

# Multiperiod Portfolio Optimization with General Transaction Costs

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We analyze the properties of the optimal portfolio policy for a multiperiod mean-variance investor facing multiple risky assets in the presence of general transaction costs such as proportional, market impact, and quadratic transaction costs. For proportional transaction costs, we find that a buy-and-hold policy is optimal: if the starting portfolio is outside a parallelogram-shaped no-trade region, then trade to the boundary of the no-trade region at the first period, and hold this portfolio thereafter. For market impact costs, we show that the optimal portfolio policy at each period is to trade to the boundary of a state-dependent rebalancing region. Moreover, we find that the rebalancing region shrinks along the investment horizon, and as a result the investor trades throughout the entire investment horizon. Finally, we show numerically that the utility loss associated with ignoring transaction costs or investing myopically may be large.

*Key words:* Portfolio optimization; Multiperiod utility; No-trade region; Market impact

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## 1. Introduction

Markowitz (1952) showed that an investor who only cares about the mean and variance of single-period portfolio returns should choose a portfolio on the efficient frontier. Two assumptions underlying Markowitz's analysis are that the investor only cares about single-period returns and that she is not subject to transaction costs. In this manuscript we consider the case where these two assumptions fail to hold, and study the optimal portfolio policy of a multiperiod mean-variance investor subject to general transaction costs, including proportional, market impact, and quadratic transaction costs.

There is an extensive literature that generalizes Markowitz's work to the multiperiod case. Mossin (1968), Samuelson (1969), and Merton (1969, 1971) show how an investor should optimally choose her portfolio in a dynamic environment in the absence of transaction costs. In practice, however, implementing a dynamic portfolio policy requires one to rebalance the portfolio weights frequently, and this may result in high

transaction costs. To address this issue, researchers have tried to characterize the optimal portfolio policies in the presence of transaction costs.

The case with a single-risky asset and proportional transaction costs is well understood. In a multiperiod setting, Constantinides (1979) shows that the optimal trading policy is characterized by a no-trade interval, such that if the risky-asset portfolio weight is inside this interval, then it is optimal not to trade, and if the portfolio weight is outside, then it is optimal to trade to the boundary of this interval. Later, Constantinides (1986) and Davis and Norman (1990) extend this result to a continuous-time setting with a single risky asset.

The case with multiple risky assets is much harder to characterize, and the existent literature is scarce. Akian et al. (1996) consider a multiple risky-asset version of the continuous time model by Davis and Norman (1990), and using *simulations* they suggest that the optimal portfolio policy is characterized by a multi-dimensional no-trade region. Leland (2000) develops a relatively simple *numerical* procedure to compute the no-trade region based on the existence results of Akian et al. (1996). The only paper that provides an analytical characterization of the no-trade region for the case with multiple risky assets is the one by Liu (2004), who shows that under the assumption that asset returns are *uncorrelated*, the optimal portfolio policy is characterized by a separate no-trade interval for each risky asset.<sup>1</sup>

None of the aforementioned papers characterizes analytically the no-trade region for the general case with multiple risky assets with correlated returns and transaction costs. The reason for this is that the analysis in these papers relies on modeling the *asset return* distribution, and as a result they must take portfolio growth into account, which renders the problem intractable analytically. Recently, Garleanu and Pedersen (2012) consider a setting that relies on modeling *price changes*, and thus they are able to give closed-form expressions for the optimal dynamic portfolio policies in the presence of *quadratic* transaction cost. Arguably, modeling price changes is not very different from modeling stock returns, at least for daily or higher trading frequencies, yet the former approach renders the problem tractable.

We make three contributions. Our first contribution is to use the multiperiod framework proposed by Garleanu and Pedersen (2012) to characterize the optimal portfolio policy for the general case with multiple risky assets and proportional transaction costs. Specifically, we characterize analytically that there exists a no-trade region, shaped as a parallelogram, such that if the starting portfolio is inside the no-trade region, then it is optimal not to trade at any period. If, on the other hand, the starting portfolio is outside the no-trade region, then it is optimal to trade to the boundary of the no-trade region in the first period, and not to trade thereafter. Furthermore, we show that the size of the no-trade region grows with the level of proportional

<sup>1</sup> Other papers provide *numerical solutions* to multiperiod investment problems in the presence of transaction costs, see for instance Muthuraman and Kumar (2006), Lynch and Tan (2010), and Brown and Smith (2011).

transaction costs and the discount factor, and shrinks with the investment horizon and the risk-aversion parameter.

Our second contribution is to study analytically the optimal portfolio policy in the presence of market impact costs, which arise when the investor makes large trades that distort market prices.<sup>2</sup> Traditionally, researchers have assumed that the market *price* impact is linear on the amount traded (see Kyle (1985)), and thus that market impact costs are quadratic. Under this assumption, Garleanu and Pedersen (2012) derive closed-form expressions for the optimal portfolio policy within their multiperiod setting. However, Torre and Ferrari (1997), Grinold and Kahn (2000), and Almgren et al. (2005) show that the square root function is more appropriate for modeling market price impact, thus suggesting market impact costs grow at a rate slower than quadratic. Our contribution is to extend the analysis by Garleanu and Pedersen (2012) to a *general case* where we are able to capture the distortions on market price through a power function with an exponent between one and two. For this general formulation, we show that there exists an analytical *rebalancing region* for every time period, such that the optimal policy at each period is to trade to the boundary of the corresponding rebalancing region. Moreover, we find that the rebalancing regions shrink throughout the investment horizon, which means that, unlike with proportional transaction costs, it is optimal for the investor to trade at every period when she faces market impact costs.

Finally, our third contribution is to study numerically the utility losses associated with ignoring transaction costs and investing myopically, as well as how these utility losses depend on relevant parameters. We find that the losses associated with either ignoring transaction costs or behaving myopically can be large. Moreover, the losses from ignoring transaction costs increase in the level of transaction costs, and decrease with the investment horizon, whereas the losses from behaving myopically increase with the investment horizon and are concave unimodal on the level of transaction costs.

Our work is related to Dybvig (2005), who considers a single-period setting with mean-variance utility and proportional transaction costs. For the case with multiple risky assets, he shows that the optimal portfolio policy is characterized by a no-trade region shaped as a parallelogram, but the manuscript does not provide a detailed analytical proof. Like Dybvig (2005), we consider proportional transaction costs and mean-variance utility, but we extend the results to a multi-period setting, and show how the results can be rigorously proven analytically. In addition, we consider the case with market impact costs.

This manuscript is organized as follows. Section 2 describes the multiperiod framework under general transaction costs. Section 3 studies the case with proportional transaction costs, Section 4 the case with market impact costs, and Section 5 the case with quadratic transaction costs. Section 6 characterizes numer-

<sup>2</sup>This is particularly relevant for optimal execution, where institutional investors have to execute an investment decision within a fixed time interval; see Bertsimas and Lo (1998) and Engle et al. (2012)

ically the utility loss associated with ignoring transaction costs, and with behaving myopically. Section 7 concludes. Appendix A contains the figures, and Appendix B contains the proofs for all results in the paper.

## 2. General Framework

Our framework is closely related to that proposed by Garleanu and Pedersen (2012); herein G&P. Like G&P, we consider a multiperiod setting, where the investor tries to maximize her discounted mean-variance utility net of transaction costs by choosing the number of shares to hold of each of the  $N$  risky assets. There are three main differences between our model and the model by G&P. First, we consider a more general class of transaction costs that includes not only quadratic transaction costs, but also proportional and market impact costs. Second, we assume price changes in excess of the risk-free rate are independent and identically distributed with mean  $\mu$  and covariance matrix  $\Sigma$ , while G&P consider the more general case in which these price changes are predictable. Third, we consider both finite and infinite investment horizons, whereas G&P focus on the infinite horizon case.

The investor's decision in our framework can be written as:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1-\rho)^t (x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t) - (1-\rho)^{t-1} \kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p \right], \quad (1)$$

where  $x_t \in \mathbb{R}^N$  contains the number of shares of each of the  $N$  risky assets held in period  $t$ ,  $T$  is the investment horizon,  $\rho$  is the discount factor, and  $\gamma$  is the risk-aversion parameter. The term  $\kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p$  is the transaction cost for the  $t$ th period, where  $\kappa \in \mathbb{R}$  is the transaction cost parameter,  $\Lambda \in \mathbb{R}^{N \times N}$  is the transaction cost matrix, and  $\|s\|_p$  is the  $p$ -norm of vector  $s$ ; that is,  $\|s\|_p^p = \sum_{i=1}^N |s_i|^p$ . The transaction cost matrix  $\Lambda$  is a symmetric positive-definite matrix that captures the distortions to asset prices originated by the interaction between the multiple assets. G&P argue that it can be viewed as a multi-dimensional version of Kyle's lambda, see Kyle (1985).

For small trades, which do not impact market prices, the transaction cost is usually assumed to be proportional to the amount traded ( $p = 1$ ), and the natural choice for the transaction cost matrix is  $\Lambda = I$ , where  $I$  is the identity matrix. For larger trades, which result in market impact costs, the literature suggests that the transaction costs grow as a power function with an exponent  $p \in (1, 2]$ . Moreover, G&P consider the case with quadratic transaction costs ( $p = 2$ ) and argue that a sensible choice for the transaction cost matrix is  $\Lambda = \Sigma$ . We consider this case as well as the case with  $\Lambda = I$  to facilitate the comparison with the case with proportional costs.<sup>3</sup>

Our analysis relies on the following assumption.

<sup>3</sup> To simplify the exposition, we focus on the case where the transaction costs associated with trading all  $N$  risky assets are symmetric. It is straightforward, however, to extend our results to the asymmetry case, where the transaction costs associated with trading the  $j$ th asset at the  $t$ th period is:  $\kappa_j |(\Lambda^{1/p} (x_t - x_{t-1}))_j|^p$ .

ASSUMPTION 1. *Price changes in excess of the risk-free rate are independently and identically distributed with mean vector  $\mu$  and covariance matrix  $\Sigma$ .*

### 3. Proportional Transaction Costs

In this section we consider the case where transaction costs are proportional to the amount traded ( $p = 1$ ). These so-called proportional transaction costs are appropriate to model the cost associated with trades that are small, and thus the transaction cost originates from the bid-ask spread and other brokerage commissions. For exposition purposes, we first study the single-period case, and show that for this case the optimal portfolio policy is *analytically characterized* by a no-trade region shaped as a parallelogram.<sup>4</sup> We then study the general multiperiod case, and again show that there is a no-trade region shaped as a parallelogram. Moreover, if the starting portfolio is inside the no-trade region, then it is optimal not to trade at any period. If, on the other hand, the starting portfolio is outside the no-trade region, then it is optimal to trade to the boundary of the no-trade region in the first period, and not to trade thereafter. Furthermore, we study how the no-trade region depends on the level of proportional transaction costs, the correlation in asset returns, the discount factor, the investment horizon, and the risk-aversion parameter.

#### 3.1. The Single-Period Case

For the single-period case, the investor's decision is

$$\max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - \kappa \|x - x_0\|_1, \quad (2)$$

where  $x_0$  is the starting portfolio. Because the term  $\|x - x_0\|_1$  is not differentiable, it is not possible to obtain closed-form expressions for the optimal portfolio policy for the case with proportional transaction costs. The following proposition, however, demonstrates that the optimal trading policy is characterized by a no-trade region shaped as a parallelogram.

