Strengthened Bounds for the Probability of k-Out-Of-n Events

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Abstract

Given a set of n random events in a probability space, represented by n Bernoulli variables (not necessarily independent,) we consider the probability that at least k out of n events occur. When partial distribution information, i.e., individual probabilities and all joint probabilities of up to m ($m < n$) events, are provided, only an upper or lower bound can be computed for this probability. Recently Prèkopa and Gao (Discrete Appl. Math. 145 (2005) 444) proposed a polynomial-size linear program to obtain strong bounds for the probability of union of events, i.e., $k = 1$. In this work, we propose inequalities that can be added to this linear program to strengthen the bounds. We also show that with a slight modification of the objective function this linear program and the inequalities can be used for the more general case where k is any positive integer less than or equal to n. We use the strengthened linear program to compute probability bounds for the examples used by Prèkopa and Gao, and the comparison shows significant improvement in the bound quality.

Keywords: linear programming, probability bound, k-of-n event

1 Introduction

Let $\{A_j : j \in N\}$ be a set of events, where N is the index set $\{1, \dots, n\}$. Define random variable $X_j : A_j \to \{0, 1\}$ as $X_j = 1$ if A_j occurs, and $X_j = 0$, otherwise. Define μ as a random variable that represents the number of events, A_js, that occur, i.e., $\mu = \sum_j X_j$. Let k be a positive integer between 1 and n; then the probability that (at least) k out of n events in $\{A_1, ..., A_n\}$ occur is denoted by $\mathbb{P}(\mu \geq k)$.

Computation of $\mathbb{P}(\mu \geq k)$ is often needed in applications. For example, in a maximum availability location problem [3], the probability that the population in a subregion is covered by at least k facilities is used to calculate the expected

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coverage. Another example is the reliability problem of communication networks, where each arc fails with a certain probability and we want to compute or approximate the node-to-node reliability of the system.

Accurate computation of $\mathbb{P}(\mu \geq k)$ is not easy. A complete distribution function for a system of Bernoulli events involves an exponential size of data, which is difficult to handle when n is large. Furthermore, in practice, the complete distribution function for $(X_1, ..., X_n)$ is often not available, unless the Bernoulli random numbers X_j are independent from each other. The available information is often the marginal distributions and joint distributions up to level m ($m \ll n$). In this situation, it is desirable/preferable to compute a lower or upper bound using only a limited amount of information.

A number of bounding results that utilize different amounts of information under different settings have been proposed. The classic Boole inequality yields a lower bound for the union of events with only individual marginal probabilities, i.e., $k = 1$ and $m = 1$. Dawson and Sanko [4] provide a sharp lower bound for the probability of the union of events using marginal and pairwise joint probabilities, i.e., $m = 2$; Kwerel [5, 6] developed bounds that can utilize one more degree of joint probabilities, i.e., $m = 3$; Prèkopa, Boros and other researchers [7, 8, 2] employed certain linear programs to derive lower and upper bounds for a general case where m can be any positive integer less than or equal to n. Most of these works use aggregation of the individual joint probabilities of the same degree in their formulations, called binomial moments. As a consequence of summation, individual probability information is lost, and the given information is not fully utilized. To make better use of the available information, Prèkopa and Gao [9] derived LP-based bounds for a union of events using partially aggregated information. We call this LP model as the *partially aggregated model* (PAM). The numerical examples showed that their bounds were at least as strong as other results using binomial moments.

In this paper, we first observe that the results in [9] which is stated for the case of $k = 1$, generalizes in a straightforward way to the case where k is any positive integer less than or equal to n. Our key contribution is to identify inequalities that can be appended to PAM to strengthen the bounds. We call the resulting LP model as *strengthened partially aggregated model* (SPAM). These results are presented in Section 2. We test the strength of the extra inequalities using the instances in [9], and the results show significant improvement of the bound quality of SPAM over PAM. We also computationally identify a family of probability distributions for which the bounds provided by SPAM has the largest improvement over the bounds provided by PAM model. We report these computational results in Section 3.

