SECOND-ORDER CHARACTERIZATIONS OF TILT STABILITY WITH APPLICATIONS TO NONLINEAR PROGRAMMING[1]

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Abstract. The paper is devoted to the study of tilt-stable local minimizers of general optimization problems in finite-dimensional spaces and its applications to classical nonlinear programs with twice continuously differentiable data. The importance of tilt stability has been well recognized from both theoretical and numerical aspects of optimization, and this notion has been extensively studied in the literature. Based on advanced tools of second-order variational analysis and generalized differentiation, we develop a new approach to tilt stability, which allows us to derive not only qualitative but also quantitative characterizations of tilt-stable minimizers with calculating the corresponding moduli. The implementation of this approach and general results in the classical framework of nonlinear programming provides complete characterizations of tilt-stable minimizers under new second-order qualification and optimality conditions.

1 Introduction

The notion of *tilt-stable local minimizers* introduced by Poliquin and Rockafellar [19] and then studied in [1, 9] among other publications has recently attracted strong attention in the literature; see, e.g., [2, 4, 10, 14, 16, 17]. Roughly speaking, tilt stability postulates single-valued Lipschitzian behavior of local minimizers with respect to a special class of "tilt" perturbations. This property of local minimizers is important not only for theoretical aspects of optimization but also plays a fundamental role in the justification of numerical algorithms, which was the original motivation in [19]. The authors of [19] studied tilt stability in the unconstrained format of optimization described by extended-real-valued functions and established a characterization of tilt-stable minimizers under rather general requirements via the *second-order subdifferential/generalized Hessian* in the sense of Mordukhovich [12].

The *first goal* of this paper is to develop a new approach to tilt stability in the *general optimization framework* deriving in this way comprehensive second-order characterizations of tilt stable minimizers that improve, in particular, the major results of [19]. In contrast to [19] and other publications on tilt stability, we establish not only qualitative but also quantitative characterizations of tilt stability. By *qualitative* results we mean those, which establish relationships between tilt stability and other notions *without* involving *numerical quantities* (constants, moduli) in these results; see, e.g., the tilt stability characterization (3.19) in Theorem 3.6 and the sufficient conditions for tilt stability in Corollary 4.4. The *quantitative* results are those, which *explicitly* indicate relationships between the corresponding *constant/moduli* of tilt stability and its verifiable characterizations; see, our main results in Theorem 3.2, Theorem 3.5, Theorem 3.6, and Theorem 4.3.

The *second goal* is to study in detail tilt stability in classical *nonlinear programs* with $\mathcal{C}^2$ initial data. In this setting we provide complete qualitative and quantitative characterizations of tilt-stable minimizers entirely in terms of the initial data via new second-order

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optimality conditions under the weakest constraint qualifications allowing us, in particular, to treat *non-unique* Lagrange multipliers.

Note that our approach and main results also hold, under appropriate modifications and using more involved tools of variational analysis, for optimization problems in infinite-dimensional spaces. This is contrary to [19] and other publications. We select here the finite-dimensional framework for simplicity and the reader's convenience.

The rest of the paper is organized as follows. To make the paper is largely self-contained, Section 2 recalls some basic definitions and facts from variational analysis widely used below. Besides known constructions, we give here a new notion of the *combined second-order subdifferential* of extended-real-valued functions important for subsequent results.

Section 3 presents several *qualitative and quantitative* characterizations (in the aforementioned sense) of tilt stability in the general extended-real-value format of finite-dimensional optimization including those given via second-order growth and second-order subdifferential conditions. As a consequence, we recover here the main result of [19] with a new, much simpler proof and a *precise formula* for calculating the exact bound of tilt stability moduli, which has never been done in the literature.

Section 4 is devoted to applications of the general results obtained to classical *nonlinear programs* (NLP) with $\mathcal{C}^2$ data. We introduce here the new *uniform second-order sufficient condition* (USOSC), which is strictly weaker than the more conventional strong second-order sufficient condition (SSOSC), and use it for characterizing tilt-stable local minimizers in NLP under additional qualification conditions. In particular, we show that tilt stability of local minimizers in this setting is *equivalent* to USOSC under the simultaneous validity of Mangasarian-Fromovitz constraint qualification (MFCQ) and constant rank constraint qualification (CRCQ) without imposing the essentially more restrictive linear independence constraint qualification. On the other hand, the example given in this section demonstrates that the validity of both MFCQ and CRCQ does not ensure the fulfillment of SSOSC for a three-dimensional NLP with linear constraints and a quadratic cost function.

Finally, in Section 5 we present concluding remarks on the results obtained and further developments and discuss some open questions of the future research.

Our notation is basically standard in variational analysis and generalized differentiation; cf. [13, 21]. Everywhere $I\!\!R^n$ stands for the n-dimensional Euclidian space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. We denote by $I\!\!B$ the closed unit ball in the space in question and by $I\!\!B_\eta(x) := x + \eta I\!\!B$ the closed ball centered at x with radius $\eta > 0$. Given a set-valued mapping $F\colon I\!\!R^n \rightrightarrows I\!\!R^m$, the symbol

$$\mathop{\rm Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ y \in I\!\!R^m \Big|\ \exists \text{ sequences } x_k \to \bar{x},\ y_k \to y \text{ such that } y_k \in F(x_k) \text{ for all } k \in I\!\!N := \{1, 2, \ldots\} \Big\} \tag{1.1}$$

signifies the *Painlevé-Kuratowski outer limit* of $F(x)$ as $x \to \bar{x}$.

2 Preliminaries from Variational Analysis

It has been realized in convex and variational analysis that it is convenient to consider *extended-real-valued* functions $f : I\!\!R^n \to \overline{I\!\!R} := (-\infty, \infty]$ that unify, in particular, standard functions with sets and allow to incorporate constraints in the unconstrained framework. We always assume that f is *proper*, i.e., $\mathrm{dom}\, f := \{x \in X \mid f(x) < \infty\} \neq \emptyset$. The *regular* *subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ (known also as the presubdifferential and as the Fréchet or viscosity subdifferential) is

$$\widehat{\partial} f(\bar{x}) := \Big\{ v \in I\!R^n \Big| \ \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \Big\}. \tag{2.1}$$

Then the *limiting subdifferential* of f at $\bar{x}$ (known also as the general/basic or Mordukhovich subdifferential) is defined via the outer limit (1.1) by

$$\partial f(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x), \tag{2.2}$$

where $x \xrightarrow{f} \bar{x}$ signifies that $x \to \bar{x}$ with $f(x) \to f(\bar{x})$. Observe that both regular and limiting subdifferentials reduce to the subdifferential of convex analysis for convex functions.

Given a set $\Omega \subset I\!R^n$ with its indicator function $\delta_\Omega(x)$ equal to 0 for $x \in \Omega$ and to ∞ otherwise, the regular and limiting *normal cones* to Ω at $\bar{x} \in \Omega$ are defined, respectively, via the corresponding subdifferentials (2.1) and (2.2) by

$$\widehat{N}(\bar{x}; \Omega) := \widehat{\partial} \delta_\Omega(\bar{x}) \ \text{ and } \ N(\bar{x}; \Omega) := \partial \delta_\Omega(\bar{x}). \tag{2.3}$$

Consider further a set-valued mapping $F : I\!R^n \rightrightarrows I\!R^m$ and associate with it the *domain* $\operatorname{dom} F$ and the *graph* $\operatorname{gph} F$ given by

$$\operatorname{dom} F := \big\{ x \in I\!R^n \big| \ F(x) \neq \emptyset \big\} \ \text{ and } \ \operatorname{gph} F := \big\{ (x, y) \in I\!R^n \times I\!R^m \big| \ y \in F(x) \big\}.$$

Then the *regular coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is defined by

$$\widehat{D}^* F(\bar{x}, \bar{y})(w) := \big\{ z \in I\!R^n \big| \ (z, -w) \in \widehat{N}((\bar{x}, \bar{y}); \operatorname{gph} F) \big\} \ \text{ for all } \ w \in I\!R^m \tag{2.4}$$

and the *limiting coderivative* of F at $(\bar{x}, \bar{y})$ is

$$D^* F(\bar{x}, \bar{y})(w) := \big\{ z \in I\!R^n \big| \ (z, -w) \in N((\bar{x}, \bar{y}); \operatorname{gph} F) \big\} \ \text{ for all } \ w \in I\!R^m. \tag{2.5}$$

It has been well recognized that the coderivative constructions (2.4) and (2.5) are appropriate tools for the study and characterizations of well-posedness and sensitivity in variational analysis; see, e.g., [13, Chapter 4] and [21, Chapter 9] for more details and references. The following property of this type is used in the paper: a set-valued mapping $F \colon I\!R^n \rightrightarrows I\!R^m$ is *Lipschitz-like* with modulus $\ell > 0$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ (it is also known as the pseudo-Lipschitz or Aubin property) if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subset F(u) + \ell \|x - u\| I\!B \ \text{ for all } \ x, u \in U. \tag{2.6}$$

The infimum of all such $\{\ell\}$, denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$, is called the *exact Lipschitzian bound* of F at $(\bar{x}, \bar{y})$. When F is single-valued around $\bar{x}$, we omit $\bar{y}$ in the above notation.

