Family Constraining of Iterative Algorithms

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Abstract In constraining iterative processes, the algorithmic operator of the iterative process is pre-multiplied by a constraining operator at each iterative step. This enables the constrained algorithm, besides solving the original problem, also to find a solution that incorporates some prior knowledge about the solution. This approach has been useful in image restoration and other image processing situations when a single constraining operator was used. In the field of image reconstruction from projections a priori information about the original image, such as smoothness or that it belongs to a certain closed convex set, may be used to improve the reconstruction quality. We study here constraining of iterative processes by a family of operators rather than by a single operator.

Keywords constraining strategy · strictly nonexpansive operators · fixed points set · least squares problems · image reconstruction from projections

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1 Introduction

This paper is about *constraining of iterative processes* which has the following meaning. When dealing with a real-world problem it is sometimes the case that we have some prior knowledge about features of the solution that is being sought after. If possible, such prior knowledge may be formulated as an additional constraint and added to the original problem formulation. But sometimes, when we have already at our disposal a "good" algorithm for solving the original problem without such an additional constraint, it is beneficial to modify the algorithm, rather than the problem, so that it will, in some way, "take care" of the additional constraint (or constraints) without loosing its ability to generate (finitely or asymptotically) a solution to the original problem. This is called constraining of the original iterative algorithm. Given an (algorithmic) operator $Q : R^n \rightarrow R^n$ between Euclidean spaces, the original iterative process may have the form

$$x^{k+1} = Q(x^k), \text{ for all } k \geq 0, \tag{1}$$

under various assumptions on Q. Constraining such an algorithm with a family of operators means that we desire to use instead of (1) the iterative process

$$x^{k+1} = S_k Q(x^k), \text{ for all } k \geq 0, \tag{2}$$

where $\{S_k\}_{k=0}^{\infty}$ is a family of operators $S_k : R^n \rightarrow R^n$, henceforth called the *constraining operators*.

Our purpose is to study the possibility to constrain an algorithm with a family of operators and to analyze the asymptotic behavior of such family-constrained algorithms. We extend earlier results on this topic that were limited to a single constraining operator, i.e., $S_k = S$ for all $k \geq 0$, see, e.g., [7, 9, 12, 18–20, 22, 30, 33], introducing a family of strictly nonexpansive operators $\{S_k\}_{k=0}^{\infty}$ and proving the convergence of the family-constrained algorithms in a more general setting.

The paper is organized as follows. In Section 2, for a family of strictly nonexpansive operators with nonempty common fixed points set and a supplementary image reconstruction condition we adapt some results from [11] for our purpose and prove our main convergence result. We present the *family-constrained algorithm* (FCA) in Section 3 and we prove that the series expansion methods and the smoothing matrices used in [11] obey all our hypotheses. In Section 4 we show that the general iterative method introduced in [24], which includes the algorithms of Kaczmarz, Cimmino and Diagonal Weighting (see, e.g., [32], [24] and [29], respectively) as special cases, is itself an algorithmic operator of the form required here and we give an example of a family of nonlinear constraining operators which satisfy our assumptions.

1.1 Relation with previous work

Some earlier works on this topic were limited to a single constraining operator, i.e., $S_k = S$ for all $k \geq 0$, see, e.g., [7, 9, 12, 18–20, 22, 30, 33]. As seen in these

works, the algorithm constraining approach is successfully applied to problems of image restoration, to smoothing in image reconstruction from projections (see also [14, Subsection 12.3]), and to constraining of linear iterative processes in general. In [14, Section 11.4] there is a discussion of all kinds of, so called, "tricks" that give rise to what we call here "constraining operators." This includes the trick of selective smoothing, that is illustrated in detail in [14, Section 5.3]. Historical references can be found in [14, p. 216]. The paper [15] is the original source of tricks in the field of image reconstruction from projections. A constraining strategy which applies a single strictly nonexpansive idempotent operator at every iteration of the classical Kaczmarz algorithm has been presented in [22]. Recently [24], the third author proposed a generalization of this result by replacing Kaczmarz's algorithm by a more general iterative process. Under the assumption that a family of strictly nonexpansive operators $\{T_k\}_{k=0}^{\infty}$ has a nonempty common fixed points set and an additional condition, reasonable in image reconstruction problems, we proved that the sequence generated by the iterative scheme

$$x^0 \in R^n \text{ and } x^{k+1} = T_{k+1}(x^k), \text{ for all } k \geq 0, \tag{3}$$

converges to a common fixed point of the operators $\{T_k\}_{k=0}^{\infty}$.

The particular problem of finding a common fixed point of nonlinear mappings is an important topic in fixed point theory, see, e.g., the excellent recent monograph [8]. We will denote by F the set of common fixed points. For finitely many paracontracting operators $T_1, T_2, \ldots, T_p$ the following algorithm, proposed in [13],

$$x^0 \in R^n \text{ and } x^{k+1} = T_{j_{k+1}}(x^k), \text{ for all } k \geq 0, \text{ with } \{j_k\}_{k=0}^{\infty} \text{ admissible, } \tag{4}$$

converges if and only if F is nonempty. Moreover, in this case the limit point of the sequence is an element of F. The authors introduced also a generalization of their result for a family $\{T_k\}_{k=0}^{\infty}$ consisting of finitely many subsequences convergent to paracontracting operators. Our result in Theorem 1 is similar in spirit to, but not identical with, [13, Theorem 3].

