manuscript No.
(will be inserted by the editor)

A direct splitting method for nonsmooth variational inequalities

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Received: date / Accepted: date

Abstract We propose a direct splitting method for solving nonsmooth variational inequality problems in Hilbert spaces. The weak convergence is established, when the operator is the sum of two point-to-set and monotone operators. The proposed method is a natural extension of the incremental subgradient method for nondifferentiable optimization, which explores strongly the structure of the operator using projected subgradient-like techniques. The advantage of our method is that any nontrivial subproblem must be solved, like the evaluation of the resolvent operator. The necessity to compute proximal iterations is the main difficult of others schemes for solving this kind of problem.

Keywords Maximal monotone operators · Monotone variational inequalities · Projection methods · Splitting methods

Mathematics Subject Classification (2010) 90C29 · 90C30

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1 Introduction

In this paper, we introduce an especial direct method for solving nonsmooth variational inequality problem where the operator is sum of two maximal monotone operators. This problem has been a classical subject in economics, operations research and mathematical physics; see [1–3]. It is closely related with many problems of nonlinear analysis, such as optimization, complementarity and equilibrium problems and finding fixed points; see [1, 4–6]. An excellent survey of methods for variational inequality problems can be found in [7]. Many methods have been proposed to solve problem (1), for point-to-point operators; see [8–13], and for point-to-set operators; see [14–16].

Here we are interested in methods that explore the structure of the main operator. These kind of methods are called splitting, since which each iteration involves only the individual operators, but not the sum.

The variational inequality problems are related with the inclusion problems, in fact, when the feasible set is the whole space, the variational problems becomes to the inclusion problems. In the case have been proposed many algorithms solving this special inclusion problem; see [17–24]. However in all of them, the resolvent operator of any individual operator, must be evaluated in each iteration. It is important to mention that this proximal-like iteration is a nontrivial problem, which demands a hard work in the computational point of view. Our algorithm avoids this difficulty replacing it by subgradient-like projection steps, for which the computational cost is negligible compared with proximal-like step. This represent a significative advantage in the implementation and theoretical sense.

This work is inspired by the incremental subgradient method for nondifferentiable optimization, proposed in [25] and it uses a similar idea exposed in [26, 27]. For the case of one operator is known that a natural extension of the subgradient iteration (one step) fails for monotone operators; see [27, 28]. However, as will shown, an extra step is an option in order to prove the weak convergence of the sequence generated by the proposed algorithm.

This paper is organized as follows. The next section provides some notation and preliminary results that will be used in the remainder of this paper. The direct Algorithm A is presented in Section 3 and Subsection 3.1 contains the convergence analysis of the algorithm. Finally in fourth section, the final remarks are presented.

2 Some notation and preliminaries

We begin introducing our notation. The inner product in the Hilbert space $\mathcal{H}$ is denoted by $\langle\cdot,\cdot\rangle$ and the norm induced by the inner product by $\|\cdot\|$. For C a nonempty, convex and closed subset of $\mathcal{H}$, we define the orthogonal projection of x onto C by $P_C(x)$, as the unique point in C, such that $\|P_C(x)-y\| \leq \|x-y\|$ for all $y\in C$. Recall that an operator $T : dom(T) \subseteq \mathcal{H} \rightrightarrows \mathcal{H}$ is monotone if, for all $(x,u),(y,v)\in Gr(T)$, we have $\langle x-y,u-v\rangle \geq 0$, and it is maximal if T has no proper monotone extension in the graph inclusion sense.

We consider the nonsmooth variational inequality problem for T and C, where T is point-to-set and sum of two maximal monotone operators, i.e, $T = T_1+T_2$ where $T_i : dom(T_i) \subseteq \mathcal{H} \rightrightarrows \mathcal{H}$ for $i = 1,2$ and $C \subseteq dom(T_1)\cap dom(T_2)$. The variational inequality problem for T and C consists in:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \exists u^* \in T(x^*), \text{ with } \ \langle u^*,x-x^*\rangle \geq 0 \qquad \forall x\in C. \tag{1}$$

The solution set of problem (1) is denoted by S^*. The main aim of this paper is proposed a direct splitting method for solving problem (1), i.e., any nontrivial subproblem must be solved.

Now, we present some definitions and results that are needed for the convergence analysis of the proposed method. First, we state two well known facts on orthogonal projections.