PROPOSITION 1. *Let Assumption 1 hold, then:*

1. *The investor's decision problem (2) can be equivalently rewritten as:*

$$\min_x (x - x_0)^\top \Sigma (x - x_0), \quad (3)$$

$$s.t. \quad \|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma}, \quad (4)$$

where  $x^* = \Sigma^{-1}\mu/\gamma$  is the optimal portfolio in the absence of transaction costs (the Markowitz or target portfolio), and  $\|s\|_\infty = \max_i \{|s_i|\}$  is the infinity norm of  $s$ .

<sup>4</sup> Our analysis for the single-period case with proportional transaction costs is similar to that by Dybvig (2005), but we provide a detailed analytical proof.

2. Constraint (4) defines a no-trade region shaped as a parallelogram centered at the target portfolio  $x^*$ , such that if the starting portfolio  $x_0$  is inside this region, then it is optimal not to trade, and if the starting portfolio is outside this no-trade region, then it is optimal to trade to the point in the boundary of the no-trade region that minimizes the objective function in (3).

Proposition 1 shows that it is optimal *not* to trade if the marginal mean-variance utility from trading on any of the  $N$  risky assets is smaller than the transaction cost parameter  $\kappa$ . To see this, note that inequality (4), which defines the no-trade region, can be equivalently rewritten as

$$\|(1 - \rho)\gamma\Sigma(x - x^*)\|_\infty \leq \kappa. \quad (5)$$

Moreover, it is easy to show that the vector  $(1 - \rho)\gamma\Sigma(x - x^*)$  gives the marginal mean-variance utility from trading in each of the assets, because it is the gradient (first derivative) of the static mean-variance utility  $(1 - \rho)(x^\top \mu - \frac{\gamma}{2}x^\top \Sigma x)$  with respect to the portfolio  $x$ .

For a two-asset example, Figure 1 depicts the parallelogram-shaped no-trade region defined by inequality (4), together with the level sets for the objective function given by (3). The optimal portfolio policy is to trade to the intersection between the no-trade region and the tangent level set, at which the marginal utility from trading is equal to the transaction cost parameter  $\kappa$ .

It is easy to see that the size of no-trade region defined by Equation (4) decreases with the risk aversion parameter  $\gamma$ . Intuitively, the more risk averse the investor, the larger her incentives to trade and diversify her portfolio. Also, it is clear that the size of the no-trade region increases with the proportional transaction parameter  $\kappa$ . This makes sense intuitively because the larger the transaction cost parameter, the less attractive to the investor is to trade in order to move closer to the target portfolio. Moreover, the following proposition shows that there exists a finite transaction cost parameter  $\kappa^*$  such that if the transaction cost parameter  $\kappa > \kappa^*$ , then it is optimal not to trade.

**PROPOSITION 2.** *The no-trade region is unbounded when  $\kappa \geq \kappa^*$ , where  $\kappa^* = \|\phi\|_\infty$  and  $\phi$  is the vector of Lagrange multipliers associated with the constraint at the unique maximizer for the following optimization problem:*

$$\max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2}x^\top \Sigma x), \quad (6)$$

$$s.t. \quad x - x_0 = 0. \quad (7)$$

### 3.2. The Multiperiod Case

In this section, we show that similar to the single-period case, the optimal portfolio policy for the multiperiod case is also characterized by a no-trade region shaped as a parallelogram and centered around the target

portfolio. If the starting portfolio at the first period is inside this no-trade region, then it is optimal not to trade at any period. If, on the other hand, the starting portfolio at the first period is outside this no-trade region, then it is optimal to trade to the boundary of the no-trade region in the first period, and not to trade thereafter.

The investor's decision for this case can be written as:

$$\max_{\{x_t\}_{t=1}^T} \left\{ \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \right\}. \quad (8)$$

The following theorem demonstrates that the optimal trading policy is characterized by a no-trade region shaped as a parallelogram.

**THEOREM 1.** *Let Assumption 1 hold, then:*

1. *It is optimal not to trade at any period other than the first period; that is,*

$$x_1 = x_2 = \dots = x_T. \quad (9)$$

2. *The investor's optimal portfolio for the first period  $x_1$  (and thus for all subsequent periods) is the solution to the following constrained optimization problem:*

$$\min_{x_1} (x_1 - x_0)^\top \Sigma (x_1 - x_0), \quad (10)$$

$$\text{s.t.} \quad \|\Sigma(x_1 - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T}. \quad (11)$$

where  $x_0$  is the starting portfolio, and  $x^* = \Sigma^{-1}\mu/\gamma$  is the optimal portfolio in the absence of transaction costs (the Markowitz or target portfolio).

3. *Constraint (11) defines a no-trade region shaped as a parallelogram centered at the target portfolio  $x^*$ , such that if the starting portfolio  $x_0$  is inside this region, then it is optimal not to trade at any period, and if the starting portfolio is outside this no-trade region, then it is optimal to trade at the first period to the point in the boundary of the no-trade region that minimizes the objective function in (10), and not to trade thereafter.*

Similar to the single-period case, Theorem 1 shows that it is optimal to trade only if the marginal discounted multiperiod mean-variance utility from trading in one of the assets is larger than the transaction cost parameter  $\kappa$ . To see this, note that inequality (11), which gives the no-trade region, can be rewritten as

$$\|(\gamma(1-\rho)(1 - (1-\rho)^T)/\rho)\Sigma(x_1 - x^*)\|_\infty \leq \kappa. \quad (12)$$

Moreover, because it is only optimal to trade at the first period, the vector inside the infinite norm on the left hand-side of inequality 12 is the gradient (first derivative) of the discounted multiperiod mean variance utility  $\sum_{t=1}^T (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right)$  with respect to the portfolio  $x_t$ .

The following corollary establishes how the size of the no-trade region for the multiperiod case depends on the problem parameters.

COROLLARY 1. *The no-trade region for the multiperiod investor satisfies the following properties:*

- *The no-trade region expands as the proportional transaction parameter  $\kappa$  increases.*
- *The no-trade region expands as the discount factor parameter  $\rho$  increases.*
- *The no-trade region shrinks as the investment horizon  $T$  increases.*
- *The no-trade region shrinks as the risk-aversion parameter  $\gamma$  increases.*

Similar to the single-period case, we observe that the size of the no-trade region grows with the transaction cost parameter  $\kappa$ . This is intuitive as the larger the transaction costs, the less willing the investor is to trade in order to diversify. This is illustrated in Panel (a) of Figure 2, which depicts the no-trade regions for different values of the transaction cost parameter  $\kappa$  for a case with two stocks. In addition, it is easy to show following the same argument used to prove Proposition 2 that there is a  $\kappa^*$  such that the no-trade region is unbounded for  $\kappa \geq \kappa^*$ .

The size of the no-trade region increases with the discount factor  $\rho$ . Again, this makes sense intuitively because the larger the discount factor, the less important the utility for future periods and thus the smaller the incentive to trade today. This is illustrated in Figure 2, Panel (b). The size of the no-trade region decreases with the investment horizon  $T$ . To see this intuitively, note that we have shown that the optimal policy is to trade at the first period and hold this position thereafter. Then, a multiperiod investor with shorter investment horizon will be more concerned about the transaction costs incurred at the first stage, compared with the investor who has a longer investment horizon. Finally, when  $T \rightarrow \infty$ , the no-trade region shrinks to the parallelogram bounded by  $\kappa\rho/((1-\rho)\gamma)$ , which is much closer to the center  $x^*$ . Of course, when  $T = 1$ , the multiperiod problem reduces to the static case. This is illustrated in Figure 2, Panel (c).

In addition, the no-trade region shrinks as the risk aversion parameter  $\gamma$  increases. Intuitively, as the investor becomes more risk averse, the optimal policy is to move closer to the diversified (safe) position  $x^*$ , despite the transaction costs associated with this. This is illustrated in Figure 2, Panel (d), which also shows that the target portfolio changes with the risk-aversion parameter, and therefore the no-trade regions are centered at different points for different risk-aversion parameters. The no-trade region also depends on the correlation between assets. Figure 2, Panel (e) shows the no-trade regions for different correlations. When two assets are positively correlated, the parallelogram leans to the left, because the advantages from diversification are small, whereas with negative correlation it leans to the right. In the absence of correlations the no-trade region becomes a rectangle.



## 4. Market Impact Costs

In this section we consider market impact costs, which arise when the investor makes large trades that distort market prices. Traditionally, researchers have assumed that the market *price* impact is linear on the amount traded (see Kyle (1985)), and thus that market impact costs are quadratic. Under this assumption, Garleanu and Pedersen (2012) derive closed-form expressions for the optimal portfolio policy within their multiperiod setting. However, Torre and Ferrari (1997), Grinold and Kahn (2000), Almgren et al. (2005), and Gatheral (2010) show that the square root function may be more appropriate for modelling market price impact, thus suggesting market impact costs grow at a rate slower than quadratic. Therefore in this section we consider a *general case*, where the transaction costs are given by the  $p$ -norm with  $p \in (1, 2)$ , and where we capture the distortions on market price through the transaction cost matrix  $\Lambda$ . For exposition purposes, we first study the single-period case.

### 4.1. The Single-Period Case

For the single-period case, the investor's decision is:

$$\max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - \kappa \|\Lambda^{1/p}(x - x_0)\|_p^p, \quad (13)$$

where  $1 < p < 2$ . Problem (13) can be solved numerically, but unfortunately it is not possible to obtain closed-form expressions for the optimal portfolio policy. The following proposition, however, shows that the optimal portfolio policy is to trade to the boundary of a *rebalancing region* that depends on the starting portfolio and contains the target or Markowitz portfolio.

**PROPOSITION 3.** *Let Assumption 1 hold, then if the starting portfolio  $x_0$  is equal to the target or Markowitz portfolio  $x^*$ , the optimal policy is not to trade. Otherwise, it is optimal to trade to the boundary of the following rebalancing region:*

$$\frac{\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q}{p\|\Lambda^{1/p}(x - x_0)\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}, \quad (14)$$

where  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

Comparing Propositions 1 and 3, we identify three main differences between the cases with proportional and market impact costs. First, for the case with market impact costs it is always optimal to trade (except in the trivial case where the starting portfolio coincides with the target or Markowitz portfolio), whereas for the case with proportional transaction costs it may be optimal not to trade if the starting portfolio is inside the no-trade region. Second, the rebalancing region depends on the starting portfolio  $x_0$ , whereas the no-trade region is independent of it. Third, the rebalancing region contains the target or Markowitz portfolio, but it is not centered around it, whereas the no-trade region is centered around the Markowitz portfolio.

Note that, as in the case with proportional transaction costs, the size of the rebalancing region increases with the transaction cost parameter  $\kappa$ , and decreases with the risk-aversion parameter. Intuitively, the more risk averse the investor, the larger her incentives to trade and diversify her portfolio. Also, the rebalancing region grows with  $\kappa$  because the larger the transaction cost parameter, the less attractive to the investor is to trade to move closer to the target portfolio.