Under suitable assumptions, the convergence of the following algorithm, proposed in [2],

$$a, x^0 \in R^n \text{ and } x^{k+1} = \alpha_{k+1}a + (1 - \alpha_{k+1})T_{k+1}(x^k), \text{ for all } k \geq 0, \tag{5}$$

was investigated for a finite number of nonexpansive operators $T_1, T_2, \ldots, T_p$ activated cyclically, when $F \neq \emptyset$ and the sequence $\{\alpha_k\}_{k=0}^{\infty}$ satisfies $\alpha_k \to 0$, $\sum_k |\alpha_k - \alpha_{k+n}| < +\infty$ and $\sum_k \alpha_k = +\infty$. Bauschke [2] showed that the limit point of any orbit generated by this algorithm is the projection $P_F(a)$ of a onto F. The question of finding $P_F(a)$ for a given a is known as the best approximation problem with respect to F.

Another approach in determining a common fixed point for a finite pool of nonexpansive mappings was studied in [21]. The authors examined the convergence of an acceleration technique under various hypotheses. Their method

employs the construction of halfspaces at every iteration. The next approximation is calculated by projecting the current one on a surrogate halfspace.

The following general algorithm was analyzed in [10] for a family $\{T_k\}_{k=0}^{\infty}$ of firmly nonexpansive operators

$$x^0 \in R^n \text{ and } x^{k+1} = x^k + \alpha_{k+1}(T_{k+1}(x^k) - x^k), \text{ for all } k \geq 0. \quad (6)$$

Weak and corresponding strong convergence was established under various assumptions on the sequence $\{\alpha_k\}_{k=0}^{\infty}$.

Hirstoaga [16] extended the results of [2] and showed under suitable hypotheses the convergence of the algorithm

$$x^{k+1} = \alpha_k Q(S_k x^k) + (1 - \alpha_k) T_k x^k, \text{ for all } k \geq 0, \quad (7)$$

where $\{T_k\}_{k=0}^{\infty}$ and $\{S_k\}_{k=0}^{\infty}$ are quasi-nonexpansive, Q is a strict contraction and $\{\alpha_k\}_{k=0}^{\infty}$ satisfies $\alpha_k \to 0$ and $\sum_k \alpha_k = +\infty$.

When solving the best approximation problem with respect to F for a uniformly asymptotically regular semigroup of nonexpansive operators, [1] introduced an algorithm similar to (5). Assuming that C is a convex subset of a real Hilbert space H, that G is an unbounded subset of R_+, that $\{T_t \mid t \in G \text{ and } T_t : H \to H\}$ is a uniformly asymptotically regular semigroup of nonexpansive operators with $F \neq \emptyset$, that $\{\alpha_k\}_{k=0}^{\infty}$ is a steering sequence, i.e., $\alpha_k \to 0$, $\sum_k |\alpha_k - \alpha_{k+1}| < +\infty$ and $\sum_k \alpha_k = +\infty$, and that $\{r_k\}_{k=0}^{\infty}$ is an increasing unbounded sequence such that

$$\sum_k \sup_{x \in C} \left\| T_s T_{r_{k+1}}(x) - T_{r_{k+1}}(x) \right\| < +\infty \quad (8)$$

holds for all $s \in G$, it is proved in [1] that the algorithm

$$x^0 \in H \text{ and } x^{k+1} = \alpha_{k+1} a + (1 - \alpha_{k+1}) T_{r_{k+1}}(x^k), \text{ for all } k \geq 0, \quad (9)$$

yields, for a given $a \in C$, an approximation of $P_F(a)$, where $\| \cdot \|$ is the induced norm.

2 Convergence for a Family of Strictly Nonexpansive Operators

We will prove in this section that, under two special hypotheses, an iterative scheme which employs a family of *strictly nonexpansive (SNE) operators*, i.e., operators that obey the next definition, converges to a common fixed point.

In the rest of the paper $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the Euclidean scalar product and norm, respectively, in the n-dimensional Euclidean space R^n.

Definition 1 We say that an operator $T : R^n \to R^n$ is strictly nonexpansive if, for all $x, y \in R^n$,
$$\|T(x) - T(y)\| \leq \|x - y\|, \quad (10)$$
and
$$\text{if } \|T(x) - T(y)\| = \|x - y\|, \text{ then } T(x) - T(y) = x - y. \quad (11)$$

For a family $\{T_k\}_{k=0}^\infty$ of strictly nonexpansive operators we define the fixed points sets and their intersection by

$$\text{Fix}(T_k) = F_k := \{x \in R^n \mid T_k(x) = x\} \text{ and } F := \cap_{k=0}^\infty F_k, \qquad (12)$$

respectively, and assume that

$$F \neq \emptyset. \qquad (13)$$

Consider the following algorithm.

