New RIC Bounds via ℓ_q-minimization with $0 < q \leq 1$ in Compressed Sensing

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***Abstract*—The restricted isometry constants (RICs) play an important role in exact recovery theory of sparse signals via $\ell_q(0 < q \leq 1)$ relaxations in compressed sensing. Recently, Cai and Zhang [6] have achieved a sharp bound $\delta_{tk} < \sqrt{1-1/t}$ for $t \geq \frac{4}{3}$ to guarantee the exact recovery of k sparse signals through the ℓ_1 minimization. This paper aims to establish new RICs bounds via $\ell_q(0 < q \leq 1)$ relaxation. Based on a key inequality on ℓ_q norm, we show that (i) the exact recovery can be succeeded via $\ell_{1/2}$ and ℓ_1 minimizations if $\delta_{tk} < \sqrt{1-1/t}$ for any $t > 1$, (ii)several sufficient conditions can be derived, such as for any $q \in (0, \frac{1}{2})$, $\delta_{2k} < 0.5547$ when $k \geq 2$, for any $q \in (\frac{1}{2}, 1)$, $\delta_{2k} < 0.6782$ when $k \geq 1$, (iii) the bound on δ_k is given as well for any $0 < q \leq 1$, especially for $q = \frac{1}{2}, 1$, we obtain $\delta_k < \frac{1}{3}$ when $k(\geq 2)$ is even or $\delta_k < 0.3203$ when $k(\geq 3)$ is odd.**

***Index Terms*—compressed sensing, restricted isometry constant, bound, ℓ_q minimization, exact recovery**

I. Introduction

The concept of compressed sensing (CS) was initiated by Donoho [13], Candès, Romberg and Tao [7] and Candès and Tao [8] with the involved essential idea–recovering some original n-dimensional but sparse signal\image from linear measurement with dimension far fewer than n. Large numbers of researchers, including applied mathematicians, computer scientists and engineers, have paid their attention to this area owing to its wide applications in signal processing, communications, astronomy, biology, medicine, seismology and so on, see, e.g., survey papers [1], [19] and a monograph [14].

To recover a sparse solution $x \in \mathbb{R}^n$ of the underdetermined system of the form $\Phi x = y$, where $y \in \mathbb{R}^m$ is the available measurement and $\Phi \in \mathbb{R}^{m\times n}$ is a known measurement matrix (with $m \ll n$), the underlying model is the following ℓ_0 *minimization*:

$$\min \|x\|_0, \quad \text{s.t. } \Phi x = y, \tag{1}$$

where $\|x\|_0$ is ℓ_0-norm of the vector $x \in \mathbb{R}^n$, i.e., the number of nonzero entries in x (this is not a true norm, as $\|\cdot\|_0$ is not positive homogeneous). However (1) is combinatorial and computationally intractable.

One natural approach is to solve (1) via convex ℓ_1 *minimization*:

$$\min \|x\|_1, \quad \text{s.t. } \Phi x = y. \tag{2}$$

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The other way is to relax (1) through the nonconvex $\ell_q(0 < q < 1)$ *minimization*:

$$\min \|x\|_q^q, \quad \text{s.t. } \Phi x = y, \tag{3}$$

where $\|x\|_q^q = \sum_j |x_j|^q$. Motivated by the fact $\lim_{q\to 0^+} \|x\|_q^q = \|x\|_0$, it is shown that there are several advantages of using this approach to recover the sparse signal [18]. This model for recovering the sparse solution is widely considered, see [9], [10], [11], [12], [15], [16], [17], [18], [20].

One of the most popular conditions for exact sparse recovery via ℓ_1 or ℓ_q minimization is related to the *Restricted Isometry Property* (RIP) introduced by Candès and Tao [8], which was recalled as follows.

Definition I.1. *For $k \in \{1, 2, \cdots, n\}$, the restricted isometry constant is the smallest positive number δ_k such that*

$$(1-\delta_k)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\delta_k)\|x\|_2^2 \tag{4}$$

holds for all k-sparse vector $x \in \mathbb{R}^n$, i.e., $\|x\|_0 \leq k$.

It is known that δ_k has the monotone property for k (see, e.g., [2], [3]), i.e.,

$$\delta_{k_1} \leq \delta_{k_2}, \quad \text{if} \quad k_1 \leq k_2 \leq n. \tag{5}$$

Current upper bounds on the restricted isometry constants (RICs) via $\ell_q(0 < q < 1)$ minimization for exact signal recovery were emerged in many studies [9], [12], [15], [16], [17], [18], [20], such as $\delta_{2k} < 0.4531$ for any $q \in (0, 1]$ in [16], $\delta_{2k} < 0.4531$ for any $q \in (0, q_0]$ with some $q_0 \in (0, 1]$ in [18] and $\delta_{2k} < 0.5$ for any $q \in (0, 0.9181]$ in [20]. Comparing with those RIC bounds, Cai and Zhang [6] recently have given a sharp bound $\delta_{2k} < \frac{\sqrt{2}}{2}$ via ℓ_1 minimization.

Motivated by results above, we make our concentrations on improving RIC bounds via ℓ_q relaxation with $0 < q \leq 1$. The main contributions of this paper are the following three aspects:

(i) If the restricted isometry constant of Φ satisfies $\delta_{tk} < \sqrt{(t-1)/t}$ for $t > 1$, which implies $\delta_{2k} < \frac{\sqrt{2}}{2}$, then exact recovery can be succeeded via $\ell_{\frac{1}{2}}$ and ℓ_1 minimizations.

