Competitive Equilibrium in Piecewise Linear and Concave Exchange Economies and the non-symmetric Nash Bargaining Solution

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Abstract

In this paper we show that for concave piecewise linear exchange economies every competitive equilibrium satisfies the property that the competitive allocation is a non-symmetric Nash bargaining solution with weights being the initial income of individual agents evaluated at the equilibrium price vector. We prove the existence of competitive equilibrium for concave piecewise linear exchange economies by obtaining a non-symmetric Nash bargaining solution with weights being defined appropriately. In a later section we provide a simpler proof of the same result using the Brouwer's fixed point theorem, when all utility functions are linear. In both cases the proofs pivotal step is the concave maximization problem due to Eisenberg and Gale and minor variations of it. We also provide a proof of the same results for economies where agents' utility functions are concave, continuously differentiable and homogeneous. In this case the main argument revolves around the concave maximization problem due to Eisenberg. Unlike previous results, we do not require all initial endowments to lie on a fixed ray through the origin.

1. Introduction: Eisenberg and Gale (1959) showed that for linear exchange economies with individual initial endowment bundles lying on a fixed ray through the origin, the proof of existence of a competitive equilibrium reduces to finding a solution for a concave programming problem. In particular they showed that the corresponding competitive allocation is a non-symmetric Nash bargaining solution (see Kalai (1977)) with weights being given by the share of each individual in the aggregate initial endowment. Moore (2007) refers to the non-symmetric Nash bargaining solution as a solution to the Cobb-Douglas-Eisenberg (CDE) aggregator function.

In this paper we first show that for concave piecewise linear exchange economies every competitive equilibrium satisfies the property that a competitive allocation is a non-symmetric

Nash bargaining solution with weights being the initial incomes evaluated at the corresponding equilibrium price vector. We then proceed to provide a proof of the existence of competitive equilibrium when utility functions are piecewise linear and concave by finding a non-symmetric Nash bargaining solution with weights determined endogenously. Thus the competitive equilibrium is obtained as the output of an optimization exercise although we need to use a fixed point theorem argument as well. The aggregate initial endowment could be distributed arbitrarily among the agents and are not restricted to lie on any fixed ray through the origin. However, each individual is required to be initially endowed with something and all goods are required to be positively endowed with one or more individuals. In a subsequent section we provide a simpler proof of the existence result using the Brouwer's fixed point theorem, when all utility functions are linear. In both cases the proofs pivotal step is the concave maximization problems due to Eisenberg and Gale (1959) and minor variations of it.

Using methods similar to that mentioned in the previous paragraph, we obtain identical results for pure exchange economies where utility functions are concave, continuously differentiable and homogeneous. Such economies have been studied by Eisenberg (1961) for the special case where all initial endowments lie on a single ray passing through the origin. We are able to extend the results to situations where initial endowments may be arbitrarily distributed. At this juncture it is worth noting that in section 5.3.2 of his thesis Gamp (2012) asks the question that we have tried to answer in this paper and the first sentence of the section summarizes the problem: "If the initial endowments are not proportionally distributed there is the problem that it is not clear for which α the α-symmetric Nash bargaining solution could be applied." The α symmetric weights refer to the weights of the corresponding non-symmetric Nash bargaining solution.

The fact that all competitive equilibria (for the kind of economies that we discuss here) satisfy the property that the competitive allocation is a non-symmetric Nash bargaining solution with weights being the individual incomes evaluated at the equilibrium price vector is an important and meaningful equivalence between competitive equilibrium allocations and non-symmetric Nash bargaining solutions.

There are alternative approaches to Nash bargaining beginning with the work of Trockel (1996). Trockel (1996) shows that "bargaining games induce in a canonical way an Arrow Debrue economy with production and private ownership. The unique Walras stable competitive equilibrium of this economy is shown to coincide with a non-symmetric Nash bargaining solution of the underlying bargaining game with weights corresponding to the shares in production." Brangewitz and Gamp (2013) show that for every possible vector of weights of an asymmetric Nash bargaining solution there exists a market that has this asymmetric Nash bargaining solution as its unique competitive payoff vector. In a related endeavor Sertel and Yildiz (2003) pose the following question: "Is there a bargaining solution that pays out theWalrasian welfare for exchange economies?" They show that there is no such bargaining solution, for there are distinct exchange economies whose Walrasian equilibrium welfare payoffs disagree but which define the same bargaining problem and should have hence determined the same bargaining solution and its payoffs. In a related but different line of investigation implementation of the Nash solution based on its Walrasian characterization and the core-equivalence of the Nash bargaining solution have been studied in Trockel (2000) and Trockel (2005) respectively. In Trockel (2005) it is shown that Nash solution coincides with the core of a large bargaining coalitional production economy with equal production possibilities for all agents. Given this, it is worth emphasizing that our investigation is an attempt to establish an entirely different result. We show that for all possible distribution of the initial endowments (possibly non-proportional), there is complete equivalence between the set of competitive equilibrium allocations and non-symmetric Nash bargaining solutions with weights being individual incomes evaluated at equilibrium prices. This result does not follow from any of the works that we have discussed above.

As such, the problem concerning existence of competitive equilibrium has already been solved under conditions more general than ours. The purpose of our paper is therefore not to restate special cases of results that already exist in the literature. The real contribution of our paper is to interpret the existence problem as finding a solution to a tractable concave programming problem, in particular finding a non-symmetric Nash bargaining problem. It is true that for a more general class of concave utility functions there already exists a celebrated proof due to Negishi (1960) which centers around the maximization of a social welfare function. However, nowhere in that paper is it indicated that a much simpler proof for the sub-class of piecewise linear and concave utility may exist. Nor is such an indication available for the sub-class of economies where agents' utility functions are concave, continuously differentiable and homogeneous. Further the endogenous determination of the parameters for the optimization problem is far more complicated and time consuming than in our case, although both use Kakutani's fixed point theorem to do so. Finally, a simpler proof with a considerably more meaningful characterization of the existence theorem for piecewise linear and concave utility functions has "stand-alone" value- using smooth utility functions to represent consumer preferences is probably just a convenient approximation for what should actually or realistically be represented by a piecewise linear utility function. Hence our results may be interesting in their own right.

2. The Model: Following Gale (1957, 1976), Eaves (1976), Cornet (1989), Codenotti and Varadrajan (2004), Ye (2007) and many others we define a piecewise linear and concave exchange economy as follows.

There are L > 1 goods indexed by j∈ {1,…,L} and H > 0 consumers indexed by h∈ {1,…,H}. Each consumer h is initially endowed with a bundle ω^h of the L goods. The j^{th} coordinate of ω^h denoted ω_j^h is equal to the amount of commodity j that consumer h is initially endowed with. It is assumed that for all h∈ {1,…,H}, $\omega^h \in \mathbb{R}_+^L\backslash\{0\}$, where $\mathbb{R}_+^L$ denotes the set of non-negative real vectors in L-dimensional Euclidean space. We also assume that the units of measurement of the quantities of the L goods are so chosen that for all j∈ {1,…,L}, $\sum_{h=1}^{H} \omega_j^h = 1$.

We shall use e to denote the L-vector all whose coordinates are equal to one. Thus we assume that $\sum_{h=1}^{H} \omega^h = e$.

Each consumer h has a utility function u^h: $\mathbb{R}_+^L \to \mathbb{R}$ that represents the preferences of h over the set of all consumption bundles $\mathbb{R}_+^L$. In this paper we assume for all h∈{1,…,H} there exists a non-empty finite set B(h) of vectors belonging to $\mathbb{R}_{++}^L$, such that for all x∈ $\mathbb{R}_+^L$, $u^h(x) = \min_{b \in B(h)} < b, x >$. Such utility functions are called piecewise linear and concave.

Observe that for all h∈{1,…,H}, u^h is strictly increasing. Let x,y∈ $\mathbb{R}_+^L$, with x-y∈ $\mathbb{R}_+^L$ and x≠ y.

Suppose $u^h(x)$ = <b', x>. Then the assumption that b' is a strictly positive vector implies that <b', x> > <b', y> ≥ $u^h(y)$. Thus, $u^h(x) > u^h(y)$.

Since preferences (as well as all the results discussed in this paper) are linearly invariant (i.e. replacing the utility function $u^h(.)$ by the utility function $ku^h(.)$ for some k > 0 does not affect our results) and b∈ $\mathbb{R}_{++}^L$ for all b∈B(h), h= 1,…,H, we may without loss of generality assume that $\max_{b \in B(h)} (\max_{j \in \{1,\ldots,L\}} b_j) = 1$.