Next we formulate two significant concepts of variational analysis taken from [18, 21]. A lower semicontinuous (l.s.c.) function $f : X \to \overline{I\!R}$ is *prox-regular* at $\bar{x} \in \operatorname{dom} f$ for $\bar{v} \in \partial f(\bar{x})$ if there are $r, \varepsilon > 0$ such that for all $x, u \in I\!B_\varepsilon(\bar{x})$ with $|f(u) - f(\bar{x})| \leq \varepsilon$ we have

$$f(x) \geq f(u) + \langle v, x - u \rangle - \frac{r}{2} \|x - u\|^2 \ \text{ whenever } \ v \in \partial f(u) \cap I\!B_\varepsilon(\bar{v}). \tag{2.7}$$

We say that f is *subdifferentially continuous* at $\bar{x} \in \operatorname{dom} f$ for $\bar{v} \in \partial f(\bar{x})$ if the function $(x, v) \mapsto f(x)$ is continuous relative to the subdifferential graph $\operatorname{gph} \partial f$ at $(\bar{x}, \bar{v})$.

When f is subdifferentially continuous at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$, it is easy to observe that the condition $|f(u) - f(\bar{x})| \le \varepsilon$ can be omitted in the definition of prox-regularity. Moreover, the graph of ∂f is closed near $(\bar{x}, \bar{v})$ in the latter setting. The class of prox-regular and subdifferentially continuous functions is rather broad including, in particular, *strongly amenable* functions, convex l.s.c. functions etc.; see [18, 21] for further details.

As shown in [18], the limiting subdifferential of prox-regular functions is strongly connected to monotonicity; in particular, the mapping M_γ in (2.8) below is monotone. Recall that $T : I\!R^n \rightrightarrows I\!R^n$ is *monotone* if

$$\langle y - v, x - u \rangle \ge 0 \ \text{ whenever } \ (x, y), (u, v) \in \operatorname{gph} T.$$

Furthermore, $T : I\!R^n \rightrightarrows I\!R^n$ is said to be *maximal monotone* if $T = S$ for any monotone mapping $S : I\!R^n \rightrightarrows I\!R^m$ with $\operatorname{gph} T \subset \operatorname{gph} S$. For any neighborhood $U \times V \subset I\!R^n \times I\!R^n$ we say T is monotone *relative* to $U \times V$ if its localization relative to $U \times V$ is monotone. Recall also that $\widehat{T}$ is a *localization* of T relative to $U \times V$ if $\operatorname{gph} \widehat{T} = \operatorname{gph} T \cap (U \times V)$. Moreover, T is maximal monotone *relative* to $U \times V$ if for any monotone mapping $S : I\!R^n \rightrightarrows I\!R^n$ with $\operatorname{gph} T \cap (U \times V) \subset \operatorname{gph} S$ we have the equality $\operatorname{gph} T \cap (U \times V) = \operatorname{gph} S \cap (U \times V)$.

Now we are ready to formulate the main optimization property studied in this paper.

Definition 2.1 (tilt stability, [19]). *Given $f \colon I\!R^n \to \overline{I\!R}$, a point $\bar{x} \in \operatorname{dom} f$ is a* TILT-STABLE LOCAL MINIMIZER *of f if there is a number $\gamma > 0$ such that the mapping*

$$M_\gamma : v \mapsto \operatorname{argmin}\big\{ f(x) - \langle v, x \rangle \big|\, x \in I\!B_\gamma(\bar{x}) \big\} \tag{2.8}$$

is single-valued and Lipschitz continuous on some neighborhood of $0 \in I\!R^n$ with $M_\gamma(0) = \bar{x}$.

We also consider in what follows a *quantitative* version of this notion that specifies a modulus of tilt stability. Namely, $\bar{x}$ is a tilt-stable minimizer of f with *modulus* $\kappa > 0$ if the mapping M_γ is Lipschitz continuous with constant κ in the framework of Definition 2.1.

As mentioned above, Poliquin and Rockafellar characterized in [19] tilt-stable minimizers of extended-real-valued l.s.c. functions via the *second-order subdifferential* (generalized Hessian) by Mordukhovich [12]. Let us recall this construction and define its new counterpart crucial for the major results of the paper.

Definition 2.2 (second-order subdifferentials). *Let $f : I\!R^n \to \overline{I\!R}$ with $\bar{x} \in \operatorname{dom} f$, and let $\bar{v} \in \partial f(\bar{x})$. Then we say that:*

(i) *The* SECOND-ORDER SUBDIFFERENTIAL *of f at $\bar{x}$ relative to $\bar{v}$ is the set-valued mapping $\partial^2 f(\bar{x}, \bar{v}) \colon I\!R^n \rightrightarrows I\!R^n$ with the values*

$$\partial^2 f(\bar{x}, \bar{v})(w) := \big(D^* \partial f\big)(\bar{x}, \bar{v})(w) \ \text{ for all } \ w \in I\!R^n. \tag{2.9}$$

(ii) *The* COMBINED SECOND-ORDER SUBDIFFERENTIAL *of f at $\bar{x}$ relative to $\bar{v}$ is the set-valued mapping $\breve{\partial}^2 f(\bar{x}, \bar{v}) \colon I\!R^n \rightrightarrows I\!R^n$ with the values*

$$\breve{\partial}^2 f(\bar{x}, \bar{v})(w) := \big(\widehat{D}^* \partial f\big)(\bar{x}, \bar{v})(w) \ \text{ for all } \ w \in I\!R^n. \tag{2.10}$$

Note that a version of $\breve{\partial}^2 f(\bar{x}, \bar{v})$ with the normal cone $\partial f(\cdot) = N(\cdot; \Omega)$ in (2.10) has been recently used in [5, 6] for different purposes. When f is $\mathcal{C}^2$ around $\bar{x}$ with $\bar{v} = \nabla f(\bar{x})$, both $\breve{\partial}^2 f(\bar{x}, \bar{v})$ and $\partial^2 f(\bar{x}, \bar{v})$ reduce to the classical symmetric single-valued Hessian operator:

$$\breve{\partial}^2 f(\bar{x}, \bar{v})(w) = \partial^2 f(\bar{x}, \bar{v})(w) = \big\{ \nabla^2 f(\bar{x}) w \big\} \ \text{ for all } \ w \in I\!R^n.$$

3 Second-Order Characterizations of Tilt Stability

We begin with the following lemma from convex analysis given in [9, Lemma 5.2]; see also [17, Lemma 3.7] for more details.

Lemma 3.1 (convex functions with smooth conjugates). *Let $h : I\!R^n \to \overline{I\!R}$ be convex and l.s.c., and let its conjugate h^* be differentiable on a neighborhood $\mathcal{O}$ of $\bar{v} \in \operatorname{dom} h^*$. Moreover, assume that ∇h^* is Lipschitz continuous on O with constant $\kappa > 0$. Then there are neighborhoods U of $\bar{x}$ and V of $\bar{v}$ with $\bar{x} := \nabla h^*(\bar{v})$ such that*

$$h(x) \ge h(u) + \langle v, x - u\rangle + \frac{1}{2\kappa}\|x-u\|^2 \;\; \text{for all} \;\; x \in U,\, (u,v) \in \operatorname{gph} \partial h \cap (U \times V). \quad (3.1)$$

Now we derive a characterization of tilt stability with a *precise modulus relationship* via a certain *uniform second-order growth condition* formulated in the next theorem. To the best of our knowledge, this condition first appeared in [1] under the name of "uniform second-order growth condition with respect to the tilt parameterization" and was used in [1, Theorem 5.36] to characterize tilt-stable local minimizers for conic programs with $\mathcal{C}^2$ data. Quite recently [2, Theorem 3.3] this condition has also been used to characterize tilt stability in the general extended-real-valued framework under consideration with a different proof while without establishing the important modulus relationship as in the next theorem.

Theorem 3.2 (tilt stability via uniform second-order growth). *Let $f : I\!R^n \to \overline{I\!R}$ be an l.s.c. function such that $0 \in \partial f(\bar{x})$. Assume that f is both prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{v} = 0$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of the function f with modulus $\kappa > 0$.*

(ii) *There are neighborhoods U of $\bar{x}$ and V of $\bar{v}$ such that the mapping $(\partial f)^{-1}$ admits a single-valued localization $\vartheta : V \to U$ around $(\bar{v}, \bar{x})$ and that for any pair $(v,u) \in \operatorname{gph} \vartheta =$ $\operatorname{gph}(\partial f)^{-1} \cap (V \times U)$ we have the* UNIFORM SECOND-ORDER GROWTH CONDITION

$$f(x) \ge f(u) + \langle v, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2 \;\; \text{whenever} \;\; x \in U. \quad (3.2)$$

Proof. To justify (i)$\Longrightarrow$(ii), suppose that $\bar{x}$ is a tilt-stable local minimizer of f with modulus $\kappa > 0$. Thus there is $\gamma > 0$ such that the mapping M_γ in (2.8) is single-valued and Lipschitz continuous on a neighborhood V of $\bar{v}$ with $M_\gamma(\bar{v}) = \bar{x}$. By shrinking V if necessary, we have that $M_\gamma(V) \subset U$ with $U := \operatorname{int} I\!B_\gamma(\bar{x})$. It is easy to check from (2.8) that M_γ is monotone. Due to the Lipschitzian continuity of M_γ on V, it is maximal monotone relative to $V \times U$. Consider next the (Fenchel) *conjugate* of $f + \delta_{\overline{U}}$ given by

$$g(v) := (f + \delta_{\overline{U}})^*(v) = \sup_{x \in \overline{U}} \big\{\langle v, x\rangle - f(x)\big\} \;\; \text{for all} \;\; v \in I\!R^n,$$

which is a proper l.s.c. convex function. For any $v \in V$ we get from (2.8) the representation $g(v) = \langle v, M_\gamma(v)\rangle - f(M_\gamma(v))$ and observe that

$$g(w) - g(v) \ge \langle w, M_\gamma(v)\rangle - f(M_\gamma(v)) - \big(\langle v, M_\gamma(v)\rangle - f(M_\gamma(v))\big) = \langle w - v, M_\gamma(v)\rangle$$

for all $w \in I\!R^n$. This ensures that $M_\gamma(v) \in \partial g(v)$. Since g is convex, its subdifferential ∂g is monotone. This together with the maximal monotonicity of M_γ relative to $V \times U$ implies that $g(v) = M_\gamma(v)$ for all $v \in V$. Hence ∂g is single-valued and Lipschitz continuous on

