Polynomial Solvability of Variants of the Trust-Region Subproblem*

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Abstract

We consider an optimization problem of the form

$$\begin{array}{lll} \min & x^T Q x + c^T x & \\ \text{s.t.} & \|x - \mu_h\| \le r_h, & h \in S, \\ & \|x - \mu_h\| \ge r_h, & h \in K, \\ & x \in P, & \end{array}$$

where $P \subseteq \mathbb{R}^n$ is a polyhedron defined by m inequalities and Q is general and the $\mu_h \in \mathbb{R}^n$ and the r_h quantities are given; a strongly NP-hard problem. In the case $|S| = 1$, $|K| = 0$ and $m = 0$ one obtains the classical trust-region subproblem which is polynomially solvable, and has been the focus of much interest because of applications to combinatorial optimization and nonlinear programming

We prove that for each fixed pair $|S|$ and $|K|$ our problem can be solved in polynomial time provided that either (1) $|S| > 0$ and the number of faces of P that intersect $\bigcap_h \{x \in \mathbb{R}^n \, : \, \|x - \mu_h\| \le r_h, \; h \in S\}$ is polynomially bounded, or (2) $|S| = 0$ and m is bounded.

1 Introduction

The trust-region subproblem concerns the minimization of a general quadratic over the unit (Euclidean) ball:

$$(1.1) \quad \min\{x^T Q x + c^T x \, : \, \|x\| \le 1, \; x \in \mathbb{R}^n\}.$$

Surprisingly, even though the objective may not be convex, this problem can be solved in polynomial time; in particular it can be formulated as a semidefinite program (also see [21]).

This result has been a starting point for the study of many extensions obtained by adding constraints to formulation (1.1); part of the motivation for studying these variants is their role in semidefinite relaxations of combinatorial optimization problems. A question of fundamental importance therefore concerns polynomial-time solvability of the extensions.

It is known that the extension obtained by adding *one* linear constraint can be solved in polynomial time, as is the problem where one additional ball constraint $\|x - x^0\| \le r$ is imposed (Sturm and Zhang [19]). Ye and Zhang [23] obtain a polynomial-time algorithm for the extension where two parallel linear inequalities are imposed in addition to the unit ball constraint.[1] They also consider several problems where there are in total two quadratic inequalities and under various conditions polynomial-time algorithms exist.

Recently, Burer and Anstreicher [5] proved that if two parallel linear constraints are added in (1.1) the resulting problem still can be formulated as a polynomially solvable convex program – the formulation includes a positive-semidefiniteness constraint, conic and linear constraints. Even more recently, Burer and Yang [6] studied the case of a general family of linear side-constraints:

$$(1.2) \qquad \min\{x^T Q x + c^T x \, : \, \|x\| \le 1, \; x \in P\},$$

where P is the polyhedron $\{x \in \mathbb{R}^n \, : \, Ax \le b\}$. They considered problem (1.2) under a 'non-intersecting' assumption, that is to say the set $\{x \in \mathbb{R}^n \, : \, \|x\| \le 1, \; x \in P\}$ does not contain a point x satisfying $a_i^T x = b_i$ and $a_j^T x = b_j$ for any pair $1 \le i < j \le m$, where m is the number of rows of A and a_i^T is the i^{th} row of A. Even though problem (1.2) is (strongly) NP-hard, they presented an elegant construction which, under the non-intersecting assumption, reduces the problem to a semidefinite program with conic and linear side-constraints. Thus under the assumption (1.2) can be solved in time polynomial in n, m, L and $\log \epsilon^{-1}$, where L is the number of bits in the data, and $0 < \epsilon < 1$ is the desired tolerance.

In this paper we consider a broad generalization of problem (1.2) namely

$$\begin{array}{llll} \mathcal{T} = \mathcal{T}(Q, c, G, P): & \min & x^T Q x + c^T x & \\ & \text{s.t.} & \|x - \mu_h\| \le r_h, & h \in S, \\ & & \|x - \mu_h\| \ge r_h, & h \in K, \\ & & x \in P, & \end{array}$$

where $P = \{x \in \mathbb{R}^n \, : \, a_i^T x \le b_i, \; 1 \le i \le m\}$, S and K

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[1] It may be possible to obtain from the results in [23] an extension to the case of two general linear inequalities [22].

are sets of indices, and

$$G = \left(\bigcap_h \{x \in \mathbb{R}^n \,:\, \|x - \mu_h\| \le r_h, \quad h \in S\} \right) \bigcap$$

$$\left(\bigcap_h \{x \in \mathbb{R}^n \,:\, \|x - \mu_h\| \ge r_h, \quad h \in K\} \right). \tag{1.3}$$

Define a face of P to be *intersecting* if it has nonempty intersection with the set $\bigcap_j \{x \in \mathbb{R}^n \,:\, \|x - \mu_h\| \le r_h, \quad h \in S\}$ and denote by $\mathcal{F}^*$ the number of intersecting faces. Thus, in the previously considered cases for problem $\mathcal{T}(Q, c, G, P)$ we have $|S| = 2$, $|K| = m = 0$ (Ye-Zhang), $|S| = 1$, $|K| = 0$ and $m = 2$ and so $|\mathcal{F}^*| = 3$ (Ye-Zhang and Burer-Anstreicher) and $|S| = 1$, $|K| = 0$ and also $|\mathcal{F}^*| \le m + 1$ (Burer-Yang). To present our results we introduce the following notation.

Definition 1.1. *The* **type** *of a problem instance $\mathcal{T}(Q, c, G, P)$ is the pair $(|S|, |K|)$.*

In this paper we present two results concerning problems $\mathcal{T}$:

Theorem 1. For each fixed $|S| \ge 1$ and $|K| \ge 0$ there is an algorithm that solves problem $\mathcal{T}(Q, c, G, P)$ of type $(|S|, |K|)$ over general P in time polynomial in n, m, $|\mathcal{F}^*|$, L and $\log \epsilon^{-1}$. The set $\mathcal{F}^*$ is not given as part of the input; rather it will be computed as part of the procedure.

Unlike several of the procedures cited above, ours does not formulate the optimization problem as a convex program; instead our method amounts to a combinatorial enumeration algorithm which produces a list of vectors in $\mathbb{R}^n$ of size polynomial in m, n and $|\mathcal{F}^*|$.

The case of a single nonconvex quadratic constraint (i.e. $|S| = 0$, $|K| = 1$ and $m = 0$ is of interest, given that it, also, can be solved in polynomial time if no other side constraints are present – this follows from the S-Lemma [20], [15]. We generalize this result:

Theorem 2. For each fixed $|K|$ and fixed m, there is an algorithm that solves any problem $\mathcal{T}$ of type $(0, |K|)$ in time polynomial in n, L and $\log \epsilon^{-1}$.

This paper is organized as follows. Section 2 presents background on quadratic formulations for combinatorial optimization problems. Section 3 contains some notation and outlines our algorithm for problem $\mathcal{T}$, with the main construction in Section 4. In the Appendix we describe some technical constructions that we rely on in our proofs.

2 Background: relaxations of combinatorial and nonconvex optimization problems

Semidefinite programming has long been recognized as a fundamental technique for combinatorial optimization problems. In the context of approximation algorithms, the celebrated results of Goemans and Williamson [9] (also see [8]) show that semidefinite relaxations of combinatorial optimization problems can yield provably good results. At the other end of the spectrum, the Lóvasz-Schrijver N_+ reformulation for integer programs (a semidefinite programming relaxation) [12] provably yields, in some cases, stronger results than possible with purely linear formulations.