(ii) For any $k \geq 1$, the bound for δ_{2k} is an nondecreasing function on $q \in (0, \frac{1}{2})$ and $q \in (\frac{1}{2}, 1)$. Moreover, several sufficient conditions are derived, such as for any $q \in (0, \frac{1}{2})$, $\delta_{2k} < 0.5547$ when $k \geq 2$, for any $q \in (\frac{1}{2}, 1)$, $\delta_{2k} < 0.6782$ when $k \geq 1$. The detailed can be seen in Tab. 2 of the Section III, which are all better bounds than current ones in terms of $\ell_q(0 < q < 1)$ minimization.

(iii) The bound on δ_k is given as well for any $0 < q \leq 1$. Especially for $q = \frac{1}{2}, 1$, we obtain $\delta_k < \frac{1}{3}$ when k is even or $\delta_k < 0.3203$ when $k(\geq 3)$ is odd.

The organization of this paper is as follows. In the next section, we establish several key lemmas. Our main results on δ_{tk} with $t > 1$ and δ_k will be presented in Sections III and IV respectively. We make some concluding remarks in Section V and give the proofs of all lemmas and theorems in the last section.

II. Key Lemmas

This section will propose several technical lemmas, which play an important role in the sequel analysis. We begin with recalling the lemma of the sparse representation of a polytope stated by Cai and Zhang [6]. Here, we define $\|x\|_\infty := \max_i \{|x_i|\}$ and $\|x\|_{-\infty} := \min_i \{|x_i|\}$ (In fact, $l_{-\infty}$ is not a norm since the triangle inequality fails).

Lemma II.1. *For a positive number α and a positive integer s, define the polytope $T(\alpha, s) \subset \mathbb{R}^n$ by*

$$T(\alpha, s) = \{v \in \mathbb{R}^n \mid \|v\|_\infty \le \alpha, \|v\|_1 \le s\alpha\}.$$

For any $v \in \mathbb{R}^n$, define the set $U(\alpha, s, v) \subset \mathbb{R}^n$ of sparse vectors by

$$U(\alpha, s, v) = \{u \in \mathbb{R}^n \mid supp(u) \subseteq supp(v), \|u\|_0 \le s, \|u\|_1 = \|v\|_1, \|u\|_\infty \le \alpha\}.$$

Then $v \in T(\alpha, s)$ if and only if v is in the convex hull of $U(\alpha, s, v)$. In particular, any $v \in T(\alpha, s)$ can be expressed as $v = \sum_{i=1}^N \lambda_i u_i$, where

$$0 \le \lambda_i \le 1, \sum_{i=1}^N \lambda_i = 1, u_i \in U(\alpha, s, v), i = 1, 2, \cdots, N.$$

Next we establish an interesting and important inequality in the following lemma, which gives a sharpened estimation of ℓ_1 with $\ell_0, \ell_q, \ell_\infty$ and $\ell_{-\infty}$.

Lemma II.2. *For $q \in (0, 1]$ and $x \in \mathbb{R}^n$, we have*

$$\|x\|_1 \le \frac{\|x\|_q}{n^{1/q-1}} + p_q n(\|x\|_\infty - \|x\|_{-\infty}), \tag{6}$$

where

$$p_q := q^{\frac{q}{1-q}} - q^{\frac{1}{1-q}}. \tag{7}$$

Moreover, p_q is a nonincreasing and convex function of $q \in [0, 1]$ with

$$p_0 := \lim_{q\to 0^+} p_q = 1 \text{ and } p_1 := \lim_{q \to 1^-} p_q = 0.$$

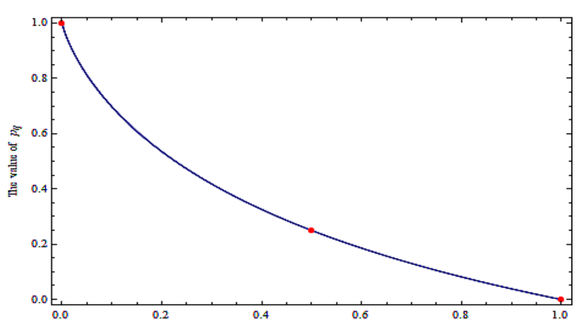

Fig. 1. Plot of $p_q \in [0, 1]$ as a function of $q \in [0, 1]$, and $p_{\frac{1}{2}} = \frac{1}{4}$.

Remark II.3. *Actually, we can substitute n with $\|x\|_0$ in inequality (6), which leads to*

$$\|x\|_1 \le \frac{\|x\|_q}{\|x\|_0^{1/q-1}} + p_q \|x\|_0(\|x\|_\infty - \|x\|_{-\infty}). \tag{8}$$

Moreover, combining with the Hölder Inequality and (8), we have

Proposition II.4. *For $q \in (0, 1]$ and $x \in \mathbb{R}^n$, we have*

$$\|x\|_0^{1-\frac{1}{q}} \|x\|_q \le \|x\|_1 \le \left(\|x\|_0^{1-\frac{1}{q}} + p_q\|x\|_0\right) \|x\|_q. \tag{9}$$

Here, (9) is an interesting inequality. Although (9) will not be applied in our proof, it manifests the relationship between ℓ_1 and ℓ_q norm.

In order to analyze a sequent useful function more clearly, we first observe the function $q^{\frac{q}{q-1}}$ of $q \in (0, 1)$, whose figure is plotted below.

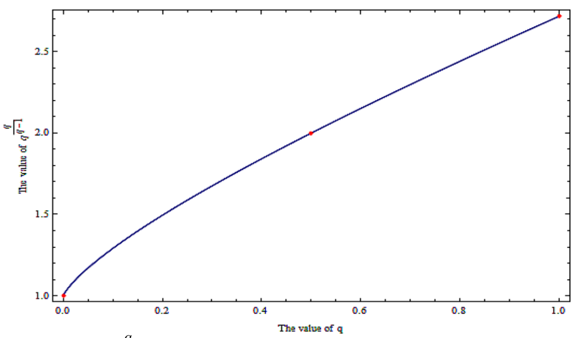

Fig. 2. Plot of $q^{\frac{q}{q-1}}$ as a function of $q \in [0, 1]$.