V. Therefore g is differentiable and Lipschitz continuous with constant κ on V. By Fermat rule we have that $\nabla g(v) = M_\gamma(v) \subset (\partial f)^{-1}(v)$ for all $v \in V$. Since f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{v}$, by shrinking U and V, we may assume that

$$f(x) \geq f(u) + \langle v, x-u\rangle - \frac{r}{2}\|x-u\|^2 \text{ for all } x \in U,\ (u,v) \in \operatorname{gph}\partial f \cap (U \times V)$$

with some $r > 0$. Taking now a localization T of ∂f relative to $U \times V$ and the identity mapping $I : I\!R^n \to I\!R^n$, we get from the above inequality that $T + rI$ is monotone; see [21, Theorem 13.26]. Since M_γ is maximal monotone relative to $V \times U$ with $\operatorname{gph} M_\gamma^{-1} \subset \operatorname{gph} T$ and $T + rI$ is monotone, it is not hard to check (see, e.g., [19, Lemma 3.1]) that M_γ^{-1} and T must coincide locally around $(\bar{x}, \bar{v})$; thus we suppose without loss of generality that

$$\operatorname{gph}\partial f \cap (U \times V) = \operatorname{gph}(\nabla g)^{-1} \cap (U \times V) = \operatorname{gph} M_\gamma^{-1} \cap (U \times V). \tag{3.3}$$

Define further $h := g^*$ and deduce from the biconjugate theorem of convex analysis [21, Theorem 11.1] that $h^* = g$. It follows from Lemma 3.1 that there are neighborhoods $U_1 \subset U$ of $\bar{x}$ and $V_1 \subset V$ of $\bar{v}$ such that

$$h(x) \geq h(u) + \langle v, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2 \text{ for all } x \in U_1,\ (u,v) \in \operatorname{gph}\partial h \cap (U_1 \times V_1). \tag{3.4}$$

With $(u, v) \in \operatorname{gph}\partial f \cap (U_1 \times V_1) := \operatorname{gph}\vartheta$ we note from (3.3) that $u = \nabla g(v) = M_\gamma(v)$. Therefore $v \in \partial g^*(u) = \partial h(u)$ in the sense of convex analysis and that

$$h(u) = g^*(u) = \langle u,v\rangle - g(v) = \langle u,v\rangle - \big(\langle v, M_\gamma(v)\rangle - f(M_\gamma(v))\big) = \langle u,v\rangle - \langle v,u\rangle + f(u) = f(u).$$

Combining this with (3.4) gives us for any $x \in U_1$ and $(u, v) \in \operatorname{gph}\vartheta$ that

$$f(x) \geq (f + \delta_{\overline{U}})^{**}(x) = h(x) \geq f(u) + \langle v, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2,$$

which readily ensures the single-valuedness of ϑ and inequality (3.2), and thus justifies (ii).

To verify (ii)$\Longrightarrow$(i), take any $(u, v), (x, y) \in \operatorname{gph}\vartheta$ and get from (3.2) that

$$\begin{aligned} \|y-v\|\cdot\|x-u\| &\geq \langle y-v, x-u\rangle = \langle y, x-u\rangle + \langle v, u-x\rangle \\ &\geq \Big[f(x) - f(u) + \frac{1}{2\kappa}\|x-u\|^2\Big] + \Big[f(u) - f(x) + \frac{1}{2\kappa}\|u-x\|^2\Big] \\ &\geq \frac{1}{\kappa}\|x-u\|^2. \end{aligned} \tag{3.5}$$

Observe from (3.2) that $f(x) \geq f(\bar{x})$ for all $x \in U$, which ensures that $\bar{x}$ is a local minimizer. Let us show $\vartheta(v) = M_\gamma(v)$ for all $v \in I\!B_\delta(\bar{v})$ with $\delta, \gamma > 0$ satisfying $\gamma > 2\kappa\delta$, $I\!B_\gamma(\bar{x}) \subset U$, and $I\!B_\delta(\bar{v}) \subset V$, where $\kappa > 0$ and the neighborhoods U and V are taken from (ii). Indeed, picking any $v \in I\!B_\delta(\bar{v})$ and any $u \in M_\gamma(v)$, we deduce from (2.8) and (3.2) that

$$f(\bar{x}) - \langle v, \bar{x}\rangle \geq f(u) - \langle v, u\rangle \geq f(\bar{x}) + \frac{1}{2\kappa}\|u-\bar{x}\|^2 - \langle v, u\rangle,$$

which gives us in turn the estimates

$$\|v\|\cdot\|u-\bar{x}\| \geq \langle v, u-\bar{x}\rangle \geq \frac{1}{2\kappa}\|u-\bar{x}\|^2.$$

This yields $\gamma > 2\kappa\delta > \|u - \bar{x}\|$, which ensures that $u \in \operatorname{int} I\!B_\gamma(\bar{x})$. By the Fermat rule we have $v \in \partial f(u)$, or equivalently $u \in (\partial f)^{-1}(v) \cap U = \vartheta(v)$. Since ϑ is single-valued, it follows that $\vartheta(v) = u = M_\gamma(v)$ for all $v \in I\!B_\delta(\bar{v})$. By (3.5) this tells us that M_γ is single-valued and Lipschitz continuous on $I\!B_\delta(\bar{v})$ with modulus κ, which verifies (i) and thus completes the proof of the theorem. △

The following observation is useful in several proofs of the subsequent results.

Remark 3.3 (on single-valued localization). Observe from the proof of (ii)$\Longrightarrow$(i) in the above theorem that whenever the uniform second-order growth condition (3.2) is satisfied, the mapping M_γ is a *single-valued localization* of $(\partial f)^{-1}$ around $(0, \bar{x})$ for some number $\gamma > 0$ sufficiently small.

Next we present a lemma of its own interest, which gives a useful sufficient condition for tilt stability and is employed in the proofs of the main results below.

Lemma 3.4 (tilt stability via the Lipschitz-like property of the inverse subdifferential). *Let $f : I\!R^n \to I\!R$ be l.s.c. with $\bar{v} := 0 \in \partial f(\bar{x})$. Assume that $(\partial f)^{-1}$ is Lipschitz-like with modulus $\kappa > 0$ around $(0, \bar{x})$ and that there is some $\delta > 0$ such that*

$$f(x) \ge f(u) + \langle v, x - u\rangle \quad \text{for all} \;\; x \in I\!B_\delta(\bar{x}),\; (u, v) \in \operatorname{gph} \partial f \cap I\!B_\delta(\bar{x}, \bar{v}). \tag{3.6}$$

Then $\bar{x}$ is a tilt-stable local minimizer of the function f with modulus κ.

Proof. Let T be a localization of $(\partial f)^{-1}$ relative to $\operatorname{int} I\!B_\delta(\bar{v}, \bar{x})$. It is easy to check from (3.6) that T is monotone. Since T is Lipschitz-like around $(\bar{v}, \bar{x})$, the classical Kenderov theorem [7] on the single-valuedness of monotone operators tells us that T is single-valued around $\bar{v}$. Thus we may consider T as Lipschitz continuous with constant κ on $V := \operatorname{int} I\!B_\delta(\bar{v})$. Define now M_δ as in (2.8) and get from (3.6) that $T(v) \subset M_\delta(v)$ for $v \in V$. Noting by (2.8) that M_δ is monotone while T is maximal monotone relative to $\operatorname{int} I\!B_\delta(\bar{v}) \times \operatorname{int} I\!B_\delta(\bar{x})$ due to its Lipschitz continuity, we conclude that $T(v) = M_\delta(v)$ when $v \in V$. Hence $\bar{x}$ is a tilt-stable local minimizer of f with modulus κ, which completes the proof. △

As mentioned above, the main result of [19, Theorem 1.3] provides a characterization of tilt stability via the second-order subdifferential (2.9). In the next theorem we derive a new characterization of tilt stability with calculating the exact bound of stability moduli via the *combined second-order subdifferential* (2.10).

Theorem 3.5 (characterization of tilt-stable minimizers via the combined second-order subdifferential). *Let $f : I\!R^n \to \overline{I\!R}$ be a l.s.c. function having $0 \in \partial f(\bar{x})$. Assume that f is both prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{v} = 0$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of the function f with modulus $\kappa > 0$.*

(ii) *There is a constant $\eta > 0$ such that for all $w \in I\!R^n$ we have*

$$\langle z, w\rangle \ge \frac{1}{\kappa}\|w\|^2 \quad \text{whenever} \;\; z \in \breve{\partial}^2 f(u, v)(w) \;\; \text{with} \;\; (u, v) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x}, 0). \tag{3.7}$$

Furthermore, the exact Lipschitzian bound of the mapping M_γ from (2.8) for all $\gamma > 0$ sufficiently small is calculated by the formula

$$\operatorname{lip} M_\gamma(0) = \inf_{\eta > 0} \sup\Big\{\frac{\|u\|^2}{\langle u^*, u\rangle}\;\Big|\; u^* \in \breve{\partial}^2 f(x, x^*)(u),\; (x, x^*) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x}, 0)\Big\} \tag{3.8}$$

with the convention that $0/0 = 0$.