Algorithm 1

Initialization: $x^0 \in R^n$ is arbitrary.

Iterative step: For every $k \geq 0$, given the current iterate x^k calculate the next iterate x^{k+1} by

$$x^{k+1} = T_{k+1}(x^k). \qquad (14)$$

Remark 1 Any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 1, is Fejér monotone with respect to F (see, e.g., [10]).

The following two well-known results (see, e.g., [4,10]) will lead us to the proof of convergence of Algorithm 1.

Proposition 1 *Let $\{T_k\}_{k=0}^\infty$ be a family of strictly nonexpansive operators for which (13) holds and $z \in F$, then for any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 1, the sequence $\{\|x^k - z\|\}_{k=0}^\infty$ is decreasing.*

Corollary 1 *Under the assumptions of Proposition 1, any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 1, is bounded.*

We will make use of the following additional condition.

Condition 1 *Under the assumptions of Proposition 1, if $\{x^k\}_{k=0}^\infty$ is any sequence, generated by Algorithm 1, then for every $\ell \geq 0$, there exists an index $k(\ell) \geq 0$ such that*

$$\left\|T_{k+1}(x^k) - z\right\| \leq \left\|T_\ell(x^k) - z\right\|, \qquad (15)$$

for all $z \in F$ and all $k \geq k(\ell)$.

Remark 2 Condition 1 induces a kind of "monotonicity" concerning the sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 1. This becomes clearer in Lemma 3 in Section 4 below, where the assumption (83) is invoked. It differs from the *strong attractivity with respect to F* of a nonexpansive operator T which is defined in [3, page 372] by

$$k \left\|Tx - x\right\|^2 \leq \left\|x - f\right\|^2 - \left\|Tx - f\right\|^2, \qquad (16)$$

where k is a positive constant.

Proposition 2 *Under the assumptions of Proposition 1 and the assumption that Condition 1 holds, if $\overline{x}$ is an accumulation point of a sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 1, then $\overline{x} \in F$.*

Proof The boundedness of $\{x^k\}_{k=0}^{\infty}$ that follows from Corollary 1, guarantees the existence of $\overline{x}$. Let $\{x^{k_s}\}_{s=0}^{\infty} \subseteq \{x^k\}_{k=0}^{\infty}$ such that $\lim_{s \to \infty} x^{k_s} = \overline{x}$. Take any $\ell \geq 0$ and let $k(\ell)$ be as in Condition 1. There exists an $s(\ell) \geq 0$ such that $k_s \geq k(\ell)$, for all $s \geq s(\ell)$. As $\{x^{k_s}\}_{s=0}^{\infty}$ is a subsequence of $\{x^k\}_{k=0}^{\infty}$, we have for every $s \geq 0$

$$k_{s+1} \geq k_s + 1 \geq k_s. \tag{17}$$

From (13) there exists $z \in F$. Then, for $s \geq s(\ell)$, we have from (17), Proposition 1, (14), Condition 1 and (10),

$$\left\| x^{k_{s+1}} - z \right\| \leq \left\| x^{k_s+1} - z \right\| = \left\| T_{k_s+1}(x^{k_s}) - z \right\| \leq \left\| T_\ell(x^{k_s}) - z \right\|$$
$$= \left\| T_\ell(x^{k_s}) - T_\ell(z) \right\| \leq \left\| x^{k_s} - z \right\|. \tag{18}$$

By taking limits in the last inequality, as $s \to \infty$, we get

$$\|\overline{x} - z\| \leq \|T_\ell(\overline{x}) - z\| \leq \|\overline{x} - z\|, \tag{19}$$

therefore $\|T_\ell(\overline{x}) - z\| = \|\overline{x} - z\|$ and, using (11), it follows that $T_\ell(\overline{x}) = \overline{x}$ implying $\overline{x} \in F_l$. Since ℓ was arbitrarily chosen we obtain

$$\overline{x} \in F, \tag{20}$$

which completes the proof.

We can now state our main convergence result, which follows directly from Proposition 2.

Theorem 1 *Under the assumptions of Proposition 1 and the assumption that Condition 1 holds, any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 1, converges to an element of F.*

Remark 3 Replacing the strict nonexpansivity of the operators $\{T_k\}_{k=0}^{\infty}$ with the assumption that they belong to the wider class of paracontracting operators (see [13, Definition 1]), the results stated in Proposition 1, Proposition 2 and, consequently, Theorem 1 still hold.

We present in the next section the case when every T_k, with $k \geq 0$, is the composition of a constraining operator S_k with an algorithmic operator Q.