It is easy to check that

$$\lim_{q \to 0^+} q^{\frac{q}{q-1}} = 1, \quad \lim_{q\to 1^-} q^{\frac{q}{q-1}} = e. \tag{10}$$

So $q^{\frac{q}{q-1}}$ can be defined as a function of q on $[0, 1]$, and it is a nondecreasing function.

In addition, for any given integer $k \ge 1$, it is trivial that if $q^{\frac{q}{q-1}}$ is an integer, then $q^{\frac{q}{q-1}} k$ apparently is an integer as well for instance $q = 1/2$. However, the integrity of $q^{\frac{q}{q-1}}$ is not necessary to ensure the integrity of $q^{\frac{q}{q-1}} k$, such as $q = 2/3$ and $k = 4$.

Based on analysis above, we now define a real valued function $g(q, k) : (0, 1) \times \{1, 2, 3, \cdots\} \to \mathbb{R}$ by

$$g(q, k) := \lceil q^{\frac{q}{q-1}} k\rceil^{1-1/q} k^{1/q} + p_q \lceil q^{\frac{q}{q-1}} k\rceil, \quad q \in (0, 1), \ k \in \{1, 2, 3, \cdots\}, \tag{11}$$

where p_q is defined as in (7) and $\lceil a\rceil$ denotes the smallest integer that is no less than a.

Lemma II.5. *Let $g(q, k)$ be defined as in (11). Then $g(q, k) = k$ when $q^{\frac{q}{q-1}} k$ is an integer and otherwise $g(q, k) \le k + p_q$. Moreover,*

$$g(0, k) := \lim_{q\to 0^+} g(q, k) = k + 1,$$
$$g(1, k) := \lim_{q \to 1^-} g(q, k) = k.$$

Therefore, $g(q, k)$ can be regarded as a function of q on $[0, 1]$, and the image of $g(q, k)$ with the special case $k = 1$, where $g(0, 1) = 2, g(\frac{1}{2}, 1) = 1, g(1, 1) = 1$, is plotted in Fig.3.

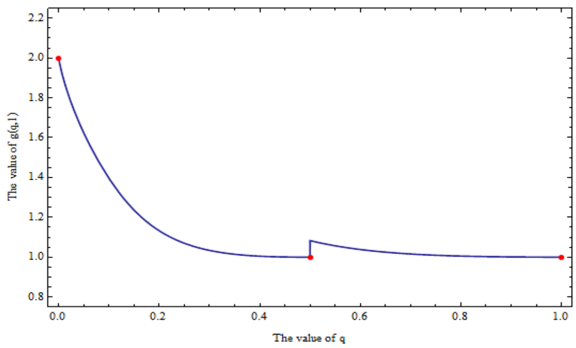

Fig. 3. Plot of $g(q,1)$ as a function of $q \in [0,1]$.

Another two useful functions are introduced and analyzed in the following lemma, which will ease sequent analysis of our main results.

Lemma II.6. *For $t > 1$ and $\theta \geq 0, \rho \geq 0$, we define*

$$\mu(t,\theta) := \frac{\sqrt{(t+\theta-1)(t-1)}+1-t}{\theta}, \tag{12}$$

$$\gamma(\rho,\theta) := \frac{\rho-\rho^2}{\frac{1}{2}-\rho+\rho^2(1+\frac{\theta}{2(t-1)})}. \tag{13}$$

Then $\gamma(\mu(t,\theta),\theta)$ is a nonincreasing function on θ when t is fixed while a nondecreasing function on t when θ is fixed.

III. Main Results on δ_{tk} with $t > 1$

Now we give our main results on δ_{tk} with $t > 1$:

Theorem III.1. *For any $q \in (0,1]$, if*

$$\delta_{g(q,k)(t-1)+k} < \gamma\left(\mu\left(t, \frac{g(q,k)}{k}\right), \frac{g(q,k)}{k}\right) \tag{14}$$

holds for some $t > 1$, then each k-sparse minimizer of the ℓ_q minimization (3) is the sparse solution of (1). Furthermore, setting $t = 1+\frac{(\tau-1)k}{g(q,k)}$ with $\tau > 1$, then the sufficient condition (14) of exact signal recovery can be reformulated as

$$\delta_{\tau k} < \gamma\left(\mu\left(1+\frac{(\tau-1)k}{g(q,k)}, \frac{g(q,k)}{k}\right), \frac{g(q,k)}{k}\right). \tag{15}$$

From Lemma II.5, when $q = 1$ or $q^{\frac{q}{q-1}}k$ is an integer (such as $q = \frac{1}{2}$), it follows that $g(q,k) = k$. Associating with (14) in Theorem III.1, we have $\delta_{tk} = \delta_{g(q,k)(t-1)+k} < \gamma(\mu(t,1),1) = \sqrt{\frac{t-1}{t}}$. Therefore, a corollary can be elicited as below.

Corollary III.2. *For $q = 1$ or $q \in (0,1)$ such that $q^{\frac{q}{q-1}}k$ is an integer, if $\delta_{tk} < \sqrt{\frac{t-1}{t}}$ holds with some $t > 1$ and $k \geq 1$, then each k-sparse minimizer of the ℓ_q minimization (3) is the sparse solution of (1).*

In particular, taking $t = 2,3,4$, we obtain $\delta_{2k} < \frac{\sqrt{2}}{2} \approx 0.7071$, $\delta_{3k} < 0.8164$, $\delta_{4k} < 0.8660$ respectively. It is worth mentioning that $\delta_{tk} < \sqrt{\frac{t-1}{t}}$ is the sharp bound for ℓ_1 minimization which has been proved by Cai and Zhang [6]. Because exact recovery can fail for any $q \in (0,1]$ if the bound of δ_{2k} is no less than $\frac{\sqrt{2}}{2}$ (see [12]), $\delta_{2k} < \frac{\sqrt{2}}{2}$ is also the sharp bound for $\ell_{\frac{1}{2}}$ minimization.