The following corollary gives the rebalancing region for two important particular cases. First, the case where the transaction cost matrix  $\Lambda = I$ , which is a realistic assumption when the amount traded is small, and thus the interaction between different assets, in terms of market impact, is small. This case also facilitates the comparison with the optimal portfolio policy for the case with proportional transaction costs. The second case corresponds to the transaction cost matrix  $\Lambda = \Sigma$ , in analogy with the analysis by G&P in the context of quadratic transaction costs.

**COROLLARY 2.** *For the single-period investor defined in (13):*

1. *When the transaction cost matrix is  $\Lambda = I$ , then the rebalancing region is*

$$\frac{\|\Sigma(x - x^*)\|_q}{p\|x - x_0\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (15)$$

2. *When the transaction cost matrix is  $\Lambda = \Sigma$ , then the rebalancing region is*

$$\frac{\|\Sigma^{1/q}(x - x^*)\|_q}{p\|\Sigma^{1/p}(x - x_0)\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (16)$$

Note that, in both particular cases, the Markowitz strategy  $x^*$  is contained in the rebalancing region.

To gain intuition about the form of the rebalancing regions characterized in (15) and (16), Panel (a) in Figure 3 depicts the rebalancing region and the optimal portfolio policy for a two-asset example when  $\Lambda = I$ , while Panel (b) depicts the corresponding rebalancing region and optimal portfolio policy when  $\Lambda = \Sigma$ . The figure shows that, in both cases, the rebalancing region is a convex region containing the Markowitz portfolio. Moreover, it shows how the optimal trading strategy moves to the boundary of the rebalancing region.

## 4.2. The Multiperiod Case

The investor's decision for this case can be written as:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1 - \rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p \right]. \quad (17)$$

As in the single-period case, it is not possible to provide closed-form expressions for the optimal portfolio policy, but the following theorem illustrates the analytical properties of the optimal portfolio policy.

**THEOREM 2.** *Let Assumption 1 hold, then:*

1. *If the starting portfolio  $x_0$  is equal to the target or Markowitz portfolio  $x^*$ , then the optimal policy is not to trade at any period.*
2. *Otherwise it is optimal to trade at every period. Moreover, at the  $t$ th period it is optimal to trade to the boundary of the following rebalancing region:*

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Lambda^{-1/p} \Sigma (x_s - x^*)\|_q}{p \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}, \quad (18)$$

where  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

Theorem 2 shows that for the multiperiod case with market impact costs it is optimal to trade at every period (except in the trivial case where the starting portfolio coincides with the Markowitz portfolio). Moreover, at every period it is optimal to trade to the boundary of a rebalancing region that depends not only on the starting portfolio, but also on the portfolio for every subsequent period. Finally, note that the size of the rebalancing region for period  $t$ , assuming the portfolios for the rest of the periods are fixed, increases with the transaction cost parameter  $\kappa$  and decreases with the discount factor  $\rho$  and the risk-aversion parameter  $\gamma$ .

The following proposition shows that the rebalancing region for period  $t$  contains the rebalancing region for every subsequent period. Moreover, the rebalancing region converges to the Markowitz portfolio as the investment horizon grows, and thus the optimal portfolio  $x_T$  converges to the target portfolio  $x^*$  in the limit when  $T$  goes to infinity.

**PROPOSITION 4.** *Let Assumption 1 hold, then:*

1. *The rebalancing region for the  $t$ -th period contains the rebalancing region for every subsequent period,*
2. *Every rebalancing region contains the Markowitz portfolio,*
3. *The rebalancing region converges to the Markowitz portfolio in the limit when the investment horizon goes to infinity.*

The next corollary gives the rebalancing region for the two particular cases of transaction cost matrix we consider.

**COROLLARY 3.** *For the multiperiod investor defined in (17):*

1. *When the transaction cost matrix is  $\Lambda = I$ , then the rebalancing region is*

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Sigma (x_s - x^*)\|_q}{p \|x_t - x_{t-1}\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (19)$$

2. *When the transaction cost matrix is  $\lambda = \Sigma$ , then the rebalancing region is*

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Sigma^{1/q} (x_s - x^*)\|_q}{p \|\Sigma^{1/p} (x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (20)$$

To gain intuition about the shape of the rebalancing regions characterized in (19) and (20), Panel (a) in Figure 4 shows the optimal portfolio policy and the rebalancing regions for a two-asset example with an investment horizon  $T = 3$  when  $\Lambda = I$ , whereas Panel (b) depicts the corresponding optimal portfolio policy and rebalancing regions when  $\Lambda = \Sigma$ . The figure shows, in both cases, how the rebalancing region for each period contains the rebalancing region for subsequent periods. Moreover, every rebalancing region contains, but is not centered at, the Markowitz portfolio  $x^*$ . In particular, for each stage, any trade is to the boundary of the rebalancing region and the rebalancing is towards the Markowitz strategy  $x^*$ . Note also that the figure shows that, for the case with  $\Lambda = I$ , it is optimal for the investor to buy the second asset in the first period, and then sell it in the second period. This may appear suboptimal from the point of view of market impact costs, but it turns out to be optimal when the investor considers the trade off between multiperiod mean-variance utility and market impact costs.

Finally, we study numerically the impact of the market impact cost growth rate  $p$  on the optimal portfolio policy. Figure 5 shows the rebalancing regions and trading trajectories for investors with different transaction growth rates  $p = 1, 1.25, 1.5, 1.75, 2$ . When the transaction cost matrix  $\Lambda = I$ , Panel (a) shows how the rebalancing region depends on  $p$ . In particular, for  $p = 1$  we recover the case with proportional transaction costs, and hence the rebalancing region becomes a parallelogram-shaped no-trade region. For  $p = 2$ , the rebalancing region becomes an ellipse. And for values of  $p$  between 1 and 2, the rebalancing regions become superellipses but not centered at the target portfolio  $x^*$ . On the other hand, Panel (b) in Figure 5 shows how the trading trajectories depend on  $p$  for a particular investment horizon of  $T = 10$  days. We observe that, as  $p$  grows, the trading trajectories become less curved and the investor converges towards the target portfolio at a slower rate. We have also considered the case  $\Lambda = \Sigma$ , but the insights are similar and thus we do not include the figure.

## 5. Quadratic Transaction Costs

We now consider the case with quadratic transaction costs. The investor's decision is:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|\Lambda^{1/2} (x_t - x_{t-1})\|_2^2 \right]. \quad (21)$$

As explained in Section 2, our general framework differs from that considered by G&P in three respects: (i) G&P assume price changes are predictable, whereas we assume price changes are iid, (ii) G&P consider an infinite horizon, whereas we allow for a finite investment horizon, and (iii) we consider a general transaction costs matrix  $\Lambda$  whereas G&P focus on the particular case  $\Lambda = \Sigma$ .

The next theorem provides an extension of the results in G&P to obtain an explicit characterization of the optimal portfolio policy.

THEOREM 3. *Let Assumption 1 hold, then:*

1. *The optimal portfolio  $x_t, x_{t+1}, \dots, x_{t+T-1}$  satisfies the following equations:*

$$x_t = A_1 x^* + A_2 x_{t-1} + A_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (22)$$

$$x_t = B_1 x^* + B_2 x_{t-1}, \quad \text{for } t = T. \quad (23)$$

where

$$A_1 = (1 - \rho)\gamma [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1}\Sigma,$$

$$A_2 = 2\kappa [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1}\Lambda,$$

$$A_3 = 2(1 - \rho)\kappa [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1}\Lambda,$$

with  $A_1 + A_2 + A_3 = I$ , and

$$B_1 = (1 - \rho)\gamma [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda]^{-1}\Sigma,$$

$$B_2 = 2\kappa [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda]^{-1}\Lambda,$$

with  $B_1 + B_2 = I$ .

2. *The optimal portfolio converges to the Markowitz portfolio as the investment horizon  $T$  goes to infinity.*

Theorem 3 shows that the optimal portfolio for each stage is a combination of the Markowitz strategy (the target portfolio), the previous period portfolio, and the next period portfolio.

The next corollary shows the specific optimal portfolios for two particular cases of transaction cost matrix. Like G&P, we consider the case where the transaction costs matrix is proportional to the covariance matrix, that is  $\Lambda = \Sigma$ . In addition, we also consider the case where the transaction costs matrix is proportional to the identity matrix; that is  $\Lambda = I$ .

COROLLARY 4. *For a multiperiod investor with objective function (21):*

1. *When the transaction cost matrix is  $\Lambda = I$ , then the optimal trading strategy satisfies*

$$x_t = A_1 x^* + A_2 x_{t-1} + A_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (24)$$

$$x_t = B_1 x^* + B_2 x_{t-1}, \quad \text{for } t = T \quad (25)$$

where

$$A_1 = (1 - \rho)\gamma [(1 - \rho)\gamma\Sigma + 2\kappa I + 2(1 - \rho)\kappa I]^{-1}\Sigma,$$

$$A_2 = 2\kappa [(1 - \rho)\gamma\Sigma + 2\kappa I + 2(1 - \rho)\kappa I]^{-1},$$

$$A_3 = 2(1 - \rho)\kappa [(1 - \rho)\gamma\Sigma + 2\kappa I + 2(1 - \rho)\kappa I]^{-1},$$

with  $A_1 + A_2 + A_3 = I$ , and

$$B_1 = (1 - \rho)\gamma[(1 - \rho)\gamma\Sigma + 2\kappa I]^{-1}\Sigma,$$

$$B_2 = 2\kappa[(1 - \rho)\gamma\Sigma + 2\kappa I]^{-1}.$$

with  $B_1 + B_2 = I$ .

2. When the transaction cost matrix is  $\Lambda = \Sigma$ , then the optimal trading strategy satisfies

$$x_t = \alpha_1 x^* + \alpha_2 x_{t-1} + \alpha_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (26)$$

$$x_t = \beta_1 x^* + \beta_2 x_{t-1}, \quad \text{for } t = T. \quad (27)$$

where  $\alpha_1 = (1 - \rho)\gamma/((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$ ,  $\alpha_2 = 2\kappa/((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$ ,  $\alpha_3 = 2(1 - \rho)\kappa/((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$  with  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ , and  $\beta_1 = (1 - \rho)\gamma/((1 - \rho)\gamma + 2\kappa)$ ,  $\beta_2 = 2\kappa/((1 - \rho)\gamma + 2\kappa)$  with  $\beta_1 + \beta_2 = 1$ .

3. When the transaction cost matrix is  $\Lambda = \Sigma$ , then the optimal portfolios for periods  $t = 1, 2, \dots, T$  lay on a straight line.

Corollary 4 shows that, when  $\Lambda = \Sigma$ , the solution becomes simpler and easier to interpret than when  $\Lambda = I$ . Note that when  $\Lambda = \Sigma$ , matrices  $A$  and  $B$  in Theorem 3 become scalars  $\alpha$  and  $\beta$ , respectively, and hence the optimal portfolio at period  $t$  can be expressed as a linear combination of the Markowitz portfolio, the previous period portfolio and the next period portfolio. For this reason, it is intuitive to observe that the optimal trading strategies for all the periods must lay on a straight line.

