A pseudo-polynomial size formulation for 2-stage 2-dimensional knapsack problems

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Abstract

Two dimensional cutting problems are about obtaining a set of rectangular items from a set of rectangular stock pieces and are of great relevance in industry, whenever a sheet of wood, metal or other material has to be cut. In this paper, we consider the *2-stage two-dimensional knapsack* (2TDK) problem which requires finding the maximum profit subset of rectangular items obtainable through 2-stage guillotine cuts in a rectangular panel. We propose a formulation having a pseudo-polynomial number of variables and constraints which can still be enumerated for the instances present in the literature. We compare the proposed formulation with the previous best known polynomial size one. Extensive computational experiments show that the new model is characterized by a stronger linear programming relaxation and can be effectively solved with a general-purpose MIP solver.
Keywords: 2-stage two-dimensional Knapsack problem, Mixed-Integer Programming models, computational experiments.

1 Introduction

In this paper we consider the *2-stage two-dimensional knapsack* (2TDK) problem. We are given one stock rectangle, also called a *panel*, with height H and width W, and a list of m rectangular shapes, also called *items*, to be cut. Each item i ($i = 1, \ldots, n$) is characterized by a height h_i, a width w_i, a profit p_i, and is available in d_i copies. The problem is to cut a subset of items of largest profit which can fit into the panel. Cuts are performed according to the following rules:

- items are obtained from the panel through guillotine cuts, i.e., cuts that are parallel to the sides of the stock and cross the stock from one side to the other;
- cuts are restricted to be 2-stage, where each stage consists of a set of parallel guillotine cuts performed on the shape obtained in the previous stage;
- third stage cuts (trimming) can be used to remove a waste area;
- without loss of generality, we consider the first stage cut to be horizontal.

The 2TDK is an NP-hard problem and it has great relevance in industry whenever a sheet of paper, wood, glass, metal or other material has to be cut (see for example [12], for further details about 2-stage two-dimensional cutting real-world applications). On the left part of Figure 1 we give an example of 2-stage guillotine cuts on a rectangular panel: first-stage horizontal cuts are represented with a thick line, second-stage vertical cuts with a thin line, and trimming horizontal cuts with a dotted line. Waste area is represented by a dashed surface. On the right part of the figure we represent a cutting pattern which cannot be obtained through guillotine cuts.

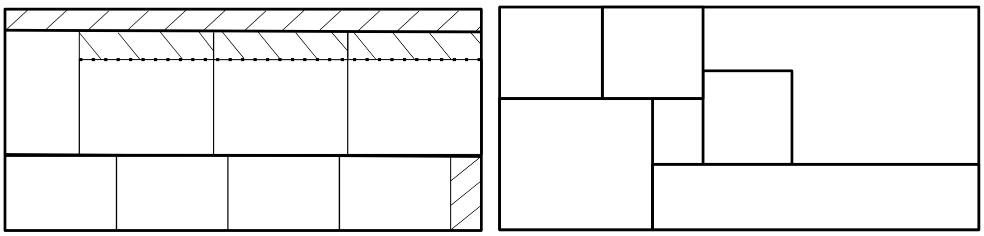


Figure 1: Guillotine cut (left) and non-guillotine cut (right).

Two-dimensional cutting problems, i.e., multiple problems of obtaining sets of (rectangular) items from sets of (rectangular) stock pieces, have received a large amount of attention in the last few decades (for a comprehensive survey on two-dimensional cutting and packing problems we refer the reader to [11] and [17]). [6] is the seminal paper which introduced this stream of research, where the $k - stage$ two-dimensional version of the classical Cutting Stock problem is tackled via column generation and the 2TDK coincides with the pricing problem to solve during the branch-and-price scheme. Since then, many variants of the 2TDK have been studied, in particular these are the most common ones:

- 2TDK is said to be *Unconstrained* (U-2TDK) if there is no limit to the number of items of each type which can be cut in the panel. Otherwise, the problem is said to be *Constrained* (C-2TDK);
- if a *90° degree Rotation* of the items is allowed the problem is referred as R-2TDK; otherwise, if the orientation of the items is *Fixed*, the problem is F-2TDK;
- 2TDK is said to be *unweighted* if the profit of all items is equal to their area; otherwise, if the profits are given in input, the problem is said to be *weighted*.