Proof. To justify (i)$\Longrightarrow$(ii), note first that Theorem 3.2 ensures the existence of a single-valued localization ϑ of $(\partial f)^{-1}$ relative to a neighborhood $V \times U$ of $(\bar{v}, \bar{x})$ such that (3.2) holds. Due to Remark 3.3 we find $\gamma > 0$ so small that $M_\gamma(v) = \vartheta(v)$ for v around $\bar{v}$, and thus we assume that it holds for all $v \in V$. Similarly to (3.5) it follows from (3.2) that

$$\langle y - v, x - u\rangle \geq \frac{1}{\kappa}\|x - u\|^2 \;\text{ whenever }\; (x, y),\, (u, v) \in \operatorname{gph}\partial f \cap (U \times V). \tag{3.9}$$

To verify (3.7), pick any $z \in \breve{\partial}^2 f(u, v)(w)$ with $(u, v) \in \operatorname{gph}\partial f \cap (U \times V)$ and $w \in I\!R^n$ and deduce from (2.10) that for any $\varepsilon > 0$ there is some $\delta > 0$ with $I\!B_\delta(u, v) \subset U \times V$ such that

$$\langle z, x - u\rangle - \langle w, y - v\rangle \leq \varepsilon\big(\|x - u\| + \|y - v\|\big) \text{ whenever } (x, y) \in \operatorname{gph}\partial f \cap I\!B_\delta(u, v). \tag{3.10}$$

For any $t > 0$ sufficiently small define $u_t := \vartheta(v_t) = M_\gamma(v_t)$ with $v_t := v + t(z - 2\kappa^{-1}w) \in V$ and get due to the Lipschitzian continuity of M_γ that $(u_t, v_t) \to (u, v)$ as $t \downarrow 0$. Suppose with no loss of generality that $(u_t, v_t) \in I\!B_\delta(u, v)$ for all $t > 0$. Replacing (x, y) in (3.10) by (u_t, v_t) and using (3.9) give us that

$$\begin{aligned} \varepsilon\big(\|u_t - u\| + \|v_t - v\|\big) &\geq \langle z, u_t - u\rangle - \langle w, v_t - v\rangle \\ &= \langle t^{-1}(v_t - v) + 2\kappa^{-1}w, u_t - u\rangle - t\langle w, z - 2\kappa^{-1}w\rangle \\ &\geq (\kappa t)^{-1}\|u_t - u\|^2 + 2\kappa^{-1}\langle w, u_t - u\rangle - t\langle w, z - 2\kappa^{-1}w\rangle \\ &\geq (\kappa t)^{-1}\|u_t - u\|^2 - 2\kappa^{-1}\|w\| \cdot \|u_t - u\| - t\langle w, z - 2\kappa^{-1}w\rangle \\ &\geq -t\kappa^{-1}\|w\|^2 - t\langle w, z - 2\kappa^{-1}w\rangle = -t\langle z, w\rangle + t\kappa^{-1}\|w\|^2. \end{aligned} \tag{3.11}$$

Observe further from the Lipschitz continuity of M_γ with modulus κ that

$$\begin{aligned} \varepsilon\big(\|u_t - u\| + \|v_t - v\|\big) &= \varepsilon\big(\|M_\gamma(v_t) - M_\gamma(v)\| + \|v_t - v\|\big) \leq \varepsilon\big(\kappa\|v_t - v\| + \|v_t - v\|\big) \\ &= \varepsilon(\kappa + 1)\|v_t - v\| = \varepsilon(\kappa + 1)t\|z - 2\kappa^{-1}w\|, \end{aligned}$$

which together with (3.11) yields $\langle z, w\rangle + \varepsilon(\kappa + 1)\|z - 2\kappa^{-1}w\| \geq \kappa^{-1}\|w\|^2$, and so $\langle z, w\rangle \geq \kappa^{-1}\|w\|^2$ while taking $\varepsilon \downarrow 0$. This ensures (3.7) and completes the first part of the proof.

To verify (ii)$\Longrightarrow$(i), let (3.7) hold with some $\eta, \kappa > 0$. Since f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{v}$, there are $r, \varepsilon > 0$ with $\varepsilon < \eta$ satisfying

$$f(x) \geq f(u) + \langle v, x - u\rangle - \frac{r}{2}\|x - u\|^2 \;\text{ if }\; x \in I\!B_\varepsilon(\bar{x}),\, (u, v) \in \operatorname{gph}\partial f \cap I\!B_\varepsilon(\bar{x}, \bar{v}). \tag{3.12}$$

For $g(x) := f(x) + \frac{r}{2}\|x - \bar{x}\|^2$ as $x \in I\!R^n$, we have $\partial g(x) = \partial f(x) + r(x - \bar{x})$. Define further $W := J(I\!B_\varepsilon(\bar{x}, \bar{v}))$ with $J(u, v) := (u, v + r(u - \bar{x}))$ for $(u, v) \in I\!R^n \times I\!R^n$ and observe that W contains $I\!B_{2\delta}(\bar{x}, \bar{v})$ for some $\delta > 0$ sufficiently small. It is easy to check from (3.12) that

$$g(x) \geq g(u) + \langle v, x - u\rangle \;\text{ whenever }\; x \in I\!B_\delta(\bar{x}),\, (u, v) \in \operatorname{gph}\partial g \cap I\!B_\delta(\bar{x}, \bar{v}). \tag{3.13}$$

Since $\operatorname{gph}\partial f$ is closed around $(\bar{x}, \bar{v})$ due to the prox-regularity and subdifferential continuity of f at $\bar{x}$ for $\bar{v}$, we have that $\operatorname{gph}\partial g$ is also closed around this point. Moreover, it follows from (2.10) and [13, Theorem 1.62(i)] that

$$z - rw \in \breve{\partial}^2 f(u, v - r(u - \bar{x}))(w) \;\text{ if }\; z \in \breve{\partial}^2 g(u, v)(w) \;\text{ with }\; (u, v) \in \operatorname{gph}\partial g \cap I\!B_\delta(\bar{x}, \bar{v}).$$

Taking into account that $(u, v - r(u - \bar{x})) = J^{-1}(u, v) \subset J^{-1}(W) = I\!B_\varepsilon(\bar{x}, \bar{v}) \subset I\!B_\eta(\bar{x}, \bar{v})$, this implies together with (3.7) that $\langle z - rw, w\rangle \geq \kappa^{-1}\|w\|^2$. Thus

$$\|z\| \cdot \|w\| \geq \langle z, w\rangle \geq (r + \kappa^{-1})\|w\|^2, \tag{3.14}$$

which ensures that $\|z\| \geq (r+\kappa^{-1})\|w\|$. By the regular coderivative criterion from [13, Theorem 4.7] we conclude that $(\partial g)^{-1}$ is Lipschitz-like around $(\bar v, \bar x)$.

Next we show that $(\partial g)^{-1}$ is Lipschitz continuous around $\bar v$ with *constant* $\ell := (r+\kappa^{-1})^{-1}$ around $(\bar v, \bar x)$. Indeed, let T be a localization of $(\partial g)^{-1}$ relative to $\operatorname{int} I\!B_\delta(\bar v) \times \operatorname{int} I\!B_\delta(\bar x)$. It follows from (3.13) that T is a monotone mapping. By the classical Kenderov theorem [7] on the single-valuedness of monotone operators, the mapping T is single-valued around $\bar v$, and thus it is Lipschitz continuous around this point. It follows from the mean value inequality in [13, Corollary 3.50] that for any $z \in I\!B$ and $v_1, v_2 \in V_1 := \operatorname{int} I\!B_{\frac{\delta}{2}}(\bar v)$ we have

$$\|\langle z, T(v_1)\rangle - \langle z, T(v_2)\rangle\| \leq \|v_1 - v_2\| \sup\big\{\|w\| \big|\ w \in \widehat\partial\langle z, T(\cdot)\rangle(v),\ v \in V_1\big\}. \tag{3.15}$$

Furthermore, the Lipschitz continuity of T on $I\!B_\varepsilon(\bar v)$ easily ensures the representations

$$\widehat\partial\langle z, T(\cdot)\rangle(v) = \widehat D^* T(v)(z) = \widehat D^*(\partial g)^{-1}(v)(z), \quad z \in I\!B,$$

which gives us together with (3.14) and (3.15) that

$$\|T(v_1) - T(v_2)\| = \sup_{z \in I\!B} \|\langle z, T(v_1) - T(v_2)\rangle\| \leq (r+\kappa^{-1})^{-1}\|v_1 - v_2\| \tag{3.16}$$

for all $v_1, v_2 \in V_1$, and hence $(\partial g)^{-1}$ is Lipschitz continuous around $\bar v$ with constant ℓ.

Taking into account that g satisfies (3.13), the above conclusion together with Lemma 3.4 and Remark 3.3 shows that $\bar x$ is a tilt-stable local minimizer of g with modulus ℓ. Employing now Theorem 3.2 allows us to find neighborhoods U of $\bar x$ and V of $\bar v$ such that

$$g(x) \geq g(u) + \langle v, x-u\rangle + \frac{1}{2\ell}\|x-u\|^2 \ \text{ if } \ x \in U,\ (u,v) \in \operatorname{gph}\partial g \cap (U \times V). \tag{3.17}$$

Since $\partial f(x) = \partial g(x) - r(x-\bar x)$ and $f(x) = g(x) - \frac{r}{2}\|x-\bar x\|^2$, we get from (3.17) that for all $(u,v) \in \operatorname{gph}\partial f \cap J^{-1}(U \times V)$ and $x \in U$

$$\begin{aligned} f(x) &= g(x) - \frac{r}{2}\|x-\bar x\|^2 \geq g(u) + \langle v + r(u-\bar x), x-u\rangle + \frac{r+\kappa^{-1}}{2}\|x-u\|^2 - \frac{r}{2}\|x-\bar x\|^2 \\ &= g(u) + \langle v, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2 + \langle r(u-\bar x), x-u\rangle + \frac{r}{2}\|x-u\|^2 - \frac{r}{2}\|x-\bar x\|^2 \\ &= g(u) + \langle v, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2 - \frac{r}{2}\|u-\bar x\|^2 \\ &= f(u) + \langle v, x-u\rangle + \frac{1}{2\kappa}\|x-u\|^2. \end{aligned}$$

Applying Theorem 3.2 again verifies (i). It is easy to observe that the exact bound formula (3.8) follows directly from (3.7). △

As a consequence of the results above, we derive now a characterization of tilt stability of local minimizers for extended-real-valued functions via the *second-order subdifferential* (2.9). From one hand, it gives a new and easier proof of the main result (3.19) of [19, Theorem 1.3] while from the other hand, it provides a new quantitative information including condition (3.18) with the modulus relationship in (ii) as well as the exact bound formula (3.20).