To conclude this section, Figure 6 provides a comparison of the optimal portfolio policy for the case with quadratic transaction costs (when  $\Lambda = \Sigma$ ), with those for the cases with proportional and market impact costs (when  $\Lambda = I$ ), for a multiperiod investor with  $T = 4$ . We have also considered other transaction cost matrices, but the insights are similar. The figure confirms that, for the case with quadratic transaction costs, the optimal portfolio policy is to trade at every period along a straight line that converges to the Markowitz portfolio. It can also be appreciated that the investor trades more aggressively at the first periods compared to the final periods. For the case with proportional transaction costs, it is optimal to trade to the boundary of the no-trade region shaped as a parallelogram in the first period and not to trade thereafter. Finally, for the case with market impact costs, the investor trades at every period to the boundary of the corresponding rebalancing region. The resulting trajectory is not a straight line. In fact, as in the example discussed in Section 4, it is optimal for the investor to buy the second asset in the first period and then sell it in the second period; that is, the optimal portfolio policy is inefficient in terms of market impact costs, but it is optimal in terms of the tradeoff between market impact costs and discounted utility.

## 6. Numerical Analysis

In this section, we study numerically the utility loss associated with ignoring transaction costs and investing myopically, as well as how these utility losses depend on the transaction cost parameter, the price-change correlation, the investment horizon, and the risk-aversion parameter. We first consider the case with proportional transaction costs, and then study how the monotonicity properties of the utility losses change when transaction costs are quadratic. We have also considered the case with market impact costs ( $p = 1.5$ ), but the insights are similar to those from the case with quadratic transaction costs and thus we do not report the results to conserve space.

For each type of transaction cost (proportional or quadratic), we consider three different portfolio policies. First, we consider the *target* portfolio policy, which consists of trading to the target or Markowitz portfolio in the first period and not trading thereafter. This is the optimal portfolio policy for an investor in the absence of transaction costs. Second, the *static* portfolio policy, which consists of trading at each period to the solution to the single-period problem subject to transaction costs. This is the optimal portfolio policy for a myopic investor who takes into account transaction costs. Third, we consider the *multiperiod* portfolio policy, which is the optimal portfolio policy for a multiperiod investor who takes into account transaction costs.

Finally, we evaluate the utility of each of the three portfolio policies using the appropriate multiperiod framework; that is, when considering proportional transaction costs, we evaluate the investor's utility from each portfolio with the objective function in equation (8); and when considering quadratic transaction costs, we evaluate the investor's utility using the objective function (21).

### 6.1. Proportional Transaction Costs

**6.1.1. Base Case.** We consider a base case with proportional transaction costs of 30 basis points, risk-aversion parameter  $\gamma = 10^{-5}$ , which corresponds to a relative risk aversion of one for an investor managing  $M = 10^5$  dollars, annual discount factor  $\rho = 5\%$ <sup>5</sup>, and an investment horizon of  $T = 22$  days (one month). We consider four risky assets ( $N = 4$ ), with starting price of one dollar, asset price change correlations of 0.2, and we assume the starting portfolio is equally-weighted across the four risky assets with a total number of  $M = 10^5$  shares. We randomly draw the annual average price changes from a uniform distribution with support  $[0.1, 0.25]$ , and the annual price change volatilities from a uniform distribution with support  $[0.1, 0.4]$ .

<sup>5</sup> Garleanu and Pedersen (2012) consider, for a larger investor facing quadratic transaction costs, an absolute risk aversion  $\gamma = 10^{-9}$ , which corresponds to an investor managing  $M = 10^9$  dollars, and a discount factor  $\rho = 1 - \exp(-0.02/260)$  which corresponds with an annual discount of 2%.

For our base case, we observe that the utility loss associated with investing myopically (that is, the difference between the utility of the *multiperiod* portfolio policy and the *static* portfolio policy) is 70.01%. The utility loss associated with ignoring transaction costs altogether (that is, the difference between the utility of the *multiperiod* portfolio policy and the *target* portfolio policy) is 13.90%. Hence we find that the loss associated with either ignoring transaction costs or behaving myopically can be substantial. The following subsection confirms this is also true when we change relevant model parameters.

**6.1.2. Comparative statics.** We study numerically how the utility losses associated with ignoring transaction costs (i.e., with the static portfolio), and investing myopically (i.e., with the target portfolio) depend on the transaction cost parameter, the price-change correlation, the investment horizon, and the risk-aversion parameter.

Panel (a) in Figure 7 depicts the utility loss associated with the target and static portfolios for values of the proportional transaction cost parameter  $\kappa$  ranging from 0 basis point to 120 basis points. As expected, the utility loss associated with ignoring transaction costs is zero in the absence of transaction costs and increases monotonically with transaction costs. Moreover, for large transaction costs parameters, the utility loss associated with ignoring transaction costs grows linearly with  $\kappa$  and can be very large. The utility losses associated with behaving myopically are concave unimodal in the transaction cost parameter, being zero for the case with zero transaction costs (because both the single-period and multiperiod portfolio policies coincide with the target or Markowitz portfolio), and for the case with large transaction costs (because both the single-period and multiperiod portfolio policies result in little or no trading). The utility loss of behaving myopically reaches a maximum of 80% for a level of transaction costs of around 5 basis points

Panel (b) in Figure 7 depicts the utility loss associated with the target and static portfolio policies for values of price change correlation ranging from  $-0.3$  to  $0.4$ . The graph shows that while the utility loss associated with the static portfolio is monotonically decreasing in correlation, the utility loss of the target portfolio increases with the correlation. The explanation for this stems from the fact that the static portfolio is *less* diversified than the multiperiod portfolio (because it assigns a higher weight to transaction costs), whereas the target portfolio is more diversified than the multiperiod portfolio (because it ignores transaction costs). Hence, because the benefits from diversification are greater for smaller correlations, the utility loss from the static portfolio decreases with correlation, and the utility loss of the target portfolio increases for large correlations.

Panel (c) in Figure 7 depicts the utility loss associated with investing myopically and ignoring transaction costs for investment horizons ranging from  $T = 5$  (one week) to  $T = 22$  (one month). Not surprisingly, the utility loss associated with behaving myopically grows with the investment horizon. Also, the utility



loss associated with ignoring transaction costs is very large for short-term investors, and decreases monotonically with the investment horizon. The reason for this is that the size of the no-trade region for the multiperiod portfolio policy decreases monotonically with the investment horizon, and thus the target and multiperiod policies become similar for long investment horizons. This makes sense intuitively: by adopting the Markowitz portfolio, a multiperiod investor incurs transaction cost losses at the first period, but makes mean-variance utility gains for the rest of the investment horizon. Hence, when the investment horizon is long, the transaction losses are negligible compared with the utility gains.

Finally, we find that the utility losses associated with investing myopically and ignoring transaction costs do not depend on the risk-aversion parameter, because the utility for these three portfolio policies decrease at the same relative rate as  $\gamma$  increases.

## 6.2. Quadratic Transaction Costs

In this section we study whether and how the presence of quadratic transaction costs (as opposed to proportional transaction costs) affects the utility losses of the static and target portfolios.

**6.2.1. The Base Case.** We consider the same parameters as in the base case with proportional transaction costs, plus we assume the transaction cost matrix  $\Lambda = \Sigma$ , and the transaction cost parameter  $\kappa = 1.5 \times 10^{-4}$ .

Similar to the case with proportional transaction costs, we find that the losses associated with either ignoring transaction costs or behaving myopically are substantial. For instance, for the base case with find that the utility loss associated with investing myopically is 30.81%, whereas the utility loss associated with ignoring transaction costs is 116.20%. Moreover, we find that the utility losses associated with the target portfolio are relatively larger, compared to those of the static portfolio, for the case with quadratic transaction costs. The explanation for this is that the target portfolio requires large trades in the first period, which are penalized heavily in the context of *quadratic* transaction costs. The static portfolio, on the other hand, results in smaller trades over successive periods and this will result in overall smaller *quadratic* transaction costs.

**6.2.2. Comparative Statics.** Panel (a) in Figure 8 depicts the utility loss associated with investing myopically and ignoring transaction costs for values of the quadratic transaction cost parameter  $\kappa$  ranging from  $5 \times 10^{-5}$  to  $5 \times 10^{-4}$ . Our findings are very similar to those for the case with proportional transaction costs. Not surprisingly, the utility losses associated with ignoring transaction costs are small for small transaction costs and grow monotonically as the transaction cost parameter grows. Also, the utility loss associated with the static portfolio policy is concave unimodal in the level of transaction costs. The intuition behind these results is similar to that provided for the case with proportional transaction costs.

Panel (b) in Figure 8 depicts the utility loss associated with investing myopically and ignoring transaction costs for values of price change correlation ranging from -0.3 to 0.4. Similar to the case with proportional transaction costs, the utility loss associated with the static portfolio is monotonically decreasing with correlation. The reason for this is again that the static portfolio is less diversified than the multiperiod portfolio, and this results in larger losses when correlations are small because the potential gains from diversification are larger. Unlike in the case with proportional transaction costs, however, the utility loss of the target portfolio is monotonically decreasing with correlation. This is surprising as the target portfolio is more diversified than the multiperiod portfolio and thus its losses should be relatively smaller when correlations are low. We find, however, that because the target portfolio requires large trades in the first period and thus it incurs very high quadratic transaction costs, the benefits of diversification are swamped by the quadratic transaction costs. Consequently, the utility loss of the target portfolio is larger for small (negative) correlations, because in this case the target portfolio takes extreme long and short positions that result in very large trades in the first period, and thus very large quadratic transaction costs.

Panel (c) in Figure 8 depicts the utility loss associated with investing myopically and ignoring transaction costs for values investment horizon  $T$  ranging from 5 days to 22 days. Our findings are similar to those for the case with proportional transaction costs. The target portfolio losses are monotonically decreasing with the investment horizon as these two portfolios become more similar for longer investment horizons, where the overall importance of transaction costs is smaller. The static portfolio losses are monotonically increasing with the investment horizon because the larger the investment horizon the faster the multiperiod portfolio converges to the target, whereas the rate at which the static portfolio converges to the target does not change with the investment horizon.

Finally, we find that the utility loss associated with investing myopically and ignoring transaction costs is monotonically decreasing for values of the risk-aversion level  $\gamma$  ranging from  $5 \times 10^{-6}$  to  $5 \times 10^{-5}$ . The explanation for this is that when the risk-aversion parameter is large, the mean-variance utility is relatively more important compared to the quadratic transaction costs, and thus the target and static portfolios are more similar to the multiperiod portfolio. This is in contrast to the case with proportional transaction costs, where the losses did not depend on the risk-aversion parameter. The reason for this difference is that with quadratic transaction costs, the mean-variance utility term and the transaction cost term are both quadratic, and thus the risk-aversion parameter does have an impact on the overall utility loss.

Finally, we have repeated our analysis for a case with market impact costs ( $p = 1.5$ ) and we find that the monotonicity properties of the utility losses are roughly in the middle of those for the case with proportional transaction costs ( $p = 1$ ) and quadratic transaction costs ( $p = 2$ ), and thus we do not report the results to conserve space.

## 7. Conclusions

We study the optimal portfolio policy for a multiperiod mean-variance investor facing multiple risky assets subject to proportional, market impact, or quadratic transaction costs. We demonstrate analytically that, in the presence of proportional transaction costs, the optimal strategy for the multiperiod investor is to trade in the first period to the boundary of a no-trade region shaped as a parallelogram, and not to trade thereafter. For the case with market impact costs, the optimal portfolio policy is to trade to the boundary of a state-dependent rebalancing region. In addition, the rebalancing region converges to the Markowitz portfolio as the investment horizon grows large.