In the literature, many studies specifically addressed 2TDK problems. In particular we refer the reader to these seminal articles: [9], [13], and [7]. In addition [8] proposed both exact and heuristic algorithms for the specific case FC-2TDK. It is worth noticing that the 2TDK has been also called the *two-dimensional 2-stage guillotine constrained cutting problem* in [2], where an effective exact algorithm based on column generation has been developed, and the *cutting knapsack problem* in [6]. In addition, both terms *2-stage* and *2-staged* can be found in the literature, we adopt the term 2-stage as in the paper [6].

The principal contribution of this paper is to propose a pseudo-polynomial size model for the 2TDK. Our formulation has been inspired by the techniques introduced in [3], where a model for the Cutting Stock (CS) problem has been proposed based on the concept of cuts and residual elements. The idea is the following: each time a one-dimensional stock of length L is cut in order to produce an item i of length l_i, an element of size $L - l_i$ is obtained, which can be further used to produce additional items. The objective function then minimizes the total number of stocks used and the constraints impose a balance on the number of residual elements. This idea has been extended by [14] to the 2-stage two-dimensional Cutting Stock problems (2TCS). In [4], we computationally tested the model from [14] for the case of multiple stock panels.

The second contribution of this paper is the comparison with the best polynomial size formulation present in the literature. To the best of our knowledge, the most effective formulation for the 2TDK has been introduced in [10], called *LM Model* in the following. Other polynomial size formulations exist, e.g., those studied in [1] and [5], but LM Model is characterized by a better computational performance on average (we refer the reader to the quoted papers for further details on this topic). Finally, we concentrate our study on formulations that can be directly solved by a general-purpose MIP solver, without the implementation of specific algorithms (e.g. branch-and-price or branch-and-cut approaches).

The paper is organized as follows. In the next two sections (Sections 2 and 3 respectively), we first revise the LM Model and then we introduce the new pseudo-polynomial formulation, called *FM Model*. In Section 4, we perform an extensive computational comparison between these two formulations and, finally, we conclude the paper in Section 5, where we summarize the results and describe further developments.

2 LM Model from the literature

In [10], the authors proposed two interconnected polynomial size models for the 2TDK, the first one, denoted as M1, is more suited from problem instances where each items is available in one or limited number of copies and a second one, denoted as M2, is more suited for problem instances where each item is available in several copies. Thanks to preliminary tests on the instances described in Section 4, we found that M2 computationally outperforms M1 on the considered instances, and thus we focus our computational comparison only on this model. In the following to avoid the confusion of a numerical nomenclature, the M2 model will be called LM model.

A shelf is a slice of the stock rectangle with width W, and height coincident with the height of the tallest item cut off from it. Two stage cutting can be seen as a cut process where in the first stage the plate is slices into shelves, and in the second stage items are obtained from the shelves.

Model M2 is characterized by two separate sets of variables. The first set is composed by the integer variables x_{ik}:

$$x_{ik} = \begin{cases} \text{number of items } i \text{ cut from shelf } k & \text{if i} \neq \beta_k \\ \text{number of additional items } i \text{ cut from shelf } k & \text{if i} = \beta_k. \end{cases}$$

The second set is composed by the binary variables q_k:

$$q_k = \begin{cases} 1 & \text{if shelf } k \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

- Assume $h_1 \geq h_2 \geq \cdots \geq h_n$, and $\tilde{n} = \sum_{i=1}^{n} d_i$.
- Any item i may be cut from shelves in $[1, \alpha_i]$, with $\alpha_i = \sum_{s=1}^{i} d_s$ ($i = 1, \ldots, n$) and $\alpha_0 = 0$.
- Any shelf k can be used to cut items in $[\beta_k, n]$, with $\beta_k = \min\{r : 1 \leq r \leq n, \alpha_r \geq k\}$ ($k = 1, \ldots, \tilde{n}$). Thus, β_k ($k = 1, \ldots, \tilde{n}$) denotes the item type initializing shelf k.