Theorem 3.6 (characterization of tilt-stable minimizers via the second-order subdifferential). *Let $f: I\!R^n \to \overline{I\!R}$ be l.s.c. with $0 \in \partial f(\bar x)$ and such that f is prox-regular and subdifferentially continuous at $\bar x$ for $\bar v = 0$. Consider the statements:*

(i) *The point $\bar x$ is a tilt-stable local minimizer of f with modulus $\kappa > 0$.*

(ii) $\partial^2 f(\bar{x},0)$ *is* POSITIVE-DEFINITE WITH MODULUS $\mu > 0$ *in the sense that*

$$\langle z, w\rangle \ge \mu \|w\|^2 \ \text{ whenever } \ z \in \partial^2 f(\bar{x},0)(w). \tag{3.18}$$

Then implication (i) $\Longrightarrow$ (ii) *holds with* $\mu = \kappa^{-1}$ *while implication* (ii) $\Longrightarrow$ (i) *is satisfied with any* $\kappa > \mu^{-1}$. *Furthermore, the validity of* (i) *with some modulus* $\kappa > 0$ *is equivalent to* POSITIVE-DEFINITENESS *of* $\partial^2 f(\bar{x},0)$ *in the sense that*

$$\langle z, w\rangle > 0 \ \text{ whenever } \ z \in \partial^2 f(\bar{x},0)(w), \ w \ne 0. \tag{3.19}$$

Finally, the exact Lipschitzian bound of the mapping M_γ *in* (2.8) *is calculated by*

$$\operatorname{lip} M_\gamma(0) = \max\Big\{ \frac{\|w\|^2}{\langle z, w\rangle} \Big| \ z \in \partial^2 f(\bar{x},0)(w)\Big\}. \tag{3.20}$$

for all $\gamma > 0$ *sufficiently small, with the convention that* $0/0 = 0$, *provided that* $\bar{x}$ *is a tilt-stable local minimizer of* f.

Proof. Implication (i)$\Longrightarrow$(ii) with $\mu = \kappa^{-1}$ follows from Theorem 3.5 by passing to the limit in (3.7) as $\eta \downarrow 0$ and using definition (2.9) of the second-order subdifferential.

To justify the converse implication (ii)$\Longrightarrow$(i), we proceed similarly to the proof of (ii)$\Longrightarrow$(i) in Theorem 3.5 with some modifications. Since f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{v}$, inequality (3.12) holds for some $r, \varepsilon > 0$. Defining $g(x) := f(x) + \frac{r}{2}\|x-\bar{x}\|^2$ for $x \in I\!R^n$, we have $\partial g(x) = \partial f(x) + r(x-\bar{x})$.

Claim. *Suppose that the following condition holds with some* $\nu \ge 0$*:*

$$\langle z, w\rangle \ge \nu \|w\|^2 \ \text{ whenever } \ z \in \partial^2 f(\bar{x},0)(w). \tag{3.21}$$

Then for any $\lambda \in (0, r+\nu)$ *there are some neighborhoods* U *of* $\bar{x}$ *and* W *of* $(\bar{x},\bar{v})$ *such that*

$$f(x) \ge f(u) + \langle v, x-u\rangle + \frac{\nu-\lambda}{\lambda}\|x-u\|^2 \ \text{ if } \ x \in U, \ (u,v) \in \operatorname{gph}\partial f \cap W. \tag{3.22}$$

To prove this claim, we verify the Lipschitz-like property of $(\partial g)^{-1}$ around $(\bar{v},\bar{x})$ with modulus $(r+\nu-\lambda)^{-1}$ as $\lambda \in (0, r+\nu)$. Given $z \in \partial^2 g(\bar{x},\bar{v})(w)$, it follows from [13, Theorem 1.62(ii)] that $z - rw \in \partial^2 f(\bar{x},\bar{v})(w)$. By (3.21) we have $\langle z - rw, w\rangle \ge \nu\|w\|^2$, which yields

$$\|z\| \cdot \|w\| \ge \langle z, w\rangle \ge (r+\nu)\|w\|^2.$$

Thus the Mordukhovich/coderivative criterion of [21, Theorem 9.40] tells us that for any $\lambda \in (0, r+\nu)$ the mapping $(\partial g)^{-1}$ is Lipschitz-like around $(\bar{v},\bar{x})$ with modulus $(r+\nu-\lambda)^{-1}$. Furthermore, since g also satisfies (3.13), combining Lemma 3.4 and Theorem 3.2 shows that there are neighborhoods U of $\bar{x}$ and V of $\bar{v}$ such that

$$g(x) \ge g(u) + \langle v, x-u\rangle + \frac{r+\nu-\lambda}{2}\|x-u\|^2 \ \text{ for all } \ x \in U, \ (u,v) \in \operatorname{gph}\partial g \cap (U \times V).$$

This yields (3.22) with $W := J^{-1}(U \times V)$ and $J(u,v) := (u, r(v-\bar{x}))$ and justifies the claim.

Let us go back to the main proof of (ii)$\Longrightarrow$(i). By (3.18) the above Claim shows that inequality (3.22) holds with $\nu = \mu$. Since $\lambda > 0$ can be chosen arbitrarily small, Theorem 3.2

tells us that $\bar{x}$ is the tilt-stable local minimizer of f with modulus $\kappa := (\mu - \lambda)^{-1}$, which can be any number larger than μ^{-1}. This verifies the claimed implication (ii)$\Longrightarrow$(i).

Next we prove the equivalence between (i) with *some* modulus $\kappa > 0$ and condition (3.19). Implication (i)$\Longrightarrow$(3.19) is trivial due to (3.18). To check the converse implication, observe from (3.19) that $D^*(\partial f)^{-1}(\bar{v}, \bar{x})(0) = \{0\}$, which tells us by [13, Corollary 4.11] and [21, Theorem 9.40] that $(\partial f)^{-1}$ is Lipschitz-like around $(\bar{v}, \bar{x})$ with some modulus $\ell > 0$. Moreover, it is clear by (3.19) that (3.21) holds with $\nu = 0$. For each $\lambda \in (0, \min\{r, \ell^{-1}\})$ we find from the above Claim such neighborhoods U of $\bar{x}$ and W of $(\bar{x}, \bar{v})$ that

$$f(x) \geq f(u) + \langle v, x - u\rangle - \frac{\lambda}{2}\|x - u\|^2 \ \text{ for all } \ x \in U, \ (u, v) \in \operatorname{gph} \partial f \cap W. \tag{3.23}$$

Define $h(x) := f(x) + \frac{\lambda}{2}\|x - \bar{x}\|^2$ with $\partial h(x) = \partial f(x) + \lambda(x - \bar{x})$. It is similar to (3.13) seeing that inequality (3.23) implies the existence of $\delta > 0$ sufficiently small with

$$h(x) \geq h(u) + \langle v, x - u\rangle \ \text{ whenever } \ x \in I\!B_\delta(\bar{x}), \ (u, v) \in \operatorname{gph} \partial h \cap I\!B_\delta(\bar{x}, \bar{v}). \tag{3.24}$$

Picking any $(z, w) \in I\!R^n \times I\!R^n$ with $z \in \partial^2 h(\bar{x}, \bar{v})(w)$, we get from [13, Theorem 1.62(ii)] that $z - \lambda w \in \partial^2 f(\bar{x}, \bar{v})(w)$ for all $w \in I\!R^n$. Since $(\partial f)^{-1}$ is Lipschitz-like around $(\bar{v}, \bar{x})$ with modulus ℓ, it follows from [13, Theorem 1.44] that $\ell\|z - \lambda w\| \geq \|w\|$, which yields

$$\ell\|z\| \geq \ell\|z - \lambda w\| - \ell\lambda\|w\| \geq \|w\| - \ell\lambda\|w\| = (1 - \ell\lambda)\|w\|.$$

Again, the aforementioned coderivative criterion ensures that $(\partial h)^{-1}$ is Lipschitz-like around $(\bar{v}, \bar{x})$ with modulus $\frac{\ell}{1-\ell\lambda} + \lambda$. Since h satisfies inequality (3.24), combining Lemma 3.4 and Theorem 3.2 gives us the existence of neighborhoods U_2 of $\bar{x}$ and V_2 of $\bar{v}$ with

$$h(x) \geq h(u) + \langle u, x - u\rangle + \frac{1}{2\frac{\ell}{1-\ell\lambda} + 2\lambda}\|x - u\|^2 \ \text{ for all } \ x \in U_2, \ (u, v) \in \operatorname{gph} \partial h \cap (U_2 \times V_2).$$

By $f(x) = h(x) - \frac{\lambda}{2}\|x - \bar{x}\|^2$ it easily implies that

$$f(x) \geq f(u) + \langle u, x - u\rangle + \left[\frac{1}{2\frac{\ell}{1-\ell\lambda} + 2\lambda} - \frac{\lambda}{2}\right]\|x - u\|^2 \ \text{ for all } \ x \in U_2, \ (u, v) \in \operatorname{gph} \partial f \cap W_2$$

with $W_2 := J_\lambda^{-1}(U_2 \times V_2)$ and $J_\lambda(u, v) := (u, v + \lambda(u - \bar{x}))$ as $(u, v) \in I\!R^n \times I\!R^n$. Choosing now $\lambda > 0$ with $\big(\frac{\ell}{1-\ell\lambda} + \lambda\big)^{-1} - \lambda > 0$ and applying Theorem 3.2 ensure that $\bar{x}$ is a tilt-stable local minimizer of f and thus justify implication (3.19)$\Longrightarrow$(i).