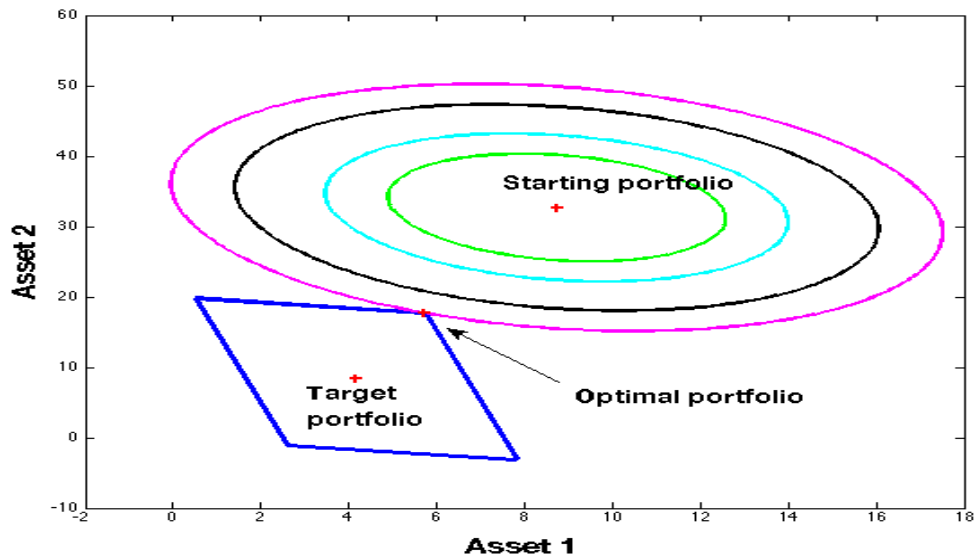
We contribute to the literature by characterizing the no-trade region for a multiperiod investor facing proportional transaction costs, and studying the analytical properties of the optimal trading strategy for the model with market impact costs. Finally, we also show numerically that the utility losses associated with ignoring transaction costs or investing myopically may be large, and study how they depend on the relevant parameters.

## Acknowledgement

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**Appendix A: Figures****Figure 1 No-trade region and level sets.**

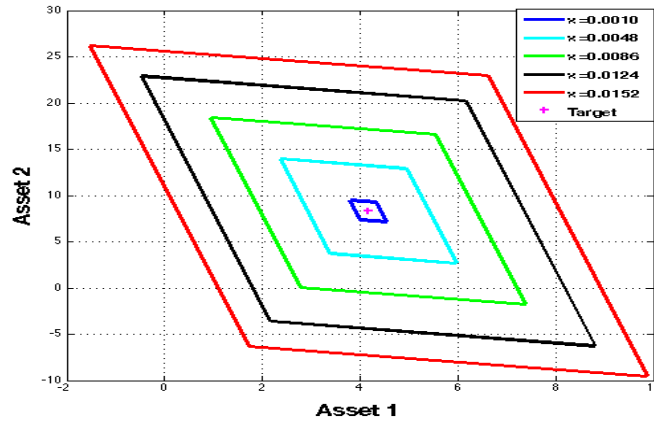
This figure plots the no-trade region and the level sets for a single-period investor facing proportional transaction costs with  $\kappa = 0.005$ , discount factor  $\rho = 0.5$ , risk-aversion parameter  $\gamma = 10$ , price-change mean  $\mu = [0.02; 0.01]$ , price-change volatilities  $[0.02; 0.01]$  and price-change correlation 0.2.



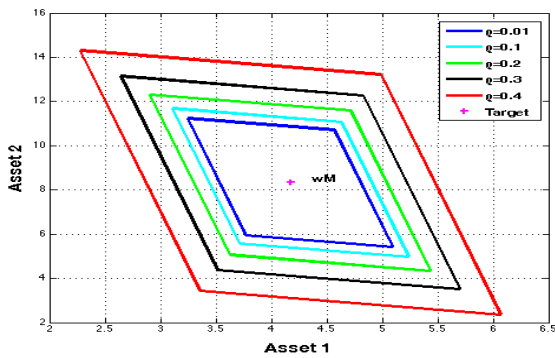
**Figure 2 No-trade region: comparative statics.**

This figure shows how the no-trade region for a multiperiod investor subject to proportional transaction costs depends on relevant parameters. In the base case, we consider the proportional transaction cost parameter  $\kappa = 0.005$ , discount factor  $\rho = 0.5$ , risk-aversion parameter  $\gamma = 10$ , price-change means  $\mu = [0.02; 0.01]$ , price-change volatilities  $[0.02; 0.01]$  and price-change correlation 0.2.

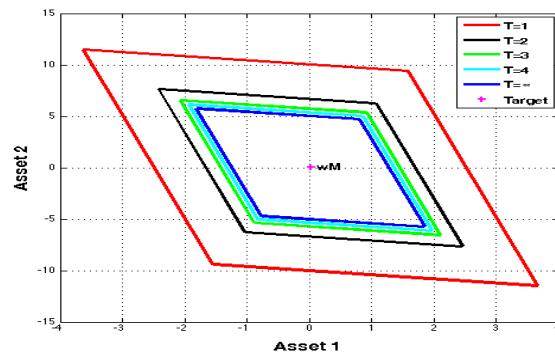
(a) No-trade regions for different  $\kappa$



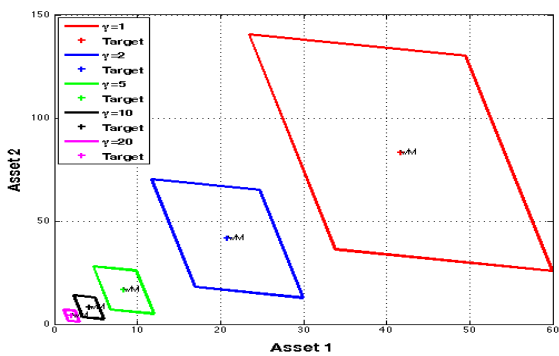
(b) No-trade regions for different  $\rho$



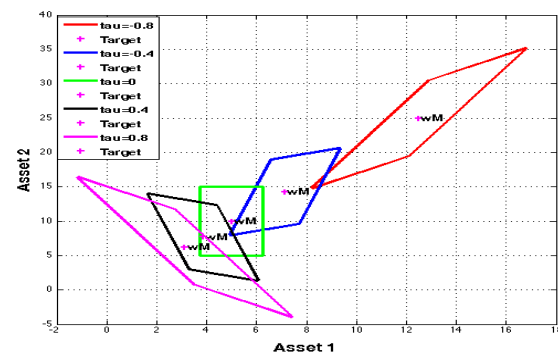
(c) No-trade regions for different  $T$



(d) No-trade regions for different  $\gamma$

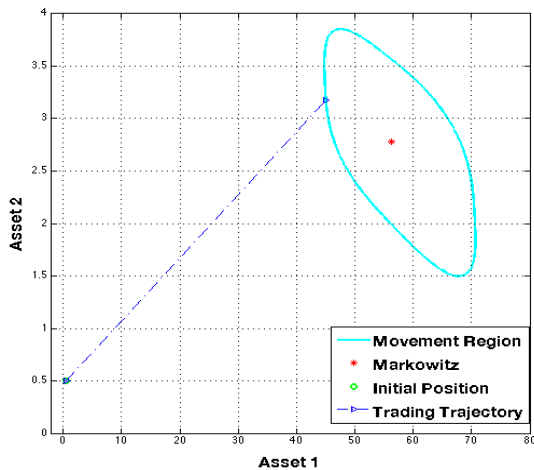
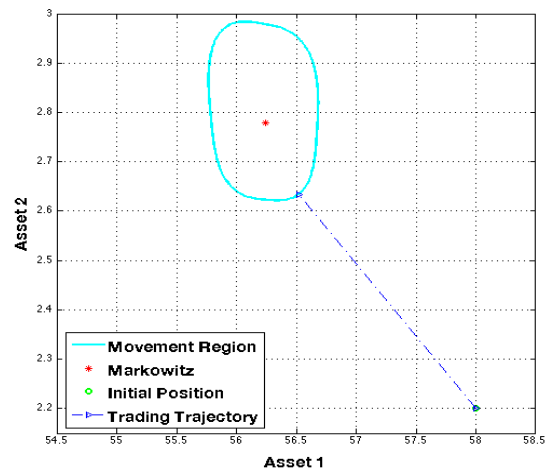


(e) No-trade regions for different correlations

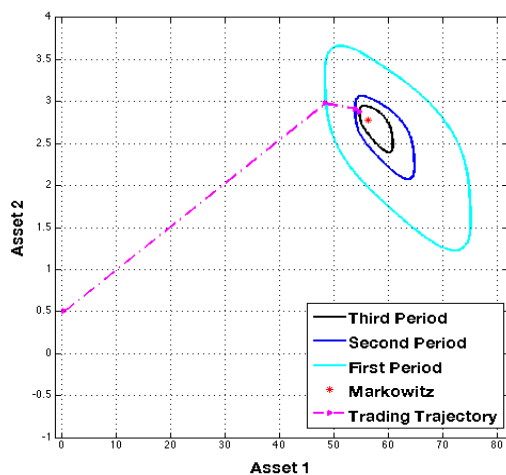
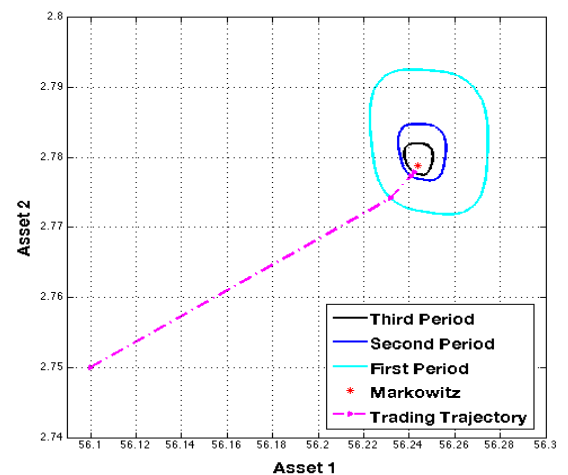


**Figure 3 Rebalancing region for single-period investor**

This figure depicts the rebalancing region for a single-period investor subject to market impact costs. Panel (a) depicts the rebalancing region when  $\Lambda = I$ , with  $p = 1.5$ , risk-aversion parameter  $\gamma = 1$ , transaction cost parameter  $\kappa = 0.0015$ , price-change correlation 0.2, and annual discount factor  $\rho = 0.02$ . Panel (b) depicts the rebalancing region when  $\Lambda = \Sigma$ , with  $\kappa = 0.15$ . In both cases, we consider the price-change means  $[0.08; 0.12]$  and price-change volatilities  $[0.037; 0.146]$ .

