Some Remarks for a Decomposition of Linear-Quadratic Optimal Control Problems for Two-Steps Systems

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Abstract

In this paper we obtained new approach for the problem, which it is described in reference[1,2]. In the references [1], the authors are studied Decomposition of Linear-Quadratic optimal Control problems for Two-Steps Systems. In [1], the authors assumed the switching point t_1 is fixed and it is given algorithm for solving Linear-Quadratic optimal Control problem. But in presented paper author assumed more general case, in the case of switching point is unknown and by using transformation, it is reduced to the problem which is defined in the ref. [1,2]. The unknown switching point case is more practical than the case of known switching point (see, ref. [3-12])
Keywords: Optimal control, switching system.

1 Introduction

In the paper [1], it is studied following following minimizing optimal control problem:
ProblemI: Minimizing the functional

$$J(u,t_1) = \frac{1}{2}\langle C_1x_1(t_1) - C_2x_2(t_1), F\left(C_1x_1(t_1) - C_2x_2(t_1)\right)\rangle \qquad (1.1)$$
$$+\sum_{j=1}^{2}\int_{t_{j-1}}^{t_j}(\langle x_j(t), W_j(t)x_j(t)\rangle + \langle u_j(t), R_j(t)u_j(t)\rangle)dt$$

, where $u=(u_1,u_2)$, (in the reference [1], the cost functional is not depend from the point t_1, because intermediate point is fixed) with respect to trajectories of the system

$$\dot{x}_j(t) = A_j(t)x_j(t) + B_j(t)u_j(t), \quad t_{j-1} \le t \le t_j, \; j = 1,2 \tag{1.2}$$

$x_1(0) = x^0, \; x_2(T) = x^T$.
Here, $0 = t_0 < t_1 < t_2 = T$, the values $t_1, \; t_2$ are fixed; $x_j(t) \in X_j, \; u_j(t) \in U_j, \; A_j(t), W_j \in L(X_j), B_j(t) \in L(U_j, X_j), \; R_j(t) \in L(U_j)$ for all $t \in [t_{j-1}, t_j], j = 1,2$; $C_1 \in L(X_1, Y), C_2 \in L(X_2, Y), F \in L(Y), X_j, U_j, Y$ are real finite dimensional Euclidean spaces, the operators $F, W_j(t) \ge 0, R_j(t) > 0$ for all $t \in [t_{j-1}, t_j]; x^0 \in X_1, \; x^T \in X_2$ are given and symmetric, the operators F, C_1, C_2 are independent of t, but the other operators depend continually on t in the corresponding segment $[t_{j-1}, t_j], j = 1, 2, < .,. >$ means an inner product in appropriate spaces.
Note1. In the reference [1], it is assumed the intermediate point t_1 is fixed. For this, the minimization functional has the form
$J(u) = \frac{1}{2}\langle C_1x_1(t_1) - C_2x_2(t_1), F\left(C_1x_1(t_1) - C_2x_2(t_1)\right)\rangle$
$+ \sum_{j=1}^{2} \int_{t_{j-1}}^{t_j} (\langle x_j(t), W_j(t)x_j(t)\rangle + \langle u_j(t), R_j(t)u_j(t)\rangle)dt$,
i,e., in the paper [1,2], minimization functional is not depend from the switching point t_1 because t_1 is fixed. But in the present paper, it is considered the point t_1 is unknown.
Let us make following substitution $u(t) = (u_1(t), u_2(t))$ and $x(t) = (x_1(t), x_2(t))$.
DefinitionI: The triple $w = (t_1, u(t), x(t)$ is called admissible, if it satisfies all constraints of the *ProblemI* (about the constraints see, ref.[1])
DefinitionII: The triple $w^0 = (t_1, u(t), x(t))$ is called optimal control, if $J(w^0) \le J(w)$ for all admissible process w.