Lemma 1 *Let C be any nonempty closed and convex set in $\mathcal{H}$. For all $x,y\in\mathcal{H}$ and all $z\in C$, the following properties hold:*

i) $\|P_C(x)-P_C(y)\| \leq \|x-y\|$.

ii) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$.

Proof See Lemma 1.1 and 1.2 in [29].

We next deal with the so called quasi-Fejér convergence and its properties.

Definition 1 Let C be a nonempty convex and closed subset of $\mathscr{H}$. A sequence $\{x^k\}$ is said to be quasi-Fejér convergent to C, if and only if, for all $x \in C$, there exist $k_0 \geq 0$ and a summable sequence $\{\delta_k\} \subset \mathbb{R}_+$, such that

$$\|x^{k+1} - x\|^2 \leq \|x^k - x\|^2 + \delta_k,$$

for all $k \geq k_0$.

This definition originates in [30] and has been further elaborated in [31]. A useful result on quasi-Fejér sequences is the following.

Proposition 1 *If $\{x^k\}$ is quasi-Fejér convergent to C then:*

i) *$\{x^k\}$ is bounded;*

ii) *for each $x \in C$, $\{\|x^k - x\|\}$ converges;*

iii) *if all weak cluster point of $\{x^k\}$ belong to C, then the sequence $\{x^k\}$ is weakly convergent.*

Proof See Proposition 1 in [32].

Now, we remind one property on quasi-Fejér sequences, which will be useful for proving that the sequence generated by our algorithm converges weakly to some point belong to S^*.

Lemma 2 *If $\{x^k\}$ is quasi-Fejér convergent C, then $\{P_C(x^k)\}$ is strongly convergent.*

Proof See Lemma 2 in [26].

We also need the following results on maximal monotone operators and monotone variational inequalities.

Lemma 3 *Let $T : \mathscr{H} \rightrightarrows \mathscr{H}$ be a maximal monotone operator and C a closed and convex set. Then S^*, if nonempty, is closed and convex.*

Proof See Lemma 2.4(ii) in [33].

The next lemma will be useful for proving that all weak cluster points of the sequence generated by our algorithm belong to S^*.

Lemma 4 *If $T : \mathscr{H} \rightrightarrows \mathscr{H}$ is maximal monotone, then*

$$S^* = \{x \in C : \langle v, y-x\rangle \geq 0, \forall y \in C, \ \forall v \in T(y)\}. \tag{2}$$

Proof See Lemma 3 in [34].

Finally, we need the following elementary result on sequence averages.

Proposition 2 *Let $\{p^k\} \subset \mathscr{H}$ be a sequence strongly convergent to $\tilde{p}$. Take nonnegative real numbers $\zeta_{k,j}$ $(k \geq 0,\ 0 \leq j \leq k)$ such that $\lim_{k\to\infty} \zeta_{k,j} = 0$ for all j and $\sum_{j=0}^{k} \zeta_{k,j} = 1$ for all k. Define*

$$x^k := \sum_{j=0}^{k} \zeta_{k,j} p^j.$$

Then, $\{x^k\}$ also converges strongly to $\tilde{p}$.

Proof See Proposition 3 in [26].

3 A splitting direct method

Our algorithm requires an exogenous sequence $\{\alpha_k\} \subset \mathbb{R}_{++}$ satisfying

$$\sum_{k=0}^{\infty} \alpha_k = \infty, \qquad \sum_{k=0}^{\infty} \alpha_k^2 < \infty. \tag{3}$$

This selection rule has been considered several times in the literature; see, for instance, [26–28,32,35,36]. The algorithm is defined as:

Algorithm A

Initialization step: Take $x^0 \in C$. Define $z^0 := x^0$ and $\sigma_0 := \alpha_0$.

Iterative step: Given x^k, z^k and σ_k. Compute:

$$y^k = P_C\left(z^k - \alpha_k w_1^k\right) \tag{4}$$

and

$$z^{k+1} = P_C\left(y^k - \alpha_k v_2^k\right), \tag{5}$$

where $w_1^k \in T_1(z^k)$, $v_2^k \in T_2(y^k)$. Set x^{k+1} as

$$x^{k+1} = \left(1 - \frac{\alpha_{k+1}}{\sigma_{k+1}}\right) x^k + \frac{\alpha_{k+1}}{\sigma_{k+1}} z^{k+1}, \tag{6}$$

with $\sigma_{k+1} := \sigma_k + \alpha_{k+1}$.

Stop criteria: If $z^{k+1} = y^k = z^k$, then stop.

We assume the following boundedness property for the operators T_1 and T_2.