3 The Family-Constrained Algorithm (FCA)

Many iterative algorithms are of, or can be cast into, the form of *one-step stationary iterations* (see, e.g., [25, Chapter 10]). Given an algorithmic operator $Q : R^n \to R^n$, the original iterative process may have the form

$$x^{k+1} = Q(x^k), \text{ for all } k \geq 0, \tag{21}$$

under various assumptions on Q. Constraining such an algorithm with a family of operators means that we desire to use instead of (21) the iterative process

$$x^{k+1} = S_k Q(x^k), \text{ for all } k \geq 0, \tag{22}$$

where $\{S_k\}_{k=0}^{\infty}$ is a family of operators $S_k : R^n \rightarrow R^n$, henceforth called *constraining operators*.

If $Q : R^n \rightarrow R^n$ and $S_k : R^n \rightarrow R^n$, with $k \geq 0$, are strictly nonexpansive, we define the operators $T_k : R^n \rightarrow R^n$ by

$$T_k(x) := S_k Q(x), \text{ for all } k \geq 0, \tag{23}$$

and prove that they are also strictly nonexpansive. The following result extends [11, Proposition 4].

Proposition 3 *For any $k \geq 0$, an operator T_k as in (23), in which Q and S_k are strictly nonexpansive, has the following properties:*

$$\|T_k(x) - T_k(y)\| \leq \|x - y\|, \text{ for all } x, y \in R^n, \tag{24}$$

and

$$\text{if } \|T_k(x) - T_k(y)\| = \|x - y\|, \text{ then } T_k(x) - T_k(y) = Q(x) - Q(y) = x - y. \tag{25}$$

Proof To prove (24) we use (10) to obtain

$$\|T_k(x) - T_k(y)\| = \|S_k(Q(x)) - S_k(Q(y))\| \leq \|Q(x) - Q(y)\| \leq \|x - y\|. \tag{26}$$

To prove (25) suppose that we have equalities in (26). Using (11) we obtain

$$T_k(x) - T_k(y) = Q(x) - Q(y) = x - y, \tag{27}$$

which completes the proof.

For $\{T_k\}_{k=0}^{\infty}$ defined according to (23), with $\{S_k\}_{k=0}^{\infty}$ and Q strictly non-expansive, Algorithm 1 may be written as a constrained algorithm.

Algorithm 2 *The Family-Constrained Algorithm (FCA)*
    Initialization: $x^0 \in R^n$ is arbitrary.
    Iterative step: For every $k \geq 0$, given the current iterate x^k calculate the next iterate x^{k+1} by

$$x^{k+1} = S_{k+1} Q(x^k). \tag{28}$$

Proposition 3 and Theorem 1 yield that if assumptions (13) and Condition 1 hold, then any sequence generated by the Algorithm 2 converges to an element of F.

We prove next that if $Q \in \mathcal{F}_2$ (see Definition 3 below) and $S_k = S$, for all $k \geq 0$, is a smoothing matrix, such as the one used in [11], then the family defined by (23) satisfies all our hypotheses. We use the following definitions.

Definition 2 [11, Definition 1] Let $\mathcal{F}_1$ be the set of continuous operators $Q : R^n \to R^n$ that satisfy

$$\|Q(x) - Q(y)\| \leq \|x - y\|, \text{ for all } x, y \in R^n. \tag{29}$$

and

If $\|Q(x) - Q(y)\| = \|x - y\|$, then
$$Q(x) - Q(y) = x - y \text{ and } \langle x - y, Q(y) - y \rangle = 0. \tag{30}$$

Definition 3 [11, Definition 2] Let $\mathcal{F}_2$ be the set of operators $Q \in \mathcal{F}_1$ with the property that for all $S \in R^{n \times n}$ the function $g : R^n \to R$ defined by

$$g(x) := \|x - SQ(x)\|^2 \tag{31}$$

attains its unconstrained global minimum.

It is clear that if $Q \in \mathcal{F}_2$, then Q is strictly nonexpansive. We show next that the family $\{S_k\}_{k=0}^{\infty}$ with $S_k = S$, for all $k \geq 0$, where S is a symmetric, stochastic, with positive diagonal matrix, is strictly nonexpansive. Such a matrix S satisfies the two following properties

$$\|Sx\| \leq \|x\|, \text{ for all } x \in R^n, \tag{32}$$

and

$$\|Sx\| = \|x\| \text{ implies that } Sx = x, \tag{33}$$

(see [11, Corollary 1]). From (32) it follows that

$$\|Sx - Sy\| = \|S(x - y)\| \leq \|x - y\|, \text{ for all } x, y \in R^n. \tag{34}$$

Consequently, for $x, y \in R^n$, if

$$\|Sx - Sy\| = \|x - y\|, \tag{35}$$

then, from (33),

$$\|S(x - y)\| = \|x - y\| \text{ implies that } S(x - y) = Sx - Sy = x - y. \tag{36}$$

If $Q \in \mathcal{F}_2$ and S is a symmetric, stochastic, with positive diagonal matrix, then the set of fixed points of the operator $T := SQ$ is not empty (see [11, Lemma 1]). Finally, Condition 1 is trivial in the context of a single operator used at every iteration of the Algorithm 2.