2 Transformation

Let us take following transformation. Assume new parameter x_{n+1} such us satisfies following differential equation with initial condition in $[t_0, t_2]$,
$\frac{dx_{n+1}(t)}{dt} = 0$ with initial condition $x_{n+1}(0) = t_1$. It means x_{n+1} is constant in $[t_0, t_2]$.
Next, a new independent time variable τ is introduced as:

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, 0 \le \tau < 1; \\ x_{n+1} + (t_2 - x_{n+1})(\tau - 1), 1 \le \tau \le 2. \end{cases} \tag{2.1}$$

Then we can write that

$$dt = \begin{cases} (x_{n+1} - t_0)d\tau, 0 \le \tau < 1; \\ (t_2 - x_{n+1})d\tau, 1 \le \tau \le 2. \end{cases} \tag{2.2}$$

Clearly, $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ corresponds to $t = t_1$, and $\tau = 2$ to $t = t_2$. By introducing x_{n+1} and τ, and substitutions $y_i(\tau) = x_i(t(\tau)), \; v_i(\tau) = u_i(t(\tau)), i = 1,2$ main problem is transcribed into the following equivalent form.

ProblemII:

$$\frac{dy_1(\tau)}{d\tau} = (x_{n+1} - t_0)\,(A_1(\tau)y_1(\tau) + B_1(\tau)v_1(\tau)) \tag{2.3}$$

$$\frac{dx_{n+1}}{d\tau} = 0, x_{n+1}(0) = t_1, \tau \in [0,1) \tag{2.4}$$

and

$$\frac{dy_2(\tau)}{d\tau} = (t_2 - x_{n+1})\,(A_2(\tau)y_2(\tau) + B_2(\tau)v_2(\tau)) \tag{2.5}$$

$$\frac{dx_{n+1}}{d\tau} = 0, x_{n+1}(0) = t_1, \tau \in [1,2] \tag{2.6}$$

and minimizing functional takes the form

$$\tilde{J}(v, x_{n+1}) = \frac{1}{2}\langle C_1y_1(1) - C_2y_2(1), F(C_1y_1(1)) - C_2y_2(1))\rangle +$$
$$\int_0^1 (x_{n+1} - t_0)(\langle y_1(\tau), W_1(\tau)y_1(\tau)\rangle + \langle v_1(\tau), R_1(\tau)v_1(\tau)\rangle)d\tau +$$
$$\int_1^2 (t_2 - x_{n+1})(\langle y_2(\tau), W_2(t)y_2(\tau)\rangle + \langle v_2(\tau), R_2(\tau)v_2(\tau)\rangle)d\tau \tag{2.7}$$

After this transformation, we reduce *ProblemI*, to the *ProblemII*. In the *ProblemII*, state trajectory is $y(\tau) = (y_1(\tau), y_2(\tau))$ and control is
$v(\tau) = (v_1(\tau), v_2(\tau), x_{n+1}), 0 \leq \tau \leq 2$

Note2: Since x_{n+1} is unknown constant (parameter) in the interval $[0, 2]$ (see, (2.4) and (2.6)), after the transformation, the dimensional of the *ProblemII* will be same as the dimensional of the *ProblemI*. There is one-to-one corresponding between admissible process $(t_1, x(t), u(t))$ and the admissible process $(y(\tau), v(\tau))$.

In fact by using transformation from the admissible process $(t_1, x(t), u(t))$, we obtained admissible process $(y(\tau), v(\tau))$. Let us prove inverse opinion. If $(y(\tau), v(\tau)$ is admissible process (where $v(\tau) = (v_1(\tau), v_2(\tau))$ in problem (2.3)-(2.6), then by using relation(2.1) we can say, if we take $\tau = 0$ then $t = t_0$, $\tau = 1$ then $t = x_{n+1}$ (in fact $x_{n+1}(0) = t_1$), and for $\tau = 2$ then $t = t_2$. It means we obtained intervals $[t_0, t_1]$ and interval $[t_1, t_2]$. From the relation (2.1), we can $\tau = \frac{t-t_0}{x_{n+1}-t_0}$, $0 \leq \tau \leq 1$ and $\tau = \frac{t-x_{n+1}}{t_2-x_{n+1}}$, $1 \leq \tau \leq 2$. Then, if we denote, $x_1(t) = y_1(\tau(t))$ and $x_2(t) = y_2(\tau(t))$ then we obtain, $\dot{x}_1 = \dot{y}_1(\tau(t))\frac{1}{x_{n+1}-t_0}$ and $\dot{x}_2 = \dot{y}_2(\tau(t))\frac{1}{t_2-x_{n+1}}$. If we consider this in the equations (2.3) and (2.5), we can come to the point that, $(t_1, x(t), u(t))$ is the admissible process for the equations (2.1).