These results have spurred research on other nonlinear relaxations for combinatorial optimization problems. Part of the interest in this approach is motivated by the fact that semidefinite programs, although polynomially solvable (to tolerance) can prove numerically challenging. From the Sherali-Adams (level-1) reformulation for 0-1 integer programs [17] we can obtain an example. This operator proceeds in two steps: given a constraint

$$\sum_j a_{ij} x_j \le b_i$$

for an integer program, suppose we multiply this constraint by x_k. We then obtain the valid quadratic inequality

$$a_{ik} x_k + \sum_{j \ne k} a_{ij} x_j x_k \; - \; b_i x_k \;\le\; 0. \tag{2.4}$$

The Sherali-Adams approach then linearizes this constraint by introducing new variables $w_{jk} = x_j x_k$ for all $j \ne k$ and substituting into (2.4). By performing this combined operation for all rows i (and for all variables x_k and their complements $1 - x_k$) one obtains the Sherali-Adams reformulation. Clearly, however, if we bypass the linearization step and we use (2.4) directly we will obtain a stronger formulation, albeit one with quadratic inequalities.

Unfortunately, as is well known, quadratically constrained optimization is NP-hard. Perhaps the max-cut problem provides one of the earliest examples. A simple, generic argument is as follows. Consider any NP-hard $\{-1, +1\}$-linear optimization problem:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x_j = 1 \,\text{or}\, -1, \quad 1 \le j \le n. \end{aligned} \tag{2.5}$$

This is equivalently rewritten as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & -1 \le x_j \le 1, \quad 1 \le j \le n, && (2.6) \\ & \|x\|^2 \ = \ n. && (2.7) \end{aligned}$$

A similar result is obtained by removing the quadratic constraint and adding to the objective a term of the form $-M\|x\|^2$ for large enough M.

In spite of these negative results, a great deal of research has recently focused on the general class of *quadratically constrained quadratic programs* or QCQPs. As the name indicates, these are optimization problems of the form

$$QCQP: \quad \min \quad \min F(x) \qquad \text{s.t.} \quad G_i(x) \le 0, \quad 1 \le i \le m$$

where F and the G_i are quadratics, which can thus be regarded as a direct generalization of linear programming problems. In special cases they are polynomially solvable (e.g. convex constraints and objective). However, as we have just seen these problems are (strongly) NP-hard, and in fact they generalize fundamental mathematical problems such as geometric programming, fractional programming and polynomial optimization. For some recent results and for background see [2].

The trust-region subproblem, and polynomially solvable extensions (such as those we present here) constitute some of the few examples of QCQPs where positive results exist. However, little work exists on approximate versions. In fact, it is safe to state that QCQPs are a fundamental family of problems whose analysis could benefit from using techniques familiar to the discrete optimization community, such as bit-scaling and randomization, used to produce provably good approximate solutions, with 'provably' appropriately defined (more below).

Quadratically constrained relaxations also arise in a practical context. An active literature exists on a number of problem families which are at the interface between combinatorial and continuous mathematics. These problems are nominally continuous, but very prominently display combinatorial behavior (in some cases one might say that they are problems in combinatorial geometry). Some of the problems listed in the preceding paragraph fall in this category; however an excellent example is provided by the *cardinality-constrained convex quadratic programming* problem:

$$\begin{aligned} \min \quad & x^T Q x + c^T x && (2.8) \\ \text{s.t.} \quad & Ax \le b && (2.9) \\ & x \in \mathbb{R}^n, \quad \|x\|_0 \ \le \ K. && (2.10) \end{aligned}$$

Here, $Q \succeq 0$, $Ax \le b$ is a linear system, $\|x\|_0$ is the number of nonzero entries in x, and $K > 0$. Often, Q is of high rank (or positive definite). This problem, which is strongly NP-hard, arises in many applications (experiment design, portfolio optimization, sparse vector recovery) and in practice can prove very challenging. In this problem, the fact that the objective is strictly convex or nearly so is precisely what makes the problem especially hard.

Roughly speaking, in the cardinality-constrained problem the solution of a convex relaxation will likely fail even if the relaxation is exact, i.e. the convex hull of feasible solutions, when the objective is strictly convex. In this case an optimal solution x^* will likely be in the convex hull of feasible solutions, but will not itself be feasible. Clearly, this is a difficulty that would arise in many of the problems described above. This observation suggests a generic iterative approach, which for brevity we detail in the case of the cardinality constrained problem. This approach maintains a convex relaxation for the problem which is updated at each iteration. A typical iteration is as follows:

1. Solve the relaxation, with solution x^*.

2. If $\|x^*\|_0 \le K$, STOP. We have solved problem (2.8)-(2.10).

3. Otherwise, find a ball $B = \{x \in \mathbb{R}^n : \|x - \mu\| \le r\}$ such that $x^* \in \text{int}(B)$ and $\|x\|_0 > K$ for all $x \in \text{int}(B)$ ("int" denotes interior).

4. Add the (nonconvex) constraint $\|x - \mu\| \ge r$ to the relaxation.

Beginning from the constraints $Ax \le b$, and iterating using **1 - 4**, each Step **1** requires the solution of a problem of the family $\mathcal{T}$ as introduced in Section 1 which we study in this paper. In experiments, even a few iterations of **1 - 4** yield a much stronger relaxation than possible by other methodologies. We stress that this framework is fairly generic, modulo the identification of the ball B in step **3**.

As discussed above, the generic QCQP is strongly NP-hard. However, a question of interest concerns the existence of approximate solutions. Here we would relax the i^{th} constraint to $G_i(x) \le \epsilon_i$, where $\epsilon_i > 0$ depends on the data. If the ϵ_i are chosen large enough we may be able to bypass the NP-hardness result while still proving useful in applications. It is possible that techniques rooted in discrete optimization may prove useful in this context.

3 Basics

In what follows, a *ball* is a subset $\{x \in \mathbb{R}^N : \|x-\mu\| \leq r\}$ for some $\mu \in \mathbb{R}^N$ and $r \geq 0$.

Next we introduce some notation in the context of a specific problem $\mathcal{T}(Q,c,G,P)$. A face F of P will be represented in the form

$$F^J \ = \{x \in \mathbb{R}^n \ : \ a_i^T x = b_i \, (\forall \, i \in J), \, x \in P\},$$

where $J \subseteq \{1,\ldots,m\}$ is the set of indices of rows of $Ax \leq b$ that hold as equations for every $x \in F$; in that case the relative interior of F^J is

$$\text{ri}(F^J) = \{x \in \mathbb{R}^n \ : \ a_i^T x = b_i \, (i \in J), \ a_i^T x < b_i \, (i \notin J)\}.$$

Given a face F^J, we define

$$(3.11) \quad \mathcal{T}^J = \mathcal{T}^J(Q,c,G,P) : \min\{x^TQx + c^Tx \ : \ x \in G, \ x \in \text{ri}(F^J)\}.$$

Note that this problem may not be well-defined. When $\text{ri}(F^J) \neq \emptyset$, this will be the case if every optimal solution y to the problem

$$(3.12) \quad \min\{x^TQx + c^Tx \ : \ x \in G, \ x \in F^J\}$$

satisfies $a_i^T y = b_i$ for some $i \notin J$ (in other words, in (3.12) we should use an "inf" rather than a "min"). However, the following is clear:

Remark. Let x^* be an optimal solution to problem $\mathcal{T}$. Then there exists a subset $J \subseteq \{1,\ldots,m\}$ such that x^* is an optimal solution to problem $\mathcal{T}^J$.