If B(h) is a singleton then u^h is said to be linear. If for all h∈{1,…,H}, u^h is linear then we call the corresponding economy a linear exchange economy. For a linear exchange economy we write $u^h(x)$ = <b^h,x> where {b^h} = B(h) for all h∈{1,…,H}.

The assumption that B(h) consists of vectors belonging to $\mathbb{R}_{++}^L$is slightly stronger than the corresponding one assumed in the papers mentioned above that says that b^h is a non-zero and non-negative L-vector. However, this stronger assumption is required for the proof strategy we have in mind.

An allocation is a array X = <x^h| h = 1,…,H> such that for all h∈{1,…,H}, x^h∈ $\mathbb{R}_+^L$. x^h denotes the L-vector of commodities allocated for consumption to consumer h. The j^{th} coordinate of x^h denoted x_j^h is the quantity of good j consumed by consumer h. An allocation X = <x^h| h = 1,…,H> is said to be feasible if for all j∈{1,…,L}: $\sum_{h=1}^H x_j^h = 1$.

Let Δ = {p∈ $\mathbb{R}_+^L$| $\sum_{j=1}^L p_j = 1$}. An element p∈Δ is a price vector and its j^{th} coordinate p_j denotes the price of commodity j.

The initial income vector evaluated at price vector p, denoted w(p) = <$w^h(p)$|h=1,…,H>, where for h = 1,…, H, $w^h(p)$ = <p, ω^h> = $\sum_{j=1}^L p_j \omega_j^h$.

Since p∈Δ and $\sum_{h=1}^H \omega^h = e$, it follows that $\sum_{h=1}^H w^h(p) = 1$.

A competitive equilibrium (CE) is a price allocation pair (p*, X*) with X*= <x^{*h}| h = 1,…,H> such that:

(i) X* is a feasible allocation.

(ii) For all h, x^{*h} maximizes $u^h(x)$ subject to <p,x> ≤ <p, ω^h> and x∈ $\mathbb{R}_+^L$.

Computational aspects of such problems have been studied by Codenotti and Varadrajan (2004) for the kind of economy considered by Eisenberg and Gale (1959) and for Arrow Debreu economies by Ye (2007).

The following concept is due to Kalai (1977).

Let w $\in \mathbb{R}^H_{++}$ be such that $\sum_{h=1}^{H} w^h = 1$. A non-symmetric Nash bargaining solution with w as weights is an allocation X such that for some v, (v,X) is a solution to the following concave maximization problem:

Maximize $\prod_{h=1}^{H} (u^h)^{w^h}$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$ for all j = 1,…,L,

$u^h - \langle b, \xi^h \rangle \leq 0$ for all $b \in B(h)$ and h = 1,…, H,

$u^h, \xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

3. Market Equilibrium: We begin this section with the following proposition which will be of use throughout the paper.

Proposition 1: Let (p,X) be a CE. Then $p \in \mathbb{R}^L_{++}$.

Proof: Follows directly from the assumption that all utility functions are strictly increasing. Q.E.D.

For $p \in \Delta$, a p-market equilibrium is a price allocation (q, X) with X= $\langle x^h | h = 1,\ldots,H \rangle$ such that:

(i) X is a feasible allocation.

(ii) For all h, x^h maximizes $u^h(x)$ subject to $\langle q,x \rangle \leq w^h(p)$.

The proof of the following proposition is immediate.

Proposition 2: The pair (p,X) is a CE if and only if (p.X) is a p-market equililibrium.

The original version of the following theorem is well known.

Theorem 1 (Eisenberg and Gale (1959), Codenotti and Varadrajan (2004)): Let $p \in \Delta$ and suppose $w^h(p) > 0$ for all h = 1,…,H. Then, there is a solution (v,X)= ($\langle v^h | h=1,\ldots,H \rangle$, $\langle x^h | h = 1,\ldots,H \rangle$) to the following concave programming problem.

Maximize $\prod_{h=1}^{H} (u^h)^{w^h(p)}$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$ for all j = 1,…,L,

u^h – <b, ξ^h> ≤ 0 for all b∈B(h) and h = 1,…, H,

u^h, ξ_j^h≥ 0 for all j = 1,…,L and h = 1,…,H.

(i) For all (v,X) = (<v^h|h=1,…,H>,<x^h| h = 1,…,H>) and (v', Y) = (<v'^h|h=1,…,H>,<y^h| h = 1,…,H>) that solves the above concave programming problem, $v^h = u^h(x^h) = u^h(y^h) = v'^h > 0$ for all h = 1,…,H.

(ii) A necessary and sufficient condition for (v,X) to solve the above concave programming problem, is the following:

There exists a q∈Δ and for h = 1,…, H, there exists <$\pi^h(b) \in \mathbb{R}_+$|b∈B(h)> such that:

(a) for all h= 1,…,H and j = 1,…,L, $\sum_{b\in B(h)} \pi^h (b) b_j \leq q_j$ and $(\sum_{b\in B(h)} \pi^h (b) b_j - q_j)\, x_j^h = 0$.

(b) for all h = 1,…,H and b∈B(h), $\pi^h(b)(\langle b, x^h\rangle - v^h) = 0$.

(c) for all h = 1,..., H, $v^h \sum_{b\in B(h)} \pi^h (b) = w^h(p)$.

(iii) The pair (q,X) where q is as defined in (ii) is a p-market equilibrium.

Proof: Since (i) follows directly from the strict concavity of the logarithmic function, let us prove (ii). For simplicity in what follows we will denote $w^h(p)$ by w^h.

A (v,X) solves the concave programming problem in the statement of theorem 2 if and only if it solves the following problem.

Maximize $\sum_{h=1}^{H} w^h \log u^h$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$ for all j = 1,…,L,

u^h – <b, ξ^h> ≤ 0 for all b∈B(h) and h = 1,…, H,

u^h, ξ_j^h≥ 0 for all j = 1,…,L and h = 1,…,H.

(a), (b) and (c) of (ii) are the standard first order necessary and sufficient conditions for (v,X) to solve the above concave programming problem and their proofs are standard.

Now let us prove (iii).

Note that by the second half of (a) of (ii), (b) and (c), $\langle q,x^h\rangle = v^h \sum_{b\in B(h)} \pi^h (b) = w^h(p)$.

Let x∈ $\mathbb{R}_+^L$ be such that <q,x> ≤ $w^h(p)$. Since $w^h(p) > 0$ and $v^h > 0$, it must be that $\sum_{b\in B(h)} \pi^h (b) > 0$.

For first half of (a) it follows that $\langle q,x\rangle \geq \sum_{b\in B(h)} \pi^h (b) < b, x > \geq u^h(x) \sum_{b\in B(h)} \pi^h (b)$.

But $v^h \sum_{b \in B(h)} \pi^h(b) = w^h(p) \geq \langle q,x \rangle$.

Thus $v^h \sum_{b \in B(h)} \pi^h(b) \geq u^h(x) \sum_{b \in B(h)} \pi^h(b)$.

Since $\sum_{b \in B(h)} \pi^h(b) > 0$, we get that $v^h \geq u^h(x)$.

This proves (iii). Q.E.D.

4. Competitive equilibrium and the non-symmetric Nash bargaining solution: In this section we establish the main theoretical contribution of this paper for concave piecewise linear exchange economies.

Theorem 2: Let (p, X) be a CE for a concave piecewise linear exchange economy. Then X is a non-symmetric Nash bargaining solution with weights given by individual incomes evaluated at p, i.e. $\langle w^h(p) | h = 1,\ldots,H \rangle$.

Proof: Let (p,X) be a CE where $X = \langle x^h | h = 1,\ldots,H \rangle$. By proposition 1, $p \in \mathbb{R}^L_{++}$. Thus, $w^h(p) > 0$ for all h = 1,…,H and $p \in \Delta$. Thus $\sum_{h=1}^{H} x_j^h = 1$ for all j = 1,…,L, $u^h(x^h) > 0$ for all h = 1,…,H, and for all h = 1,…, H, the pair $(u^h(x^h), x^h)$ solves the following linear programming problem:

Maximize u^h

Subject to $u^h - \langle b, \xi^h \rangle \leq 0$ for all $b \in B(h)$,

$\langle p, \xi^h \rangle \leq w^h(p)$

$u^h \geq 0$ and $\xi_j^h \geq 0$ for all j = 1,…,L.