To complete the proof of the theorem, it remains to verify the exact bound formula (3.20). If $\operatorname{dom} \partial^2 f(\bar{x}, 0)(\cdot) = \{0\}$, then we always have (3.18) for any $\mu > 0$. It follows that $\bar{x}$ is a tilt-stable local minimizer for any modulus $\kappa > 0$, which implies that $\operatorname{lip} M_\gamma(\bar{x}) = 0$. Thus we derive (3.20) in which the maximum is attained at $(0, 0)$ due to $0 \in \partial^2 f(\bar{x}, 0)(0)$ and the convention $0/0 = 0$. Now suppose that $\operatorname{dom} \partial^2 f(\bar{x}, 0)(\cdot) \neq \{0\}$ and observe from the quantitative relationship between moduli in (i) and (ii) that

$$\operatorname{lip} M_\gamma(0) = \sup\left\{\frac{\|w\|^2}{\langle z, w\rangle}\,\Big|\, z \in \partial^2 f(\bar{x}, 0)(w)\right\}, \tag{3.25}$$

and thus we only need to show that the maximum is achieved in (3.25). By (3.25) there is a sequence (z_k, w_k) with $z_k \in \partial^2 f(\bar{x}, 0)(w_k)$ and $\frac{\|w_k\|^2}{\langle z_k, w_k\rangle} \to \operatorname{lip} M_\gamma(\bar{x})$ as $k \to \infty$. If $z_k = 0$

for large $k \in I\!\!N$, we have $w_k = 0$ and thus $\mathrm{lip}\, M_\gamma(\bar{x}) = 0$ by the convention that $0/0 = 0$, which verifies that the maximum in (3.20) is achieved at (w_k, z_k). Otherwise, suppose that $\|z_k\| > 0$ for all $k \in I\!\!N$ and then define $\bar{z}_k := z_k\|z_k\|^{-1}$ and $\bar{w}_k := w_k\|z_k\|^{-1}$. This gives

$$\frac{\|w_k\|^2}{\langle z_k, w_k\rangle} = \frac{\|\bar{w}_k\|^2}{\langle \bar{z}_k, \bar{w}_k\rangle} \geq \frac{\|\bar{w}_k\|}{\|\bar{z}_k\|} = \|\bar{w}_k\|, \quad k \in I\!\!N.$$

Hence the sequence $\{\bar{w}_k\}$ is bounded, we suppose that $\bar{z}_k \to \bar{z}$ and $\bar{w}_k \to \bar{w}$ as $k \to \infty$. It follows from $\bar{z}_k \in \partial^2 f(\bar{x}, \bar{v})(\bar{w}_k)$ that $\bar{z} \in \partial^2 f(\bar{x}, \bar{v})(\bar{w})$, which shows that the supremum in (3.25) is attained at $(\bar{w}, \bar{z})$ and thus completes the proof of the theorem. $\triangle$

4 Applications to Tilt Stability in Nonlinear Programming

This section is devoted to applications of the general results obtained in the previous sections to problems of nonlinear programming (NLP) given by:

$$\begin{cases} \text{minimize } \ \varphi_0(x) \ \text{ subject to } x \in I\!\!R^n, \\ \varphi_i(x) \leq 0 \ \text{ for } \ i = 1, \ldots, m, \end{cases} \tag{4.1}$$

where all φ_i are $\mathcal{C}^2$ around the reference point $\bar{x}$. Define the feasible solution set to (4.1) by

$$\Omega := \big\{x \in X \big|\, \varphi(x) \in \Theta\big\} \quad \text{ with } \quad \Theta := I\!\!R^m_- \text{ and } \varphi(x) := \big(\varphi_1(x), \ldots, \varphi_m(x)\big) \tag{4.2}$$

and observe that problem (4.1) can be written in the equivalent unconstrained format:

$$\text{minimize } \ f(x) := \varphi_0(x) + \delta_\Omega(x) \text{ with } \ x \in I\!\!R^n. \tag{4.3}$$

If $\bar{x} \in \Omega$ is a local minimizer of (4.1), it satisfies the following first-order optimality condition via the normal cone to the feasible set Ω (see, e.g., [13, Proposition 5.1]):

$$0 \in \partial f(\bar{x}) = \nabla\varphi_0(\bar{x}) + N(\bar{x}; \Omega). \tag{4.4}$$

We say that $\bar{x}$ is a *tilt-stable local minimizer* of the nonlinear program (4.1) with *modulus* $\kappa > 0$ if it satisfies all the requirements of Definition 2.1 with respect to the extended-real-valued function f in (4.3) and the Lipschitz constant κ for the mapping M_γ therein.

Let us now recall some well-known qualification conditions used in this section; see [13, 21] for more details. The first one is the *linear independent constraint qualification* (LICQ) for (4.1) at $\bar{x} \in \Omega$, which means that the gradients $\big\{\nabla\varphi_i(\bar{x})\big|\, i \in I(\bar{x})\big\}$ are linearly independent in $I\!\!R^n$ along the set of active constraint indices $I(\bar{x}) := \{i \in \{1, \ldots, m\}|\, \varphi_i(\bar{x}) = 0\}$. The second condition strictly weaker than LICQ is the *Mangasarian-Fromovitz constraint qualification* (MFCQ) for (4.1) at $\bar{x} \in \Omega$ meaning that there is $d \in I\!\!R^n$ such that

$$\langle \nabla\varphi_i(\bar{x}), d\rangle < 0 \ \text{ for } \ i \in I(\bar{x}). \tag{4.5}$$

It is worth noting that both LICQ and MFCQ are *robust* in the sense that if either MFCQ or LICQ holds at $\bar{x}$, then it must be satisfied at all x in a neighborhood $\mathcal{O}$ of $\bar{x}$. In these cases the normal cone to Ω at $x \in \mathcal{O} \cap \Omega$ is equivalently calculated by the formulas

$$N(x; \Omega) = \nabla\varphi(x)^* N\big(\varphi(x); \Theta\big) = \big\{\nabla\varphi(x)^*\lambda\big|\, \langle\lambda, \varphi(x)\rangle = 0, \ \lambda \in I\!\!R^m_+ \ \text{ for } \ i \in I(x)\big\}. \tag{4.6}$$

Let us further consider the standard *Lagrange function*

$$L(x,\lambda):=\varphi_0(x)+\sum_{i=1}^m\lambda_i\varphi_i(x)\ \text{ with }\ x\in I\!R^n\ \text{ and }\ \lambda\in I\!R^m$$

and then define the set-valued mapping $\Psi: I\!R^n \rightrightarrows I\!R^n$ by

$$\Psi(x):=\big\{\nabla_x L(x,\lambda)\big|\ \lambda\in N\big(\varphi(x);\Theta\big)\big\}. \tag{4.7}$$

It is well known that if $\bar{x}$ is a local minimizer of problem (4.1), then we have the *stationary condition* $0\in\Psi(\bar{x})$ written equivalently in the form of the *KKT system*

$$0\in\nabla\varphi_0(\bar{x})+\nabla\varphi(\bar{x})^*\lambda=\nabla_x L(\bar{x},\lambda)\ \text{ with some }\ \lambda\in N\big(\varphi(\bar{x});\Theta\big) \tag{4.8}$$

provided that MFCQ holds at $\bar{x}$. Taking into account the explicit form of Θ in (4.6) allows us to describe the set of *Lagrange multipliers* satisfying (4.8) as

$$\Lambda(\bar{x}):=\big\{\lambda\in I\!R^m_+\times I\!R^r\big|\ 0\in\nabla_x L(\bar{x},\lambda),\ \langle\lambda,\varphi(\bar{x})\rangle=0\big\}. \tag{4.9}$$

Based on (4.9), we introduce the *parameterized* set of multipliers useful in the sequel:

$$\Lambda(x,v):=\big\{\lambda\in I\!R^m_+\big|\ v\in\nabla_x L(x,\lambda),\ \langle\lambda,\varphi(x)\rangle=0\big\}\ \text{ with }\ v\in\Psi(x), \tag{4.10}$$

where the mapping Ψ is defined in (4.7). It is clear that $\Lambda(x)=\Lambda(x,0)$ and that $\Lambda(x,v)$ is singleton for any $v\in\Psi(x)$ with $x\in\mathcal{O}$ provided that LICQ holds at $\bar{x}$.

In the second-order framework, a well-recognized condition for NLP (4.1) was introduced by Robinson [20] under the name of *strong second-order sufficient condition* (SSOSC). Recall that SSOSC holds at $\bar{x}$ if for all $\lambda\in\Lambda(\bar{x})$ we have

$$\langle w,\nabla^2_{xx}L(\bar{x},\lambda)w\rangle>0\ \text{ whenever }\ \langle\nabla\varphi_i(\bar{x}),w\rangle=0\ \text{ for }\ i\in I_+(\bar{x},\lambda),\ w\neq 0, \tag{4.11}$$

where $I_+(\bar{x},\lambda):=\big\{i\in\{1,\ldots,m\}\big|\ \lambda_i>0\big\}$.