In the next section we will describe an algorithm that uses this observation to solve problem $\mathcal{T}$ of type $(|S|,|K|)$ with complexity as per Theorems 1 and 2. In the case $|S| > 0$ (which handles Theorem 1 in the introduction) the algorithm assumes that the set $\mathcal{F}^*$ has been previously generated; for completeness we show in Section 4.5 how to obtain $\mathcal{F}^*$ in time polynomial in the size of the problem and linear in $|\mathcal{F}^*|$. When $|S| = 0$ we adopt the convention that $\mathcal{F}^*$ is the set of all faces of P, which, in the context of Theorem 2 is of polynomial size.

Our algorithm will process each member of $\mathcal{F}^*$; upon consideration of a face $F^J \in \mathcal{F}^*$, the algorithm will run in polynomial time, possibly by recursing to a problem $\mathcal{T}(Q',c',G',P')$ of type $(|S'|,|K'|)$ with $|S'|+|K'| < |S|+|K|$, where P' is defined by a system with $\leq m$ rows and whose number of intersecting faces is $\leq |\mathcal{F}^*|$. We will prove that when problem $\mathcal{T}^J$ is well-defined, at least one of following outcomes is guaranteed to hold:

- **(I)** The algorithm returns a vector $\tilde{x}^J$ which optimally solves $\mathcal{T}^J$ (to tolerance ϵ – in other words $\tilde{x}^J$ is both ϵ-feasible and ϵ-optimal).
- **(II)** There exists a strict superset J' of J such that $\mathcal{T}_{J'}$ is also well-defined, and the value of problem $\mathcal{T}_{J'}$ is a lower bound to the value of $\mathcal{T}_J$.[2]

As a result, after examining all elements of $\mathcal{F}^*$, the minimum-objective value vector from among those returned by the algorithm is an optimal solution to problem $\mathcal{T}$, within tolerance. To see this, let $\breve{x}$ be optimal for $\mathcal{T}$, and let J be such that $|J|$ is maximum subject to $\breve{x}$ solving $\mathcal{T}^J$. Then the algorithm necessarily must produce outcome (a) when considering $\mathcal{F}^J$.

4 Main construction

Returning to problem $\mathcal{T}(Q,c,G,P)$, in this section we consider a given subset $J \subseteq \{1,\ldots,m\}$ such that $F^J \in \mathcal{F}^*$, and describe our algorithm to process F^J as indicated at the end of last section. Our procedure given below, in Sections 4.1 and 4.3 will be proved to attain conditions **I** and **II** listed above. Section 4.1 describes a transformation of problem $\mathcal{T}^J$ to be used in our proof. Section 4.2 considers the case where the unit norm constraint in $\mathcal{T}^J$ is not binding, and Section 4.3 considers the case where it is.

4.1 Preliminaries.

For $h \in S \cup K$ define

$$\mu_h^J \ \doteq \ \text{argmin}\{\|x - \mu_h\| \ : \ x \in F^J\},$$

e.g. the closest point to μ_h in F^J. Note that if $\|\mu_h - \mu_h^J\| > r_h$ for some $h \in S$ problem $\mathcal{T}^J$ is infeasible, and in what follows we assume we assume that $\|\mu_h - \mu_h^J\| \leq r_h$ for all $h \in S$.

We can now rewrite problem $\mathcal{T}^J$ in the form

$$\begin{array}{ll} \min & x^TQx + c^Tx \\ \text{s.t.} & \|x - \mu_h^J\| \leq \bar{r}_h, \quad h \in S, \\ & \|x - \mu_h^J\| \geq \bar{r}_h, \quad h \in K, \\ & x \in \text{ri}(F^J), \end{array}$$

where

$$\bar{r}_h^2 \ \doteq \ r_h^2 - \|\mu_h - \mu_h^J\|^2$$

[2] And therefore both problems have the same value.

for each h. However this representation of problem $\mathcal{T}^J$ is not the most convenient for our analysis. Instead, our algorithm for processing F^J relies on an alternate formulation which, essentially, is a null space representation. The description of this reformulation uses the following notation:

- A^J is the submatrix of A induced by the rows indexed by J.
- $\rho = \text{rank}(A^J)$.
- $y^J \doteq \text{argmin}\{\|x\| \,:\, x \in F^J\}$.

Consider the affine mapping $\mathcal{L} \,:\, \mathbb{R}^n \to \mathbb{R}^{n-\rho}$ given by

$$z \;=\; \tilde{V}^T(x - y^J) \tag{4.13}$$

where $\tilde{V}$ is an $n \times (n-\rho)$ matrix, whose columns form an orthonormal basis for the subspace $F^J - y^J$, i.e., for $\text{Null}(A^J)$. Below we will make a specific choice for this basis. Regardless of the choice, however, $\mathcal{L}$ is a composition of two following steps: translation by $-y^J$, followed by projection onto $\text{Null}(A^J)$, followed by a change of coordinates using the columns in $\tilde{V}$ as basis. Consequently, on $\text{Aff}(F^J)$ (the affine subspace generated by F^J), $\mathcal{L}$ is one-to-one and onto.

Mapping $\mathcal{L}$ has several useful properties that are worth reviewing. Suppose we extend the vector z in (4.13) to a vector $\hat{z} \in \mathbb{R}^n$ by appending ρ zeros. Likewise, suppose we extend $\tilde{V}$ to an $n \times n$ matrix V by appending ρ columns, so that altogether the columns of V form an orthonormal basis of $\mathbb{R}^n$. Then from (4.13) we obtain, for any $x \in \text{Aff}(F^J)$,

$$\begin{aligned} \hat{z} \;&=\; V^T(x - y^J) \quad \text{and so} \\ x \;&=\; V\hat{z} + y^J \;=\; \tilde{V}z + y^J \quad \text{and thus} \\ Ax \;&=\; A\tilde{V}z + Ay^J. \end{aligned} \tag{4.14}$$

Moreover, suppose $x \in \text{Aff}(F^J)$. Then for any $h \in S \cup K$, by (4.14)

$$\|x - \mu_h^J\|^2 \;=\; \|\mathcal{L}(x) - \mathcal{L}(\mu_h^J)\|^2$$

and so $\mathcal{L}$ bijectively maps $\{x \in \text{Aff}(A^J) \,:\, \|x - \mu_h\| \lesseqgtr r_h\}$ onto $\{z \in \mathbb{R}^{n-\rho} \,:\, \|z - Ł(\mu_h)\|^2 \lesseqgtr \bar{r}_h\}$. To summarize, write $\bar{\mu}_h = \mathcal{L}(\mu_h)$ ($h \in S \cup K$) and (recall definition (1.3) $\bar{G} \;=\; \mathcal{L}(G)$, in other words

$$\begin{aligned} \bar{G} \;=\; &\left(\bigcap_h \{z \in \mathbb{R}^{n-\rho} \,:\, \|z - \bar{\mu}_h\| \leq \bar{r}_h, \quad h \in S\}\right) \bigcap \\ &\left(\bigcap_h \{z \in \mathbb{R}^{n-\rho} \,:\, \|z - \bar{\mu}_h\| \geq \bar{r}_h, \quad h \in K\}\right). \end{aligned} \tag{4.15}$$

Thus, $\mathcal{L}$ maps

$$\{x \in F^J \,:\, x \in G\} \text{ onto } \{z \in \mathbb{R}^{n-\rho} \,:\, z \in \bar{G},\, \hat{A}_J z \leq \hat{b}_J\},$$

and

$$\{x \in \text{ri}(F^J) \,:\, x \in G\} \text{ onto } \{z \in \mathbb{R}^{n-\rho} \,:\, z \in \bar{G},\, \hat{A}_J z < \hat{b}_J\}$$

(in a one-to-one fashion in both cases) where $\hat{A}_J$ is the submatrix of $A\tilde{V}$ made up of the first $n - |J|$ rows, and $\hat{b}_J$ is the corresponding subvector of $b - Ay^J$.