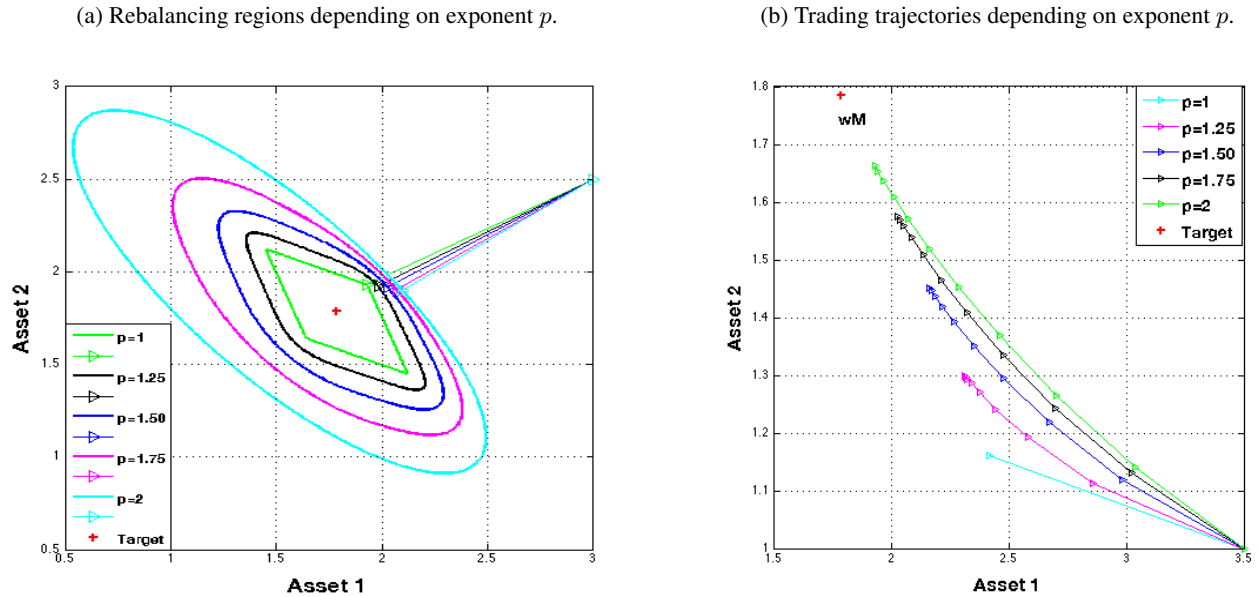
(a) Rebalancing region when  $\Lambda = I$ (b) Rebalancing region when  $\Lambda = \Sigma$ **Figure 4 Rebalancing region for multiperiod investor**

This figure depicts the rebalancing region for a multiperiod investor subject to market impact costs, and investment horizon  $T = 3$ . Panel (a) depicts the rebalancing region when  $\Lambda = I$ , with  $p = 1.5$ , risk-aversion parameter  $\gamma = 1$ , transaction cost parameter  $\kappa = 0.0015$ , price-change correlation 0.2, and annual discount factor  $\rho = 0.02$ . Panel (b) depicts the rebalancing region when  $\Lambda = \Sigma$ , with  $\kappa = 0.09$ . In both cases, we consider the price-change means  $[0.08; 0.12]$  and price-change volatilities  $[0.037; 0.146]$ .

(a) Rebalancing region when  $\Lambda = I, T = 3$ (b) Rebalancing region when  $\Lambda = \Sigma, T = 3$ 

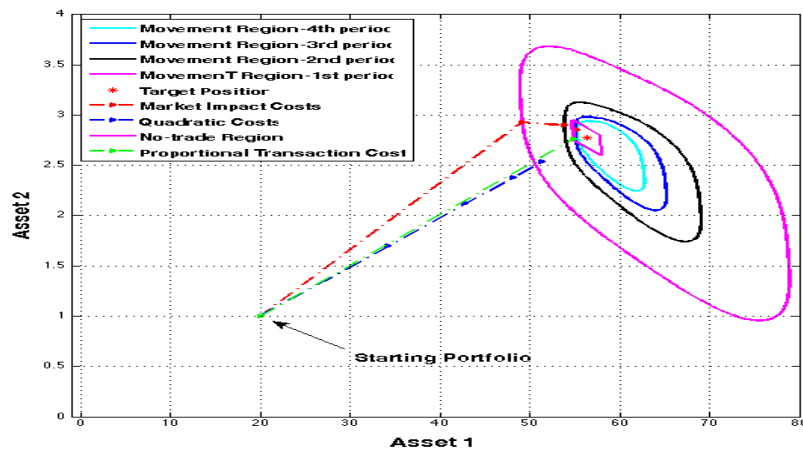
**Figure 5 Rebalancing regions and trading trajectories for different exponents  $p$ .**

This figure depicts the rebalancing regions and trading trajectories for the market impact costs model depending on exponent  $p$ . Panel (a) depicts the rebalancing regions for the single-period portfolio, with transaction cost parameter  $\kappa = 0.008$ , and annual discount factor  $\rho = 0.09$ . Panel (b) depicts the multiperiod optimal trading trajectories for the multiperiod portfolio when  $T = 10$ , with transaction cost parameter  $\kappa = 0.15$ , and annual discount factor  $\rho = 0.05$ . In both cases, we consider transaction costs matrix  $\Lambda = I$ , the risk-aversion parameter  $\gamma = 1$ , correlation 0.4, price-change means  $[0.1; 0.1]$  and price-change volatilities  $[0.2; 0.2]$ .



**Figure 6 Trading trajectories for different transaction costs.**

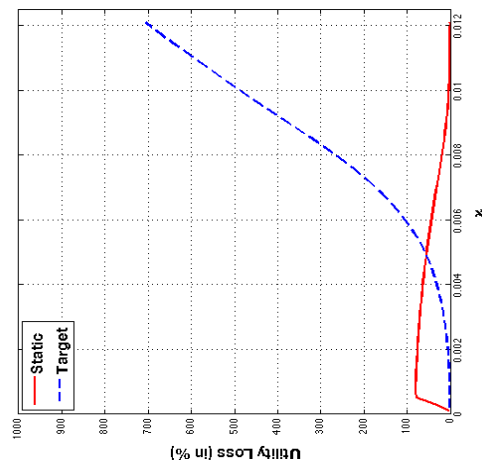
This figure depicts the multiperiod trading trajectories for a multiperiod investor facing different transaction costs and an investment horizon  $T = 4$ . We consider a risk-aversion parameter  $\gamma = 1$ , price-change correlation 0.2, annual discount factor  $\rho = 0.05$ . In the proportional transaction cost case, the transaction costs parameter  $\kappa = 0.008$ . In the market impact cost case,  $\kappa = 0.002$  and  $\Lambda = I$ . In the quadratic transaction cost case,  $\kappa = 2$  and  $\Lambda = \Sigma$ . For all cases, we consider the price-change means  $[0.08; 0.12]$  and price-change volatilities  $[0.037; 0.146]$ .



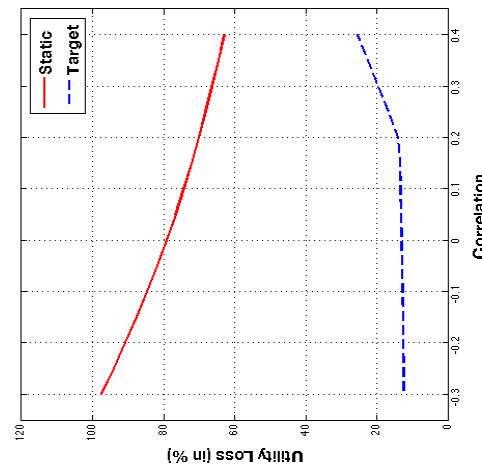
**Figure 7 Utility losses with proportional transaction costs.**

This figure depicts the utility loss depending on relevant model parameters. Panel (a) depicts the utility loss as a function of the proportional transaction costs parameter  $\kappa$ . Panel (b) depicts the utility loss as a function of the price-change correlation. Panel (c) depicts the utility loss as a function of the investment horizon  $T$ . In the base case, we consider number of assets  $N = 4$ , proportional transaction costs parameter  $\kappa = 0.0030$ , risk-aversion parameter  $\gamma = 1e - 5$ , annual discount factor  $\rho = 0.05$ , price-change correlations equal to  $0.2$ , and investment horizon  $T = 22$ . The annual price-change means are randomly generated from interval  $[0.1, 0.25]$  and the price-change volatilities are generated from interval  $[0.1, 0.4]$ .

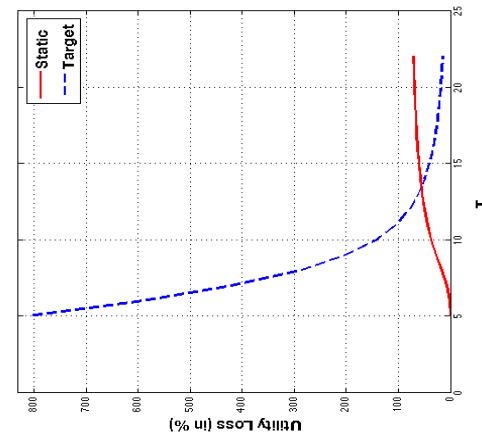
(a) Utility losses depending on  $\kappa$ .



(b) Utility losses depending on correlation.



(c) Utility losses depending on investment horizon  $T$ .

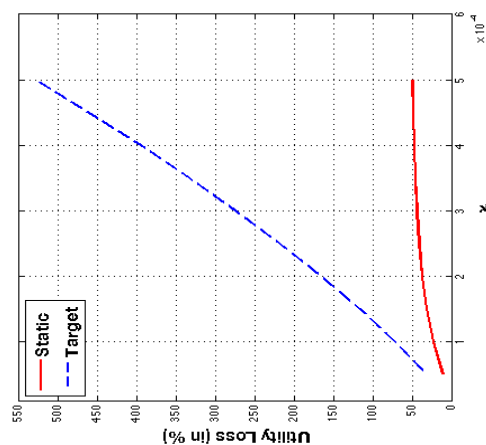




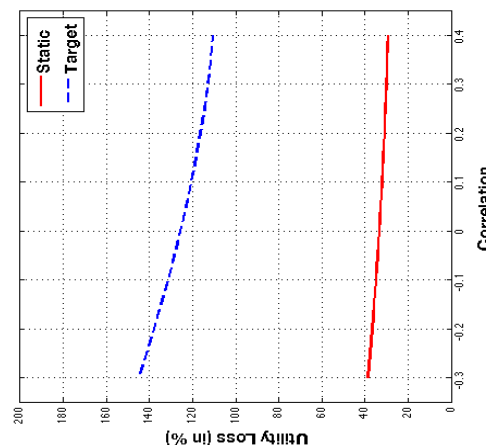
**Figure 8 Utility losses with quadratic transaction costs.**

This figure depicts the utility loss as a function of relevant model parameters. Panel (a) depicts the utility loss as a function of the quadratic transaction cost parameter  $\lambda$ . Panel (b) depicts the utility loss as a function of the price-change correlation. Panel (c) depicts the utility loss as a function of the investment horizon parameter  $T$ . In the base case, we consider number of assets  $N = 4$ , quadratic transaction costs parameter  $\kappa = 1.5e - 4$ , risk-aversion parameter  $\gamma = 1e - 5$ , annual discount factor  $\rho = 0.05$ , price-change correlations equal to 0.2, and investment horizon  $T = 22$ . The annual price-change means are randomly generated from interval  $[0.1, 0.25]$  and price-change volatilities are generated from interval  $[0.1, 0.4]$ .