Actually, besides $q = \frac{1}{2}$, $k \geq 1$, there are several other (q,k)s satisfying that $q^{\frac{q}{q-1}}k$ are integers, for instance $(0.2025, 2)$, $(\frac{2}{3}, 4)$. Thus $\delta_{tk} < \sqrt{\frac{t-1}{t}}$ is also a sharp RIC bound for such (q,k)s.

Remark III.3. *(i) For any $k \geq 1$, we can check*

$$g(q,1) \geq \frac{g(q,k)}{k}.$$

Then from Lemma II.6 and (15) in Theorem III.1, for $k \geq 1$ and any $q \in (0,1]$, it yields that

$$\delta_{\tau k} < \gamma\left(\mu\left(1+\frac{\tau-1}{g(q,1)}, g(q,1)\right), g(q,1)\right), \tag{16}$$

whose figure (with $\tau = 2$) is plotted as follows.

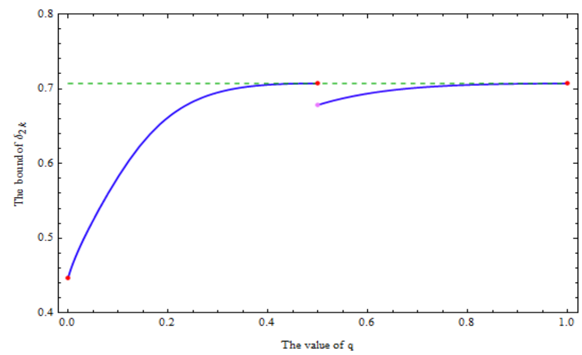

Fig. 4. Plot of bounds on δ_{2k} as a function of $q \in (0,1]$ when $k \geq 1$.

(ii) Moreover, under some assumptions $k \geq k_0(k_0 = 1,2,3,\cdots)$, since for $q \in (\frac{1}{2}, 1]$

$$\lim_{q\to\frac{1}{2}^+} \frac{g(q,k_0)}{k_0} \geq \max\{\lim_{q\to\frac{1}{2}^+} \frac{g(q,k)}{k}, \frac{g(q,k_0)}{k_0}\}$$

and for $q \in (0, \frac{1}{2}]$

$$\lim_{q\to 0^+} \frac{g(q,k_0)}{k_0} \geq \max\{\lim_{q\to 0^+} \frac{g(q,k)}{k}, \frac{g(q,k_0)}{k_0}\}.$$

Then from Lemma II.6, we have Tab. 2 by calculating limits for cases $q \to 0^+$ and $q \to \frac{1}{2}^+$ of the right-hand side of (15) with $k = k_0$.

	δ_{2k}		δ_{3k}		δ_{4k}	
	$q \in (0,\frac{1}{2})$	$q \in (\frac{1}{2},1)$	$q \in (0,\frac{1}{2})$	$q \in (\frac{1}{2},1)$	$q \in (0,\frac{1}{2})$	$q \in (\frac{1}{2},1)$
$k \geq 1$	0.4472	0.6782	0.5773	0.7938	0.6546	0.8478
$k \geq 2$	0.5547	0.6983	0.6859	0.8096	0.7559	0.8605
$k \geq 3$	0.6000	0.7029	0.7276	0.8132	0.7924	0.8634
$k \geq 4$	0.6246	0.7046	0.7492	0.8146	0.8108	0.8645

Tab. 2: Bounds on $\delta_{2k}, \delta_{3k}, \delta_{4k}$ for any $q \in (0, \frac{1}{2})$ and $q \in (\frac{1}{2}, 1)$.

IV. Main Results on δ_k

In this section, we state the bound on δ_k for any $q \in (0,1]$ in the following results:

Theorem IV.1. *For any $q \in (0,1]$, if*

$$\delta_k < \begin{cases} \dfrac{1}{1+2\lceil g(q,k)\rceil/k}, & \textit{for even number } k \geq 2, \\ \dfrac{1}{1+2\lceil g(q,k)\rceil/\sqrt{k^2-1}}, & \textit{for odd number } k \geq 3, \end{cases}$$

holds, then each k-sparse minimizer of the ℓ_q minimization (3) *is the sparse solution of* (1).

Particularly, for the case $q = 1$ or $q^{\frac{q}{q-1}}k$ to be an integer (such as $q = \frac{1}{2}$), we have the corollary below by applying Lemma II.5.

Corollary IV.2. *For $q = 1$ or $q \in (0,1)$ such that $q^{\frac{q}{q-1}}k$ is an integer, if*

$$\delta_k < \begin{cases} 1/3, & \textit{for even number } k \geq 2, \\ \dfrac{1}{1+2k/\sqrt{k^2-1}}, & \textit{for odd number } k \geq 3, \end{cases}$$

hold, then each k-sparse minimizer of the ℓ_q minimization (3) *is the sparse solution of* (1).

Taking $q = \frac{1}{2}, 1$, then $g(q,k) = k$ from Lemma II.5, which produces the bound $\delta_k < \frac{1}{3}$ if $k \geq 2$ is even. Meanwhile $\delta_k < \frac{1}{3}$ for $k \geq 2$ is the sharp bound for ℓ_1 minimization that has been gotten by Cai and Zhang [4]. From Theorem IV.1 and Corollary IV.2, we list the following table.