The discussion just concluded applies to *any* matrix $\tilde{V}$ whose columns form an orthonormal basis of $\text{Null}(A^J)$. In Section A.1 we will show how to make a specific choice:

Theorem 4.1. *In polynomial-time we can compute a vector $\pi \in \mathbb{R}^{n-\rho}$, a vector $\hat{c} \in \mathbb{R}^{n-\rho}$, a real C, and a matrix $\tilde{V}$ whose columns form an orthonormal basis for* $\text{Null}(A^J)$ *so that the resulting map $\mathcal{L}$ has the following property: for all $x \in \text{Aff}(A^J)$,*

$$x^TQx + c^Tx \;=\; \sum_{j=1}^{n-\rho} \pi_j z_j^2 - 2\hat{c}^Tz + C,$$

where $z = \mathcal{L}(x)$.

Full details are provided in Section A.1; however they can be summarized as follows. Denoting by $\tilde{P}^J$ the matrix representing projection onto $\text{Null}(A^J)$, the matrix V to be chosen is that used in the spectral decomposition $V\Lambda V^T$ of the "projected quadratic" $\tilde{Q} \;\doteq\; \tilde{P}^JQ\tilde{P}^J$. It is easily checked that for any $w \in \text{Null}(A^J)$, $w^TQw = w^T\tilde{Q}w$; furthermore, for any $w \in \mathbb{R}^n$, $w^T\tilde{Q}w = (V^Tw)^T\Lambda Vw$. Thus, the change in coordinates in (4.13) allows one to obtain a diagonalized quadratic.

Pending the proof of Theorem 4.1, we have:

Corollary 4.2. *Problem $\mathcal{T}^J$ can be equivalently restated as:*

$$\mathcal{Z}^J: \quad \min \quad \sum_{j=1}^{n-\rho} \pi_j z_j^2 - 2\hat{c}^Tz$$

$$\text{s.t.} \quad \|z - \bar{\mu}_h\| \leq \bar{r}_h, \quad h \in S, \tag{4.16}$$

$$\|z - \bar{\mu}_h\| \geq \bar{r}_h, \quad h \in K, \tag{4.17}$$

$$\hat{A}_J z < \hat{b}_J. \tag{4.18}$$

We will assume the representation of $\mathcal{T}^J$ as $\mathcal{Z}^J$, below. We will analyze problem $\mathcal{Z}^J$ by considering the case where none of the constraints (4.16)-(4.17) are binding, or at least one such constraint is binding (Sections 4.2

and 4.3, respectively).

Next we provide some definitions to be used below. Write

$$\mathcal{N} \doteq \{j \,:\, \pi_j \neq 0\}.$$

DEFINITION 4.3. *A root vector with respect to the quadruple* $(\pi,\, \hat{c}, \bar{G}, \hat{A}_J z \leq \hat{b}_J)$*, is any point* $\hat{z}$ *satisfying*

$$\hat{z} \in \bar{G},\;\; \hat{A}_J \hat{z} \leq \hat{b}_J,\;\; \textit{and}\; \hat{z}_j = \hat{c}_j/\pi_j\, \forall j\, \in \mathcal{N}.$$

Root vectors will play a critical role in the development of the algorithm below. The next set of results address the computation of root vectors.

LEMMA 4.4. *Suppose* $|K| = 0$*. Then a root vector can be computed in polynomial time.*

Proof. Trivial. ∎

In order to handle the computation of a root vector when $|K| > 0$ we make some observations. Namely, we can use the equations $z_j = \hat{c}_j/\pi_j$, $j \in \mathcal{N}$ to eliminate $\mathcal{N}$ variables. Clearly, this change of variables maps balls into balls (preserving convexity) and polyhedra into polyhedra, and therefore the computation of a root vector now amounts to finding a point $w \in \mathbb{R}^{n-\rho-|\mathcal{N}|}$ such that

$$\begin{aligned} \text{FEAS(S,K):} \qquad & \|w - \tilde{\mu}_h\| \leq \tilde{r}_h, \quad h \in S,\\ & \|w - \tilde{\mu}_h\| \geq \tilde{r}_h, \quad h \in K,\\ & \tilde{A}_J w \leq \tilde{b}_J, \end{aligned}$$

for appropriate $\tilde{\mu}_h$, $\tilde{\mu}_h$, $\tilde{A}_J$ and $\tilde{b}_J$ (the last two with $m - |J|$ rows). Let $\tilde{P}_J = \{z \in \mathbb{R}^{n-\rho-|\mathcal{N}|} \,:\, \tilde{A}_J z \leq \tilde{b}_J\}$.

LEMMA 4.5. *(a) Suppose* $|S| = 0$ *and* $|K| = 1$*. Then a root vector can be computed in polynomial time for each fixed* m*. (b) Suppose* $|K| > 0$*. FEAS(S,K) is a problem* $\mathcal{T}(Q', c', G', P')$ *of type* $(|S|, |K| - 1)$*, where* P' *is defined by a system with* $\leq m$ *inequalities and whose number of intersecting faces is* $\leq |\mathcal{F}^*|$*.*

Proof. (a) Here we have $K = \{k\}$ for some singleton k. We can enumerate, in time polynomial in $n - \rho - |\mathcal{N}|$ (but exponential in m) all of the extreme points of the polyhedron. Assuming all satisfy $\|w - \tilde{\mu}_k\| < r_k$, a root vector exists exists if and only if $\tilde{P}$ is unbounded, which can be easily checked (and a root vector, produced) in polynomial time.
(b) Problem FEAS(S,K) given above can be solved by choosing any $k \in K$, and solving

$$\begin{aligned} \max \quad & \|w - \tilde{\mu}_k\|^2\\ \text{s.t.} \quad & \|w - \tilde{\mu}_h\| \leq \tilde{r}_h, \quad h \in S,\\ & \|w - \tilde{\mu}_h\| \geq \tilde{r}_h, \quad h \in K - k,\\ & \tilde{A}_J w \leq \tilde{b}_J, \end{aligned}$$

which is an instance of $\mathcal{T}$, of type $(|S|, |K| - 1)$. Moreover, the number of rows of $\tilde{A}_J$ is at most $\leq m$. Finally, by construction, each distinct face of the polyhedron defined by $\tilde{A}_J w \leq \tilde{b}_J$ arises from a distinct face of the polyhedron defined by $\hat{A}_J z \leq \hat{b}_J$; thus the number of intersecting faces remains $\leq |\mathcal{F}^*|$, as desired. ∎

As a consequence of this Lemmas 4.4-4.5, assuming (by induction) that we have solved all problems FEAS(S',K') with with $|S'| + |K'| < |S| + |K|$, we will have that a root vector can be computed in polynomial time.