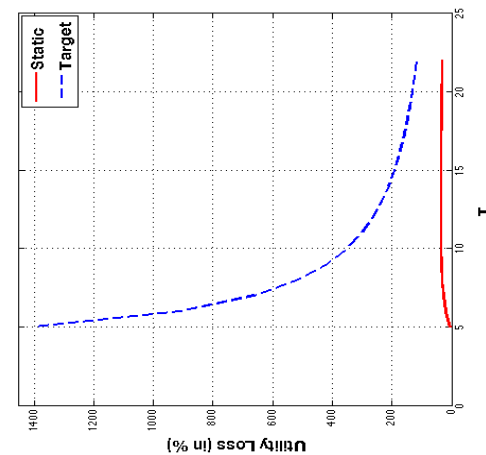
(a) Utility losses depending on  $\lambda$



(b) Utility losses depending on correlation.



(c) Utility losses depending on investment horizon  $T$



## Appendix B: Proofs of all results

### Proof of Proposition 1

Define the subdifferential  $\Omega$  of  $\kappa\|x - x_0\|_1$  as

$$s \in \Omega = \{u \mid u^\top(x - x_0) = \kappa\|x - x_0\|_1, \|u\|_\infty \leq \kappa\}, \quad (28)$$

where  $s$  denotes a subgradient of  $\kappa\|x - x_0\|_1$ . If we write  $\kappa\|x - x_0\|_1 = \max_{\|s\|_\infty \leq \kappa} s^\top(x - x_0)$ , then objective function (2) can be sequentially rewritten as:

$$\begin{aligned} & \max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - \kappa\|x - x_0\|_1 \\ &= \max_x \min_{\|s\|_\infty \leq \kappa} (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - s^\top(x - x_0) \\ &= \min_{\|s\|_\infty \leq \kappa} \max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - s^\top(x - x_0). \end{aligned} \quad (29)$$

Differentiating the inner objective function in (29) with respect to  $x$  gives

$$0 = (1 - \rho)(\mu - \gamma \Sigma x) - s, \quad (30)$$

and hence  $x = \frac{1}{\gamma} \Sigma^{-1}(\mu - \frac{1}{1 - \rho} s)$  for  $s \in \Omega$ . Plugging  $x$  into (29),

$$\begin{aligned} & \min_{\|s\|_\infty \leq \kappa} \left[ \frac{1}{\gamma} \Sigma^{-1}(\mu - \frac{1}{1 - \rho} s) \right]^\top \mu - \frac{\gamma}{2} \left[ \frac{1}{\gamma} \Sigma^{-1}(\mu - \frac{1}{1 - \rho} s) \right]^\top \Sigma \left[ \frac{1}{\gamma} \Sigma^{-1}(\mu - \frac{1}{1 - \rho} s) \right] \\ & \quad - s^\top \left[ \frac{1}{\gamma} \Sigma^{-1}(\mu - \frac{1}{1 - \rho} s) - x_0 \right] \\ & \equiv \min_{\|s\|_\infty \leq \kappa} \frac{1}{2\gamma} (\mu - \frac{1}{1 - \rho} s)^\top \Sigma^{-1} (\mu - \frac{1}{1 - \rho} s) + s^\top x_0. \end{aligned} \quad (31)$$

Note that from (30), we can attain  $s = (1 - \rho)(\mu - \gamma \Sigma x) = (1 - \rho)[\gamma \Sigma(x^* - x)]$  as well as  $\|s\|_\infty \leq \kappa$ , where  $x^* = \frac{1}{\gamma} \Sigma^{-1} \mu$  denoting Markowitz strategy. Plugging  $s$  back into (31), we conclude that problem (31) is equivalent to the following

$$\min_x \frac{\gamma}{2} x^\top \Sigma x - \gamma x^\top \Sigma x_0. \quad (32)$$

$$s.t. \quad \|(1 - \rho)\gamma \Sigma(x - x^*)\|_\infty \leq \kappa. \quad (33)$$

If we now add a constant term  $\frac{\gamma}{2} x_0^\top \Sigma x_0$  to the objective function (32), it follows immediately that

$$\begin{aligned} & \min_x \frac{\gamma}{2} x^\top \Sigma x - \gamma x^\top \Sigma x_0 \\ & \equiv \min_x \frac{\gamma}{2} x^\top \Sigma x - \gamma x^\top \Sigma x_0 + \frac{\gamma}{2} x_0^\top \Sigma x_0 \\ & \equiv \min_x (x - x_0)^\top \Sigma (x - x_0). \end{aligned} \quad (34)$$

Rearranging terms in constraint (33), we conclude the objective function (2) is equivalent to

$$\min_x (x - x_0)^\top \Sigma (x - x_0), \quad (35)$$

$$s.t. \quad \|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (36)$$

To show that constraint (4) defines a no-trade region, note that when the initial position  $x_0$  satisfies constraint (4), then  $x = x_0$  minimizes the objective function and it is feasible with respect to the constraint. On the other hand, when

$x_0$  is not in the region defined by the constraint, the optimal  $x$  must be the point on the boundary of the feasible region that minimizes the objective. Finally, note that constraint  $\|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma}$  is equivalent to  $-\frac{\gamma}{(1-\rho)\kappa}e \leq \Sigma(x - x^*) \leq \frac{\gamma}{(1-\rho)\kappa}e$ , hence we conclude that the no-trade region defined by constraint (4) is a parallelogram centered around  $x^*$ .  $\square$

### Proof of Proposition 2

Consider the following optimization problem

$$\min_x \quad -(1-\rho)(x^\top \mu - \frac{\gamma}{2}x^\top \Sigma x) \quad (37)$$

$$s.t. \quad x - x_0 = 0 \quad (38)$$

Noting that the optimal solution to the above optimization is  $x = x_0$ . Define  $\phi$  as the Lagrange multipliers corresponding to constraint (38) and further the  $\ell_1$  penalty function for optimization problem (37)-(38) as

$$\phi(x, \kappa) = -(1-\rho)(x^\top \mu - \frac{\gamma}{2}x^\top \Sigma x) + \kappa \|x - x_0\|_1, \quad (39)$$

which is also equivalent to objective (2). Let  $\kappa^* = \|\phi\|_\infty$ , when  $\kappa > \kappa^*$ , the solution to problem (37)-(38)  $x = x_0$  is also the minimizer of penalty function (39) (see Theorem 17.3 of Nocedal and Wright (2006)).

We conclude that when  $\kappa > \kappa^*$ , the no-trade region will become unbounded.  $\square$

### Proof of Theorem 1

**Part 1.** Define subdifferential  $\Omega_t$  of  $\kappa \|x_t - x_{t-1}\|_1$  as

$$s_t \in \Omega_t = \{u_t \mid u_t^\top (x_t - x_{t-1}) = \kappa \|x_t - x_{t-1}\|_1, \|u_t\|_\infty \leq \kappa\}, \quad (40)$$

where  $s_t$  denotes a subgradient of  $\kappa \|x_t - x_{t-1}\|_1$ ,  $t = 1, 2, \dots, T$ . If we write  $\kappa \|x_t - x_{t-1}\|_1 = \max_{\|s_t\|_\infty \leq \kappa} s_t^\top (x_t - x_{t-1})$ , objective function (8) can be sequentially rewritten as

$$\begin{aligned} & \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \\ &= \max_{\{x_t\}_{t=1}^T} \min_{\|s_t\|_\infty \leq \kappa} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} s_t^\top (x_t - x_{t-1}) \right] \\ &= \min_{\|s_t\|_\infty \leq \kappa} \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} s_t^\top (x_t - x_{t-1}) \right]. \end{aligned} \quad (41)$$

Differentiating the inner objective function of (41) with respect to  $x_t$  gives

$$0 = (1-\rho)(\mu - \gamma \Sigma x_t) - s_t + (1-\rho)s_{t+1}. \quad (42)$$

hence

$$x_t = \frac{1}{\gamma} \Sigma^{-1} (\mu + s_{t+1}) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_t, \quad \text{for } s_t \in \Omega_t, s_{t+1} \in \Omega_{t+1} \quad (43)$$

Denote  $x_t^*$  as the optimal solution for stage  $x_t^*$ , then there exists  $s_t^*$  and  $s_{t+1}^*$  so that

$$x_t^* = \frac{1}{\gamma} \Sigma^{-1} (\mu + s_{t+1}^*) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_t^*, \quad \forall t \quad (44)$$

If we let  $s_t^* = \frac{1-(1-\rho)^{T-t+2}}{\rho} s_T^*$ , for  $t = 1, 2, \dots, T-1$  and  $s_T^* = (1-\rho)(\mu - \gamma \Sigma x_T^*)$  and rewrite  $x_t^*$  as

$$\begin{aligned} x_t^* &= \frac{1}{\gamma} \Sigma^{-1} (\mu + s_{t+1}^*) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_t^* \\ &= \frac{1}{\gamma} \Sigma^{-1} (\mu + s_{r+1}^*) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_r^* = x_r^*, \quad \forall t, r \end{aligned} \quad (45)$$

where  $\|s_t\|_\infty \leq \kappa$ . By this means, we find the values of  $s_t^*$  such that  $x_t^* = x_r^*$  for all  $t \neq r$ . we conclude that  $x_1 = x_2 = \dots = x_T$  satisfies the optimality conditions.

**Part 2.** Rewrite objective function (8) into the following by combining the fact that  $x_1 = x_2 = \dots = x_T$ ,

$$\begin{aligned} & \max_{\{x_t\}_{t=1}^T} \left\{ \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \right\} \\ &= \max_{x_1} \left\{ \sum_{t=1}^T \left[ (1-\rho)^t \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) \right] - \kappa \|x_1 - x_0\|_1 \right\} \\ &= \max_{x_1} \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - \kappa \|x_1 - x_0\|_1. \end{aligned} \quad (46)$$

Define subgradient of  $\kappa \|x_1 - x_0\|_1$  as  $s$  and the subdifferential  $\Omega$  as

$$s \in \Omega = \{u \mid u^\top (x_1 - x_0) = \kappa \|x_1 - x_0\|_1, \|u\|_\infty \leq \kappa\}. \quad (47)$$

Substitute  $(1-\rho)$  in Theorem 1 with  $\frac{(1-\rho) - (1-\rho)^{T+1}}{\rho}$ , it follows immediately that objective function (46) is equivalent to

$$\min_{x_1} (x_1 - x_0)^\top \Sigma (x_1 - x_0), \quad (48)$$

$$s.t. \quad \|\Sigma(x_1 - x^*)\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T}. \quad (49)$$

**Part 3.** Note that constraint (11) is equivalent to

$$-\frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T} e \leq \Sigma(x_1 - x^*) \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T} e,$$

which is a parallelogram centered at  $x^*$ . To show that constraint (11) defines a no-trade region, note that when the starting portfolio  $x_0$  satisfies constraint (11), then  $x_1 = x_0$  minimizes the objective function (10) and is feasible with respect to the constraint. On the other hand, when  $x_0$  is not inside the region defined by (11), the optimal solution  $x_1$  must be the point on the boundary of the feasible region that minimizes the objective. We then conclude that constraint (11) defines a no-trade region.  $\square$