	$q \in (0,1)\setminus\left\{\frac{1}{2}\right\}$		$q = \frac{1}{2}, 1$
k is even	$k \geq 2$	$\delta_k < 0.2500$	$\delta_k < \frac{1}{3}$
	$k \geq 4$	$\delta_k < 0.2857$	
k is odd	$k \geq 3$	$\delta_k < 0.2612$	$\delta_k < 0.3203$
	$k \geq 5$	$\delta_k < 0.2898$	$\delta_k < 0.3288$

Tab. 3: Upper bounds on δ_k for different q.

V. Concluding Remarks

In this paper, we have generalized the upper bounds for RICs from ℓ_1 minimization to $\ell_q (0 < q \leq 1)$ minimization, and established new RIC bounds through ℓ_q minimization with $q \in (0,1]$ for exact sparse recovery. An interesting issue which deserves future research would be: how to improve these new bounds for some $q \in (0,1]$ when $q^{\frac{q}{q-1}}k$ is not an integer.

VI. Proofs

Proof of Lemma II.2

Stimulated by the approach in [20], without loss of generality, we only need to prove the case $x \in \Omega := \{(x_1, x_2, \cdots, x_n) \neq 0 \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$ due to the symmetry of components $|x_1|, |x_2|, \cdots, |x_n|$. Clearly, $x_1 \neq 0$. Notice that if the inequality (6) holds for any $(1, x_2, \cdots, x_n) \in \Omega$, then we can immediately generalize the conclusion to all $x \in \Omega$ through substituting $x/x_1, x \in \Omega$ into (6) and eliminating the common factor $1/x_1$. Henceforth, it remains to show

$$\|x\|_1 \leq \frac{\|x\|_q}{n^{1/q-1}} + p_q n(1-x_n), \tag{17}$$

with $x \in \{(1, x_2, \cdots, x_n) \mid 1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$, where p_q is a function of q specified in (7).

First, for any given $q \in (0,1]$ define that

$$f(x) := \|x\|_1 - n^{1-1/q}\|x\|_q.$$

It is easy to verify that $f(x)$ is a convex function on $\mathbb{R}^n_+$. Since the maximum of a convex function always arrives on the boundary, we have

$$\begin{aligned} h(x_n) : &= \max_{1 \geq x_2 \geq x_3 \geq \cdots \geq x_n} f(1, x_2, x_3, \cdots, x_n) \\ &= f(1, \cdots, 1, x_n, \cdots, x_n), \quad x_n \in [0,1] \end{aligned}$$

Letting the distribution of 1 appear for r times $(1 \leq r \leq n)$ in the maximum solution of f, we have

$$h(x_n) = r(1-x_n) + nx_n - \frac{(r(1-x_n^q) + nx_n^q)^{1/q}}{n^{1/q-1}}.$$

By the convexity of h and $h(1) = 0$, it follows that

$$h(x_n) \leq (1-x_n)h(0) + x_n h(1) = (1-x_n)h(0).$$

Then it holds that

$$\begin{aligned} f(x) &\leq h(x_n) \leq (1-x_n)h(0) \\ &= (1-x_n)(r - n^{1-1/q}r^{1/q}) \\ &\leq (1-x_n) \max_{r \in \{1,2,\cdots,n\}} \{r - n^{1-1/q}r^{1/q}\} \\ &\leq (1-x_n) \max_{0 < r_1 \leq n} \{r_1 - n^{1-1/q}r_1^{1/q}\} \\ &= (1-x_n)p_q n, \end{aligned}$$

where p_q is defined as (7) and the last equality holds when $r_1 = q^{\frac{q}{1-q}}n \in (0,n]$ for any $q \in (0,1]$.

By computing the first and second order partial derivatives of p_q on q, it is easy to verify that p_q is a nonincreasing convex function of $q \in (0,1]$ and

$$\lim_{q \to 0^+} p_q = 1 \text{ and } \lim_{q \to 1^-} p_q = 0.$$

Thus the proof is completed. □

Proof of Lemma II.5

If $q^{\frac{q}{q-1}}k$ is an integer, then

$$\begin{aligned} g(q,k) &= (q^{\frac{q}{q-1}}k)^{1-1/q}k^{1/q} + p_q(q^{\frac{q}{q-1}}k) \\ &= qk^{1-1/q}k^{1/q} + (q^{\frac{q}{1-q}} - q^{\frac{1}{1-q}})(q^{\frac{q}{q-1}}k) \\ &= qk + (1-q)k = k. \end{aligned}$$

If $q^{\frac{q}{q-1}}k$ is not an integer, then

$$\begin{aligned} g(q,k) &\leq (q^{\frac{q}{q-1}}k)^{1-1/q}k^{1/q} + p_q(q^{\frac{q}{q-1}}k + 1) \\ &= qk^{1-1/q}k^{1/q} + (q^{\frac{q}{1-q}} - q^{\frac{1}{1-q}})(q^{\frac{q}{q-1}}k + 1) \\ &= qk + (1-q)k + p_q = k + p_q. \end{aligned}$$

Due to $\lim_{q \to 1^-} q^{\frac{q}{q-1}} = e$ and $\lim_{q \to 1^-} p_q = 0$, we have

$$\begin{aligned} g(1,k) : &= \lim_{q \to 1^-} g(q,k) \\ &= \lim_{q \to 1^-} \left\{ \lceil q^{\frac{q}{q-1}}k \rceil^{1-1/q}k^{1/q} + p_q \lceil q^{\frac{q}{q-1}}k \rceil \right\} \\ &= k + 0 = k. \end{aligned}$$

Now we prove the remaining part $\lim_{q\to 0^+} g(q,k) = k+1$. Since $\lim_{q\to 0^+} q^{\frac{q}{q-1}} = 1$ and $q^{\frac{q}{q-1}} \in (1,e]$ is a nondecreasing function on $q \in (0,1]$, for any fixed k, we can set $q^{\frac{q}{q-1}} = 1+\varepsilon(q)$ with sufficient small $0 < \varepsilon(q) < \frac{1}{k}$. Thus

$$\lceil q^{\frac{q}{q-1}} k \rceil = \lceil (1+\varepsilon(q))k \rceil = k+1, \quad \text{as } q(\neq 0) \to 0^+,$$