The dual of the above problem is

Minimize $\mu^h w^h(p)$

Subject to $\sum_{b \in B(h)} \rho^h(b) \geq 1$,

$-\sum_{b \in B(h)} \rho^h(b) b_j + \mu^h p_j \geq 0$.

By the duality theorem of linear programming the optimal values of $\langle \rho^h(b) | b \in B(h) \rangle$ and μ^h must satisfy $\mu^h w^h(p) = u^h(x^h) > 0$, i.e. $\mu^h = \frac{u^h(x^h)}{w^h(p)}$. Let $\pi^h(b) = \frac{\rho^h(b)}{\mu^h}$ for all $b \in B(h)$.

By the complementary slackness condition, $u^h(x^h)[\sum_{b \in B(h)} \rho^h(b) - 1] = 0$, for all j = 1,…,L, $[-\sum_{b \in B(h)} \rho^h(b) b_j + \mu^h p_j] x_j^h = 0$, $\rho^h(b)[u^h(x^h) - \langle b, x^h \rangle] = 0$ for all $b \in B(h)$. Since $u^h(x^h) > 0$, we get $\sum_{b \in B(h)} \rho^h(b) = 1$.

Using the fact that $\pi^h(b) = \frac{\rho^h(b)}{\mu^h} = \frac{\rho^h(b)w^h(p)}{u^h(x^h)}$ for all $b \in B(h)$, we get the following:

$u^h(x^h) \sum_{b\in B(h)} \pi^h(b) = w^h(p)$, $\sum_{b\in B(h)} \pi^h(b)b_j - p_j \leq 0$ and $[\sum_{b\in B(h)} \pi^h(b)b_j - p_j]\, x_j^h = 0$ for j = 1,…, L and $\pi^h(b)[u^h(x^h) - \langle b, x^h\rangle] = 0$ for all $b \in B(h)$.

But these are precisely conditions (a), (b) and (c) of (ii) of Theorem 1. Since these conditions along with $\sum_{h=1}^H x_j^h = 1$ for all j = 1,…,L, are sufficient for X to solve the concave programming problem in the statement of theorem 1, with q = p, we get our desired result. Q.E.D.

5. Existence of Competitive equilibrium in the general case: Having established that every CE for concave piecewise linear exchange economies is a non-symmetric Nash bargaining solution with weights being given by individual incomes evaluated at the equilibrium price vector, it is appropriate to seek existence of a CE for such an economy by obtaining a non-symmetric Nash bargaining solution with the kind of weights mentioned earlier.

In what follows we will need the following proposition.

Proposition 3: Suppose that for all $h \in \{1,…,H\}$, $\omega^h \in \mathbb{R}_{++}^L$. For $X^0 \in \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,…,L\}$: $\sum_{h=1}^H x_j^h = 1\}$ and $p \in \Delta$, consider the concave maximization problem

Maximize $\sum_{h=1}^H w^h(p) \log u^h - \|\xi - X^0\|^2$

Subject to $\sum_{h=1}^H \xi_j^h = 1$ for all j = 1,…,L,

$u^h - \langle b, \xi^h\rangle \leq 0$ for all $b \in B(h)$ and h = 1,…, H,

$u^h, \xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

(i) The above problem has a unique solution (v,X).

(ii) There exists a $q \in \Delta$ and for h = 1,…, H, there exists $\langle \pi^h(b) \in \mathbb{R}_+ | b \in B(h)\rangle$ and $\beta > 0$ such that:

(a) for all h= 1,…,H and j = 1,…,L, $\sum_{b\in B(h)} \pi^h(b)b_j - 2\beta(x_j^h - x_j^{0h}) \leq q_j$ and $(\sum_{b\in B(h)} \pi^h(b)b_j - q_j - 2\beta(x_j^h - x_j^{0h}))\, x_j^h = 0$.

(b) for all h = 1,…,H and $b \in B(h)$, $\pi^h(b)(\langle b, x^h\rangle - v^h) = 0$.

(c) for all h = 1,..., H, $v^h \sum_{b\in B(h)} \pi^h(b) = \beta w^h(p)$.

Proof: Since $\omega^h \in \mathbb{R}_{++}^L$ for all $h \in \{1,…,H\}$, we have $w^h(p) > 0$ for all h = 1,…,H.

Now, (i) follows from the continuity and strict concavity of the objective function and the non-emptiness, compactness and convexity of the constraint set.

(ii) The list of the first order necessary and sufficient conditions for optimality of the above concave programming problem is as follows:

There exists $q \in \mathbb{R}_+^L$ and for h = 1,…, H, there exists $<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>$ such that:

(a) for all h= 1,…,H and j = 1,…,L, $\sum_{b \in B(h)} \pi^h(b) b_j - 2(x_j^h - x_j^{0h}) \leq q_j$ and $(\sum_{b \in B(h)} \pi^h(b) b_j - q_j - 2(x_j^h - x_j^{0h}))\ x_j^h = 0$.

(b) for all h = 1,…,H and $b \in B(h)$, $\pi^h(b)(<b, x^h> - v^h) = 0$.

(c) for all h = 1,..., H, $v^h \sum_{b \in B(h)} \pi^h(b) = w^h(p)$.

All that we need to show is that $\sum_{j=1}^L q_j > 0$. Then if we set $\beta = \frac{1}{\sum_{j=1}^L q_j}$ we will obtain a proof of the proposition.

By the first half of (ii) (a) above in this proof, $\sum_{j=1}^L q_j \geq \sum_{j=1}^L (\sum_{b \in B(h)} \pi^h(b) b_j) \geq \min_{b \in B(h)} (\min_{j \epsilon \{1,\dots,L\}} b_j) \sum_{b \in B(h)} \pi^h(b) = \min_{b \in B(h)} (\min_{j \epsilon \{1,\dots,L\}} b_j) \frac{w^h(p)}{v^h} > 0$, since $\min_{b \in B(h)} (\min_{j \epsilon \{1,\dots,L\}} b_j) > 0$ and $\frac{w^h(p)}{v^h} > 0$.

Multiplying all the inequalities and equations in (a), (b), (c) of (ii) in this proof by $\frac{1}{\sum_{j=1}^L q_j}$ (and replacing $\frac{\pi^h(b)}{\sum_{j=1}^L q_j}$ by $\pi^h(b)$) we get the result. Q.E.D.

Note: The idea behind the objective function used in the concave programming problem in the statement of Proposition 3, comes from the kind of objective functions used in Geanakopolos (2003).

Let $(v(p,X^0), \mathcal{X}(p,X^0))$ denote the unique solution to the concave maximization problem in Proposition 1.

Lemma 1: The function $\mathcal{X}$ from $\Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\dots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$ to $\{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\dots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$ is continuous.

The proof of continuity of $\mathcal{X}$ follows from the maximum theorem of Berge (1963).

Let $\mathcal{P}(p,X^0) = \{q \in \Delta |$ there exists $\beta > 0$ and for h = 1,…, H, there exists $<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>$ which along with q and $(v(p,X^0), \mathcal{X}(p,X^0))$ satisfies (a), (b), (c) of (ii) of proposition 3$\}$.

Lemma 2: The correspondence $\mathcal{P}$ from $\Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\dots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$ to $\{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\dots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$ is non-empty valued, compact valued, convex valued and upper semicontinuous.

Proof: It follows from the maximum theorem due to Berge (1963) that the function v $\Delta\times\{X\in(\mathbb{R}^L_+)^H|$ for all $j\in\{1,\ldots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$ to $\mathbb{R}^H_+$ is continuous.

Let us show that the set $\{(<\pi^h(b)\in\mathbb{R}_+|b\in B(h)>)_{h=1}^H$ |for some (p,X^0) there exists q such that $(<\pi^h(b)\in\mathbb{R}_+|b\in B(h)>)_{h=1}^H$ along with q satisfies (a), (b), (c) in the proof of proposition 3} is bounded.

It follows from (c) of the proof of proposition 1, that for all h = 1,…,H, $\sum_{b\in B(h)}\pi^h(b) = \frac{w^h(p)}{v^h} > 0$. Since the function $(p,X^0)\mapsto\frac{w^h(p)}{v^h}$ is continuous on the compact set $\Delta\times\{X\in(\mathbb{R}^L_+)^H|$ for all $j\in\{1,\ldots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$, it attains a maximum and minimum both of which have positive value. Hence there exists real numbers with $K_2 > K_1 > 0$ such that for all $(p, X^0)\in\Delta\times\{X\in(\mathbb{R}^L_+)^H|$ for all $j\in\{1,\ldots,L\}$: $\sum_{h=1}^H x_j^h = 1\}$, h = 1,…,H, $K_2 \geq \sum_{b\in B(h)}\pi^h(b) \geq K_1$. This in particular implies that the set $\{(<\pi^h(b)\in\mathbb{R}_+|b\in B(h)>)_{h=1}^H$ |for some (p,X^0) there exists q such that $(<\pi^h(b)\in\mathbb{R}_+|b\in B(h)>)_{h=1}^H$ along with q satisfies (a), (b), (c) in the proof of proposition 3} is bounded.