### Proof of Proposition 3

Differentiating objective function (13) with respect to  $x_t$  gives

$$(1-\rho)(\mu - \gamma \Sigma x) - \kappa p \Lambda^{1/p} |\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0)) = 0, \quad (50)$$

where  $|a|^{p-1}$  denotes the absolute value to the power of  $p-1$  for each component:

$$|a|^{p-1} = (|a_1|^{p-1}, |a_2|^{p-1}, \dots, |a_N|^{p-1}),$$

and  $\text{sign}(\Lambda^{1/p}(x - x_0))$  is the vector containing the sign of each component for  $\Lambda^{1/p}(x - x_0)$ . Given that  $\Lambda$  is symmetric, rearranging (50)

$$(1 - \rho)\Lambda^{-1/p}\gamma\Sigma(x^* - x) = \kappa p |\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0)). \quad (51)$$

Keep in mind that  $x = x_0$  can not be the optimal solution unless the initial position  $x_0$  satisfies  $x_0 = x^*$ . Otherwise, take  $q$ -norm on both sides of (51):

$$\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q = \frac{\kappa}{(1 - \rho)\gamma} p \|\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0))\|_q, \quad (52)$$

where  $q$  is the value such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Note that  $\|\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0))\|_q = \|x - x_0\|_p^{p-1}$ , we conclude that the optimal trading strategy satisfies

$$\frac{\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q}{p \|\Lambda^{1/p}(x - x_0)\|_p^{p-1}} = \frac{\kappa}{(1 - \rho)\gamma}. \quad (53)$$

It follows immediately Proposition 3.  $\square$

## Proof of Theorem 2

Differentiating objective function (17) with respect to  $x_t$  gives

$$\begin{aligned} (1 - \rho)^t (\mu - \gamma\Sigma x_t) - (1 - \rho)^{t-1} p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_t - x_{t-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_t - x_{t-1})) \\ + (1 - \rho)^t p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_{t+1} - x_t)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_{t+1} - x_t)) = 0. \end{aligned} \quad (54)$$

Specifically, given that  $\Lambda$  is symmetric, the optimality condition for stage  $T$  reduces to

$$(1 - \rho)\Lambda^{-1/p}(\mu - \gamma\Sigma x_T) = p \kappa |\Lambda^{1/p}(x_T - x_{T-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_T - x_{T-1})), \quad (55)$$

Keep in mind that  $x_T = x_{T-1}$  can not be the optimal solution unless  $x_{T-1} = x^*$ . Otherwise, take  $q$ -norm on both sides of (55) and rearrange terms,

$$\frac{\|\Lambda^{-1/p}\Sigma(x_T - x^*)\|_q}{p \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (56)$$

Iterating the optimal condition recursively gives

$$p \kappa |x_t - x_{t-1}|^{p-1} \cdot \text{sign}(x_t - x_{t-1}) = \sum_{s=t}^T (1 - \rho)^{s-t+1} \gamma \Lambda^{-1/p} \Sigma(x^* - x_s). \quad (57)$$

Keep in mind that  $x_t = x_{t-1}$  can not be the optimal solution unless that  $x_{t-1} = x^*$ . Otherwise, take  $q$ -norm on both sides, it follows straightforwardly that

$$\frac{\|\sum_{s=t}^T (1 - \rho)^{s-t} \Lambda^{-1/p} \Sigma(x_s - x^*)\|_q}{p \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^{p-1}} = \frac{\kappa}{(1 - \rho)\gamma}. \quad (58)$$

We conclude that the optimal trading strategy for period  $t$  satisfies (58) whenever the initial position is not  $x^*$ .  $\square$

## Proof of Proposition 4

**Part 1.** Define  $g(x) = (1 - \rho)\Lambda^{-1/p}\Sigma(x - x^*)$ . For period  $T$  it holds that  $\|g(x_T)\|_q \leq p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$ .

Moreover, for the following last period it holds  $\|g(x_{T-1}) + (1 - \rho)g(x_T)\|_q \leq p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}$ . Noting that  $\|A + B\|_q \geq \|A\|_q - \|B\|_q$ , it follows immediately that

$$\begin{aligned} p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1} &\geq \|g(x_{T-1}) + (1 - \rho)g(x_T)\|_q \\ &\geq \|g(x_{T-1})\|_q - (1 - \rho)\|g(x_T)\|_q \\ &\geq \|g(x_{T-1})\|_q - (1 - \rho)p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}, \end{aligned}$$

where the last inequality holds because  $\|g(x_T)\|_q \leq p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$ .

Rearranging terms,

$$\|g(x_{T-1})\|_q \leq p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1} + (1 - \rho)p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1},$$

it implies that

$$\frac{\|g(x_{T-1})\|_q}{p \|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1 - \rho) \frac{\kappa}{\gamma} \frac{\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}}{\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}},$$

which is a wider area than the rebalancing region defined for  $x_T$ :  $\frac{\|g(x_T)\|_q}{p \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}} \leq \frac{\kappa}{\gamma}$ .

Similarly,  $\|g(x_{T-2}) + (1 - \rho)g(x_{T-1}) + (1 - \rho)^2g(x_T)\|_q \leq \frac{\kappa}{\gamma} p \|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}$ , it follows

$$\begin{aligned} \frac{\kappa}{\gamma} p \|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1} &\geq \|g(x_{T-2}) + (1 - \rho)g(x_{T-1}) + (1 - \rho)^2g(x_T)\|_q \\ &\geq \|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p - (1 - \rho)^2\|g(x_T)\|_q, \\ &\geq \|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p - (1 - \rho)^2 p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}, \end{aligned}$$

where the last inequality holds because  $\|g(x_T)\|_q \leq p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$ .

rearranging terms

$$\|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p \leq \frac{\kappa}{\gamma} p \|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1} + (1 - \rho)^2 p \frac{\kappa}{\gamma} \|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1},$$

It implies that,

$$\frac{\|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p}{p \|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1 - \rho)^2 \frac{\kappa}{\gamma} \frac{\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}}{\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}},$$

which is a region wider than the rebalancing region defined by  $\frac{\|g(x_{T-1}) + (1 - \rho)\|g(x_T)\|_p}{p \|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}} \leq \frac{\kappa}{\gamma}$  for  $x_{T-1}$ .

Recursively, we can deduce the rebalancing region corresponding to each period shrinks along  $t$ .

**Part 2.** Note that the rebalancing region for period  $t$  relates with the trading strategies thereafter. Moreover, The condition  $x_1 = x_2 = \dots = x_T = x^*$  satisfies inequality (18), we then conclude that the rebalancing region for stage  $t$  contains Markowitz strategy  $x^*$ .

**Part 3.** The optimality condition for period  $T$  satisfies

$$(1 - \rho)(\mu - \gamma \Sigma x_T) - p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_T - x_{T-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_T - x_{T-1})) = 0. \quad (59)$$

Let  $\omega$  to be the vector such that  $\lim_{T \rightarrow \infty} x_T = \omega$ . Taking limit on both sides of (59):

$$(1 - \rho)(\mu - \gamma \Sigma \omega) - p \kappa \Lambda^{1/p} |\Lambda^{1/p}(\omega - \omega)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(\omega - \omega)) = 0, \quad (60)$$

Noting that  $\lim_{T \rightarrow \infty} x_T = \lim_{T \rightarrow \infty} x_{T-1} = \omega$ , it follows

$$(1 - \rho)(\mu - \gamma \Sigma \omega) = 0,$$

which gives that  $\omega = \frac{1}{\gamma} \Sigma^{-1} \mu = x^*$ . We conclude the investor will move to Markowitz strategy  $x^*$  in the limit case.  $\square$

### Proof of Theorem 3

**Part 1.** For  $t = 1, 2, \dots, T-1$ , differentiating objective function (21) with respect to  $x_t$  gives

$$(1 - \rho)(\mu - \gamma \Sigma x_t) - \kappa(2\Lambda x_t - 2\Lambda x_{t-1}) - \kappa(1 - \rho)(2\Lambda x_t - 2\Lambda x_{t+1}) = 0, \quad (61)$$

rearranging terms

$$[(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda] x_t = (1 - \rho)\mu + 2\kappa\Lambda x_{t-1} + 2(1 - \rho)\kappa\Lambda x_{t+1}. \quad (62)$$

The solution can be written explicitly as following

$$\begin{aligned} x_t = & (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda]^{-1} \Sigma x^* \\ & + 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda]^{-1} \Lambda x_{t-1} \\ & + 2(1 - \rho)\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2\kappa(1 - \rho)\Lambda]^{-1} \Lambda x_{t+1}. \end{aligned} \quad (63)$$

Define

$$\begin{aligned} A_1 &= (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Sigma, \\ A_2 &= 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Lambda, \\ A_3 &= 2(1 - \rho)\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Lambda, \end{aligned}$$

where  $A_1 + A_2 + A_3 = I$  to conclude the result.

For  $t = T - 1$ , the optimality condition is

$$(1 - \rho)(\mu - \gamma \Sigma x_T) - \kappa(2\Lambda x_T - 2\Lambda x_{T-1}) = 0, \quad (64)$$

the explicit solution is straightforwardly as following

$$x_T = (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Sigma x^* + 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Lambda x_{T-1}. \quad (65)$$

Define

$$\begin{aligned} B_1 &= (1 - \rho)\gamma [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Sigma, \\ B_2 &= 2\kappa [(1 - \rho)\gamma \Sigma + 2\kappa\Lambda]^{-1} \Lambda, \end{aligned}$$

where  $B_1 + B_2 = I$ , we conclude the results.

**Part 2.** As  $T \rightarrow \infty$ , taking limit on both sides of (65) we then conclude that  $\lim_{T \rightarrow \infty} x_T = x^*$ .  $\square$

**Proof of Corollary 4**

Substituting  $\Lambda$  with  $\Lambda = I$  and  $\Lambda = \Sigma$  respectively we obtain the optimal trading strategy.

To show that the trading trajectory of the case  $\Lambda = \Sigma$  follows a straight line, noting that  $x_T$  is a linear combination of  $x_{T-1}$  and  $x^*$ , which indicates that  $x_T$ ,  $x_{T-1}$  and  $x^*$  are on a straight line. Contrarily, assuming that  $x_{T-2}$  is not on the same line, then  $x_{T-2}$  can not be expressed as linear combination of  $x_T$ ,  $x_{T-1}$  and  $x^*$ . According to equation (26) when  $t = T - 1$ , it follows

$$x_{T-1} = \alpha_1 x^* + \alpha_2 x_{T-2} + \alpha_3 x_T, \quad (66)$$

rearranging terms we get

$$x_{T-2} = \frac{1}{\alpha_2} x_{T-1} - \frac{\alpha_1}{\alpha_2} x^* - \frac{\alpha_3}{\alpha_2} x_T.$$

Note that  $\frac{1}{\alpha_2} - \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_2} = 1$ , it is contradictory with the assumption that  $x_{T-2}$  is not a linear combination of  $x_T$ ,  $x_{T-1}$  and  $x^*$ . By this means, we can show recursively that all the policies corresponding to the model when  $\Lambda = \Sigma$  lay on the same straight line.  $\square$



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