Another second-order condition for the classical nonlinear programs (4.1) in finite dimensions, labeled as the *standard second-order sufficient condition* (standard SOSC), is formulated as follows: for all $\lambda\in\Lambda(\bar{x})$ we have $\langle w,\nabla^2_{xx}L(\bar{x},\lambda)w\rangle>0$ (or, equivalently, $\langle w,\nabla^2_{xx}L(\bar{x},\lambda)w\rangle\ge\alpha\|w\|^2$ with some constant $\alpha>0$) whenever $w\neq 0$ satisfies

$$\langle\nabla\varphi_i(\bar{x}),w\rangle=0\ \text{ for }\ i\in I_+(\bar{x},\lambda)\text{ and }\langle\nabla\varphi_i(\bar{x}),w\rangle\ge 0\ \text{ for }\ i\in I(\bar{x})\setminus I_+(\bar{x},\lambda).$$

It has been recognized that SSOSC (4.11) is stronger than the standard SOSC; see [20]. Now we introduce a new condition that is a uniform version of the standard SOSC, being stronger than the latter, while playing a crucial role in the characterization of tilt stability.

Definition 4.1 (uniform second-order sufficient condition) *We say that the* UNIFORM SECOND-ORDER SUFFICIENT CONDITION (USOSC) *holds for* (4.1) *at* $\bar{x}$ *with modulus* $\ell>0$ *if there is a constant* $\eta>0$ *such that*

$$\begin{array}{l}\langle w,\nabla^2_{xx}L(x,\lambda)w\rangle\ge\ell\|w\|^2\ \ \textit{whenever}\ \ (x,v)\in\operatorname{gph}\Psi\cap I\!B_\eta(\bar{x},0),\lambda\in\Lambda(x,v),\\ \langle\nabla\varphi_i(x),w\rangle=0\ \ \textit{for}\ \ i\in I_+(x,\lambda)\ \ \textit{and}\ \ \langle\nabla\varphi_i(x),w\rangle\ge 0\ \ \textit{for}\ \ i\in I(x)\setminus I_+(x,\lambda),\end{array} \tag{4.12}$$

where the mapping Ψ *and the set* $\Lambda(x,v)$ *are defined in* (4.7) *and* (4.10), *respectively.*

The next proposition shows that under the validity of MFCQ at $\bar{x}$ the introduced USOSC is implied by SSOSC (4.11).

Proposition 4.2 (SSOSC implies USOSC under MFCQ). *Let $\bar{x}$ be a feasible solution to* (4.1) *satisfying the first-order optimality condition* (4.4) *under the validity of MFCQ at $\bar{x}$. Assume also that SSOSC* (4.11) *holds at this point. Then USOSC from Definition* 4.1 *is satisfied at $\bar{x}$ with some modulus $\ell > 0$.*

Proof. Having SSOSC at $\bar{x}$, we argue by contradiction and assume that there is no number $\ell > 0$ such that USOSC holds at $\bar{x}$ with modulus ℓ. This allows us to find a sequence $\{x_k, v_k, w_k, \lambda_k\} \subset I\!R^n \times I\!R^n \times I\!R^n \times I\!R^m$ satisfying

$$\begin{array}{l} x_k \to \bar{x},\ v_k \to 0,\ \lambda_k \in \Lambda(x_k, v_k),\ \|w_k\| = 1,\ \langle w_k, \nabla^2_{xx} L(x_k, \lambda_k) w_k \rangle \le k^{-1}, \ \text{ and} \\ \langle \nabla\varphi_i(x_k), w_k \rangle = 0 \text{ for } i \in I_+(x_k, \lambda_k),\ \langle \nabla\varphi_i(x_k), w_k \rangle \ge 0 \text{ for } i \in I(x_k) \setminus I_+(x_k, \lambda_k). \end{array} \tag{4.13}$$

It follows from the inclusion $\lambda_k \in \Lambda(x_k, v_k)$ in (4.13) and construction (4.10) that

$$v_k = \nabla\varphi_0(x_k) + \sum_{i=1}^m \lambda_{k_i} \nabla\varphi_i(x_k) \ \text{ with } \ \lambda_k \in I\!R^m_+ \ \text{ and } \ \langle \lambda_k, \varphi(x_k) \rangle = 0, \tag{4.14}$$

where λ_{k_i} signifies the i^{th} component of the vector λ_k. We get from MFCQ (4.5) at $\bar{x}$ that there are $d \in I\!R^n$ and $\delta > 0$ such that

$$\langle \nabla\varphi_i(\bar{x}), d \rangle < -2\delta \ \text{ for } \ i \in I(\bar{x}). \tag{4.15}$$

Since $x_k \to \bar{x}$ as $k \to \infty$, we suppose without loss of generality that $\langle \nabla\varphi_i(x_k), d \rangle < -\delta$ for all $i \in I(x_k) \subset I(\bar{x})$ and $k \in I\!N$. Note further that $\lambda_{k_i} = 0$ for $i \in \{1, \ldots, m\} \setminus I(x_k)$. Hence we get $\lambda_{k_i} \langle \nabla\varphi_i(x_k), d \rangle \le -\delta \lambda_{k_i}$ whenever $i \in \{1, \ldots, m\}$. Combining this with the relationships in (4.14) and (4.15) gives us the estimates

$$-\|v_k\| \cdot \|d\| \le \langle v_k, d \rangle = \langle \nabla\varphi_0(x_k), d \rangle + \sum_{i=1}^m \lambda_{k_i} \langle \nabla\varphi_i(x_k), d \rangle \le \|\nabla\varphi_0(x_k)\| \cdot \|d\| - \delta \sum_{i=1}^m \lambda_{k_i},$$

which ensures that the sequence $\{\lambda_k\}$ is bounded in $I\!R^m_+$. By passing to subsequences, we assume that $\lambda_k \to \lambda \in I\!R^m_+$ and $w_k \to w \in I\!R^n$ as $k \to \infty$ with $\|w\| = 1$. Noting that $I_+(\bar{x}, \lambda) \subset I_+(x_k, \lambda_k)$ for sufficiently large $k \in I\!N$, we get from (4.13) that

$$\lambda \in \Lambda(\bar{x}),\ \langle w, \nabla^2_{xx} L(\bar{x}, \lambda) w \rangle \le 0, \ \text{ and } \ \langle \nabla\varphi_i(\bar{x}), w \rangle = 0 \ \text{ for } \ i \in I_+(\bar{x}, \lambda),$$

which contradicts (4.11) and thus completes the proof of the proposition. $\triangle$

In what follows we show (see Theorem 4.3 and Example 4.5) that the introduced USOSC is *strictly weaker* than its SSOSC counterpart (4.11) even under the simultaneous fulfillment of MFCQ and the well-known *constant rank* constraint qualification formulated below.

It is worth mentioning that SSOSC characterizes of tilt-stable minimizers of (4.1) under LICQ [16, Theorem 5.2]. Recently paper [14] has employed the combination of MFCQ and the *constant rank constraint qualification* (CRCQ), which is strictly weaker than LICQ to derive separately necessary conditions and sufficient conditions (but not characterizations) for tilt-stable minimizers in NLP. Recall that *CRCQ* holds at $\bar{x}$ if there is a neighborhood $\mathcal{W}$ of $\bar{x}$ such that the gradient system $\{\nabla\varphi_i(x)|\ i \in J\}$ has the same rank in $\mathcal{W}$ for any index $J \subset I(\bar{x})$; see, e.g., [5, 11, 14] and the references therein for historical remarks and recent developments. Note that $I(x) \subset I(\bar{x})$ provided that x is sufficiently close to $\bar{x}$, i.e., the CRCQ condition is robust.

Note further that MFCQ and CRCQ are independent in the sense that one can not imply another. It is proved in [14, Theorem 3.5] that *SSOSC* (4.11) is *sufficient* for tilt-stability of local minimizers of (4.1) under the simultaneous validity of MFCQ and CRCQ. Our next result shows that the new *USOSC* have a complete *characterization* of tilt-stable minimizers in the same setting.

Theorem 4.3 (USOSC characterization of tilt-stable minimizers under MFCQ and CRCQ). *Let $\bar{x}$ be a feasible solution to* (4.1) *satisfying* (4.4). *Assume that both MFCQ and CRCQ hold at $\bar{x}$. Then the following assertions are equivalent:*

(i) *The point $\bar{x}$ is a tilt-stable local minimizer of* (4.1) *with modulus $\kappa > 0$.*

(ii) *The USOSC from Definition* 4.1 *holds at $\bar{x}$ with modulus $\ell = \kappa^{-1}$.*

Proof. Let $\eta > 0$ be sufficiently small so that both MFCQ and CRCQ hold at all $x \in I\!B_\eta(\bar{x})$. Pick any $z \in \breve{\partial}^2 f(x,v)(w)$ with $(x,v) \in \operatorname{gph} \partial f \cap I\!B_\eta(\bar{x},0) = \operatorname{gph} \Psi \cap I\!B_\eta(\bar{x},0)$ and any $\lambda \in \Lambda(x,x^*)$. It follows from [13, Theorem 1.62] that

$$\breve{\partial}^2 f(x,v)(w) = \widehat{D}^*\big(\nabla\varphi_0 + N(\cdot;\Omega)\big)(x,v)(w) = \nabla^2\varphi_0(x)w + \widehat{D}^*N(\cdot;\Omega)\big(x, v - \nabla\varphi_0(x)\big)(w).$$