Assumption H: There exists a positive scalar M such that

$$\|u\| \le M, \qquad \forall u \in T_i(z^k) \cup T_i(y^k), \quad i = 1,2 \quad \forall k. \tag{7}$$

This assumption holds automatically in finite-dimensional spaces, when $dom(T_1) = dom(T_2) = \mathscr{H}$ or $C \subset int(dom(T_1) \cap dom(T_2))$. We also mention that Assumption H is required in the analysis of [25] for proving convergence of the incremental subgradient method and in others similar schemes; see [26, 28].

3.1 Convergence analysis

We start with the good definition of the stoping criteria.

Proposition 3 *If Algorithm A stop in the step k, then* $z^k \in S^*$.

Proof If $z^{k+1} = y^k$, then using Lemma 1(ii) in (5) we have, $\langle y^k - \alpha_k v_2^k - y^k, x - y^k\rangle \geq 0$ for all $x \in C$, hence, $\langle v_2^k, y^k - x\rangle \leq 0$. Moreover, if $z^k = y^k$ imply, using again Lemma 1(ii) in (4), that, $\langle w_1^k, y^k - x\rangle \leq 0$, for all $x \in C$. Then, $\langle v^k, x - y^k\rangle \leq 0$ for all $x \in C$ and $v^k = w_1^k + v_2^k \in T(y^k)$, showing that $y^k \in S^*$.

From now on, we assume that Algorithm A generates infinite sequences. We present an important algebraic property on the auxiliary sequence $\{z^k\}$ obtained by the Algorithm A.

Proposition 4 *Let $\{z^k\}$ be the auxiliary sequence generated by the Algorithm H. Then, for each $(x,u) \in Gr(T)$, with $x \in C$, there exists a constant $L > 0$ such that,*

$$\|z^{k+1} - x\|^2 \leq \|z^k - x\|^2 + L\alpha_k^2 - 2\alpha_k\langle u, z^k - x\rangle, \tag{8}$$

for all k.

Proof For each $x \in C$, take $u \in T(x)$, such that $u = u_1 + u_2$, with $u_1 \in T_1(x)$ and $u_2 \in T_2(x)$. Choosing M like in Assumption H, we have

$$\begin{aligned}
\|z^{k+1} - x\|^2 &= \left\|P_C\left(y^k - \alpha_k v_2^k\right) - P_C(x)\right\|^2 \leq \left\|\left(y^k - \alpha_k v_2^k\right) - x\right\|^2 \\
&\leq \|y^k - x\|^2 + M^2\alpha_k^2 - 2\alpha_k\ \langle v_2^k, y^k - x\rangle \\
&\leq \|y^k - x\|^2 + M^2\alpha_k^2 - 2\alpha_k\ \langle u_2, y^k - x\rangle \\
&= \|P_C\left(z^k - \alpha_k w_1^k\right) - P_C(x)\|^2 + M^2\alpha_k^2 - 2\alpha_k\langle u_2, y^k - x\rangle \\
&\leq \|z^k - x\|^2 + 2M^2\alpha_k^2 - 2\alpha_k\left(\langle u_2, y^k - x\rangle + \langle u_1, z^k - x\rangle\right) \\
&= \|z^k - x\|^2 + 2M^2\alpha_k^2 - 2\alpha_k\langle u, z^k - x\rangle - 2\alpha_k\langle u_2, y^k - z^k\rangle \\
&\leq \|z^k - x\|^2 + 2M^2\alpha_k^2 - 2\alpha_k\langle u, z^k - x\rangle + 2\alpha_k\|u_2\|\|y^k - z^k\| \\
&\leq \|z^k - x\|^2 + (2M^2 + M\|u_2\|)\alpha_k^2 - 2\alpha_k\langle u, z^k - x\rangle,
\end{aligned}$$

where we used Lemma 1(i) in the first inequality, the monotonicity of T_2 in the third one, Lemma 1(i) and the monotonicity of T_1 in the fourth one, and the last inequality come from

$$\|y^k - z^k\| = \|P_C\left(z^k - \alpha_k w_1^k\right) - P_C\left(z^k\right)\| \leq \alpha_k M, \tag{9}$$

using Assumption H and Lemma 1(i). Defining $L = 2M^2 + 2M\|u_2\|$, we get (8).

From now on S^* is nonempty. We prove the quasi-Fejér property on the auxiliary sequence $\{z^k\}$ generated by Algorithm A.