4 Solving The Linear Least Squares Problem

We show in this section that a commonly used, in the field of image reconstruction, algorithmic operator Q obeys the conditions set forth in Section 3. Consider the linear least squares (LLS) problem of seeking a vector $x \in R^n$ such that

$$\|Ax - b\| = \min\{\|Az - b\| \mid z \in R^n\}, \tag{37}$$

where the matrix A is $m \times n$ and $b \in R^m$. We use the notations A^T, $\mathcal{R}(A)$, $\mathcal{N}(A)$, for the transpose, range and null space of A, respectively, and $LSS(A; b)$ and x_{LS}, for the set of all least squares solutions and the minimal norm solution of (37), respectively.

We present in the sequel a general iterative method, introduced recently in [24], and prove that its algorithmic operator is strictly nonexpansive and, moreover, belongs to $\mathcal{F}_2$. Let T and R be matrices of dimensions $n \times n$ and $n \times m$, respectively, having the following three properties with respect to a given $m \times n$ matrix A:

$$T + RA = I, \tag{38}$$

where I is the identity matrix;

$$\text{for every } y \in R^m \text{ we have } Ry \in \mathcal{R}(A^T); \tag{39}$$

$$\text{defining } \tilde{T} := TP_{\mathcal{R}(A^T)} \text{ we have } \left\|\tilde{T}\right\| < 1, \tag{40}$$

where P_V and $\|\tilde{T}\|$ denote the orthogonal projection onto a linear subspace V and the induced norm of $\tilde{T}$, respectively.

The following result is known, see, e.g., [24].

Proposition 4 *When A and b are as in (37) and the matrices T, R and A have the properties (38)–(40), then the matrix T has the properties*

$$\text{if } x \in \mathcal{N}(A) \text{ then } Tx = x, \tag{41}$$

$$\text{if } x \in \mathcal{R}(A^T) \text{ then } Tx \in \mathcal{R}(A^T), \tag{42}$$

$$\|Tx\| = \|x\| \text{ if and only if } x \in \mathcal{N}(A) \tag{43}$$

and

$$\|T\| \leq 1. \tag{44}$$

Proposition 5 *When A and b are as in (37) and the matrices T, R and A have the properties (38)–(40), then the affine operator $Q : R^n \to R^n$ defined by*

$$Q(\cdot) := T(\cdot) + Rb \tag{45}$$

belongs to $\mathcal{F}_2$.

Proof We first show that $Q \in \mathcal{F}_1$. To prove (29) let $x, y \in R^n$, then, from (44)

$$\|(Tx + Rb) - (Ty + Rb)\| = \|T(x - y)\| \le \|T\| \, \|x - y\| \le \|x - y\| . \quad (46)$$

Now for the case that $\|(Tx + Rb) - (Ty + Rb)\| = \|x - y\|$ we obtain from (43) that

$$x - y \in \mathcal{N}(A). \quad (47)$$

Using (41) we get

$$T(x - y) = x - y \text{ if and only if } (Tx + Rb) - (Ty + Rb) = x - y. \quad (48)$$

Representing y as $y = P_{\mathcal{N}(A)}(y) + P_{\mathcal{R}(A^T)}(y)$ and using (41) we obtain

$$(Ty + Rb) - y = T P_{\mathcal{R}(A^T)}(y) + Rb - P_{\mathcal{R}(A^T)}(y), \quad (49)$$

which, from (42) and (39), gives us

$$(Ty + Rb) - y \in \mathcal{R}(A^T). \quad (50)$$

Therefore, from (47) and (50) it follows that

$$\langle x - y, (Ty + Rb) - y \rangle = 0. \quad (51)$$

The second derivative of $g(\cdot)$, defined by (31), is the constant function $g''(\cdot) = 2(I - SQ)^T(I - SQ)$. Since, for any $S \in R^{n \times n}$, the matrix $(I - SQ)^T(I - SQ)$ is symmetric and positive-semidefinite, it follows that g is convex and attains its global minimum.

Remark 4 The FCA Algorithm 2, with T_k as in (23), $S_k = I$, Q as in (45) with T, R as in (38)-(40) includes the Kaczmarz (see, e.g., [32]), Cimmino (see, e.g., [24]) and Diagonal Weighting (see, e.g., [29]) algorithms (for details and proofs of this statement see [28]). We will prove in the following result that another such example is the Landweber method (see, e.g., [23, 28]).

Proposition 6 *Let $\{\omega_k\}_{k=0}^{\infty} \subset R^n$ have the property that there exists a real ϵ such that*

$$0 < \epsilon \le \omega_k \le \frac{2}{\rho(A)^2} - \epsilon, \quad (52)$$

where $\rho(A)$ denotes the spectral norm of A. For any $x^0 \in R^n$ and $k \ge 0$ the Landweber iteration is defined by

$$x^{k+1} = (I - \omega_k A^T A)x^k + \omega_k A^T b. \quad (53)$$

If we denote $I - \omega_k A^T A$ by T_k and $\omega_k A^T$ by R_k, then, for every $k \ge 0$, the properties (38)-(40) hold.

