_k	the right-hand-side for the budget constraint of the k-th uncertainty set
l_i	Lower bound output of unit i
u_i	Upper bound output of unit i
Δ_+^i	Ramping up limit of unit i
Δ_-^i	Ramping down limit of unit i
m_+^i	Minimum up time limit of unit i
m_-^i	Minimum down time limit of unit i

Decision variables

y_{it}	Binary on/off status of unit i at time t
z_{it}	Binary start up of unit i at time t
x_{it}	Continuous generation of unit i at time t
s_t^+	Purchased power or load shedding at time t (continuous)
s_t^-	Sold power at time t (continuous)

APPENDIX B
THE EXPANDED ROBUST UC MODEL

In this model, we consider the unit commitment problem with random loads, which is captured by multiple uncertainty sets. For simplicity, we do not include spinning reserve constraints and assume a linear fuel cost function, which can be included or replaced by a piecewise linear function. The expanded robust UC is formulated as following.

$$\min_{\mathbf{y},\mathbf{z}} \sum_{i=0}^{I-1}\sum_{t=0}^{T-1}(r_i y_{it} + a_i z_{it}) + \sum_{k=1}^{K}\rho_k \max_{\mathbf{v}\in\mathbb{V}_k} \min_{(\mathbf{x},\mathbf{s}^+,\mathbf{s}^-)\in\Omega(\mathbf{y},\mathbf{z},\mathbf{v})} \Big(\sum_{i=0}^{I-1}\sum_{t=0}^{T-1} c_i x_{it} + \sum_{t=0}^{T-1}(q_t^+ s_t^+ - q_t^- s_t^-)\Big) \quad (21)$$

s.t.

$$-y_{i(t-1)} + y_{it} - y_{ih} \leq 0$$
$$\forall i, t \geq 1, t \leq h \leq min\{m_+^i + t - 1, T-1\}; \quad (22)$$
$$y_{i(t-1)} - y_{it} + y_{ih} \leq 1$$
$$\forall i, t \geq 1, t \leq h \leq min\{m_-^i + t - 1, T-1\}; \quad (23)$$
$$-y_{i(t-1)} + y_{it} - z_{it} \leq 0 \qquad \forall i, t \geq 1; \quad (24)$$
$$y_{it}, z_{it} \in \{0, 1\} \qquad \forall i, t; \quad (25)$$

where

$$\mathbb{V}_k = \{\bar{v}_t - \hat{v}_t \leq v_t \leq \bar{v}_t + \hat{v}_t,\ \forall t; \sum_{t=0}^{T-1}\frac{|v_t - \bar{v}_t|}{\hat{v}_t} \leq \Gamma_k\} \qquad k = 1, \dots, K; \quad (26)$$

and

$$\Omega(\mathbf{y},\mathbf{z},\mathbf{v}) = \{l_i y_{it} \leq x_{it} \leq u_i y_{it}, \qquad \forall i, t; \quad (27)$$
$$\sum_{i=0}^{I-1} x_{it} + s_t^+ - s_t^- = v_t, \qquad \forall t; \quad (28)$$
$$x_{i(t+1)} \leq x_{it} + y_{it}\Delta_+^i + (1 - y_{it})u_i$$
$$\forall i, t = 0, 1, ..., T-2; \quad (29)$$
$$x_{it} \leq x_{i(t+1)} + y_{i(t+1)}\Delta_-^i + (1 - y_{i(t+1)})u_i$$
$$\forall i, t = 0, 1, ..., T-2; \quad (30)$$
$$x_{it} \geq 0\ \forall i, t; s_t^+, s_t^- \geq 0\ \forall t; \} \quad (31)$$

The objective function in (21) is to minimize the total cost, including the first stage commitment cost and the second stage economic dispatch cost estimated by the weighted worst case costs in different uncertainty sets. Constraints in (22-25) are typical commitment constraints that define the minimum up and down times and define the start-up decision and generator status, as well as define variable type restrictions. $\mathbb{V}_k$ in (26) is defined in the same fashion as those in [8, 9] that uses *a budget constraint* to refine our uncertainty set description. The set $\Omega(\mathbf{y},\mathbf{z},\mathbf{v})$, a polyhedral set, is the feasible set of the economic dispatch problem, for the fixed $\mathbf{y},\mathbf{z},\mathbf{v}$. Constraints in (27) define the lower and upper bounds on generation level. Constraints in (28) ensure loads can be satisfied all the time. Constraints (29) and (30) are ramping up/down limits. Constraints in (31) provide variable type restrictions.