2 Strengthening the Partially Aggregated Model

2.1 The Partially Aggregated Model for $k \in \{1, \ldots, n\}$

Let S be a subset of N. Denote the joint probability that all events in the set S occur as $p_S := \mathbb{P}(\bigcap_{\ell \in S} A_\ell)$. Let $s_t^j := \sum_{\substack{S: \ |S|=t \\ j \in S}} p_S$. We first state the main result of [9] using our notation.

Theorem 1 ([9]). *Given the joint distributions up to level m, i.e., $|S| \leq m$, a lower (or upper) bound for the probability of the union of these events, i.e., $\mathbb{P}(\mu \geq 1)$, can be calculated by solving the following linear program (PAM):*

$$\min(\max) \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} \tag{1a}$$

$$\text{s.t.} \quad \sum_{i=0}^{n} \sum_{j=1}^{n} v_{ij} \quad = 1 \tag{1b}$$

$$\sum_{i=t}^{n} \binom{i}{t} v_{ij} \quad = \tfrac{1}{t} s_t^j \tag{1c}$$

$$v_{ij} \quad \geq 0. \tag{1d}$$

$\square$

The LP model above and those in [7, 8, 2, 9] are derived using probabilistic reasonings (See Appendix 1 for a brief review.)

We next show that the bounds on $\mathbb{P}(\mu \geq k)$ for $k \in \{1, \ldots, n\}$ can be obtained by solving (1) by changing the objective function (1a) to $\sum_{i=k}^{n} \sum_{j=1}^{n} v_{ij}$.

As pointed out in [9], any bound that uses p_S can be recovered as the value of the objective function of the Boolean LP model [1], which consists of the probability of each possible outcome. Let C be a subset of N, and let the probability of the outcome associated with C be denoted as

$$w_C := \mathbb{P}\left((\bigcap_{i \in C} A_i) \bigcap (\bigcap_{j \in N \setminus C} \bar{A}_j) \right),$$

where $\bar{A}_j$ represents that event A_j does not occur. Note that these outcomes are mutually exclusive. The probability of any event can be represented by using the probabilities of an appropriate set of outcomes. For example, the probability that at least k out of n events occur can be expressed as $\sum_{C \subseteq N, |C| \geq k} w_C$. Given joint distributions up to level m, i.e., p_S, $|S| \leq m$, a lower (or upper) bound of $\mathbb{P}(\mu \geq k)$ can be obtained by the following Boolean linear program [1]:

$$\min(\max) \sum_{C : |C| \geq k} w_C$$

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$$\text{s.t.} \sum_{C \subseteq N} w_C = 1 \tag{2}$$

$$\sum_{C:S \subseteq C} w_C = p_S, \quad \forall S \subseteq N, \ |S| \leq m$$

$$w_C \geq 0 \ \forall C \subseteq N.$$

Formulation (2) consists of an exponential number of variables and may not be useful in practice. In order to obtain the PAM LP model, we first duplicate each row with right-hand side p_S in (2) $|S|$-1 times, add up rows with $|S| = t$ and $j \in S$ for each t and j, and then arrive at

$$\min \sum_{i=k}^{n} \sum_{j=1}^{n} \frac{1}{i} \sum_{\substack{C:|C|=i \\ j \in C}} w_C \tag{3a}$$

$$\text{s.t.} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i} \sum_{\substack{C:|C|=i \\ j \in C}} w_C = 1 \tag{3b}$$

$$\sum_{i=t}^{n} \binom{i-1}{t-1} \sum_{\substack{C:|C|=i \\ j \in C}} w_C = \sum_{\substack{S: |S|=t \\ j \in S}} p_S \quad t = 1, ..., m \ j = 1, ..., n \tag{3c}$$

$$w_C \geq 0 \quad \forall C \subseteq N. \tag{3d}$$

Equations (3a)-(3d) are the resulting rows by duplicating and aggregating the rows in (2). Notice that the following variables share the same coefficient in each row: w_C with $|C| = i$ and $j \in C$. Therefore, we aggregate these variables into a single variable and, for notational simplicity, scale the resulting aggregated variable as

$$v_{ij} := \frac{1}{i} \sum_{\substack{C:|C|=i \\ j \in C}} w_C \quad i = 1, ..., n \ j = 1, ...n. \tag{4}$$