Proof Let $k \geq 0$ be arbitrarily fixed. From the definitions of T_k and R_k it follows that (38) and (39) are satisfied. If we denote by $A^\dagger$ the (unique) Moore-Penrose pseudoinverse of A, then we have the relation $AA^\dagger A = A$ (see, e.g., [6]) and we may write $P_{\mathcal{R}(A^T)} = A^\dagger A$ (see, e.g., [5]). Consequently, according to (40), we obtain

$$\tilde{T}_k = (I - \omega_k A^T A)A^\dagger A = A^\dagger A - \omega_k A^T AA^\dagger A = A^\dagger A - \omega_k A^T A. \qquad (54)$$

Consider the SVD decomposition $A = U\Sigma V^T$, where the matrices U, Σ and V are of dimensions $m \times m$, $m \times n$ and $n \times n$, respectively. We have that

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ with } \Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \qquad (55)$$

where r is the rank of A and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Then, the pseudoinverse has the form (for details and proofs see [6])

$$A^\dagger = V \begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T. \qquad (56)$$

After a simple computation, (54), (55) and (56) yield

$$\tilde{T}_k = VEV^T, \qquad (57)$$

where

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ with } E_1 = \text{diag}\left(1 - \omega_k\sigma_1^2, 1 - \omega_k\sigma_2^2, \ldots, 1 - \omega_k\sigma_r^2\right). \qquad (58)$$

Therefore, since $\tilde{T}_k$ is normal, we obtain

$$\left\|\tilde{T}_k\right\| = \rho(VEV^T) = \rho(V^TVE) = \rho(E) = \max_{i \in \{1,2,\ldots,r\}} |1 - \omega_k\sigma_i^2|, \qquad (59)$$

which together with (52) gives

$$\left\|\tilde{T}_k\right\| < 1, \qquad (60)$$

and the proof is complete.

Lemma 1 *Let* $\text{Fix}(Q)$ *be the fixed points set of the operator* Q *defined by (45), with* T *and* R *matrices of dimensions* $n \times n$ *and* $n \times m$, *respectively, having the properties (38)–(40). The following property then holds*

$$\text{Fix}(Q) = \{x + \Delta \mid x \in LSS(A;b)\}, \qquad (61)$$

where

$$\Delta = (I - \tilde{T})^{-1}RP_{\mathcal{N}(A^T)}(b). \qquad (62)$$

Proof For $x \in LSS(A; b)$ we know that

$$Ax = P_{\mathcal{R}(A)}(b), \tag{63}$$

thus, by using (38) we get

$$(I - T)(x + \Delta) = RA(x + \Delta) = RP_{\mathcal{R}(A)}(b) + RA\Delta. \tag{64}$$

$\|\tilde{T}\|$ is the spectral norm of $\tilde{T}$, thus a matrix norm. Since, by (40), $\|\tilde{T}\| < 1$, it follows from [17, Corollary 5.6.16 on page 301] that $I - \tilde{T}$ is invertible and

$$(I - \tilde{T})^{-1} = \sum_{k=0}^{\infty} \tilde{T}^k. \tag{65}$$

Using (63), (40), (39), (42) and (65) we obtain

$$RA\Delta = (I - T)\Delta = (I - \tilde{T} - TP_{\mathcal{N}(A)})(I - \tilde{T})^{-1}RP_{\mathcal{N}(A^T)}(b)$$

$$= RP_{\mathcal{N}(A^T)}(b) - TP_{\mathcal{N}(A)}\sum_{k=0}^{\infty}\tilde{T}^k RP_{\mathcal{N}(A^T)}(b) = RP_{\mathcal{N}(A^T)}(b). \tag{66}$$

In view of (64) we then obtain

$$x + \Delta = T(x + \Delta) + Rb, \tag{67}$$

which implies that $\{x + \Delta \mid x \in LSS(A; b)\} \subseteq \mathrm{Fix}(Q)$.

For the reverse inclusion we consider $x \in \mathrm{Fix}(Q)$, i.e., $x = Tx + Rb$, which allows us to write

$$P_{\mathcal{R}(A^T)}(x) + P_{\mathcal{N}(A)}(x) = TP_{\mathcal{R}(A^T)}(x) + TP_{\mathcal{N}(A)}(x) + Rb. \tag{68}$$

From (41) and (39) we get in the above equality

$$P_{\mathcal{R}(A^T)}(x) = \tilde{T}P_{\mathcal{R}(A^T)}(x) + Rb = \tilde{T}(\tilde{T}P_{\mathcal{R}(A^T)}(x) + Rb) + Rb$$

$$= \cdots = \tilde{T}^k P_{\mathcal{R}(A^T)}(x) + (\sum_{i=0}^{k-1}\tilde{T}^i)Rb. \tag{69}$$