In what follows, we denote by $F(z)$ the objective in problem $\mathcal{Z}^J$.

4.2 The case where the none of the ball constraints are binding.

First we consider the case where the none of the constraints in the set $\bar{G}$ is binding.

LEMMA 4.6. *Let* $\check{z}$ *with* $\|\check{z} - \mu_h\| < r_h$ *(*$h \in S$*) and* $\|\check{z} - \mu_h\| > r_h$ *(*$h \in K$*) be optimal for* $\mathcal{Z}^J$*. Then* $\hat{z}^J$ *is optimal for*

$$\min\{F(z) \,:\, z \in \bar{G},\, \hat{A}_J z \leq \hat{b}_J\}.$$

Proof. Clearly

$$\pi_j \check{z}_j \;-\; \hat{c}_j \;=\; 0, \quad \forall j \in \mathcal{N}, \tag{4.19}$$

$$\hat{c}_j \;=\; 0, \quad \forall j \notin \mathcal{N}. \tag{4.20}$$

It follows that $\check{z}$ is a root vector. Thus, for any root vector $\hat{z}^J$, by (4.19)-(4.20) we have $F(\hat{z}^J) = F(\check{z})$. ∎

4.3 The case where at least one ball constraint is binding.

We now analyze problem $\mathcal{Z}^J$ under the assumption that at least one of the constraints in the set $\bar{G}$ is binding. To this effect, our algorithm considers, for each subset $T^o \subseteq S \cup K$, the optimization problem

$$\begin{aligned} \mathcal{Z}^J(T^o): \quad \min \quad & \sum_{j=1}^{n-\rho} \pi_j z_j^2 - 2\hat{c}^T z\\ \text{s.t.} \quad & \|z - \bar{\mu}_h\| < \bar{r}_h, \quad h \in S \setminus T^o,\\ & \|z - \bar{\mu}_h\| > \bar{r}_h, \quad h \in K \setminus T^o,\\ & \|z - \bar{\mu}_h\| = \bar{r}_h, \quad h \in T^o,\\ & \hat{A}_J z < \hat{b}_J. \end{aligned}$$

We will prove that our algorithm produces one of three outcomes:

(a) It computes an optimal solution z^{J,T^o} for $\mathcal{Z}^J(T^o)$.

(b) It proves that there is a problem $\mathcal{T}$ of type $(|S'|, |K'|)$ whose value is a lower bound to the value of problem $\mathcal{Z}^J(T^o)$. Further $|S'|+|K'| < |S|+|K|$, the number of linear inequalities is $\leq m-|J|$ linear inequalities, and as before the number of intersecting faces remains and $\leq \mathcal{F}^*$.

(c) It proves that for some $\tilde{J} \supseteq J$ and $\tilde{T}^o \supseteq T^o$ with at least one of the inclusions strict, the value of problem $\mathcal{Z}^{\tilde{J}}(\tilde{T}^o)$ is a lower bound for the value of problem $\mathcal{Z}^J(T^o)$.

It will then follow that either

(1) $\text{argmin}\{F(z^{J,T^o}) \ : \ \text{case (a) applies for } T^o\}$ solves $\mathcal{Z}^J$, or

(2) For some strict superset $\tilde{J}$ of J, the value of problem $Z^{\tilde{J}}$ is a lower bound on the value of $\mathcal{Z}^J$.

Case (2) amounts to case **(II)** at the end of Section 3.

We now turn to the analysis of problem $\mathcal{Z}^J(T^o)$ for a specific T^o. The key geometrical insight, described next, was already used in [19] in a different context.

LEMMA 4.7. *Let B_1 and and B_2 be distinct balls with nonempty intersection and not contained in one another. Then there exists a hyperplane H, $w \in H$ and $\rho \geq 0$ such that*

$$H \cap B_1 \ = \ H \cap B_2 \ = \ \{x \in H \ : \ \|x - w\| \leq \rho\}.$$

The proof of this fact is routine. We can now extend this observation as follows:

COROLLARY 4.8. *For $1 \leq i \leq q$ let $v_i \in \mathbb{R}^p$ and $d_i \geq 0$, such that*

$$B \doteq \bigcap_i \{x \in \mathbb{R}^p \ : \ \|x - v_i\| = r_i\} \neq \emptyset.$$

Then there exists a polynomial-time computable affine subspace S of $\mathbb{R}^n$, point $w \in S$ and $\rho \geq 0$ such that

$$\begin{aligned} B \ &= \ \{x \in S \ : \ \|x - w\| \leq \rho\} \\ &= \ S \cap \{x \in \mathbb{R}^p \ : \ \|x - v_i\| = r_i\} \quad \text{for } 1 \leq i \leq q. \end{aligned}$$

Proof. If $q = 1$ we let $S = \mathbb{R}^p$, and otherwise apply induction on q use Lemma 4.7. ■

We can use this result to reformulate problem $\mathcal{Z}^J(T^o)$. Namely, given the subspace S in the Corollary, we will have that $\bar{G} \cap S$ is either (i) empty (and then the case corresponding to T^o can be discarded) or (ii) consists of a single point (and then the case corresponding to T^o is immediately evaluated) or (iii) is the intersection of $|S \setminus T^o|$ (convex) balls, $|K \setminus T^o|$ concave balls, and the boundary of one additional ball. Assuming that (iii) holds, we can project the problem $\mathcal{Z}^J(T^o)$ onto subspace S in the sense of Section 4.1 and Theorem 4.1. This projection involves an affine mapping

$$\tilde{L} : \ \mathbb{R}^{n-\rho} \ \to \mathbb{R}^{\dim(S)} \tag{4.21}$$

(of the form (4.13)), which on S is one-to-one and onto. We obtain an equivalent representation for problem $\mathcal{Z}^J(T^o)$ in a space of dimension $\dim(S)$. By appropriately redefining parameters, scaling and translating, this problem has the general form

$$\begin{aligned} \breve{\mathcal{Z}}^J(T^o) : \quad \min \quad & \breve{F}(z) \ \doteq \ \sum_{j=1}^{\breve{n}} \breve{\pi}_j z_j^2 - 2\breve{c}^T z \\ \text{s.t.} \quad & \|z - \breve{\mu}_h\| < \breve{r}_h, \quad h \in S \setminus T^o, \\ & \|z - \breve{\mu}_h\| > \breve{r}_h, \quad h \in K \setminus T^o, \\ & \|z\| = 1, \\ & \breve{A}_J z < \breve{b}_J. \end{aligned} \tag{4.22}$$

[Remark: the affine mapping $\tilde{L}$ has as its last step a translation, so as to get the form (4.22). Here, $\breve{n} = \dim(S)$, and $\breve{A}_J$ has the same dimensions as $\hat{A}_J$. We note that the set of balls corresponding to T^0 has been replaced by the single constraint (4.22). In the rest of this section we handle problem $\breve{\mathcal{Z}}^J(T^o)$ and again for economy of notation we write $\mathcal{N} \doteq \{j \ : \ \breve{\pi}_j \neq 0\}$.