Thus we obtain the following linear program:

$$\min \sum_{i=k}^{n} \sum_{j=1}^{n} v_{ij} \tag{5a}$$

$$\text{s.t.} \sum_{j=1}^{n} \sum_{i=0}^{n} v_{ij} = 1 \tag{5b}$$

$$\sum_{i=t}^{n} \binom{i}{t} v_{ij} = \frac{1}{t} s_t^j \quad t = 1, ..., m \ j = 1, ..., n \tag{5c}$$

$$v_{ij} = \frac{1}{i} \sum_{\substack{C:|C|=i \\ j \in C}} w_C \quad i = 1, ..., n \ j = 1, ...n \tag{5d}$$

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$$v \geq 0 \ i = 1, ..., n \ j = 1, ..., n \tag{5e}$$

$$w_C \geq 0 \ \forall C \subseteq N. \tag{5f}$$

Now observing that the variables w_C do not appear in the objective function (5a), we may *relax* (5), by *dropping* the variables w_C and the constraints (5d) and (5f), to obtain the PAM model.

2.2 Strengthened PAM

As discussed in the previous section, PAM is obtained by relaxing (5). In order to strengthen PAM, we project out the variables w_C from the set defined by the constraints (5d) and (5f) to obtain valid inequalities which we append to PAM. Again we are able to accomplish this using the fact that the w_C variables do not appear in the objective function.

We need the following lemma.

Lemma 1. *Let $S^i := \{x \in \{0,1\}^n \,|\, \sum_{j=1}^n x_j = i\}$. For $1 \leq i \leq n-1$, $\dim(S^i) = n$.*

Proof. We list n vectors from S^i that are linearly independent. Set vectors $v^1, \ldots, v^{i+1}$ as

$$v_k^j = \begin{cases} 1 & k \in \{1, \ldots, i+1\} \setminus \{j\} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Set vectors $v^{i+2}, \ldots, v^n$ as

$$v_k^j = \begin{cases} 1 & k \in \{1, \ldots, i-1\} \cup \{j\} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

It is straighforward to check that $v^1, \ldots, v^n$ belong to S^i and are linearly independent. $\qquad\square$

Proposition 2. *Let*

$$W_i = \{(..., w_C, ..., v_{ij}, ...) \in \mathbb{R}_+^{\binom{n}{i}} \times \mathbb{R}^n : v_{ij} = \frac{1}{i} \sum_{\substack{C: |C| = i \\ j \in C}} w_C \ j = 1, ...n\}. \tag{8}$$

Then the projections of W_is onto v space are the following sets:

$$Proj_v(W_i) = \{(v_{i1}, ..., v_{in}) \in \mathbb{R}^n \,|\, -(i-1)v_{ij} + \sum_{t \neq j} v_{it} \geq 0, \ v_{ij} \geq 0 \ j \in N\}$$

$$i = 2, ..., n-2, \tag{9}$$

$$Proj_v(W_{n-1}) = \{(v_{(n-1)1}, ..., v_{(n-1)n}) \in \mathbb{R}^n \,|\, -(n-2)v_{(n-1)j} + \sum_{t \neq j} v_{(n-1)t} \geq 0, \ j \in N\}$$

$$\tag{10}$$

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and

$$Proj_v(W_n) = \{(v_{n1}, ..., v_{nn}) \in \mathbb{R}^n_+ \mid v_{n1} = v_{nt} \ 2 \leq t \leq n, \ v_{n1} \geq 0\}. \qquad (11)$$

Proof. Note that $\text{Proj}_v(W_i)$ is a cone. Therefore, for simplicity of analysis, we may scale the v variables and thus without loss of generality assume that

$$W_i := \{(..., w_C, ..., v_{ij}, ...) \in \mathbb{R}^{\binom{n}{i}}_+ \times \mathbb{R}^n : v_{ij} = \sum_{\substack{C:|C|=i \\ j \in C}} w_C \ j = 1, ...n\}. \qquad (12)$$