By taking the limit as $k \to \infty$, from (40) and (65) we arrive at

$$P_{\mathcal{R}(A^T)}(x) = (I - \tilde{T})^{-1}Rb = x_{LS} + \Delta, \tag{70}$$

which, in turn, implies

$$Ax = A(P_{\mathcal{R}(A^T)}(x) + P_{\mathcal{N}(A)}(x)) = Ax_{LS} + A\Delta = P_{\mathcal{R}(A)}(b) + A\Delta, \tag{71}$$

thus, $x - \Delta \in LSS(A; b)$, i.e.,

$$\mathrm{Fix}(Q) - \Delta \subseteq LSS(A; b), \tag{72}$$

which is equivalent to

$$\mathrm{Fix}(Q) \subseteq LSS(A; b) + \Delta. \tag{73}$$

Since we have also proved that $\Delta + LSS(A; b) \subseteq \mathrm{Fix}(Q)$, the equality

$$\mathrm{Fix}(Q) = LSS(A; b) + \Delta \tag{74}$$

follows and the proof is complete.

For Q chosen as in Proposition 5, we present in the following an example of a family of strictly nonexpansive constraining operators $\{S_k\}_{k=0}^{\infty}$ such that $\{T_k\}_{k=0}^{\infty}$, defined according to (23), satisfies (13) and Condition 1, making it applicable for the convergence theory of the FCA (Algorithm 2).

The family consists of metric projection operators onto closed and convex sets in R^n with an additional "image inclusion assumption". The metric projection operator C onto the box $[a,b] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset R^n$ is defined by its i-th component, $i = 1, 2, \ldots, n$, as

$$(Cx)_i := \begin{cases} x_i, & \text{if } x_i \in [a_i, b_i], \\ a_i, & \text{if } x_i < a_i, \\ b_i, & \text{if } x_i > b_i. \end{cases} \tag{75}$$

It is known that such an operator is strictly nonexpansive (see, e.g., [11, 28]).

Lemma 2 *Let C and $\overline{C}$ be two metric projection operators onto the boxes $[a,b] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset R^n$ and $[\overline{a}, \overline{b}] = [\overline{a}_1, \overline{b}_1] \times \cdots \times [\overline{a}_n, \overline{b}_n] \subset R^n$, respectively, defined as in (75). If the image sets $[\overline{a}, \overline{b}] \subset [a, b]$, then for any $y \in [\overline{a}, \overline{b}]$ the following inequality holds*

$$\left\| \overline{C}z - y \right\| \leq \left\| Cz - y \right\|, \text{ for all } z \in R^n. \tag{76}$$

Proof From the inclusion $[\overline{a}, \overline{b}] \subset [a, b]$ we get that

$$[\overline{a}_i, \overline{b}_i] \subset [a_i, b_i] \text{ for all } i = 1, 2, \ldots, n. \tag{77}$$

If $z \in R^n$ is arbitrarily fixed, from (77) it results $P_{[\overline{a}_i, \overline{b}_i]} P_{[a_i, b_i]} z_i = P_{[\overline{a}_i, \overline{b}_i]} z_i$, for all $i = 1, 2, \ldots, n$. Therefore,

$$\overline{C}Cz = P_{[\overline{a}, \overline{b}]} P_{[a,b]} z = P_{[\overline{a}, \overline{b}]} z = \overline{C}z. \tag{78}$$

For any $y \in [\overline{a}, \overline{b}]$, we have

$$y = \overline{C}y. \tag{79}$$

Since the linear mappings C and $\overline{C}$ are strictly nonexpansive, (78) and (79) yield

$$\left\| \overline{C}z - y \right\| = \left\| \overline{C}Cz - \overline{C}y \right\| \leq \left\| Cz - y \right\|, \text{ for all } z \in R^n \tag{80}$$

and the proof is complete.

Consider now a family $\{C_k\}_{k=0}^{\infty}$ of operators, where for each $k \geq 0$, C_k is a metric projection operator onto the k-th box $[a_k, b_k] \subseteq R^n$, as defined in (75). For this family we define the sets

$$\mathcal{V}_k^* := \{z \in \text{Im}(C_k) \mid z - \Delta \in LSS(A; b)\}, \tag{81}$$

and assume that for all $k \geq 0$,

$$\mathcal{V}_k^* \neq \emptyset. \tag{82}$$

We develop next a sufficient condition for this family $\{C_k\}_{k=0}^{\infty}$ to guarantee that the sequence $\{C_k Q\}_{k=0}^{\infty}$ satisfies (13) and Condition 1, where Q is defined according to Proposition 5.

Lemma 3 *Let $\{C_k\}_{k=0}^{\infty}$ be a family of metric projection operators onto the k-th box $[a_k, b_k] \subset R^n$, as defined in (75) and assume that $\mathcal{V}_k^* \neq \emptyset$ for all $k \geq 0$. If for every $\ell \geq 0$ there exists a $k(\ell) \geq \ell$ such that*

$$Im(C_{k+1}) \subseteq Im(C_\ell), \text{ for all } k \geq k(\ell), \tag{83}$$

then the infinite intersection set

$$\mathcal{V}_\infty^* := \cap_{k=0}^{\infty} \mathcal{V}_k^*, \tag{84}$$

is nonempty.