LEMMA 4.9. *Suppose problem $\breve{\mathcal{Z}}^J(T^o)$ has an optimum solution $\breve{z}$. Then there exists a real λ such that*

$$\breve{\pi}_j \breve{z}_j \ - \ \breve{c}_j \ = \ \lambda \breve{z}_j, \quad \forall j \in \mathcal{N}, \tag{4.23}$$

$$-\breve{c}_j \ = \ \lambda \breve{z}_j, \quad \textit{otherwise}. \tag{4.24}$$

Proof. This follows from first-order optimality conditions. ■

Remark. Lemma 4.9 simply states that an optimal solution to $\breve{\mathcal{Z}}^J(T^o)$ on the boundary of the unit ball must be a stationary point of $\breve{F}(z)$ with respect to the constraint $\|z\| = 1$; λ is the corresponding Lagrange multiplier. Martínez [13] has proved that there are at most *three* such values λ; however the difficult task is to obtain, for a given λ is as in (4.23)-(4.24) a corresponding vector $\breve{z} = \breve{z}(\lambda)$ satisfying properties (a) and (b) given above.

In what follows we assume that problem $\breve{\mathcal{Z}}^J(T^o)$ has an optimum solution $\breve{z}$ and that λ is as in (4.23)-(4.24); we will show next that there are at most $3(\dim(S)) + 1$ possible values that λ could take. This analysis will be broken into three cases: $\lambda = 0$ (Section 4.3.1), $\lambda \neq 0$

but $\lambda = \pi_k$ for some k (Section 4.3.2, and $\lambda \neq 0$, $\lambda \neq \pi_k$ for all k (Section 4.3.3).

4.3.1 Case where $\boldsymbol{\lambda = 0}$.

Here we will prove the following result:

LEMMA 4.10. *Suppose $\lambda = 0$. There is a polynomial-time algorithm which computes a feasible solution to problem $\mathcal{Z}^J$, of value no more than that of problem $\mathcal{Z}^J(T^o)$.*

The analysis of the case $\lambda = 0$ is akin to that presented in Lemma 4.6 and we include it for completeness and continuity of our presentation. By Lemma 4.9, if problem $\breve{\mathcal{Z}}^J(T^o)$ is feasible, then any feasible solution to the system

$$\begin{aligned}
&(4.25) \qquad \|z - \breve{\mu}_h\| \leq \breve{r}_h, \quad h \in S \setminus T^o,\\
&\qquad\qquad \|z - \breve{\mu}_h\| \geq \breve{r}_h, \quad h \in K \setminus T^o,\\
&\qquad\qquad \|z\| = 1,\\
&(4.26) \qquad \breve{\pi}_j \breve{z}_j - \breve{c}_j = 0, \quad \forall j \in \mathcal{N},\\
&(4.27) \qquad \breve{A}_J z \leq \breve{b}_J,
\end{aligned}$$

will have objective value equal to that of problem $\breve{\mathcal{Z}}^J(T^o)$. Proceeding as in the construction prior to Lemma 4.9 we can eliminate variables using (4.26), thereby obtaining a system equivalent to (4.25)-(4.27) of the form

$$\begin{aligned}
&(4.28) \qquad \|z - \bar{\mu}_h\| \leq \bar{r}_h, \quad h \in S \setminus T^o,\\
&\qquad\qquad \|z - \bar{\mu}_h\| \geq \bar{r}_h, \quad h \in K \setminus T^o,\\
&(4.29) \qquad \|z\| \geq 1, \quad \|z\| \leq 1\\
&(4.30) \qquad \bar{A}_J z \leq \bar{b}_J.
\end{aligned}$$

Denote the composition of the affine mapping $\tilde{L}$, and the elimination of variables, by $\breve{L}$. The number of quadratic inequalities in the system (4.28)-(4.30) is $|S| + |K| - |T^0| + 2$. If $|T^0| \geq 2$, then using the same approach as in Lemma 4.5 we can indeed find a solution to this system in polynomial time, recursively. Assume, therefore, that $|T^0| = 1$. It follows that in constructing the problem $\mathcal{Z}^J(T^0)$ we required a single quadratic inequality to hold as equation. Assume, e.g. that this was an inequality in the set K, i.e., of the form $\|z - \hat{\mu}_k\| \geq \bar{r}_k$ for some $k \in K$ (the other case is omitted). Then we solve the optimization problem

$$\begin{aligned}
\text{FEAS':} \quad \max \quad & \|z\|\\
\text{s.t.} \quad & \|z - \bar{\mu}_h\| \leq \bar{r}_h, \quad h \in S \setminus T^o,\\
& \|z - \bar{\mu}_h\| \geq \bar{r}_h, \quad h \in K \setminus T^o,\\
& \bar{A}_J z \leq \bar{b}_J.
\end{aligned}$$

Notice that the two constraints (4.29) have been removed and as a result the total number of quadratic constraints is now $|S| + |K| - 1$. Thus (as in the analysis following Lemma 4.9), by induction w can be computed in polynomial time. Moreover, let $w\ (\in \mathbb{R}^{\breve{n}})$ be an optimal solution to FEAS' and let $v \in S$ be such that $\breve{L}(v) = w$. Then by (4.26) and (4.24), w and $\breve{z}$ have equal objective value. Moreover, since $\|\breve{z}\| = 1$ we must have $\|w\| \geq 1$ and so v is feasible for $\mathcal{Z}^J$.

The above discussion provides a proof Lemma 4.10, as desired. In what follows we will assume $\lambda \neq 0$.

4.3.2 Case where $\boldsymbol{\lambda \neq 0, \lambda = \breve{\pi}_k}$, for some $\boldsymbol{k \in \mathcal{N}}$.

The main result in this section, Lemma 4.11 (below) shows that there is a polynomial time algorithm that under the assumption that problem $\breve{\mathcal{Z}}^J(T^o)$ is well-defined, either obtains a optimal solution to problem $\breve{\mathcal{Z}}^J(T^o)$, or proves that there is an equivalent problem with at least one more binding constraint (linear or quadratic).

We have that

$$(4.31) \qquad E \doteq \{j \in \mathcal{N} : \breve{\pi}_j = \lambda\} \neq \emptyset.$$

For $j \notin E$ equations (4.23) and (4.24) uniquely define $\breve{z}_j$, that is to say

$$(4.32) \quad \breve{z}_j = \begin{cases} -\frac{\breve{c}_j}{\breve{\pi}_k - \breve{\pi}_j}, & j \in \mathcal{N} \setminus E\\ -\frac{\breve{c}_j}{\breve{\pi}_k}, & j \notin \mathcal{N}. \end{cases}$$

So the only undetermined values are the $\breve{z}_j$ for $j \in E$. And, further, we know that

$$\sum_{j \in E} \breve{z}_j^2 = 1 - \sum_{j \notin E} \breve{z}_j^2 =$$

$$(4.33) \quad 1 - \sum_{j \in \mathcal{N} \setminus E} \left(\frac{\breve{c}_j}{\breve{\pi}_k - \breve{\pi}_j} \right)^2 - \sum_{j \notin \mathcal{N}} \left(\frac{\breve{c}_j}{\breve{\pi}_k} \right)^2 \doteq B.$$

If $B < 0$ we must discard the given choice for λ, and if $B = 0$ then $\breve{z}$ is uniquely specified. So we will assume $B > 0$. Note that by (4.23), $\breve{c}_j = 0$ for $j \in E$. If this condition is not satisfied then, again, the case $\lambda = \breve{\pi}_k$ must be discarded.

LEMMA 4.11. *Suppose $\lambda = \breve{\pi}_k \neq 0$. Choose an arbitrary index $i \in E$. Let $z^+ \in \mathbb{R}^{\breve{n}}$ be the vector satisfying*

(1) On indices $j \notin E$, z_j^+ takes the value stipulated by the appropriate expression in (4.32).