We show that (9) holds for an arbitrary i between 2 and $n - 1$, where $|C| = i$. To simplify notations, for now, we drop subscript i in the constraint. Let $w = (..., w_C, ...) \in \mathbb{R}^{\binom{n}{i}}_+$ and $v = (..., v_{ij}, ...) \in \mathbb{R}^n$. We then rewrite (12) as follows

$$W_i = \{(w, v) \in \mathbb{R}^{\binom{n}{i}}_+ \times \mathbb{R}^n : v = Gw\},$$

where $G = (..., g_C, ...) \in \mathbb{R}^{n \times \binom{n}{i}}$ is the coefficient matrix and $g_C = (g_C^1, ..., g_C^j, ..., g_C^n)^\top$ is the column corresponding to the variable w_C. $g_C^j = 1$ if $j \in C$; $g_C^j = 0$ if $j \notin C$. G consists of all permutations of the 0-1 vector with i ones and $n - i$ zeros. Rewrite (9) as follows:

$$V_i := \{v \in \mathbb{R}^n : (e - ie_j)^\top v \geq 0, \ e_j^\top v \geq 0 \ j = 1, ..., n\},$$

where $e \in \mathbb{R}^n$ is a vector with all components equal to one and $e_j \in \mathbb{R}^n$ is the j-th unit vector.

To show that V_i is the projection of W_i into v-space, we need to show

$$\bar{v} \in V_i \Leftrightarrow \text{There is a } \bar{w} \in \mathbb{R}^{\binom{n}{i}}_+ \text{ such that } G\bar{w} = \bar{v}.$$

By Farkas' lemma, we have that

There is a $\bar{w} \in \mathbb{R}^{\binom{n}{i}}_+$ such that $G\bar{w} = \bar{v} \Leftrightarrow u \in \mathbb{R}^n$ such that $u^\top G \geq 0 \Rightarrow u^\top \bar{v} \geq 0$.

Let $\{u_\ell\}$ be the set of extreme rays of the cone$\{u : u^\top G \geq 0\}$. It is sufficient to show that the set of constraint vectors in V_i, i.e., $(e - ie_j)$ and e_j, is exactly $\{u_\ell\}$.

We first show $(e - ie_j)$ and e_j are extreme rays.

$(e - ie_j)^\top$ is a feasible solution for the cone $\{u : u^\top G \geq 0\}$:

$$(e - ie_j)^\top g_C = \begin{cases} 0 & C : j \in C \\ i - 1 & C : j \notin C. \end{cases}$$

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Furthermore, the products above have $\binom{n-1}{i-1}$ zeros, which means that $\binom{n-1}{i-1}$ constraints in $u^\top G \geq 0$ are tight. Notice that the binding constraint vectors g_C are all the permutations of vectors with the j-th position fixed to one. Therefore, by Lemma 1, they span $\mathbb{R}^{n-1}$ and we can find $n-1$ linearly independent vectors among them which have zero products with $(e - ie_j)$. Thus, $(e - ie_j)$ is an extreme ray of $\{u : u^\top G \geq 0\}$.

As for e_j, we have $e_j^\top G$ as follows

$$e_j^\top g_C = \begin{cases} 1 & C : j \in C \\ 0 & C : j \notin C. \end{cases}$$

Therefore, e_j is a feasible solution for the cone $\{u : u^\top G \geq 0\}$ and the product above has $\binom{n-1}{i}$ zeros that correspond to g_C with $j \notin C$. Among those columns that have zero products with e_j, by using Lemma 1 we can find $n - 1$ linearly independent columns. Therefore, e_j is an extreme ray of $\{u : u^\top G \geq 0\}$.

Now we show by contradiction that there are no other extreme rays. Let $\lambda \neq 0$ be a new distinct extreme ray; and let $g_{S_\ell} = (g_{S_\ell}^1, ..., g_{S_\ell}^t, ...g_{S_\ell}^n)^\top$ $\ell = 1, ..., n-1$ be the set of linear independent columns of G with $\lambda^\top g_{S_\ell} = 0$ $\ell = 1, ..., n - 1$. Since $\lambda \neq e_j$ for all $j = 1, ..., n$, we have

$$\nexists t \text{ such that } g_{S_\ell}^t = 0 \quad \forall \ell = 1, ..., n - 1 \tag{13}$$

since otherwise, e_t would be the extreme ray formed by the half planes $\{u^\top g_{S_\ell} \geq 0 \ \ell = 1, ..., n - 1\}$. Similarly, since $\lambda \neq (e - ie_j)$ for all $j = 1, ..., n$, we have

$$\nexists t \text{ such that } g_{S_\ell}^t = 1 \quad \forall \ell = 1, ..., n - 1. \tag{14}$$