It follows readily that

$$\begin{aligned} g(0,k): &= \lim_{q\to 0^+} g(q,k) \\ &= \lim_{q\to 0^+} \left\{ \lceil q^{\frac{q}{q-1}} k \rceil^{1-1/q} k^{1/q} + p_q \lceil q^{\frac{q}{q-1}} k \rceil \right\} \\ &= \lim_{q\to 0^+} \left\{ (k+1)^{1-1/q} k^{1/q} + p_q (k+1) \right\} \\ &= \lim_{q\to 0^+} \left\{ (k+1)\left(\frac{k}{k+1}\right)^{1/q} + p_q(k+1) \right\} \\ &= 0+k+1 = k+1. \end{aligned}$$

The whole proof is finished. □

Proof of Lemma II.6

We verify $\gamma\left(\mu\left(t,\theta\right),\theta\right)$ is a nonincreasing function on $\theta \geq 0$ and a nondecreasing function on $t > 1$. By directly computing the first order partial derivative of $\gamma\left(\mu\left(t,\theta\right),\theta\right)$ on $\theta \geq 0$, it yields

$$\frac{\partial}{\partial \theta}\gamma\left(\mu\left(t,\theta\right),\theta\right) = \frac{-\sqrt{(t+\theta-1)(t-1)}}{2(t+\theta-1)^2} \leq 0.$$

Likewise, by computing the first order partial derivative of $\gamma\left(\mu\left(t,\theta\right),\theta\right)$ on $t > 1$, we have

$$\frac{\partial}{\partial t}\gamma\left(\mu\left(t,\theta\right),\theta\right) = \frac{\theta}{2\sqrt{(t-1)(t+\theta-1)^3}} \geq 0.$$

Then the desired conclusions hold immediately. □

Before proving Theorem III.1, we introduce hereafter several notations. For $h \in \mathbb{R}^n$, we denote hereafter h_T the vector equal to h on an index set T and zero elsewhere. Especially, we denote $h_{max(k)}$ as h with all but the largest k entries in absolute value set to zero, and $h_{-max(k)} := h - h_{max(k)}$.

Proof of Theorem III.1

The approach of this proof is similar as [6]. First we consider the case that $g(k,q)(t-1)$ is an integer. By the Null Space Property [18] in ℓ_q minimization case, we only need to check for all $h \in \mathcal{N}(\Phi) \setminus \{0\}$,

$$\|h_{max(k)}\|_q^q < \|h_{-max(k)}\|_q^q.$$

Suppose on the contrary that there exists $h \in \mathcal{N}(\Phi)\setminus\{0\}$, such that $\|h_{max(k)}\|_q^q \geq \|h_{-max(k)}\|_q^q$. Set $\alpha = k^{-1/q}\|h_{max(k)}\|_q$ and decompose $h_{-max(k)}$ into a sum of vectors $h_{T_1}, h_{T_2}, \ldots$, where T_1 corresponds to the locations of the $\lceil q^{\frac{q}{q-1}} k \rceil$ largest coefficients of $h_{-max(k)}$; T_2 to the locations of the $\lceil q^{\frac{q}{q-1}} k \rceil$ largest coefficients of $h_{-max(k)T_1^C}$, and so on. That is

$$h_{-max(k)} = h_{T_1} + h_{T_2} + h_{T_3} + \cdots .$$

Here, the sparsity of $h_{T_j} (j \geq 1)$ is at most $\lceil q^{\frac{q}{q-1}} k \rceil$.

Clearly, $k\|h_{-max(k)}\|_\infty^q \leq \|h_{max(k)}\|_q^q = k\alpha^q$, which generates $\|h_{-max(k)}\|_\infty \leq \alpha$. From Lemma II.2, for $j \geq 1$,

$$\begin{aligned} \|h_{T_j}\|_1 \quad &\leq \quad \lceil q^{q/(q-1)} k \rceil^{1-1/q} \|h_{T_j}\|_q \\ &\quad + p_q \lceil q^{\frac{q}{q-1}} k \rceil (\|h_{T_j}\|_\infty - \|h_{T_j}\|_{-\infty}). \end{aligned} \tag{18}$$

Then we sum $\|h_{T_j}\|_1$ for $j \geq 1$ to obtain that

$$\begin{aligned} &\|h_{-max(k)}\|_1 = \sum_{j\geq 1} \|h_{T_j}\|_1 \\ \leq \quad & \lceil q^{\frac{q}{q-1}} k \rceil^{1-1/q} \sum_{j\geq 1} \|h_{T_j}\|_q \\ & + p_q \lceil q^{\frac{q}{q-1}} k \rceil \sum_{j\geq 1} \left(\|h_{T_j}\|_\infty - \|h_{T_j}\|_{-\infty} \right) \\ \leq \quad & \lceil q^{\frac{q}{q-1}} k \rceil^{\frac{q-1}{q}} (\sum_{j\geq 1} \|h_{T_j}\|_q^q)^{1/q} + p_q \lceil q^{\frac{q}{q-1}} k \rceil \|h_{T_1}\|_\infty . \\ \leq \quad & \lceil q^{\frac{q}{q-1}} k \rceil^{\frac{q-1}{q}} k^{1/q}\alpha + p_q \lceil q^{\frac{q}{q-1}} k \rceil \alpha = g(q,k)\alpha. \end{aligned} \tag{19}$$