Now let us show that there exists real numbers $k_2 > k_1 > 0$, such that the set $\{\frac{1}{\sum_{j=1}^L q_j}|$ for some (p,X^0) there exists $(<\pi^h(b)\in\mathbb{R}_+|b\in B(h)>)_{h=1}^H$ such that q along with $(<\pi^h(b)\in\mathbb{R}_+|b\in B(h)>)_{h=1}^H$ satisfies (a), (b), (c) in the proof of proposition 3} $\subset [k_1, k_2]$.

We have already show that given (p,X^0), $\sum_{j=1}^L q_j \geq \min_{b\in B(h)}(\min_{j\epsilon\{1,\ldots,L\}} b_j)\frac{w^h(p)}{v^h} > 0$.

Thus, $\sum_{j=1}^L q_j \geq K_1\min_{b\in B(h)}(\min_{j\epsilon\{1,\ldots,L\}} b_j) > 0$.

Let $k_2 = \frac{1}{K_1\min_{b\in B(h)}(\min_{j\epsilon\{1,\ldots,L\}} b_j)} > 0$. Thus, $\frac{1}{\sum_{j=1}^L q_j} \leq k_2$.

From (a) of the proof of proposition 3, we have for all h= 1,…,H and j = 1,…,L, $(\sum_{b\in B(h)}\pi^h(b)b_j - q_j - 2(x_j^h - x_j^{0h}))\ x_j^h = 0$.

Thus, $x_j^h\sum_{b\in B(h)}\pi^h(b)b_j - 2(x_j^h - x_j^{0h})\ x_j^h = q_j x_j^h$.

By our assumption on preferences, $\sum_{b\in B(h)}\pi^h(b)b_j \leq \sum_{b\in B(h)}\pi^h(b) \leq K_2$ and by our assumption that the aggregate initial endowment of each good is 1 unit we get $-2(x_j^h - x_j^{0h})\ x_j^h \leq 2$.

Thus $K_2 x_j^h + 2 \geq q_j x_j^h$ for all h= 1,…,H and j = 1,…,L.

Summing over h and appealing to the feasibility of X we get $K_2 + 2H \geq q_j$.

Let $k_1 = \frac{1}{L(K_2 + 2H)} > 0$. Thus, $\frac{1}{\sum_{j=1}^{L} q_j} \geq k_1$.

Combining all the inequalities obtained above we get that:

(i) The set $\{(<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>)_{h=1}^{H}$ |for some (p,X^0) there exists $q \in \Delta$ and $\beta > 0$ and such that $(<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>)_{h=1}^{H}$ along with q and β satisfies (a), (b), (c) in the statement of proposition 3} is a bounded set.

(ii) {β| for some (p,X^0) there exists $(<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>)_{h=1}^{H}$ such that q along with $(<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>)_{h=1}^{H}$ and β satisfies (a), (b), (c) in the statement of proposition 3}$\subset [k_1, k_2]$, where k_1 and k_2 are real numbers satisfying $k_2 > k_1 > 0$.

If $<(p^{(n)}, X^{0(n)}) | n \in \mathbb{N}>$ is a sequence converging to (p, X^0) and $<q^{(n)} \in \mathcal{P}(p^{(n)},X^{0(n)}) | n \in \mathbb{N}>$ is a sequence converging to q, then for all $n \in \mathbb{N}$, h = 1,…, H and $b \in B(h)$ there exists $<\pi^{h(n)}(b) \in \mathbb{R}_+ | n \in \mathbb{N}>$ and $<\beta^{(n)} | n \in \mathbb{N}>$ which along with $q^{(n)}$ and $(v(p^{(n)},X^{0(n)}), \mathcal{X}(p^{(n)},X^{0(n)}))$ satisfies (a), (b), (c) of (ii) of proposition 1 for all $n \in \mathbb{N}$. Since the sequences $<\pi^{h(n)}(b) \in \mathbb{R}_+ | n \in \mathbb{N}>$ for h = 1,…, H and $b \in B(h)$ as well as $<\beta^{(n)} | n \in \mathbb{N}>$, lie within a bounded set there are convergent subsequences $<\pi^{h(m(n))}(b) \in \mathbb{R}_+ | n \in \mathbb{N}>$ for h = 1,…, H and $b \in B(h)$ and $<\beta^{m(n)} | n \in \mathbb{N}>$, such that $\lim_{n\to\infty} \pi^{h(m(n))}(b) = \pi^h(b)$ for h = 1,…, H and $b \in B(h)$ and $\lim_{n\to\infty} \beta^{m(n)} = \beta$. It follows easily that $<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>$, h = 1,…,H and β along with q and $(v(p,X^0), \mathcal{X}(p,X^0))$ satisfies (a), (b), (c) of (ii) of corollary of proposition 1. Thus $q \in \mathcal{P}(p,X^0)$. This proves the upper semi-continuity of $\mathcal{P}$.

The other properties of $\mathcal{P}$ are easy to verify. Q.E.D.

Note that in the two proofs above, we distinguish between (a), (b) and (c) in the statement of proposition 1 and (a), (b) and (c) in the proof of proposition 1.

Proposition 4: Suppose that for all $h \in \{1,\ldots,H\}$, $\omega^h \in \mathbb{R}_{++}^L$. Then a CE (p,X) exists for the corresponding concave piecewise linear economy.

Proof: Consider the correspondence F from $\Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\ldots,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$ to itself defined by $F(p,X^0) = \mathcal{P}(p,X^0) \times \{\mathcal{X}(p,X^0)\}$ for all $(p,X^0) \in \Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\ldots,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$.

By lemmas 1 and 2, F is non-empty valued, compact valued, convex valued and upper-semicontinuous. Since its domain is compact, convex and the same as the co-domain, by Kakutani's fixed point theorem it has a fixed point, i.e. there exists $(p, X^0) \in \Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\ldots,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$, such that $(p, X^0) \in F(p,X^0)$.

Thus $(v(p,X^0), \mathcal{X}(p,X^0))$ solves the concave maximization problem in proposition 1 and for h = 1,…, H, there exists $\beta > 0$ and $<\pi^h(b) \in \mathbb{R}_+ | b \in B(h)>$ such that:

(a) for all h= 1,…,H and j = 1,…,L, $\sum_{b\in B(h)} \pi^h(b)b_j$ - $2\beta(x_j^{0h}$ - $x_j^{0h}) \leq p_j$ and $(\sum_{b\in B(h)} \pi^h(b)b_j$- p_j- $2\beta(x_j^{0h}$ - $x_j^{0h}))\, x_j^{0h} = 0$.

(b) for all h = 1,…,H and b∈B(h), $\pi^h(b)(<b, x^{0h}> - v^h) = 0$.

(c) for all h = 1,...., H, $v^h\sum_{b\in B(h)} \pi^h(b) = \beta w^h(p)$.

(d) for all h = 1,…,H, $v^h = v^h(p,X^0)$.

Hence,

(a) for all h= 1,…,H and j = 1,…,L, $\sum_{b\in B(h)} \pi^h(b)b_j \leq p_j$ and $(\sum_{b\in B(h)} \pi^h(b)b_j$- $p_j)\, x_j^{0h} = 0$.

(b) for all h = 1,…,H and b∈B(h), $\pi^h(b)(<b, x^{0h}> - v^h) = 0$.

(c) for all h = 1,...., H, $v^h\sum_{b\in B(h)} \pi^h(b) = \beta w^h(p)$.

(d) for all h = 1,…,H, $v^h = v^h(p,X^0)$.

From (a), $\sum_{b\in B(h)} \pi^h(b) <b, x^{0h}> = <p, x^{0h}>$.

From (b), $\pi^h(b)<b, x^{0h}> = \pi^h(b)v^h$.

Thus, $<p, x^{0h}> = v^h\sum_{b\in B(h)} \pi^h(b) = \beta w^h(p)$.

Thus, $1 = \sum_{h=1}^H < p, x^{0h} > = \beta\sum_{h=1}^H w^h(p) = \beta$, since $\sum_{h=1}^H w^h(p) = 1$ and $\sum_{h=1}^H \sum_{j=1}^L p_j x_j^{0h} = \sum_{j=1}^L p_j \sum_{h=1}^H x_j^{0h} = 1$.