Using (13) and $g_{S_\ell} \geq 0$, we obtain that $\lambda_{t^*} < 0$ for some t^* (Otherwise, $\lambda^\top g_{S_\ell} > 0$ for some ℓ by (13)). By (14), there is a ℓ^* such that $g_{S_{\ell^*}}^{t^*} = 0$. Since $\lambda^\top g_{S_{\ell^*}} = \sum_{t \neq t^*} \lambda_t g_{S_{\ell^*}}^t = 0$ and $g_{S_{\ell^*}} \neq 0$, we can find an index $\bar{t}$ such that $\lambda_{\bar{t}} \geq 0$ and $g_{S_{\ell^*}}^{\bar{t}} = 1$. Let g^* be a vector obtained by switching the components at position t^* and $\bar{t}$ in vector $g_{S_{\ell^*}}$. Note that g^* is also a vector of G. Then $\lambda^\top g^* = \lambda^\top g_{S_{\ell^*}} + \lambda_{t^*} - \lambda_{\bar{t}} < 0$. Thus, λ is not a feasible ray of the cone $\{u : u^\top G \geq 0\}$. Therefore, there are no extreme rays of $\{u : u^\top G \geq 0\}$ other than $\{(e - ie_j), e_i \ j = 1, ..., n\}$ and $\mathrm{Proj}(W_i) = V_i$ for $i = 2, ..., n - 2$.

For $\mathrm{Proj}(W_{n-1})$, we can show that $\{(e - ie_j) \ j = 1, ..., n\}$ are extreme rays of $\{u : u^\top G \geq 0\}$. However $\{e_j\}$ are not extreme rays in this case since each row of G has only one zero. Using a similar argument as that in (2), we can show that $\{(e - ie_j) \ j = 1, ..., n\}$ are the only extreme rays and that $\mathrm{Proj}(W_n) = V_n$.

The equations in (10) hold since $v_{nj} = v_N$ for all $j = 1, ...n$. $\qquad \square$

Notice that the projection of W_1 yields only non-negativity constraints on v_{1j}, $j = 1, ..., n$ and the non-negativity of $v_{(n-1),j}$ for $j = 1, ..., n$ is implied by $-(i - 1)v_{(n-1)j} + \sum_{t \neq j} v_{it} \geq 0$ $j = 1, ..., n$.

The inequalities in Proposition 2,

$$-(i - 1)v_{ij} + \sum_{t \neq j} v_{it} \geq 0, \ v_{ij} \geq 0 \quad j = 1, ..., n, \ i = 2, ..., n - 1 \tag{15}$$

and

$$v_{n1} = v_{nt} \ 2 \leq t \leq n, \tag{16}$$

are valid for model (5) and they are not implied by the constraints in PAM. We call the linear program with these additional constraints as the *Strengthened Partially Aggregated Model* or *SPAM*.

3 Numerical Examples

In the following, we calculate lower bounds with the strengthened model for the examples in [9], and compare the new bounds with those presented in [9]. Note that the lower bounds in [9] are only for the probability of the union of events, i.e., $k = 1$. Examples 1, 2, and 3 have 20 Bernoulli variables each, i.e., $X_1, ..., X_{20}$. All the outcomes and their probabilities are presented in Table 1, 2, and 3, respectively. With these tables, we can obtain the marginal probabilities, pair-wise joint probabilities, and higher-order joint probabilities by adding up appropriate rows.

Table 1: Probability Distributions in Example 1

Outcome	Bernoulli Random Variables																				Probability
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	
1	1	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	1	0.012214
2	0	1	0	1	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0.022231
3	1	0	1	0	1	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0.023287
4	0	1	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0.033976
5	1	0	1	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	1	0	0.034761
6	0	1	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	1	0.044582
7	1	0	0	1	0	1	0	1	0	0	1	0	1	0	0	1	0	0	0	0	0.045943
8	0	1	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	1	0	1	0.055185
9	1	0	1	0	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0.056404
10	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	0	0.066317
11	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	0	1	0.067685
12	0	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0.077376
13	1	0	0	1	0	1	0	0	0	1	0	0	1	0	1	0	0	1	0	1	0.078648
14	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0.088878
15	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0.292514

We compare the lower bounds corresponding to $k = 1$ produced by SPAM with the results in [9] in Table 4. Note that all of the bounds in Table 4 are calculated with only marginal probabilities and pair-wise joint probabilities, except those in the fourth column.