We again divide $h_{-max(k)}$ into two parts, $h_{-max(k)} = h^{(1)} + h^{(2)}$, where

$$h^{(1)} := h \cdot \mathbf{1}_{\{i:|h_{-max(k)}(i)| > \frac{\alpha}{t-1}\}},$$

$$h^{(2)} := h \cdot \mathbf{1}_{\{i:|h_{-max(k)}(i)| \leq \frac{\alpha}{t-1}\}}.$$

Therefore $h^{(1)}$ is $g(q,k)(t-1)$-sparse as a result of facts that $\|h^{(1)}\|_1 \leq \|h_{-max(k)}\|_1 \leq g(q,k)\alpha$ and all non-zero entries of $h^{(1)}$ has magnitude larger than $\frac{\alpha}{t-1}$. Let $\|h^{(1)}\|_0 = m$, then

$$\begin{aligned} \|h^{(2)}\|_1 \quad &= \quad \|h_{max(k)}\|_1 - \|h^{(1)}\|_1 \\ &\leq \quad [g(q,k)(t-1) - m] \frac{\alpha}{t-1}, \end{aligned} \tag{20}$$

$$\|h^{(2)}\|_\infty \quad \leq \quad \frac{\alpha}{t-1}. \tag{21}$$

Applying Lemma II.1 with $s = g(q,k)(t-1) - m$, it makes $h^{(2)}$ be expressed as a convex combination of sparse vectors: $h^{(2)} = \sum_{i=1}^N \lambda_i u_i$, where u_i is s-sparse, $\|u_i\|_1 = \|h^{(2)}\|_1, \|u_i\|_\infty \leq \frac{\alpha}{t-1}$. Henceforth,

$$\|u_i\|_2 \leq \sqrt{g(q,k)(t-1)-m}\|u_i\|_\infty \leq \sqrt{\frac{g(q,k)}{t-1}}\alpha.$$

For any $\mu \geq 0$, denoting $\eta_i = h_{max(k)} + h^{(1)} + \mu u_i$, we obtain

$$\begin{aligned} &\sum_{j=1}^N \lambda_j \eta_j - \frac{1}{2}\eta_i = h_{max(k)} + h^{(1)} + \mu h^{(2)} - \frac{1}{2}\eta_i \\ = \quad & (\frac{1}{2} - \mu)\left(h_{max(k)} + h^{(1)}\right) - \frac{1}{2}\mu u_i + \mu h, \end{aligned} \tag{22}$$

where $\eta_i, \sum_{i=1}^N \lambda_i \eta_i - \frac{1}{2}\eta_i - \mu h$ are all $(g(q,k)(t-1)+k)$-sparse vectors from the sparsity of $\|h_{max(k)}\|_0 \leq k$, $\|h^{(1)}\|_0 = m$ and $\|u_i\|_0 \leq s$.

It is easy to check the following identity,

$$\sum_{i=1}^N \lambda_i \|\Phi(\sum_{j=1}^N \lambda_j \eta_j - \frac{1}{2}\eta_i)\|_2^2 = \frac{1}{4}\sum_{i=1}^N \lambda_i \|\Phi \eta_i\|_2^2. \tag{23}$$

Since $\Phi h=0$, together with (22), we have

$$\Phi(\sum_{j=1}^{N}\lambda_j\eta_j-\frac{1}{2}\eta_i)=\Phi((\frac{1}{2}-\mu)(h_{max(k)}+h^{(1)})-\frac{1}{2}\mu u_i).$$

Setting $\mu=\mu\left(t,g(q,k)/k\right)>0$, if (14) holds, that is

$$\delta:=\delta_{g(q,k)(t-1)+k}<\gamma\left(\mu\left(t,\frac{g(q,k)}{k}\right),\frac{g(q,k)}{k}\right), \quad (24)$$

then combining (23) with (24), we get

$$\begin{aligned}
0 &= \sum_{i=1}^{N}\lambda_i\|\Phi((\frac{1}{2}-\mu)(h_{max(k)}+h^{(1)})-\frac{1}{2}\mu u_i)\|_2^2\\
&\quad -\frac{1}{4}\sum_{i=1}^{N}\lambda_i\|\Phi\eta_i\|_2^2\\
&\leq (1+\delta)\sum_{i=1}^{N}\lambda_i[(\frac{1}{2}-\mu)^2\|h_{max(k)}+h^{(1)}\|_2^2+\frac{\mu^2}{4}\|u_i\|_2^2]\\
&\quad -\frac{1-\delta}{4}\sum_{i=1}^{N}\lambda_i(\|h_{max(k)}+h^{(1)}\|_2^2+\mu^2\|u_i\|_2^2)\\
&= \sum_{i=1}^{N}\lambda_i[((1+\delta)(\frac{1}{2}-\mu)^2-\frac{1-\delta}{4})\cdot\\
&\quad \|h_{max(k)}+h^{(1)}\|_2^2+\frac{1}{2}\delta\mu^2\|u_i\|_2^2]\\
&\leq \sum_{i=1}^{N}\lambda_i\|h_{max(k)}+h^{(1)}\|_2^2\cdot\\
&\quad \left[\mu^2-\mu+\delta(\frac{1}{2}-\mu+(1+\frac{g(q,k)}{2k(t-1)})\mu^2)\right] \quad (25)\\
&= \|h_{max(k)}+h^{(1)}\|_2^2\cdot\\
&\quad \left[\mu^2-\mu+\delta(\frac{1}{2}-\mu+(1+\frac{g(q,k)}{2k(t-1)})\mu^2)\right]\\
&= \|h_{max(k)}+h^{(1)}\|_2^2(\frac{1}{2}-\mu+(1+\frac{g(q,k)}{2k(t-1)})\mu^2)\cdot\\
&\quad \left[\delta-\gamma(\mu\left(t,\frac{g(q,k)}{k}\right),\frac{g(q,k)}{k}\Big)\right]\\
&< 0,
\end{aligned}$$

where the inequality (25) is derived from the following facts:

$$\begin{aligned}
\|h_{max(k)}\|_2^2 &\geq k^{1-2/q}\|h_{max(k)}\|_q^2\\
&= k^{1-2/q}(k\alpha^q)^{2/q}=k\alpha^2, \quad (26)\\
\|u_i\|_2 &\leq \sqrt{\frac{g(q,k)}{t-1}}\alpha\leq\sqrt{\frac{g(q,k)}{k}}\frac{\|h_{max(k)}\|_2}{\sqrt{t-1}}\\
&\leq \sqrt{\frac{g(q,k)}{k}}\frac{\|h_{max(k)}+h^{(1)}\|_2}{\sqrt{t-1}}. \quad (27)
\end{aligned}$$

Obviously, this is a contradiction.