(2) $z_i^+ = B^{1/2}$.

(3) $z_j^+ = 0$ *for all* $j \in E - i$.

Likewise, define z^- *to be identical to* z^+, *with the exception that* $z_i^- = -B^{1/2}$. *Then if either vector* z^+ *or* z^- *is feasible for* $\breve{\mathcal{Z}}^J(T^o)$, *that vector is optimal for* $\breve{\mathcal{Z}}^J(T^o)$. *Otherwise, there exist sets* $\check{J} \supseteq J$ *and* $\check{T}^o \supseteq T^o$ *with at least one of the inclusions strict, such that problem* $\breve{\mathcal{Z}}^{\check{J}}(\check{T}^o)$ *is well-defined and has optimal value less than or equal to that of* $\breve{\mathcal{Z}}^J(T^o)$.

Proof. Note that z^+ and z^- agree with $\breve{z}$ on indices not in E, and by construction, $\sum_{j\in E} z_j^{+2} = \sum_{j\in E} z_j^{-2} = \sum_{j\in E} \breve{z}_j^2$. Thus $F(z^+) = F(z^-) = F(\breve{z})$. This proves the first part of the Lemma.

Assume therefore that both z^+ and z^- are infeasible for $\mathcal{Z}^J$. If $|E| = 1$, then $\breve{z}$ equals one of z^+ or z^-, a contradiction since $\breve{z}$ is feasible. So $|E| > 1$, and in that case there is a closed curve (i.e. a homeomorph of $[0, 1]$) joining $\breve{z}$ and z^+ and contained on the surface $\{z \in \mathbb{R}^{\breve{n}} : \sum_{j\in E} z_j^2 = B, z_j = \breve{z}_j \, \forall j \notin E\}$.

Note that every point on the curve has the same objective value $F(\breve{z})$ and has unit norm. Since z^+ is not feasible for $\mathcal{Z}^J$ it follows that there is a point z on this curve that satisfies $\|z - \breve{\mu}_h\| \le \breve{r}_h$, $h \in S \setminus T^o$, $\|z - \breve{\mu}_h\| \ge \breve{r}_h$, $h \in K \setminus T^o$, and $\breve{A}_J z \le \breve{b}_J$ but with at least one one more of these constraints binding than at $\breve{z}$. This completes the proof. ■

4.3.3 Case where $\lambda \neq 0$, $\lambda \neq \breve{\pi}_j$, for all $j \in \mathcal{N}$.

This case will be similar to the "secular equation" solution step of trust-region subproblems (see e.g. Section 7.3 in [7]). Also see [21]. Using (4.23) and (4.24) we thus obtain

$$\text{(4.34)} \qquad \breve{z}_j = \frac{\breve{c}_j}{\breve{\pi}_j - \lambda}, \quad \forall j, \text{ and}$$

$$\text{(4.35)} \qquad 1 = \sum_j \frac{d_j}{(\breve{\pi}_j - \lambda)^2}.$$

Write $d_j = \breve{c}_j^2$ for each j, and $M = \{j : d_j > 0\}$. By (4.35) we have $M \neq \emptyset$. We denote by $\rho_1 < \rho_2 < \ldots < \rho_T$ the distinct values $\breve{\pi}_j$ such that $j \in M$. Write

$$f(\lambda) = \sum_{j\in M} \frac{d_j}{(\breve{\pi}_j - \lambda)^2};$$

with this notation equation (4.35) becomes $f(\lambda) = 1$. We will show next that this equation has at most $2|T|$ solutions which can furthermore be computed in polynomial time.

For all $\lambda \neq \rho_i$ for $1 \le i \le T$ we have $f(\lambda) > 0$ and

$$\text{(4.36)} \qquad f'(\lambda) = \sum_{j\in M} \frac{2d_j}{(\breve{\pi}_j - \lambda)^3},$$

$$\text{(4.37)} \qquad f''(\lambda) = \sum_{j\in M} \frac{6d_j}{(\breve{\pi}_j - \lambda)^4} > 0.$$

As a consequence, we have:

(a) $f(\lambda) \to 0^+$ as $\lambda \to -\infty$, $f(\lambda) \to +\infty$ as $\lambda \to \rho_1^-$, and in the range $(-\infty, \rho_1)$, $f(\lambda)$ is strictly increasing. Thus in this range there is a unique solution to $f(\lambda) = 1$.

(b) For $1 \le i < T$, $f(\lambda) \to +\infty$ as $\lambda \to \rho_i^+$ and as $\lambda \to \rho_{i+1}^-$, and in the range (ρ_i, ρ_{i+1}), $f(\lambda)$ has a unique minimum. Hence there are at most two values of λ in (ρ_i, ρ_{i+1}) that attain $f(\lambda) = 1$.

(c) $f(\lambda) \to +\infty$ as $\lambda \to \rho_T^+$, $f(\lambda) \to 0^+$ as $\lambda \to +\infty$, and in the range $(\rho_T, +\infty)$, $f(\lambda)$ is strictly decreasing. Thus, again in this range there is a unique solution to $f(\lambda) = 1$.

In each case, it is a simple exercise to show that binary search (or golden ratio search) will compute all solutions to $f(\lambda) = 1$ to the desired tolerance in polynomial time; from each such solution we obtain, using (4.34), a candidate vector for $\breve{z}$. Formally:

LEMMA 4.12. *Suppose* $\lambda \neq \breve{\pi}_j$ *for all* j. *Then one of the candidates found by the search procedure that solves* $f(\lambda) = 1$ *is* $\breve{z}$, *within tolerance* ϵ. ■

This statement can be made more precise. Let $\check{\lambda}$ be the particular (exact) value of λ that corresponds to $\breve{z}$. The search procedure above computes an estimate λ^* of $\check{\lambda}$ and a corresponding estimate z^* (via eq. (4.23) and (4.24)) of $\breve{z}$. Since we can assume $\check{\lambda} \neq 0$ (because we have already enumerated the case $\lambda = 0$) we can then guarantee that $|\check{\lambda} - \lambda^*| \le \delta|\check{\lambda}|$, where δ is the tolerance employed in the search. Further the complexity will be polynomial in $\log \delta^{-1}$. It is easy to see that we can choose δ so that:

(1) $\log \delta^{-1} = \log \epsilon^{-1} +$ polynomial in n and L, and

(2) for any j, $|\breve{z}_j - z_j^*| \le \epsilon|\breve{z}_j|$.

Note: the constraint $\|z\| \le 1$ implies that (2) is stronger than $|\breve{z}_j - z_j^*| \le \epsilon$.

4.4 Summary.

We now complete the analysis of our algorithm. Assuming problem $\mathcal{Z}^J$ is well-defined, then, with one exception, an application of Lemmas 4.6, 4.10, 4.11 and 4.12 will produce a polynomially-computable, polynomial-size list of vectors at least one of which is ϵ-feasible and ϵ-optimal for $\min\{F(z) : z \in \bar{G}, \hat{A}_J z \leq \hat{b}_J\}$. The only exception arises in Lemma 4.11, in which case there is a strictly more constrained problem which has, however, the same optimal objective value.

Thus the algorithm has the properties claimed at the end of Section 3, as desired.