Proposition 5 *The auxiliary sequence $\{z^k\}$ generated by Algorithm A is quasi-Fejér convergent to S^* and bounded.*

Proof Take $\bar{x} \in S^*$. Thus, there exists $\bar{u} \in T(\bar{x})$ such that

$$\langle \bar{u}, x - \bar{x} \rangle \geq 0 \quad \forall x \in C. \tag{10}$$

By Proposition 4, with $x = \bar{x}$ and $u = \bar{u}$, and using that $z^k \in C$ for all k, we have

$$\|z^{k+1} - \bar{x}\|^2 \leq \|z^k - \bar{x}\|^2 + L\alpha_k^2 - 2\alpha_k \langle \bar{u}, z^k - \bar{x} \rangle \leq \|z^k - \bar{x}\|^2 + L\alpha_k^2.$$

Establishing that $\{z^k\}$ is quasi-Fejér convergent to S^*. The boundedness of $\{z^k\}$ follows from Proposition 1(i).

Corollary 1 *Let $\{x^k\}$ be the sequence generated by Algorithm A. Then,*

i) $x^k = \dfrac{1}{\sigma_k} \sum_{i=0}^{k} \alpha_i z^i$*, for all k;*
ii) $\{x^k\}$ *is bounded.*

Proof

(i) We proceed by induction on k. For $k = 0$, we have $x^0 = z^0$ by definition. By hypothesis of induction assume that

$$x^k = \frac{1}{\sigma_k} \sum_{i=0}^{k} \alpha_i z^k. \tag{11}$$

Since $\sigma_{k+1} = \sigma_k + \alpha_{k+1}$, we get

$$x^{k+1} = \frac{\sigma_k}{\sigma_{k+1}} x^k + \frac{\alpha_{k+1}}{\sigma_{k+1}} z^{k+1}.$$

By (11) and the above equation, we have

$$x^{k+1} = \frac{1}{\sigma_{k+1}}\sum_{i=1}^{k}\alpha_i z^i + \frac{\alpha_{k+1}}{\sigma_{k+1}}z^{k+1} = \frac{1}{\sigma_{k+1}}\sum_{i=0}^{k+1}\alpha_i z^i,$$

proving the assertion.

(ii) Using Proposition 5 and Proposition 1(i), we have the boundedness of $\{z^k\}$. We may assume that there exists $R > 0$ such that $\|z^k\| \le R$, for all k. By the previous item,

$$\|x^k\| \le \frac{1}{\sigma_k}\sum_{i=0}^{k}\alpha_i\|z^i\| \le R,$$

for all k.

Now we prove that the clusters points of the sequence generated by Algorithm A belong to the solution set.

Theorem 1 *All weak cluster points of* $\{x^k\}$ *belong to* S^*.

Proof Take any $x \in C$ and $u \in T(x)$. Rewriting (8) in Proposition 4, we get,

$$\|z^{i+1} - x\|^2 - \|z^i - x\|^2 - L\alpha_i^2 \le 2\alpha_i\langle u, x - z^i\rangle, \tag{12}$$

for all i. Now summing (12), from $i = 0$ to $i = k$, and dividing by σ_k, we have

$$\frac{1}{\sigma_k}\sum_{i=0}^{k}\left(\|z^{i+1} - x\|^2 - \|z^i - x\|^2 - L\alpha_i^2\right) \le 2\left\langle u, \frac{1}{\sigma_k}\sum_{i=0}^{k}\alpha_i(x - z^i)\right\rangle.$$

Using Corollary 1(i) and define $S := \sum_{i=0}^{\infty}\alpha_i^2$, we get

$$\frac{\|z^{k+1} - x\|^2 - \|z^0 - x\|^2 - LS}{\sigma_k} \le 2\langle u, x - x^k\rangle, \tag{13}$$

for all k.

Let $\bar{x}$ be any weak cluster of $\{x^k\}$, that exist, by Corollary 1(ii). Since $\{z^k\}$ is bounded and $\lim_{k\to\infty}\sigma_k = \infty$, then taking limits in (13), over any weak convergent subsequence to $\bar{x}$, we have $\langle u, x - \bar{x}\rangle \ge 0$ for all $x \in C$ and $u \in T(x)$. By Lemma 4, $\bar{x}$ belongs to the solution set. Hence, all cluster points of $\{x^k\}$ belong to S^*.

Finally, we prove our main result.