Employing this with the exact calculation of $\widehat{D}^*N(\cdot;\Omega)(x, v - \nabla\varphi_0(x))(w)$ given in [5, Theorem 6] ensures that $z \in \breve{\partial}^2 f(x,v)(w)$ if and only if

$$z - \nabla^2_{xx}L(x,\lambda)w \in K(x, v - \nabla\varphi_0(x))^* \quad\text{and}\quad -w \in K(x, v - \nabla\varphi_0(x)) \tag{4.16}$$

for some $\lambda \in \Lambda(x,v)$, where $K(x, v - \nabla\varphi_0(x)) := \widehat{N}(x;\Omega)^* \cap \{v - \nabla\varphi_0(x)\}^\perp$ for the *critical cone* to (4.2) with the notation $A^* := \big\{b \in I\!R^n \big|\ \langle b, a\rangle \le 0, a \in A\big\}$ standing for the (negative) *dual cone* of $A \subset I\!R^n$. It is well known that the assumed MFCQ ensures that

$$\widehat{N}(x;\Omega)^* = \big\{u \in I\!R^n \big|\ \langle \nabla\varphi_i(x), u\rangle \le 0,\ i \in I(x)\big\}.$$

Using this formula and the fact that $v - \nabla\varphi_0(\bar{x}) = \sum_{i\in I(x)} \lambda_i \nabla\varphi_i(x)$, we conclude that

$$-w \in K(x, v - \nabla\varphi_0(x)) \Longleftrightarrow \left\{\begin{array}{ll} \langle \nabla\varphi_i(x), w\rangle = 0 & \text{for } \ i \in I_+(x,\lambda), \\ \langle \nabla\varphi_i(x), w\rangle \ge 0 & \text{for } \ i \in I(x) \setminus I_+(x,\lambda). \end{array}\right. \tag{4.17}$$

If (ii) holds, we derive (i) from Theorem 3.5 by checking that (3.7) is satisfied due that $\langle z, w\rangle \ge \frac{1}{\kappa}\|w\|^2$. Indeed, note from (4.17) and the imposed USOSC that

$$\langle \nabla^2_{xx}L(x,\lambda)w, w\rangle \ge \frac{1}{\kappa}\|w\|^2. \tag{4.18}$$

Moreover, it follows from (4.16) and the definition of the dual cone that

$$\langle z - \nabla^2_{xx}L(x,\lambda)w, w\rangle \ge 0.$$

This together with (4.18) verifies that $\langle z, w\rangle \ge \frac{1}{\kappa}\|w\|^2$ thus ensuring (i).

Conversely, suppose that (i) is satisfied. Pick w with

$$\langle \nabla\varphi_i(x), w\rangle = 0 \ \text{ as } \ i \in I_+(x,\lambda) \ \text{ and } \ \langle \nabla\varphi_i(x), w\rangle \ge 0 \ \text{ as } \ i \in I(x) \setminus I_+(x,\lambda)$$

with any $\lambda \in \Lambda(x,v)$ and select $z := \nabla^2_{xx}L(x,\lambda)w + v - \nabla\varphi_0(x)$. Then observe from the definition of $K(x, v-\nabla\varphi_0(x))$ that $v - \nabla\varphi_0(x) \in K(x, v-\nabla\varphi_0(x))^*$. This implies together with (4.16) and (4.17) that $z \in \breve{\partial}^2 f(x,v)(w)$ and $\langle v - \nabla\varphi_0(x), w\rangle = 0$, which yields in turn by combining with (3.7) that

$$\langle \nabla^2_{xx}L(x,\lambda)w, w\rangle = \langle z, w\rangle - \langle v - \nabla\varphi_0(x), w\rangle = \langle z, w\rangle \geq \frac{1}{\kappa}\|w\|^2.$$

It ensures USOSC with modulus $\ell = \kappa^{-1}$ and thus completes the proof. △

Now we recover the aforementioned sufficient condition for tilt stability from [14, Theorem 3.5] derived by a different approach.

Corollary 4.4 (sufficiency of SSOSC for tilt stability under MFCQ and CRCQ). *Let $\bar{x} \in \Omega$ satisfy* (4.4)*, and let both MFCQ and CRCQ hold at $\bar{x}$. If SSOSC holds at $\bar{x}$, then $\bar{x}$ is a tilt-stable local minimizer of* (4.1).

Proof. We know from Proposition 4.2 that SSOSC at $\bar{x}$ implies the fulfillment of USOSC at this point with some modulus $\ell > 0$. Thus the result of the corollary is an immediate consequence of Theorem 4.3. △

The next example shows that SSOSC is *not necessary* for tilt stability of local minimizers under the validity of both MFCQ and CRCQ.

Example 4.5 (SSOSC is not a necessary condition for tilt stability under MFCQ and CRCQ). Consider the following nonlinear problem in $I\!R^3$:

$$\begin{cases} \text{minimize } \ \varphi_0(x) := x_3 + \frac{1}{4}x_1 + x_3^2 - x_1x_2 \ \text{ subject to} \\ \varphi_1(x) := x_1 - x_3 \leq 0 \\ \varphi_2(x) := -x_1 - x_3 \leq 0 \\ \varphi_3(x) := x_2 - x_3 \leq 0 \\ \varphi_4(x) := -x_2 - x_3 \leq 0, \\ x = (x_1, x_2, x_3) \in I\!R^3. \end{cases} \tag{4.19}$$

It is easy to check that both MFCQ and CRCQ hold at $\bar{x} = (0,0,0)$. Taking any vector $v = (v_1, v_2, v_3)$ with $|v_1| < \frac{1}{12}$ and $|v_2|, v_3 < \frac{1}{3}$ and writing the function f from (4.4) in this case, we have in this case $0 \leq \max\{|x_1|, |x_2|\} \leq x_3$ and

$$\begin{aligned} f(x) - \langle v, x\rangle &= x_3 + \frac{1}{4}x_1 - v_1x_1 - v_2x_2 - v_3x_3 + x_3^2 - x_1x_2 \\ &\geq \frac{1}{3}x_3 + \Big(\frac{1}{4} - v_1\Big)x_1 + \frac{1}{3}x_3 - v_2x_2 + \Big(\frac{1}{3} - v_3\Big)x_3 \\ &\geq \frac{1}{3}|x_1| + \Big(\frac{1}{4} - v_1\Big)x_1 + \frac{1}{3}|x_2| - v_2x_2 + \Big(\frac{1}{3} - v_3\Big)x_3 \\ &\geq \Big(\frac{1}{12} - |v_1|\Big)|x_1| + \Big(\frac{1}{3} - |v_2|\Big)|x_2| + \Big(\frac{1}{3} - v_3\Big)x_3 \geq 0, \end{aligned}$$

where the last equality holds due to the choice of v. It follows that $M_\gamma(v) = \{\bar{x}\}$ whenever v is sufficiently close to $0 \in I\!R^3$. Thus $\bar{x}$ is a tilt-stable local minimizer of program (4.19), and we only need to check that SSOSC does not hold at $\bar{x}$. It is easy to see that $\big(\frac{3}{8}, \frac{5}{8}, 0, 0\big) \in \Lambda(\bar{x})$, and so $w = (0,1,0) \neq 0$ satisfies the equation

$$\langle \nabla\varphi_i(\bar{x}), w\rangle = 0 \ \text{ for } \ i \in I_+(\bar{x}, \lambda) = \{1, 2\}.$$

At the same time $\langle w, \nabla^2_{xx}L(\bar{x},\lambda)w\rangle = 0$, which shows that SSOSC does not hold at $\bar{x}$.

Note that the generalized equation/KKT system associated with problem (4.19) is *not strongly regular* in the sense of Robinson [20] at the tilt-stable minimizer $\bar{x}$ and the corresponding Lagrange multiplier in Example 4.5 since the converse assertion yields LICQ and thus contradicts [3, Theorem 6]. Observe also that we do *not* have *strong stability* in the sense of Kojima [8] in this example. Indeed, it has been well recognized (see the original version in [8, Theorem 7.2] and the improved one in [1, Proposition 5.37] with the references therein) that strong stability of NLP can be characterized, under the validity of MFCQ, via a uniform quadratic growth condition equivalent in this case to SSOSC. As shown in Example 4.5, SSOSC does not hold at the tilt-stable minimizer $\bar{x}$ in problem (4.19) while MFCQ is satisfied. Thus strong stability fails in this setting.

5 Concluding Remarks

This paper presents a systematic and largely self-contained study of the important concept of tilt stability for local minimizers in general optimization problems via advanced tools of second-order variational analysis and generalized differentiation. We develop a new approach to tilt stability and establish its qualitative and quantitative characterizations in finite-dimensional spaces. The applications are given to classical nonlinear programs with twice continuously differentiable data, where the developed approach allows us to derive new characterizations of tilt stability via both conventional and novel optimality and qualification conditions expressed entirely in terms of the initial data.

The call for further research is to implement this approach in the study of other remarkable classes of problems in mathematical programming of theoretical and practical interest. Among them are problems of *conic programming* for which some results have been recently obtained in [15] for programs with $\mathcal{C}^2$ data under the conventional nondegeneracy condition ensuring the uniqueness of the corresponding Lagrange multipliers similarly to the classical LICQ in nonlinear programming. We feel that our approach is able to relax this rather restrictive assumption and to obtain new characterizations of tilt stability in conic programming. Another important topic of further research is *full stability* of optimal solutions in the sense of [9], which is a far-going extension of tilt stability. After initial developments in [9] in the extended-real-valued framework of finite-dimensional optimization, recent advances have been given in [17] for some classes of mathematical programs with polyhedral structures. The approach presented in this paper seems to be promising for further developments and applications of full stability in optimization.

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