Hence,

(a) for all h= 1,…,H and j = 1,…,L, $\sum_{b\in B(h)} \pi^h(b)b_j \leq p_j$ and $(\sum_{b\in B(h)} \pi^h(b)b_j$- $p_j)\, x_j^{0h} = 0$.

(b) for all h = 1,…,H and b∈B(h), $\pi^h(b)(<b, x^{0h}> - v^h) = 0$.

(c) for all h = 1,...., H, $v^h\sum_{b\in B(h)} \pi^h(b) = w^h(p)$.

(d) for all h = 1,…,H, $v^h = v^h(p,X^0)$.

Thus the pair (p,X^0) is a p-market equilibrium.

Thus by proposition 2, (p,X^0) is a CE>. Q.E.D.

Now we are in a position to show that a CE exists for a concave piecewise linear exchange economy without assuming that for all h∈{1,…,H}, $\omega^h \in \mathbb{R}_{++}^L$.

Theorem 3: A competitive equilibrium exists for a concave piecewise linear exchange economy.
Proof: If for all h∈{1,…,H}, $\omega^h \in \mathbb{R}^L_{++}$ then the proof follows from Proposition 4. Hence suppose that for some h∈{1,…,H}, $\omega^h \notin \mathbb{R}^L_{++}$. Now there exists a sequence $<(\omega^{(n)1},\ldots,\omega^{(n)H})| n\in\mathbb{N}>$ such that:

(i) for each $n\in\mathbb{N}$, $(\omega^{(n)1},\ldots,\omega^{(n)H}) \in (\mathbb{R}^L_{++})^H$;

(ii) for each j = 1,…,L and $n\in\mathbb{N}$, $\sum_{h=1}^{H}\omega_j^{(n)h} = 1$;

(iii) for each j = 1,…, L and h = 1,…, H, $\lim_{n\to\infty}\omega_j^{(n)h} = \omega_j^h$.

By proposition 4, for each $n\in\mathbb{N}$, there exists a CE $(p^{(n)}, X^{(n)})$ for $(\omega^{(n)1},\ldots,\omega^{(n)H})$ where $(v^{(n)}, X^{(n)})$ solves the concave maximization problem

Maximize $\prod_{h=1}^{H}(u^h)^{w^h(p^{(n)})}$

Subject to $\sum_{h=1}^{H}\xi_j^h = 1$ for all j = 1,…,L,

$u^h - <b, \xi^h> \leq 0$ for all b∈B(h) and h = 1,…, H,

$u^h, \xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

Since the sequence $<(p^{(n)}, X^{(n)})| n\in\mathbb{N}>$ lies in a closed and bounded set it has a convergent subsequence which without loss of generality can be considered to be the original sequence. Thus let $<(p^{(n)}, X^{(n)})| n\in\mathbb{N}>$ converge to (p,X). Clearly, p∈Δ and X is a feasible allocation.

Towards a contradiction suppose $p_j = 0$. Without loss of generality suppose j = 1. Then there exists some other price say p_2 which is strictly positive. Since the aggregate endowment of good 2 is 1, there exists some consumer, say consumer 1, such that $\omega_2^1 > 0$. Since $\lim_{n\to\infty}\omega_2^{(n)1} = \omega_2^1 > 0$ and since $\lim_{n\to\infty}p_2^{(n)} = p_2 > 0$, there exists a positive integer such that for all n greater than this positive integer $\omega_2^{(n)1} > \frac{\omega_2^1}{2} > 0$ and $p_2^{(n)} > \frac{p_2}{2} > 0$. Without loss of generality suppose that for all $n\in\mathbb{N}$, $\omega_2^{(n)1} > \frac{\omega_2^1}{2} > 0$ and $p_2^{(n)} > \frac{p_2}{2} > 0$.

Consider the consumption bundle $x^{(n)}$ such that for all j≠ 1, $x_j^{(n)} = 0$ and $x_1^{(n)} = \frac{\sum_{j=1}^{L}p_j^{(n)}\omega_j^{(n)1}}{p_1^{(n)}} > \frac{p_2\omega_2^1}{4p_1^{(n)}}$ for all $n\in\mathbb{N}$. Clearly this consumption bundle is affordable for consumer 1 at price vector $p^{(n)}$. Further, the sequence $<u^1(x^{(n)})|n\in\mathbb{N}>$ is a divergent sequence. However since $X^{(n)}$ is a feasible allocation for all $n\in\mathbb{N}$, the sequence $<u^1(x^{(n)1})| n\in\mathbb{N}>$ is a bounded sequence. Hence eventually, $u^1(x^{(n)}) > u^1(x^{(n)1})$, contradicting that $(p^{(n)}, X^{(n)})$ is a CE for $(\omega^{(n)1},\ldots,\omega^{(n)H})$.

Thus $p_j > 0$ for all $j \in \{1,\ldots,L\}$.

Now suppose (p,X) is not a CE. Then there exists some consumer say consumer 1 and some consumption bundle x such that $<p,x> \leq <p,\omega^1>$ and $u^1(x) > u^1(x^1)$. Since the sequence $<x^{(n)1} | n \in \mathbb{N}>$ converges to x^1, by the continuity of u^1 for all n sufficiently large $u^1(x) > u^1(x^{(n)1})$. Since for all n, $(p^{(n)}, X^{(n)})$ is a CE for $(\omega^{(n)1},\ldots, \omega^{(n)H})$ it must be the case that for all n sufficiently large, $<p^{(n)},x> > <p^{(n)},\omega^{(n)1}>$. By taking limits we get $<p,x> \geq <p,\omega^1>$. Thus $<p,x> = <p,\omega^1>$. Thus if x' is any consumption bundle with $u^1(x') > u^1(x^1)$ then $<p,x'> \geq <p,\omega^1>$.

Since $u^1(x) > u^1(x^1) \geq 0$, it must be the case that at least one coordinate of x is strictly positive. Without loss of generality suppose $x_1 > 0$. By continuity of u^1 there exists $\varepsilon > 0$ such that $x_1 - \varepsilon > 0$ and $u^1(x_1-\varepsilon, x_2,\ldots,x_L) > u^1(x^1)$. But $<p, (x_1-\varepsilon, x_2,\ldots,x_L)> < <p,x> = <p,\omega^1>$, contradicting what we obtained above.

Thus (p,X) is a CE>. Q.E.D.

Note: In theorem 3 of his paper, Ye (2007) does mention an existence result like our theorem 1 for the case where L =H, each agent h is initially endowed with exactly one unit of the h^{th} good and utility functions are of the type due by Leontief.

6. Existence of CE for linear exchange economies: In this section we provide a simple proof of the equivalent version of theorem 1, for a linear exchange economy. In this special case we do not need to use Kakutani's fixed point theorem; in fact the simpler Brouwer's fixed point theorem suffices for our purpose.

Theorem 4 (Eisenberg-Gale (1959)): Suppose that for all $h \in \{1,\ldots,H\}$, $\omega^h \in \mathbb{R}^L_{++}$ and $p \in \Delta$. Then there is a solution $X = <x^h | h = 1,\ldots,H>$ to the following concave programming problem.

Maximize $\prod_{h=1}^{L}(\sum_{j=1}^{L} b_j^h \xi_j^h)^{w^h(p)}$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$,

$\xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

(i) For all $X = <x^h | h = 1,\ldots,H>$ and $Y = <y^h | h = 1,\ldots,H>$ that solves the above concave programming problem, $<b^h, x^h> = <b^h, y^h>$ for all h = 1,…,H.

(ii) A necessary and sufficient condition for X to solve the above concave programming problem, is that there exists a $q \in \Delta$ such that for all h= 1,…,H and j = 1,…,L, $\frac{b_j^h w^h(p)}{\sum_{j=1}^{L} b_j^h x_j^h} \leq q_j$ and $(\frac{b_j^h w^h(p)}{\sum_{j=1}^{L} b_j^h x_j^h} - q_j)\, x_j^h = 0$.

(iii) The pair (q,X) with q as defined in (ii) is a p-market equilibrium.

For h = 1,…,H, let $v^h(p) = \langle b^h, x^h \rangle$, where X= $\langle x^h | h = 1,\ldots,H \rangle$ is a solution to the concave programming problem defined in the statement of theorem 3. By part (i) of theorem 3, the function $v^h: \Delta \to \mathbb{R}_+$ is well defined for h = 1,…,H. It easy to see from the strict monotonicity of individual utility functions and the positivity of $w^h(p)$ for all $p \in \Delta$ and h = 1,…,H that $v^h(p) \in \mathbb{R}_{++}$ for all $p \in \Delta$ and h = 1,…,H.