Note3: This corresponding between the admissible processes $(t_1, x(t), u(t))$ and

$(y(t), v(t))$ for the equations (1.2) and (2.3),(2.5) preserve the value of the cost functionals (1.1) and (2.7).
In fact, assume process $(t_1^0, x^0(t), u^0(t))$ is optimal control for the *ProblemI*. Let us take process $(y^0(\tau), v^0(\tau))$, which is obtained from the optimal process $(t_1^0, x^0(t), u^0(t))$ above mentioned transformation. Assume that,
$(y^0(\tau), v^0(\tau))$, is not optimal process and there exist another optimal process $(\tilde{y}(\tau), \tilde{v}(\tau))$ with $\tilde{J}(\tilde{y}(\tau), \tilde{v}(\tau)) \leq J(y^0(\tau), v^0(\tau))$. Take corresponding admissible process, which is obtained inverse transformation from the process $(\tilde{x}_{n+1}, \tilde{y}(\tau), \tilde{v}(\tau))$ and denote it by $(t_1, u(t), x(t))$. Then it is clear that,
$J(t_1, u(t), x(t)) = \tilde{J}(\tilde{y}(\tau), \tilde{v}(\tau)) \leq \tilde{J}(y^0(\tau), v^0(\tau) = \tilde{J}(t_1^0, x^0(t), u^0(t))$.
But it is contradiction of the optimality of the process $(t_1^0, x^0(t), u^0(t))$.
The inverse opinion can be prove same way.
Note4. Then we can say, if the process $(t_1^0, x^0(t), u^0(t))$ gives minimum for the *ProblemI*, then the process $(y^0(\tau), v^0(\tau))$, which is obtained after transformation, gives minimum value for the *ProblemII*, and vice versa.
Note5. If there are K numbers of switchings, then it is no difficulty in applying the previous method to the problems with several subsystems. If there exist nonfixed the switchings, $t_1, t_2, ..., t_K$ with $0 = t_0 < t_1 < t_2 <, ..., < t_K < T = 0$, then we can transcribe the problem into an equivalent problem by introducing K new state variables $x_{n+1}, x_{n+2}, ..., x_{n+K}$ which correspond to the switching instants $t_1, t_2, ..., t_K$ and satisfies,
$\frac{dx_{n+i}}{d\tau} = 0, x_{n+i}(0) = t_i, \tau \in [1, 2],\ i = 1, 2, ..., K$
The new independent time variable τ has a linear relationships with t where $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ corresponds to $t = t_1 .... \tau = K + 1$ corresponds to $t = t_T$.

3 An Example.

This example is taken from the ref. [2], but it is added the case of the switching point t_1 to be unfixed.We will try reduce the unknown switching case to the known switching case, which after this can be used all the procedure in the ref. [1].
Consider following problem of minimizing the functional,
$J(x, u, t_1) = \frac{1}{2}((x_{11}(t_1) + x_{21}(t_1)) + \int_0^{t_1}(x_{11}^2(t) + 2x_{11}(t)x_{12}(t) + x_{12}^2(t) + u_1(t))dt + \int_{t_1}^2(x_{21}^2(t) + 8x_{22}^2(t) + u_2^2(t)))dt$, where $u = (u_1, u_2)$
with respect the trajectories of the system
$\dot{x}_{11}(t) = x_{11}(t),\quad x_{12}(t) + u_1(t) = 0,\ x_{11}(0) = -1,\ t \in [0, t_1]$
$\dot{x}_{21}(t) = 0\quad x_{22}(t) - u_2(t) = 0,\ x_{21}(2) = 1,\ t \in [t_1, 2]$
Let us following transformation. For this aim, take new variable $\dot{x}_{n+1}(t) = 0,\ x_{n+1}(0) = t_1$. From this differential equation, it is clear $x_{n+1} = t_1$ is unknown constant in $[0, 2]$. Take also, $y_{i,j}(\tau) = x_{i,j}(t(\tau)),\ v_i(\tau) = u_i(t(\tau))$ where $i, j = 1, 2$. Let us use also interval transformation in (2.1) with $t_0 = 0$ and $t_2 = 2$. Then we can come the point that, if $\tau = 0$ then $t = 0$, if $\tau = 1$ then $t = x_{n+1} = t_1$, and, if $\tau = 2$ then $t = 2$. If we use all these transformation, then minimizing