4.5 Computing all intersecting faces. To conclude our description of an algorithm for problem $\mathcal{T}$ we must describe how to compute the set $\mathcal{F}^*$ when $S \neq \emptyset$ i.e. the set of all faces of the polyhedron P that intersect the $\mathcal{E} \doteq \bigcap_j \{x \in \mathbb{R}^n : \|x - \mu_h\| \leq r_h, \quad h \in S\}$. Here we describe an algorithm for generating $\mathcal{F}^*$ in time $P_1 + \mathcal{F}^* P_2$ where P_1 and P_2 are polynomials in n, m, L and $\log \epsilon^{-1}$.

This algorithm amounts to an application of breadth-first search on the graph whose vertices are the faces of P, and there is an edge (F, F') if F' is contained in F and of dimension one lower than F.

We assume that P itself does intersect the $\mathcal{E}$. Our algorithm will maintain a list $\mathcal{L}$ of faces of P which is initialized with P itself (which is a trivial face). The face P is *marked* (no other faces are marked). Note that $P = F^J$ for some J (possibly $J = \emptyset$). The algorithm proceeds as follows:

1. If $\mathcal{L}$ is empty, stop. Otherwise, let F^J be the first face in $\mathcal{L}$. Remove F^J from $\mathcal{L}$.

2. For each index $1 \leq k \leq m$ such that $k \notin J$, proceed as follows. Let F be the face of P obtained by requiring that $a_i^T x = b_i$ for all $i \in J \cup \{k\}$. If $F \neq \emptyset$, let $H \subseteq \{1, ...m\}$ be such that $F = F_H$ (H can be computed in polynomial time). If F_H is unmarked then we test whether F_H intersects $\mathcal{E}$. If so, we mark F_H and add it to $\mathcal{L}$.

3. Go to **1.**

It is a simple exercise to verify that this algorithm works correctly, and that an appropriate data structure enables us to check in Step 2 whether a face F_H was previously marked, in polynomial time.

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A Appendix – proofs

A.1 Representation of problems $\mathcal{Z}^J$ Here we will detail how to obtain the representation of the objective for problem $\mathcal{Z}^J$ given in Theorem 4.1 above. First we introduce a key technical idea to be used in our construction.

A.1.1 Projected quadratics Consider a subspace H of $\mathbb{R}^n$. Let *tildeP* be matrix denoting projection onto H, that is to say for any $z \in \mathbb{R}^n$, $\tilde{P}z \in H$ is the projection of z onto H. Then $\tilde{P}$ is symmetric, polynomial-time computable, and satisfies $\tilde{P}^2 = \tilde{P}$. See [11].

Given a quadratic $z^T Mz$ with M symmetric, the *projected quadratic* onto H is the quadratic with matrix $\tilde{P}M\tilde{P}$. See [10], Section 12.6 of [11] and [4]. We observe:

(1) For any $z \in H$, $\tilde{P}z = z$ and so $z^T\tilde{P}M\tilde{P}z = z^T Mz$.

(2) For any $z \in \mathbb{R}^n$, $\tilde{P}M\tilde{P}z = \tilde{P}(M\tilde{P}z) \in H$, and as a result any eigenvector v of $\tilde{P}M\tilde{P}$ associated with a nonzero eigenvalue satisfies $v \in H$. Therefore the number of nonzero eigenvalues of $\tilde{P}M\tilde{P}$ is at most the dimension of H.

(3) Consider a spectral decomposition $V\Lambda V^T$ of $\tilde{P}M\tilde{P}$, e.g. the columns of V are an orthonormal family of eigenvectors and $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is the diagonal matrix of eigenvalues, both of $\tilde{P}M\tilde{P}$. Without loss of generality, we list the nonzero eigenvalues first. Suppose we change coordinates so as to make the columns of V as the basis, i.e. we write $z = Vw$. Then we have, for any $z \in \mathbb{R}^n$,

$$z^T\tilde{P}M\tilde{P}z \ = \ w^T V^T \tilde{P}M\tilde{P}Vw \ = \ \sum_{1\le j\le d} \lambda_j w_j^2,$$

where d is the number of nonzero eigenvalues of $\tilde{P}M\tilde{P}$ (including multiplicities) and furthermore as noted in (2), $d \le \dim(H)$.

A.1.2 Application to problem $\mathcal{Z}^J$ To obtain Theorem 4.1 we rely on the projected quadratic method. As before, A^J is the submatrix of A corresponding to the rows indexed by J; its rank is ρ. Further,

(a) $\tilde{P}^J$ is the matrix representing projection onto $\mathrm{Null}(A^J)$.

(b) $V\Lambda V^T$ is a spectral decomposition of $\tilde{P}^J Q\tilde{P}^J$, where $\Lambda = \mathrm{diag}(\pi_1, \ldots, \pi_n)$.

(c) $\tilde{V}$ is the submatrix of V made up of the $n - \rho$ columns that are eigenvectors of $\tilde{P}^J Q\tilde{P}^J$ contained in (and thus spanning) $\mathrm{Null}(A^J)$. Without loss of generality these are the first $n - \rho$ columns of V. Thus, the last ρ columns of V are eigenvectors of $\tilde{P}^J Q\tilde{P}^J$ orthogonal to $\mathrm{Null}(A^J)$, and, so, with associated eigenvalues $\pi_j = 0$.

We next detail how this specific choice of a basis for $\mathrm{Null}(A^J)$ acts on the objective problem $\mathcal{T}^J$. To that effect, consider any $x \in \mathrm{Aff}(F^J)$. Then $x - y^J \in \mathrm{Null}(A^J)$ and so $\tilde{P}^J(x - y^J) = (x - y^J)$. Writing $w = V^T(x - y^J)$ we therefore have $w_j = 0$ for $n - \rho < j \le n$, and

$$(x - y^J)^T Q(x - y^J) = (x - y^J)^T\tilde{P}^J Q\tilde{P}^J(x - y^J) = w^T V^T V\Lambda V^T Vw \ = \ w^T\Lambda w \ = \ \sum_j \pi_j w_j^2 =$$

$$\sum_{j=1}^{n-\rho} \pi_j w_j^2 \ = \ \sum_{j=1}^{n-\rho} \pi_j z_j^2,$$

since by our ordering of the columns of V, $\pi_j = 0$ for $j > n - \rho$. Thus, $x^T Qx =$

$$(x - y^J)^T Q(x - y^J) + 2(y^J)^T Q(x - y^J) + (y^J)^T Qy^J$$
$$= \sum_{j=1}^{n-\rho} \pi_j z_j^2 \ + \ 2(y^J)^T Q\tilde{P}^J Vw + (y^J)^T Qy^J. \tag{A.1}$$

This expression yields the desired quadratic in z in Theorem 4.1, plus a linear term. To express this term as a function of z, recall that we defined $w = V^T(x - y^J)$; since $x - y^J \in \mathrm{Null}(A^J)$ we therefore have $w_j = 0$ for $n - \rho < j \le n$. Thus, w is obtained by appending ρ zeros to the vector $\tilde{V}^T(x - y^J) = z$. Consequently, in (A.1),

$$2(y^J)^T Q\tilde{P}^J Vw = 2Q\tilde{P}^J\tilde{V}z.$$

The objective in problem $\mathcal{T}^J$ contains the additional linear term $c^T x$; this equals

$$c^T(Vw + y^J) = c^T\tilde{V}z \ + \ c^T y^J.$$

The proof of Theorem 4.1 is now complete, with

$$\hat{c} = -\left[\tilde{V}^T\tilde{P}_j Q + \frac{1}{2}\tilde{V}^T c\right],$$

and $C = (y^J)^T Qy^J + c^T y^J$.