Observe that X is the non-symmetric Nash bargaining solution for the economy with weights being given by w(p).

Let $f:\Delta \to \Delta$ be the function such that for all $p \in \Delta$ and j = 1,…,L, $f_j(p) = \max_{h=1,\ldots,H}\{\frac{b_j^h w^h(p)}{v^h(p)}\}$. f(p) is the q in the statement of theorem 4. We shall refer to the pair (f(p), X) as an Eisenberg-Gale p-market equilibrium (EGpME).

Lemma 3: The function $v = (v^1, v^2, \ldots, v^H): \Delta \to \mathbb{R}_{++}^L$ is continuous. Thus the function $f:\Delta \to \Delta$ is continuous.

Proof: Let $\langle p^{(n)} | n \in \mathbb{N} \rangle$ denote a sequence of price vectors converging to the price vector p. Clearly for all h = 1,…,H, the sequence $\langle w^h(p^{(n)}) | n \in \mathbb{N} \rangle$ converges to $w^h(p) > 0$. For $n \in \mathbb{N}$, let $(f(p^{(n)}), X^{(n)})$ be an EG$p^{(n)}$ME with $X^{(n)} = \langle x^{(n)h} | h = 1,\ldots,H \rangle$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $v(p^{(n)}) = (v^1(p^{(n)}), v^2(p^{(n)}), \ldots, v^H(p^{(n)}))$. We have to show that the sequence $\langle v(p^{(n)}) | n \in \mathbb{N} \rangle$ converges to v(p). We will show that every subsequence of $\langle v(p^{(n)}) | n \in \mathbb{N} \rangle$ admits a further subsequence that converges to v(p). That will prove the lemma.

Observe that $v^h(p^{(n)}) = \langle b^h, x^{(n)h} \rangle$ for all h = 1,…,H and $n \in \mathbb{N}$. Consider a subsequence of $\langle v(p^{(n)}) | n \in \mathbb{N} \rangle$ which without loss of generality (and to save the use of cumbersome notation) we shall denote by $\langle v(p^{(n)}) | n \in \mathbb{N} \rangle$. We shall denote the corresponding subsequence of allocations by $\langle X^{(n)} | n \in \mathbb{N} \rangle$, knowing full well that it is really a subsequence of the sequence that we started of with. Since the sequence (subsequence of) $\langle X^{(n)} | n \in \mathbb{N} \rangle$ lies in a closed and bounded subset of $(\mathbb{R}^L)^H$ it has a convergent subsequence which again for the same reasons as before can be considered to be the original sequence.

Let $\langle X^{(n)} | n \in \mathbb{N} \rangle$ converge to X. Then it is easy to see that (a) X is a feasible allocation; and (b) the sequence $\langle \prod_{h=1}^{L} (\sum_{j=1}^{L} b_j^h x_j^{(n)h})^{w^h(p^{(n)})} | n \in \mathbb{N} \rangle$ converges to $\prod_{h=1}^{L} (\sum_{j=1}^{L} b_j^h x_j^h)^{w^h(p)}$. Note that the convergences in (a) and (b) follow from the continuity of the respective functions.

Towards a contradiction suppose X does not solve the concave programming problem in the statement of theorem 1. Then there exists feasible allocation $Y = \langle y^h | h = 1,\ldots,H \rangle$ such that $\prod_{h=1}^{L} (\sum_{j=1}^{L} b_j^h y_j^h)^{w^h(p)} > \prod_{h=1}^{L} (\sum_{j=1}^{L} b_j^h x_j^h)^{w^h(p)}$. Thus for large n, $\prod_{h=1}^{L} (\sum_{j=1}^{L} b_j^h y_j^h)^{w^h(p^{(n)})} > \prod_{h=1}^{L} (\sum_{j=1}^{L} b_j^h x_j^{(n)h})^{w^h(p^{(n)})}$, contradicting that $u(p^{(n)}) = (\langle b^1, x^{(n)1} \rangle, \ldots, \langle b^H, x^{(n)H} \rangle)$ for all $n \in \mathbb{N}$.

Thus X solves the concave programming problem in the statement of theorem 1 and so v(p) = ($<b^1, x^1>,\ldots,<b^H, x^H>$) $=\lim_{n\to\infty}(< b^{(n)1}, x^{n(1)} >, \ldots, < b^{(n)H}, x^{(n)H} >) = \lim_{n\to\infty} v(p^{(n)})$.

Now recalling that we have appealed to no loss of generality to replace subsequences by the original sequence (several times), what we have shown is that every subsequence of $<v(p^{(n)}) | n\in\mathbb{N}>$ contains a subsequence that converges to v(p).

Thus the sequence $<v(p^{(n)}) | n\in\mathbb{N}>$ itself converges to v(p). Thus v is continuous.

Since $v(p)\in\mathbb{R}^L_{++}$ for all $p\in\Delta$, it follows that f is continuous. Q.E.D.

Proposition 5: Suppose that for all $h\in\{1,\ldots,H\}$, $\omega^h\in\mathbb{R}^L_{++}$. Then a CE exists for the economy described in the previous section.

Proof: By lemma 3 and Brouwer's fixed point theorem, f has a fixed point $p\in\Delta$. Let (p,X) be the (corresponding) EGpME. Then it is easily verified that (p,X) is a CE. Q.E.D.

The following theorem now follows from theorem 4 and proposition 5.

Theorem 5: A competitive equilibrium exists for a linear exchange economy.

Proof: The proof is identical to the proof of theorem 3. Q.E.D.

7. Existence of CE for concave differentiable and homogeneous economies: In this section we provide a proof of existence of a CE when utility functions are concave, continuously differentiable and homogeneous.

We assume that for all h = 1,…,H:

(a) u^h is continuously differentiable and concave with $u^h(0) = 0$.

(b) $Du^h(x) >> 0$ for all $x\in\mathbb{R}^L_{++}$.

(c) For all $x\in\mathbb{R}^L_{+}\backslash\mathbb{R}^L_{++}$: $u^h(x) > 0$ implies $Du^h(x) >> 0$, where for $x\in\mathbb{R}^L_{+}\backslash\mathbb{R}^L_{++}$ with $u^h(x) > 0$, $Du^h(x)$ is defined to be the real valued vector $\lim_{n\to\infty} Du^h(x^{(n)})$ given that $<x^{(n)}/n\in\mathbb{N}>$ is a sequence in $\mathbb{R}^L_{++}$ converging to x.

(d) u^h is homogeneous, i.e. there exists $\alpha^h > 0$ such that for all $x\in\mathbb{R}^L_{+}$ and t > 0: $u^h(tx) = t^{\alpha^h}u^h(x)$.

α^h in (d) is called the degree of homogeneity of u^h. By Euler's Theorem for all $x\in\mathbb{R}^L_{+}$:

$$\alpha^h u^h(x) = \sum_{j=1}^{L} x_j \frac{\partial u^h(x)}{\partial x_j}.$$

Note that u^h is semi-strictly increasing (i.e. [$x\in\mathbb{R}^L_{++}$, $y\in\mathbb{R}^L_{+}$, x > y] implies [$u^h(x) > u^h(y)$]). By continuity of u^h, it is weakly increasing (i.e. [$x\in\mathbb{R}^L_{+}$, $y\in\mathbb{R}^L_{+}$, x > y] implies [$u^h(x) \geq u^h(y)$]).

Under our assumptions on u^h and due to its concavity, for all $x \in \mathbb{R}_+^L$: $0 = u^h(0) \leq u^h(x) + \sum_{j=1}^{L}(-x_j)\frac{\partial u^h(x)}{\partial x_j}$.

Thus for all $x \in \mathbb{R}_+^L$: $u^h(x) \geq \sum_{j=1}^{L} x_j \frac{\partial u^h(x)}{\partial x_j} = \alpha^h u^h(x)$.

Since $u^h(x) > 0$ for $x \in \mathbb{R}_{++}^L$, it must be the case that $\alpha^h \leq 1$.

Except for this alteration everything remains the same as for a concave piecewise linear exchange economy.

Propositions 1 and 2 continue to hold even in this modified setting.

Theorem 6 (Eisenberg (1961)): Suppose $\omega^h \in \mathbb{R}_{++}^L$ for all h = 1,…,H. For all $p \in \Delta$, there is a solution $X = \langle x^h | h = 1,\ldots,H\rangle$ to the following concave programming problem.