functional and state equations will take following form,
$J(y,v,t_1) = \frac{1}{2}((y_{11}(1)+y_{21}(1))^2 + t_1\int_0^1 (y_{11}^2(\tau)+2y_{11}(\tau)y_{21}(\tau)+3y_{12}^2+v_1(\tau))d\tau +$
$(2-t_1)\int_1^2 (y_{21}^2(\tau)+8y_{22}^2(\tau)+v_2^2(\tau)))d\tau$, where $v=(v_1,v_2)$,
and state equations takes the form
$\dot{y}_{11}(\tau) = t_1 y_{11}(\tau), \quad y_{11}(0)=0,\ \tau\in[0,1]$,
$\dot{y}(\tau) = (2-0)0 = 0,\ y_{22}(\tau)-v_2(\tau)=0,\ \tau\in[1,2]$.
We can see, state equations are defined in known intervals $[0,1]$ and $[1,2]$, the boundary of the integral of minimizing functional is known as in example the ref. [2]. Only difference is that, right side of the minimizing functional appears new variable t_1. But it is constant unknown all the interval $[0,1]$ and is not change dimensionality of the control v. Then we can consider the couple (v,t_1) as a new control and after this we can apply all the procedure and algorithms in (ref.1, theorem3, (8),(17),(18),(20),(22),(23),(24)).

4 Conclusion. We considered New approach for the Decomposition of Linear-Quadratic Optimal Control problems for Two-Steps Systems, which is studied in ref.[1]. But, for this system, we assumed switching point is unknown. After suitable transformation, we reduced this problem to the problem with known intermediate point. After these, we can consider the theorem3 in the ref.[1], use all the procedure and algorithms in (ref.1, theorem3, (8),(17),(18),(20),(22),(23),(24)).

References

[1] G. A. Kurina and Y. Zhou, Decomposition of Linear-Quadratic Optimal Control problems for Two-Steps Systems, Doklady Mathematics, 2011, Vol. 83, No.2, pp.275-277.

[2] G. Kurina, On Decompasition of Linear-Quadratic Optimal Control Problems for Two-Steps Descriptor Systems, 50th IEEE Conference on Decision and Control and European Control Conference, 2011, p.6705-6711.

[3] P. J. Antsaklis and A. Nerode., Special issue on hybrid system, IEEE Trans. Automat. Control 43 (1998), no.4.,

[4] A. Bensoussan and J.Menaldi, Hybrid Control and dynamic programming, 43(1997),pp.475-482.

[5] Sh. F. Maharramov, Necessary optimality Conditions for Switching Control System, American Institute for Mathematical Science, Journal of Industrial and Manegament Optimization, 6(2010), pp. 47-58.

[6] V. Boltyanskii, The maximum principle for variable structure systems, International Journal of Control. 77(2004), pp.1445-1451.

[7] Sh. F. Magerramov(Maharramov) and K.B. Mansimov, Optimization of a class of discrete step control systems. (Russian, English) Comput. Math.

Math. Phys. 41(2001)3, pp.334-339. translation from Zh. Vychisl. Mat. Mat. Fiz. 41(3),2001, pp.360-366.

[8] R. M. Caravello and B. Piccoli, Hybrid Necessary Principle, preprint SSSA 71(2002).

[9] A. v. Dmitruk and A. M. kaganovich, The Hybrid Maximum Principle is a Consequence of Pontryagin Maximum Principle, Sstem and Control Letters, vol. 57, 2008, pp. 964-970

[10] R. Li, K.L. Teo, K.H. Wong and G.R. Duan, Control parameterization enhancing transform for optimal control of switched systems,Mathematical and Computer Modelling, Volume 43, Issues 1112, June 2006, Pages 1393-1403

[11] Honglei Xu, Xinzhi Liu and Kok Lay Teo, Delay independent stability criteria of impulsive switched systems with time-invariant delays, Mathematical and Computer Modelling, Volume 47, Issues 34, February 2008, p. 372-379

[12] Azmyakov, V. V; Galvan-Guerra R and Polyakov, A. E,On the method of dynamic programming for linear-quadratic problems of optimal control in hybrid systems, Automation and Remote Control,Volume: 70 , Issue: 5 p. 787-799, 2009