Proof We construct a decreasing nested sequence of nonempty closed and bounded sets in order to apply Cantor's Intersection Theorem (see, e.g., [31]). Defining for every $\ell \in N$ the set $B_\ell := \cap_{i=0}^{\ell} \mathcal{V}_i^*$, it is clear that for every $\ell \geq 0$, $B_{\ell+1} \subseteq B_\ell$. Moreover, since $B_0 \subseteq [a_0, b_0]$, it is bounded.

Since $LSS(A; b)$ is closed and, for each $\ell \geq 0$, $Im(C_\ell)$ is closed, B_ℓ is also closed. Next we show that B_ℓ is nonempty for every $\ell \geq 0$. Take an arbitrarily fixed $\ell \in N$, and $k(i) \geq i$ for all $i \in \{0, 1, 2, \ldots, \ell\}$, as in (83), and define

$$\overline{k} := \max\{k(0), k(1), k(2), \ldots, k(\ell)\}. \tag{85}$$

Using the definition (81) we obtain

$$\mathcal{V}_{k+1}^* \subseteq \mathcal{V}_i^*, \text{ for all } k \geq \overline{k} \text{ and } i \in \{0, 1, 2, \ldots \ell\}, \tag{86}$$

which implies that

$$\cap_{i=0}^{\ell} \mathcal{V}_i^* \neq \emptyset. \tag{87}$$

Since $\cap_{k=0}^{\infty} \mathcal{V}_k^* = \cap_{\ell=0}^{\infty} B_\ell$, applying Cantor's Intersection Theorem yields $\mathcal{V}_\infty^* \neq \emptyset$.

Remark 5 The condition (83) from the previous lemma is equivalent to the following component-wise inequality on the sequences $\{a_k\}_{k=0}^{\infty}, \{b_k\}_{k=0}^{\infty} \subset R^n$: for all $\ell \geq 0$ there exists a $k(\ell) \geq \ell$ such that

$$a_\ell \leq a_{k+1} \leq b_{k+1} \leq b_\ell, \text{ for all } k \geq k(\ell). \tag{88}$$

The metric projection operators, like those in (75), are frequently used for constraining purposes in image reconstruction problems that are formulated according to (37). As mentioned at the beginning of this paper, the idea of using iteration independent constraints was previously examined, see Subsection 1.1. Our purpose is to explore a procedure of adapting the constraining function at each step of the algorithm to obtain a better approximation of the scanned image. The meaning of (83) in practice is that the image of every constraining function should be built from a priori knowledge to contain the exact solution (the original image), however, $\{Im(C_k)\}_{k=0}^{\infty}$ should not necessarily be a decreasing nested sequence.

In [26] such a family of constraining operators is used to solve a Tomographic Particle Image Velocimetry (TomoPIV) problem (see [27] for more

details), which reduces to reconstructing a binary vector. The difficulty of this problem is to find the number and the approximate location of the particles, corresponding to values of one in the solution. When applying a constant $[0, 1]^n$ constraining interval, the obtained approximation usually contains "ghost" particles. The authors observed in the aforementioned paper that, if as the iterations progress, the intervals are focused on zero or one values by using an iteration-adaptive constraining process, these "ghosts" are eliminated and the correct number of particles is found.

Proposition 7 *For a family $\{C_k\}_{k=0}^{\infty}$ of box constraining operators like those in (75) with the properties (82) and (83), and an operator Q defined by (45), with T and R matrices having the properties (38)–(40), the assumption (13) and Condition 1 are satisfied.*

Proof For the first part we use Lemma 3. Let $z \in \mathcal{V}_{\infty}^*$. From the definition (84) of $\mathcal{V}_{\infty}^*$ we get that for all $k \geq 0$

$$C_k(z) = z \tag{89}$$

and that

$$z - \Delta \in LSS(A; b)), \tag{90}$$

which is equivalent to

$$z \in \Delta + LSS(A; b)), \tag{91}$$

which, from (61), implies that

$$z \in \text{Fix}(Q), \tag{92}$$

thus $z \in F$. It follows that $\mathcal{V}_{\infty}^* \subseteq F$ and, since $\mathcal{V}_{\infty}^*$ is nonempty, that also $F \neq \emptyset$.

To prove Condition 1 we use Lemma 2. For an arbitrarily fixed $\ell \geq 1$ and $k(\ell)$ from (83) we choose $y \in F$, $k \geq k(\ell)$ and $z = Q(x^k)$, from (28). Using (83), the fact that $F \subseteq Im(C_{k+1})$ and Lemma 2 we get

$$\|C_{k+1}(Q(x^k)) - y\| \leq \|C_\ell(Q(x^k)) - y\|, \tag{93}$$

which completes the proof.

In conclusion, according to Proposition 5, Lemma 1, Proposition 7 and Theorem 1, we may solve the linear least squares problem (37) using Algorithm 2 with Q defined by (45), when T and R matrices have the properties (38)–(40) and a family $\{C_k\}_{k=0}^{\infty}$ of box constraining operators like those in (75) satisfying the properties (82) and (83).

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