The second column cites the results obtained by the formula derived in [4], which can be obtained by further aggregating the Boolean LP model. We call this the *Fully Aggregated Model* or *FAM* (See (18) in Appendix 1). The third column cites the results obtained by PAM derived in [9]. The fourth column cites the results obtained by PAM but with three binomial moments, including the triple-wise joint probabilities. Prèkopa and Gao also developed heuristics in [9] to strengthen the lower bounds by PAM, but the best results obtained are

Table 2: Probability Distributions in Example 2

Outcome	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	Probability
1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	0	1	0	0.00896463
2	1	1	0	1	0	0	0	0	0	1	0	1	0	1	1	0	0	1	0	1	0.02492217
3	1	0	1	1	1	0	0	1	0	1	0	0	1	0	0	0	0	0	1	0	0.02109813
4	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	0	0	1	0	0	0.03779353
5	0	1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	1	1	1	0	0.04632610
6	1	0	0	0	0	0	1	0	1	1	1	1	1	0	1	0	0	0	0	1	0.04284324
7	1	1	0	0	0	0	1	1	0	1	0	0	1	0	0	1	0	0	0	0	0.07804262
8	1	0	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0.02536991
9	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	1	0.01916672
10	1	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	1	0	0.06340085
11	0	1	0	1	1	0	0	0	1	1	0	1	0	0	1	1	0	0	0	1	0.07315289
12	1	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0.07732742
13	0	1	0	0	1	1	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0.02248020
14	0	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0.09164494
15	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0.36745660

Table 3: Probability Distributions in Example 3

Outcome	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	Probability
1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0.10176880
2	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0.11299200
3	0	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0.01514044
4	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0.05684733
5	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0	0.03270125
6	1	0	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0.10050750
7	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0.07306695
8	0	1	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0.01743922
9	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0.06284498
10	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0.05830101
11	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	1	0	0	0.06833096
12	1	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	1	0	0	1	0.07153743
13	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0.04503293
14	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0.03487869
15	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0.14860240

Table 4: Comparison with results in [9] for $k = 1$

Example	FAM	PAM	PAM(3)	SPAM
4	0.8275266	0.8580833	0.8864460	**0.9394167**
5	0.8658182	0.9100646	0.9354100	**0.9482229**
6	0.8985498	0.9435812	0.9587778	**0.9715460**

no better than those obtained by involving triple-wise probabilities, which are listed in the fourth column. Therefore, we ignore the results of the heuristics. The fifth column gives the results obtained by SPAM. We observe in Table 4 that the extra inequalities derived in this study significantly improve the lower bounds calculated by PAM.

Now, we use the same instances to calculate the lower bounds for the probability of 3-coverage, i.e., $k = 3$, and summarize the results in Table 5, which

Table 5: Lower Bounds for 3-Coverage ($k = 3$)

Example	FAM	PAM	PAM(3)	SPAM
1	0.6643058	0.665249	**0.6768353**	0.6745504
2	0.7298830	0.7528989	0.7847514	**0.8005835**
3	0.7387907	0.7819380	0.8482093	**0.8614323**

again shows that SPAM produces significantly tighter lower bounds than both FAM and PAM. We also calculate the upper bounds and compare the results with the results in [9]. However, we do not observe improvement in the new upper bounds.

In Table 6 we provide examples to show that for certain probability distributions, the improvement by the inequalities in Proposition 2 can be much more significant than the improvement observed in Examples 1, 2, and 3. The specific distrobution under which the lower bounds in Table 6 are calculated are obtained computationally by an optimization model provided in Appendix 2. We run this optimization model with n fixed at 10, m fixed at 3, and k varying from 2 to 4. The optimization problems are solved by the nonlinear solver BARON and the optimal solutions produce probability distributions under which the gaps between PAM and SPAM are maximal. In our experiments we enforce a solution time limit of ten hours. Columns 3 and 4 of Table 6 list the lower bounds yield by PAM and SPAM, respectively. For comparison we also calculate the lower bounds using FAM and the Boolean model (BM) and list them in Columns 2 and 5, respectively.