Maximize $\prod_{h=1}^{H}(u^h(\xi^h))^{\frac{w^h(p)}{\alpha^h}}$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$ for all j = 1,…,L,

$\xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

(i) For all $X = \langle x^h | h = 1,\ldots,H\rangle$ and $Y = \langle y^h | h = 1,\ldots,H\rangle$ that solves the above concave programming problem, $u^h(x^h) = u^h(y^h) > 0$, $x^h, y^h \in \mathbb{R}_{++}^L$ for all h = 1,…,H.

(ii) For all X that solves the above concave programming problem, there exists a $q \in \Delta$ such that for all h= 1,…,H and j = 1,…,L, $\frac{w^h(p)}{\alpha^h u^h(x^h)}\frac{\partial u^h(x^h)}{\partial x_j^h} \leq q_j$ and $\left(\frac{w^h(p)}{\alpha^h u^h(x^h)}\frac{\partial u^h(x^h)}{\partial x_j^h} - q_j\right) x_j^h = 0$.

(iii) The pair (q,X) is a p-market equilibrium.

Proof: Since (i) follows directly from the strict concavity of the objective function and our assumption on preferences, let us prove (ii). For simplicity in what follows we will denote $w^h(p)$ by w^h.

X solves the concave programming problem in the statement of theorem 6 if and only if it solves the following problem.

Maximize $\sum_{h=1}^{H}\frac{w^h}{\alpha^h}\log u^h(\xi^h)$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$ for all j = 1,…,L,

$\xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

(ii) is the standard first order necessary and sufficient conditions for X to solve the above concave programming problem. Note that the first order necessary and sufficient conditions would only guarantee that q is a non-negative L-vector. From the first order conditions we get that $<q,x^h> = \frac{w^h}{\alpha^h u^h(x^h)} \sum_{j=1}^{L} x_j^h \frac{\partial u^h(x^h)}{\partial x_j^h} = w^h$ by Euler's theorem.

Thus, $\sum_{j=1}^{L} q_j = \sum_{j=1}^{L} q_j \sum_{h=1}^{H} x_j^h = \sum_{h=1}^{H} < q, x^h > = \sum_{h=1}^{H} w^h = \sum_{h=1}^{H} < p, \omega^h > = \sum_{j=1}^{L} p_j \sum_{h=1}^{H} \omega_j^h = \sum_{j=1}^{L} p_j = 1$.

Hence $q \in \Delta$.

Now let us prove prove (iii).

Note that by (ii) and Euler's theorem, $<q,x^h> = \frac{w^h}{\alpha^h u^h(x^h)} \sum_{j=1}^{L} x_j^h \frac{\partial u^h(x^h)}{\partial x_j^h} = w^h$

Let $x \in \mathbb{R}_+^L$ be such that $<q,x> \leq w^h$.

From (ii) it follows that $<q,x> \geq \frac{w^h}{\alpha^h u^h(x^h)} \sum_{j=1}^{L} x_j \frac{\partial u^h(x^h)}{\partial x_j^h}$.

Thus, $\sum_{j=1}^{L} x_j^h \frac{\partial u^h(x^h)}{\partial x_j^h} \geq \sum_{j=1}^{L} x_j \frac{\partial u^h(x^h)}{\partial x_j^h}$, i.e. $\sum_{j=1}^{L} (x_j - x_j^h) \frac{\partial u^h(x^h)}{\partial x_j^h} \leq 0$.

Hence $u^h(x) \leq u^h(x^h) + \sum_{j=1}^{L} (x_j - x_j^h) \frac{\partial u^h(x^h)}{\partial x_j^h} \leq u^h(x^h)$.

This proves (iii). Q.E.D.

Now we show that for the kind of exchange economy that we discuss in this section, every CE is a non-symmetric Nash bargaining solution with weights given by the individual incomes evaluated at equilibrium prices.

Theorem 7: Let (p, X) be a CE for the kind of economy that we discuss in this section. Then X is a non-symmetric Nash bargaining solution with weights given by individual incomes evaluated at p divided by the degree of homogeneity of the individual's utility function, i.e. $<\frac{w^h(p)}{\alpha^h} | h = 1,\ldots,H>$.

Proof: Let (p,X) be a CE where $X = <x^h | h = 1,\ldots,H>$. Thus $\sum_{h=1}^{H} x_j^h = 1$ for all j = 1,…,L, $u^h(x^h) > 0$ for all h = 1,…,H, $p \in \mathbb{R}_{++}^L$ and x^h solves the following concave programming problem:

Maximize $u^h(\xi^h)$,

Subject to $<p, \xi^h> \leq w^h(p)$,

$\xi_j^h \geq 0$ for all j = 1,…,L.

Thus there exists $\lambda^h \geq 0$, such that $\frac{\partial u^h(x^h)}{\partial x_j^h}$ - $\lambda^h p_j \leq 0$ and [$\frac{\partial u^h(x^h)}{\partial x_j^h}$ - $\lambda^h p_j$] x_j^h = 0 for all j = 1,…,L.

Hence, $\sum_{j=1}^{L} x_j^h \frac{\partial u^h(x^h)}{\partial x_j^h} = \lambda^h \sum_{j=1}^{L} p_j x_j^{0h} = \lambda^h w^h(p)$.

By Euler's theorem, $\alpha^h u^h(x^h) = \sum_{j=1}^{L} x_j^h \frac{\partial u^h(x^h)}{\partial x_j^h}$.

Thus $\lambda^h = \frac{\alpha^h u^h(x^h)}{w^h(p)}$.

Thus $\frac{w^h(p)}{\alpha^h u^h(x^h)} \frac{\partial u^h(x^h)}{\partial x_j^h}$ - $p_j \leq 0$ and [$\frac{\partial u^h(x^h)}{\partial x_j^h}$ - $\lambda^h p_j$] x_j^h = 0 for all j = 1,…,L.

But these are precisely the conditions in (ii) of theorem 5. Along with $\sum_{h=1}^{H} x_j^h = 1$ for all j = 1,…,L, they are sufficient for X to solve the concave maximization problem in the statement of theorem 6 with q = p. This proves the theorem. Q.E.D.

Thus the equivalent of theorem 2 holds in the differentiable case as well. We will show that it is possible to obtain the existence of CE in the differentiable case by using by showing that an appropriate non-symmetric Nash bargaining solution exists.

Proposition 6: Suppose $\omega^h \in \mathbb{R}_{++}^L$ for all h = 1,…,H. For $X^0 \in \{X \in (\mathbb{R}_+^L)^H \mid$ for all $j \in \{1,…,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$ and $p \in \Delta$, consider the concave maximization problem

Maximize $\sum_{h=1}^{H} w^h(p) \log u^h(\xi^h) - \|\xi - X^0\|^2$

Subject to $\sum_{h=1}^{H} \xi_j^h = 1$ for all j = 1,…,L,

$\xi_j^h \geq 0$ for all j = 1,…,L and h = 1,…,H.

(i) The above problem has a unique solution $X \in \mathbb{R}_{++}^L$.

(ii) There exists a $\beta > 0$ and $q \in \Delta$ such that for all h= 1,…,H and j = 1,…,L, $\beta \frac{w^h(p)}{\alpha^h u^h(x^h)} \frac{\partial u^h(x^h)}{\partial x_j^h}$ -

$2\beta (x_j^h - x_j^{0h}) - q_j \leq 0$ and $(\beta \frac{w^h(p)}{\alpha^h u^h(x^h)} \frac{\partial u^h(x^h)}{\partial x_j^h} - 2\beta (x_j^h - x_j^{0h}) - q_j) x_j^h = 0$.

Proof: (i) follows from the continuity and strict concavity of the objective function and the non-emptiness, compactness and convexity of the constraint set.

(ii) the first order necessary and sufficient conditions for optimality the above concave programming problem is the following: there exists q∈ $\mathbb{R}^L_+$ such that for all h= 1,…,H and j = 1,…,L, $\frac{w^h(p)}{\alpha^h u^h(x^h)}\frac{\partial u^h(x^h)}{\partial x^h_j}$ - 2(x^h_j - x^{0h}_j) - $q_j \leq 0$ and ($\frac{w^h(p)}{\alpha^h u^h(x^h)}\frac{\partial u^h(x^h)}{\partial x^h_j}$ - 2(x^h_j - x^{0h}_j) - q_j) x^h_j = 0.