Theorem 2 *Define* $x^* = \lim_{k\to\infty} P_{S^*}(z^k)$*. Then, either* $S^* \neq \emptyset$ *and* $\{x^k\}$ *converges weakly to* x^**, or* $S^* = \emptyset$ *and* $\lim_{k\to\infty} \|x^k\| = \infty$*.*

Proof Assume that $S^* \neq \emptyset$ and define $p^k := P_{S^*}(z^k)$. Note that p^k, the orthogonal projection of z^k onto S^*, exists since the solution set S^* is nonempty by assumption, and closed and convex by Lemma 3. By Proposition 1, $\{z^k\}$ is quasi-Fejér convergent to S^*. Therefore, it follows from Lemma 2 that $\{P_{S^*}(z^k)\}$ is strongly convergent. Set

$$x^* := \lim_{k\to\infty} P_{S^*}(z^k) = \lim_{k\to\infty} p^k. \tag{14}$$

By Corollary 1(ii), $\{x^k\}$ is bounded and by Theorem 1 each of its weak cluster points belong to S^*. Let $\{x^{i_k}\}$ be any weakly convergent subsequence of $\{x^k\}$, and let $\bar{x} \in S^*$ be its weak limit. It suffices to show that $\bar{x} = x^*$ for establishing the weak convergence of $\{x^k\}$.

By Lemma 1(ii) we have that $\langle \bar{x} - p^j, z^j - p^j\rangle \leq 0$ for all j. Let $\xi = \sup_{0\leq j\leq\infty} \|z^j - p^j\|$. Since $\{z^k\}$ is bounded by Proposition 5, we get that $\xi < \infty$. Using Cauchy-Schwarz,

$$\langle \bar{x} - x^*, z^j - p^j\rangle \leq \langle p^j - x^*, z^j - p^j\rangle \leq \xi\, \|p^j - x^*\|, \tag{15}$$

for all j. Multiplying (15) by $\dfrac{\alpha_j}{\sigma_k}$ and summing from $j = 0$ to k, we get from Corollary 1(i),

$$\left\langle \bar{x} - x^*, x^k - \frac{1}{\sigma_k}\sum_{j=0}^{k} \alpha_j p^j \right\rangle \leq \frac{\xi}{\sigma_k}\sum_{j=0}^{k} \alpha_j \|p^j - x^*\|. \tag{16}$$

Define

$$\zeta_{k,j} := \frac{\alpha_j}{\sigma_k} \quad (k \geq 0, \quad 0 \leq j \leq k).$$

It follows from the definition of σ_k, that $\lim_{k\to\infty} \zeta_{k,j} = 0$ for all j and $\sum_{j=0}^{k} \zeta_{k,j} = 1$ for all k. Using (14) and Proposition 2 with $p^k = \sum_{j=0}^{k} \zeta_{k,j} p^j = \frac{1}{\sigma_k}\sum_{j=0}^{k} \alpha_j p^j$, we have

$$x^* = \lim_{k\to\infty} p^k = \lim_{k\to\infty} \frac{1}{\sigma_k} \sum_{j=0}^{k} \alpha_j p^j, \tag{17}$$

and

$$\lim_{k\to\infty} \frac{1}{\sigma_k} \sum_{j=0}^{k} \alpha_j \|p^j - x^*\| = 0. \tag{18}$$

Taking limits in (16) over the subsequence $\{i_k\}$, and using (17) and (18), we get $\langle \bar{x} - x^*, \bar{x} - x^* \rangle \le 0$, implying that $\bar{x} = x^*$.

If $S^* = \emptyset$, then by Theorem 1 no subsequence of $\{x^k\}$ can be bounded, and hence $\lim_{k\to\infty} \|x^k\| = \infty$.

4 Final Remarks

We present a direct splitting method for nonsmooth variational inequality, where the operator is a sum of two monotone operators. In the proposed scheme we do not evaluate the resolvent of any individual operator, which represents an important advantage. As a matter for future research we leave the issue of finding the stepsizes though an Armijo-type line search instead of defining them exogenously, at least in the smooth case, i.e., when T is point-to-point. Also we will consider the problem (1), when the constrains set can be expressed as the solution set of another problem, as for instance, functional inequalities, a fixed point problem, an equilibrium problems or even another variational inequality problem. The motivation for this studying such problems comes from the applications in practical problems.

Acknowledgements The authors would like to thank CNPq. Research for this paper was partially supported by PROCAD-nf - UFG/UnB/IMPA and by project CAPES-MES-CUBA 226/2012 "Modelos de Otimização e Aplicações". The second author was supported by his scholarship for his doctoral studies, granted by CAPES.

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