APPENDIX C
THE RISK CONSTRAINED ROBUST UC MODEL

In this model, we consider the unit commitment problem with bound constraints on load shedding under G-k contingencies (i.e., up to k generators in forced outages). Most notations and variables are identical to those used in the expanded robust UC model. Important differences are: spinning reserve constraints are included in the nominal situation with w_{it} representing spinning reserve of generator i and SR_t representing system reserve requirement at time t, $\mathbb{V}_k$ is a finite discrete set to describe the generator outage status, and d_t is used to represent load at time t that is certain.

$$\min_{\mathbf{y},\mathbf{z},\mathbf{x}} \sum_{i=0}^{I-1}\sum_{t=0}^{T-1} c_i x_{0,it} + r_i y_{it} + a_i z_{it} \quad (32)$$

s.t. (33)

$$- y_{i(t-1)} + y_{it} - y_{ih} \leq 0$$
$$\forall i, t \geq 1, t \leq h \leq min\{m_+^i + t - 1, T-1\}; \quad (34)$$
$$y_{i(t-1)} - y_{it} + y_{ih} \leq 1$$
$$\forall i, t \geq 1, t \leq h \leq min\{m_-^i + t - 1, T-1\}; \quad (35)$$
$$- y_{i(t-1)} + y_{it} - z_{it} \leq 0, \qquad \forall i, t \geq 1; \quad (36)$$
$$x_{0,it} \geq l_i y_{it}, \qquad \forall i, t; \quad (37)$$
$$x_{0,it} + w_{it} \leq u_i y_{it}, \qquad \forall i, t; \quad (38)$$
$$\sum_i w_{it} \geq SR_t, \qquad \forall t; \quad (39)$$
$$x_{0,i(t+1)} \leq x_{0,it} + y_{it}\Delta_+^i + (1-y_{it})u_i \quad (40)$$
$$\forall i, t = 0, 1, ..., T-2;$$
$$x_{0,it} \leq x_{0,i(t+1)} + y_{i(t+1)}\Delta_-^i + (1 - y_{i(t+1)})u_i \quad (41)$$
$$\forall i, t = 0, 1, ..., T-2;$$
$$\sum_{i=0}^{I-1} x_{0,it} \geq d_t \qquad \forall t; \quad (42)$$
$$\max_{\mathbf{v}\in\mathbb{V}_k} \min_{(\mathbf{x},\mathbf{s}^+)\in\Omega(\mathbf{y},\mathbf{z},\mathbf{v})} \sum_t s_t^+ \leq \gamma_k, \ k = 1, \ldots, K \quad (43)$$
$$y_{it}, z_{it} \in \{0, 1\}, \ x_{0,it} \geq 0, \ w_{it} \geq 0, \ \forall i, t; \quad (44)$$

where

$$\mathbb{V}_k = \{v_{it} \in \{0, 1\}, \forall i, t; \quad (45)$$
$$v_{i(t+1)} \geq v_{it}, \ \forall i, 0 \leq t \leq T-2; \quad (46)$$
$$\sum_i v_{it} \leq k, \forall t\} \qquad k = 1, \ldots, K; \quad (47)$$

and

$$\Omega(\mathbf{y},\mathbf{z},\mathbf{v}) = \{s_t^+ \geq d_t - \sum_i x_{it}, \qquad \forall t; \quad (48)$$
$$l_i y_{it}(1 - v_{it}) \leq x_{it} \leq u_i y_{it}(1 - v_{it}) \qquad \forall i, t; \quad (49)$$
$$x_{i(t+1)} \leq x_{it} + y_{it}\Delta_+^i + (1 - y_{it})u_i \quad (50)$$
$$\forall i, t = 0, 1, ..., T-2;$$
$$x_{it} \leq x_{i(t+1)} + y_{i(t+1)}\Delta_-^i + (1 - y_{i(t+1)})u_i + u_i v_{i(t+1)} \quad (51)$$
$$\forall i, t = 0, 1, ..., T-2;$$
$$s_t^+ \geq 0 \ \forall t; \ x_{it} \geq 0 \ \forall i, t; \}; \quad (52)$$

The objective function in (32) is to minimize the overall cost in the nominal situation. Constraints in (34-42) are the regular unit commitment constrains, along with variable type restrictions in (44). Constraints in (43) define the different restrictions on the overall load shedding in K contingency sets. The contingency set $\mathbb{V}_k$ in (45-47) includes all the contingencies with up to k generator outages. Specifically, constraints in (46) indicate that once generator i is in outage at time t, i.e., $v_{i,t} = 1$, it remains in outage status. Constraints in (47) ensure that at any time, no more than k generators are in outage. Finally, the set $\Omega(\mathbf{y},\mathbf{z},\mathbf{v})$ in (48-52) defines the feasible set of the economic dispatch subject to fixed $(\mathbf{y},\mathbf{z},\mathbf{v})$. Note from (49) that once generator i is in outage at time t, its generation will be zero. Also, (51) ensures that the ramping down constraint is not needed if generator i is in outage at time $t+1$.

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