When $g(k,q)(t-1)$ is not an integer, by setting

$$t'=\frac{\lceil g(k,q)(t-1)\rceil}{g(k,q)}+1,$$

we have $t'>t$ and $g(k,q)(t'-1)$ is an integer. Utilizing the nondecreasing monotonicity of $\gamma\left(\mu\left(t,\theta\right),\theta\right)$ on $t\geq 0$ for fixed θ presented in Lemma II.6, we can get

$$\begin{aligned}
\delta_{g(k,q)(t'-1)+k} &= \delta_{g(k,q)(t-1)+k}\\
&< \gamma\left(\mu\left(t,\frac{g(q,k)}{k}\right),\frac{g(q,k)}{k}\right)\\
&< \gamma\left(\mu\left(t',\frac{g(q,k)}{k}\right),\frac{g(q,k)}{k}\right),
\end{aligned}$$

which can be deduced to the former case. Hence we complete the proof. □

In order to prove the result Theorem IV.1, we need another important concept in the RIP framework the restricted orthogonal constants (ROC) proposed in [8].

Definition VI.1. *Suppose $\Phi\in\mathbb{R}^{m\times n}$, define the restricted orthogonal constants (ROC) of order k_1,k_2 as the smallest non-negative number θ_{k_1,k_2} such that*

$$|\langle\Phi h_1,\Phi h_2\rangle|\leq\theta_{k_1,k_2}\|h_1\|_2\|h_2\|_2 \quad (28)$$

for all k_1-sparse vector $h_1\in\mathbb{R}^n$ and k_2-sparse vector $h_2\in\mathbb{R}^n$ with disjoint supports.

Proof of Theorem IV.1

Similar to the proof of Theorem III.1, we only need to check for all $h\in\mathcal{N}(\Phi)\setminus\{0\}$,

$$\|h_{max(k)}\|_q^q<\|h_{-max(k)}\|_q^q.$$

Suppose there exists $h\in\mathcal{N}(\Phi)\setminus\{0\}$, such that $\|h_{max(k)}\|_q^q\geq\|h_{-max(k)}\|_q^q$. Set $\alpha=k^{-1/q}\|h_{max(k)}\|_q$. From the proof of Theorem III.1, we have $\|h_{-max(k)}\|_1\leq g(q,k)\alpha\leq\lceil g(q,k)\rceil\alpha$ and $\|h_{-max(k)}\|_\infty\leq\alpha$. Then it follows from Lemma 5.1 in [5] that

$$\begin{aligned}
&\left|\left\langle\Phi h_{max(k)},\Phi h_{-max(k)}\right\rangle\right|\\
&\leq \theta_{k,\lceil g(q,k)\rceil}\|h_{max(k)}\|_2\sqrt{\lceil g(q,k)\rceil}\alpha\\
&\leq \theta_{k,k}\sqrt{\frac{\lceil g(q,k)\rceil}{k}}\|h_{max(k)}\|_2\sqrt{\lceil g(q,k)\rceil}\alpha\\
&\leq \theta_{k,k}\frac{\lceil g(q,k)\rceil}{k}\|h_{max(k)}\|_2^2,
\end{aligned}$$

where the first inequality holds by Lemma 5.4 in [5] and the second inequality by (26). Thus from the condition

$$\delta_k+\theta_{k,k}\frac{\lceil g(q,k)\rceil}{k}<1,$$

it follows that

$$\begin{aligned}
0 &= \left|\left\langle\Phi h_{max(k)},\Phi h\right\rangle\right|\\
&\geq \left|\left\langle\Phi h_{max(k)},\Phi h_{max(k)}\right\rangle\right|-\left|\left\langle\Phi h_{max(k)},\Phi h_{-max(k)}\right\rangle\right|\\
&\geq (1-\delta_k)\|h_{max(k)}\|_2^2-\theta_{k,k}\frac{\lceil g(q,k)\rceil}{k}\|h_{max(k)}\|_2^2\\
&= (1-\delta_k-\theta_{k,k}\frac{\lceil g(q,k)\rceil}{k})\|h_{max(k)}\|_2^2\\
&> 0.
\end{aligned}$$

Obviously, this is a contradiction. By Lemma 3.1 in [5],

$$\theta_{k,k}<\begin{cases}2\delta_k, & \text{for any even } k\geq 2,\\ \dfrac{2k}{\sqrt{k^2-1}}\delta_k & \text{for any odd } k\geq 3.\end{cases}$$

Hence, when $k \geq 2$ is even, it yields that

$$\delta_k + \frac{g(q,k)}{k}\theta_{k,k} < \left(1 + \frac{2\lceil g(q,k)\rceil}{k}\right)\delta_k,$$

and when $k \geq 3$ is odd, it generates that

$$\delta_k + \frac{g(q,k)}{k}\theta_{k,k} < \left(1 + \frac{2\lceil g(q,k)\rceil}{\sqrt{k^2-1}}\right)\delta_k.$$

Therefore the theorem is proved. □

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