Table 6: Comparison of Lower Bounds

k	FAM	PAM	SPAM	BM
2	0.6296296	0.6296296	1.0000000	1.0000000
3	0.3412896	0.3432099	0.8065844	0.8065844
4	0.2108243	0.2143992	0.8599802	0.8599802

References

[1] G. Boole. An investigation into the laws of thought. *Walton and Maberly, London.[Reprinted (1951) Dover, New York.]*, 1854.

[2] E. Boros and A. Prekopa. Closed form two-sided bounds for probabilities that at least r and exactly r out of n events occur. *Mathematics of Operations Research*, pages 317–342, 1989.

[3] M.S. Daskin. A maximum expected covering location model: formulation, properties and heuristic solution. *Transportation Science*, 17(1):48–70, 1983.

[4] D. A. Dawson and D. Sankoff. An inequality for probabilities. *Proc. Amer. Math. Soc.*, 18:504–507, 1967.

[5] S. M. Kwerel. Bounds on the probability of the union and intersection of m events. *Advances in Appl. Probability*, 7:431–448, 1975.

[6] S. M. Kwerel. Most stringent bounds on the probability of the union and intersection of m events for systems partially specified by $S_1, S_2, \cdots, S_k$, $2 \leq k < m$. *J. Appl. Probability*, 12(3):612–619, 1975.

[7] A. Prékopa. Boole-bonferroni inequalities and linear programming. *Operations Research*, pages 145–162, 1988.

[8] A. Prékopa. Sharp bounds on probabilities using linear programming. *Operations Research*, pages 227–239, 1990.

[9] A. Prékopa and L. Gao. Bounding the probability of the union of events by aggregation and disaggregation in linear programs. *Discrete applied mathematics*, 145(3):444–454, 2005.

Appendix 1

We briefly motivate the partially aggregated model introduced in [9]. Let $\binom{\mu}{i}$ represent the combination of choosing i out v. We observe that [7]

$$\binom{\mu}{i} = \sum_{S \subseteq N : |S| = i} \prod_{\ell \in S} X_\ell. \tag{17}$$

Notice that both sides of above equation are random numbers. After taking expectation, we have the following equation

$$\sum_{t=1}^{n} \binom{t}{i} v_i = \sum_{S \subseteq N : |S| = i} p_S, \tag{18}$$

where $p_S = \mathbb{P}(\bigcap_{\ell \in S} A_\ell)$ and the right-hand side is the i-th binomial moment. Note that (18) are the constraints of the LP model in [7, 8, 2], which we call *Fully Aggregated Model* or FAM.

A simple consequence of (17) is as follows [9]

$$X_j \binom{\mu - 1}{i - 1} = X_j \sum_{S \subseteq N : |S| = i} \prod_{\ell \in S} X_\ell. \tag{19}$$

Taking expectation of both sides in above equation, we have the following

$$\sum_{t=1}^{n} \binom{t-1}{i-1} v_{tj} = \sum_{S \subseteq N : |S|=i, j \in S} p_S, \tag{20}$$

where $v_{tj} = \mathbb{P}(\mu = t \cap X_j = 1)$. Note that the right-hand side of (20) only consists of the probabilities of those events that A_j occurs. Since the probability of the union of $A_1, \ldots, A_n$ can be expressed as follows

Theorem 3 ([9]).

$$\mathbb{P}(\mu \geq 1) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{i} v_{ij}, \tag{21}$$

an upper bound or a lower bound on the probability of union of events can be obtained by minimizing or maximizing the right-hand side of (21) over the set of constraints in (20) and the non-negativity constraints [9].