All that we need to show is that $\sum_{j=1}^L q_j > 0$. Then if we set β = $\frac{1}{\sum_{j=1}^L q_j}$ we will obtain a proof of the proposition.

By the first half of (ii) in the proof of this proposition, $\sum_{j=1}^L q_j \geq \frac{w^h(p)}{\alpha^h u^h(x^h)}\sum_{j=1}^L \frac{\partial u^h(x^h)}{\partial x^h_j} > 0$ since $\frac{w^h(p)}{\alpha^h u^h(x^h)} > 0$ and $\sum_{j=1}^L \frac{\partial u^h(x^h)}{\partial x^h_j} > 0$.

Dividing all the equalities and inequalities in (ii) of this proof by $\sum_{j=1}^L q_j$ we get q∈ Δ and the proposition. Q.E.D.

Let $\mathcal{X}$(p,X^0) denote the unique solution to the concave maximization problem in Proposition 6.

Lemma 4: Suppose $\omega^h \in \mathbb{R}^L_{++}$ for all h = 1,…,H. The function $\mathcal{X}$ from Δ×{X∈ $(\mathbb{R}^L_+)^H$| for all j∈ {1,…,L}: $\sum_{h=1}^H x^h_j$ = 1} to {X∈ $(\mathbb{R}^L_+)^H$| for all j∈ {1,…,L}: $\sum_{h=1}^H x^h_j$ = 1} is continuous.

The proof of continuity of $\mathcal{X}$ follows from the maximum theorem of Berge (1963).

Let $\mathcal{P}$(p,X^0) = {q∈ Δ| there exists β > 0 which along with q and $\mathcal{X}$(p,X^0) satisfies (ii) of proposition 1}.

Lemma 5: Suppose $\omega^h \in \mathbb{R}^L_{++}$ for all h = 1,…,H. The correspondence $\mathcal{P}$ from Δ×{X∈ $(\mathbb{R}^L_+)^H$| for all j∈ {1,…,L}: $\sum_{h=1}^H x^h_j$ = 1} to {X∈ $(\mathbb{R}^L_+)^H$| for all j∈ {1,…,L}: $\sum_{h=1}^H x^h_j$ = 1} is non-empty valued, compact valued, convex valued and upper semicontinuous.

Proof: Note that $\sum_{j=1}^L q_j \geq \frac{w^h(p)}{\alpha^h u^h(x^h)}\sum_{j=1}^L \frac{\partial u^h(x^h)}{\partial x^h_j} \geq \frac{w^h(p)}{\alpha^h u^h(x^h)}\sum_{j=1}^L x^h_j \frac{\partial u^h(x^h)}{\partial x^h_j}$ = w^h(p) by Euler's theorem. Since $\min_{p\epsilon\Delta} w^h(p) > 0$, the set of β's that satisfy (ii) in the statement of proposition 6, is bounded above by a positive real number.

We will now show that the set of β's that satisfy (ii) in the statement of proposition 5, is bounded below by a positive real number.

By the equality in (ii) of proposition 6, $\frac{w^h(p)}{\alpha^h u^h(x^h)}\frac{\partial u^h(x^h)}{\partial x^h_j} x^h_j$ - 2(x^h_j - x^{0h}_j) x^h_j = q_j x^h_j.

Thus, q_j $x^h_j \leq \frac{w^h(p)}{\alpha^h u^h(x^h)}\frac{\partial u^h(x^h)}{\partial x^h_j} x^h_j$ + 2, since |(x^h_j - x^{0h}_j) x^h_j| ≤ 1.

Hence $q_j = q_j\sum_{h=1}^{H} x_j^h \leq \sum_{h=1}^{H} \frac{w^h(p)}{\alpha^h u^h(x^h)} \frac{\partial u^h(x^h)}{\partial x_j^h} x_j^h + 2H$.

Thus, $\sum_{j=1}^{L} q_j \leq 2LH + \sum_{j=1}^{L} \sum_{h=1}^{H} \frac{w^h(p)}{\alpha^h u^h(x^h)} \frac{\partial u^h(x^h)}{\partial x_j^h} x_j^h = 2LH + \sum_{h=1}^{H} \frac{w^h(p)}{\alpha^h u^h(x^h)} \sum_{j=1}^{L} \frac{\partial u^h(x^h)}{\partial x_j^h} x_j^h =$
$2LH + \sum_{h=1}^{H} w^h(p) = 2LH + 1$, since $\sum_{j=1}^{L} x_j \frac{\partial u^h(x)}{\partial x_j} = \alpha^h u^h(x)$ by Euler's theorem.

Thus the set of β's that satisfy (ii) in the statement of proposition 5, is bounded below by $\frac{1}{2LH+1} >$ 0.

The rest of the proof is similar to the proof of lemma 2. Q.E.D.

Proposition 7: Suppose $\omega^h \in \mathbb{R}_{++}^L$ for all h = 1,…,H. A CE (p,X) exists for the economy described in this section. Further X is a solution to the non-symmetric Nash bargaining solution with weights being given by $<\frac{w^h(p)}{\alpha^h} | h = 1,\ldots,H>$.

Proof: Consider the correspondence F from $\Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\ldots,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$ to itself defined by $F(p,X^0) = \mathcal{P}(p,X^0) \times \{\mathcal{X}(p,X^0)\}$ for all $(p,X^0) \in \Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\ldots,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$.

By lemmas 4 and 5, F is non-empty valued, compact valued, convex valued and upper-semicontinuous. Since its domain is compact, convex and the same as the co-domain, by Kakutani's fixed point theorem it has a fixed point, i.e. there exists $(p, X^0) \in \Delta \times \{X \in (\mathbb{R}_+^L)^H |$ for all $j \in \{1,\ldots,L\}$: $\sum_{h=1}^{H} x_j^h = 1\}$, such that $(p, X^0) \in F(p,X^0)$.

Thus $\mathcal{X}(p,X^0)$ solves the concave maximization problem in proposition 6 and for h = 1,…, H, there exists β > 0 such that for all h= 1,…,H and j = 1,…,L, $\beta \frac{w^h(p)}{\alpha^h u^h(x^{0h})} \frac{\partial u^h(x^{0h})}{\partial x_j^h} - 2\beta (x_j^{0h} - x_j^{0h}) - p_j \leq 0$ and $(\beta \frac{w^h(p)}{\alpha^h u^h(x^{0h})} \frac{\partial u^h(x^{0h})}{\partial x_j^h} - 2\beta (x_j^{0h} - x_j^{0h}) - p_j) x_j^h = 0$.

Hence, for all h= 1,…,H and j = 1,…,L, $\beta \frac{w^h(p)}{\alpha^h u^h(x^{0h})} \frac{\partial u^h(x^{0h})}{\partial x_j^h} - p_j \leq 0$ and $(\beta \frac{w^h(p)}{\alpha^h u^h(x^{0h})} \frac{\partial u^h(x^{0h})}{\partial x_j^h} - p_j) x_j^h = 0$.

Thus $\beta \frac{w^h(p)}{\alpha^h u^h(x^{0h})} \sum_{j=1}^{L} x_j^{0h} \frac{\partial u^h(x^{0h})}{\partial x_j^h} = \sum_{j=1}^{L} p_j x_j^{0h}$.

By Euler's theorem, $\beta w^h(p) = \sum_{j=1}^{L} p_j x_j^{0h}$.

Thus, $\beta = \beta \sum_{h=1}^{H} w^h(p) = \sum_{h=1}^{H} \sum_{j=1}^{L} p_j x_j^{0h} = \sum_{j=1}^{L} p_j \sum_{h=1}^{H} x_j^{0h} = \sum_{j=1}^{L} p_j = 1$.

Hence for all h= 1,…,H and j = 1,…,L, $\frac{w^h(p)}{\alpha^h u^h(x^{0h})}\frac{\partial u^h(x^{0h})}{\partial x_j^h}$ - $p_j \leq 0$ and ($\frac{w^h(p)}{\alpha^h u^h(x^{0h})}\frac{\partial u^h(x^{0h})}{\partial x_j^h}$ - p_j) x_j^h = 0.

Thus the pair (p,X^0) is a p-market equilibrium.

Thus, (p,X^0) is a CE>. Q.E.D.

As a result of the above we have the following theorem whose proof is the same as the proof of theorem 3.

Theorem 8: A competitive equilibrium exists for the economy described in this section.

Note: This is a revised version of an earlier paper entitled "Existence of Competitive Equilibrium in Linear Exchange Economies: A Simple Proof Using Brouwer's Fixed Point Theorem".

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