Appendix 2

The following optimization model identifies the probability distributions for which the gap between SPAM and PAM is maximal. Let z^*_{SPAM} and z^*_{PAM} be the lower bounds for the probability that at least k-out-of-n events occur, obtained by $SPAM$ model and PAM model, respectively. Let variables p_S, for all $S \subseteq N$ and $|S| \leq m$, represent the probability distributions we aim to obtain. Note that s_i^j is now a function of p_S. Let π_0 and π_i^j be the dual variables for constraint (5b) and (5c), respectively; μ_i^j be the dual variable corresponding to the constraint (15) for $i = 2, .., n-1$ and $j = 1, ..., n$, and μ_n^j be the dual variable corresponding to the constraint (16) for $j = 2, ..., n$.

$$
\begin{aligned}
\text{gap} = \quad & \max_{p_S} \; (z^*_{SPAM} - z^*_{PAM}) \\
= \quad & \max_{p_S} \; (\min_{v_{ij}} \{ \sum_{j=1}^{n} \sum_{i=k}^{n} v_{ij} : (5b), (5c), (5e), (15), \text{and } (16) \} \\
& - \min_{v_i} \{ \sum_{i=k}^{n} v_i : (5b), (5c), \text{and } (5e) \}) \\
= \quad & \max_{p_S} \; (\max_{(\pi, \mu)} \{ \pi_0 + \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{1}{i} \pi_i^j s_i^j : \\
& \pi_0 + \pi_1^j \leq e_1 \; j = 1, ..., n \\
& \pi_0 + \sum_{i=1}^{m} \binom{t}{i} \pi_i^j - (t-1)\mu_j^t + \sum_{\ell \neq j} \mu_t^\ell \leq e_t \; t = 2, ..., n-1; \; j = 1, ..., n \\
& \pi_0 + \sum_{i=1}^{m} \binom{n}{i} \pi_i^1 - \sum_{j=2}^{n} \mu_n^j \leq e_n
\end{aligned}
$$

$$\pi_0 + \sum_{i=1}^{m} \binom{n}{i} \pi_i^j + \mu_n^j \le e_n \ j = 2, ..., n$$

$$\pi_0, \pi_i^j, \mu_n^j \text{ free, }, \mu_i^j \ge 0 \ i = 2, ..., n-1\}$$

$$+ \max\{\sum_{i=k}^{n} \sum_{j=1}^{n} -v_{ij} : \sum_{i,j} v_{ij} = 1; \sum_{i=t}^{n} \binom{i}{t} v_{ij} = \frac{1}{t} s_t^j(x) \quad t = 1, ..., m \ j = 1, ..., n,$$

$$v_i^j \ge 0\})$$

$$= \max_{p_S, \pi, \mu, v_i^j} \{\pi_0 + \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{1}{i} \pi_i^j s_i^j - \sum_{i=k}^{n} \sum_{j=1}^{n} v_{ij} :$$

$$\pi_0 + \pi_1^j \le e_1 \ j = 1, ..., n$$

$$\pi_0 + \sum_{i=1}^{m} \binom{t}{i} \pi_i^j - (t-1)\mu_j^t + \sum_{\ell \ne j} \mu_t^\ell \le e_t \ t = 2, ..., n-1; \ j = 1, ..., n$$

$$\pi_0 + \sum_{i=1}^{m} \binom{n}{i} \pi_i^1 - \sum_{j=2}^{n} \mu_n^j \le e_n$$

$$\pi_0 + \sum_{i=1}^{m} \binom{n}{i} \pi_i^j + \mu_n^j \le e_n \ j = 2, ..., n$$

$$\sum_{i,j} v_{ij} = 1; \sum_{i=t}^{n} \binom{i}{t} v_{ij} = \frac{1}{t} s_t^j(x) \quad t = 1, ..., m \ j = 1, ..., n,$$

$$\pi_0, \pi_i^j, \mu_n^j \text{ free, }, \mu_i^j \ge 0 \ i = 2, ..., n-1, \ v_i^j \ge 0\},$$

where $e_t = 0$ if $t < k$; $e_t = 1$, otherwise.

The feasible set for $(..., p_S, ...)$ is as follows

$$\mathcal{P}^m = \{(..., p_S, ...) \in \mathbb{R}_+^{\binom{n}{1}+...+\binom{n}{m}} : \exists v \in \mathbb{R}_+^{2^n} \text{ s.t. } \sum_{C \subseteq N} w_C \le 1, p_S = \sum_{C:S \subseteq C} w_C \ \forall S \subseteq N, |S| \le m\}.$$