$$\max \sum_{i=1}^{n} p_i(\sum_{k=1}^{\alpha_i} x_{ik} + \sum_{k=\alpha_{i-1}+1}^{\alpha_i} q_k) \tag{1}$$

$$\sum_{k=1}^{\alpha_i} x_{ik} + \sum_{k=\alpha_{i-1}+1}^{\alpha_i} q_k \leq d_i \qquad i = 1, \ldots, n \tag{2}$$

$$\sum_{i=\beta_k}^{m} w_i x_{ik} \leq (W - w_{\beta_k}) q_k \qquad k = 1, \ldots, \tilde{n} \tag{3}$$

$$\sum_{k=1}^{\tilde{n}} h_{\beta_k} q_k \leq H \tag{4}$$

$$\sum_{s=k}^{\alpha_i} x_{is} \leq d_i - (k - \alpha_{i-1}) \qquad i = 1, \ldots, n;\ k \in [\alpha_{i-1}+1, \alpha_i] \tag{5}$$

$$0 \leq x_{ik} \leq d_i, \quad x_{ik} \in \mathbb{Z}_+ \qquad i = 1, \ldots, n;\ k \in [1, \alpha_i] \tag{6}$$

$$q_k \in \{0, 1\} \qquad k = 1, \ldots, \tilde{n}. \tag{7}$$

The linear relaxation (LR) of the model can be strengthened by suited inequalities. We refer the interested reader to [10] for a further details. Since the model defines a strip for each item, it is crucial to preprocess the instances so as to restrict the number of copies of each item to those which can fit in the KP. To this end, item demands can easily be redefined as $\tilde{d}_i = \min\{d_i, \lfloor w_i h_i / WH \rfloor\}$ ($i = 1, \ldots, n$). When the rotation of items is allowed, for each item i, having sizes l_i and w_i, the rotated counterpart i' is defined, having sizes $l_{i'} = w_i$ and $w_{i'} = l_i$, thus obtaining $2n$ items. The demand constraints (2) are modified accordingly.

3 Pseudo-polynomial size Model FM

In this Section we introduce an alternative model for the 2TDK. The model is of pseudo-polynomial size and is called FM Model in the following. As mentioned in the introduction, the idea is an extension of the model originally presented in [3] for the CS problem, and developed for the 2TCS in [14].

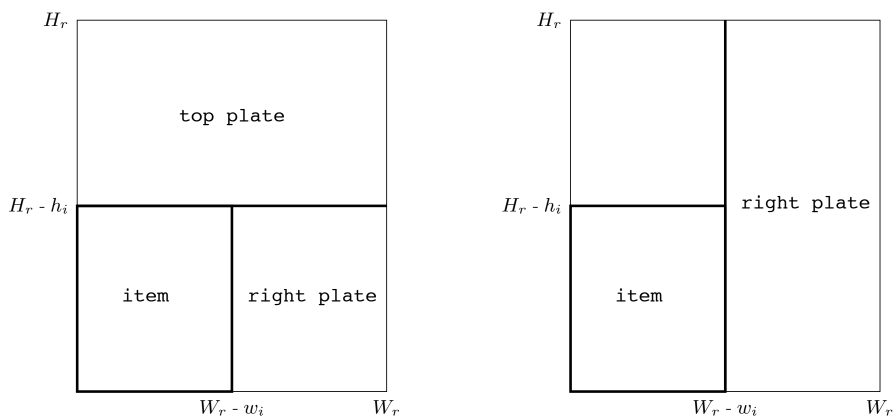
As in [14], the FM Model considers pairs of items and residual plates, where a plate can be the original panel or a plate obtained as the result of an item cut from another plate. Precisely, each time an item i is obtained through guillotine cuts from a plate r of height H_r and width W_r, we obtain two residual plates, the first one having height $H_r - h_i$ and width w_i, and the second one having height H_r and width $W_j - w_i$. The two residual plates can then be used in a subsequent cut or can be waste depending on their size and on the number of stages of the cutting procedure, as summarized in Table 1 (see [14] for more details).

In addition, in Figures 2 and 3, we give a graphical representation of the process of cutting an item i from a plate r. In Figure 2, we represent a first-stage cut while in Figure 3, we represent a second stage cut.

The model we propose for the 2TDK uses integer variables x_{ir} defined as the number of times a item i is cut from plate r. Let R be the set of all plates which can be obtained by the available items and $R(i)$ be the set of plates that can fit an item i. The original panel is indexed by $r = 0$; note that

Cuts					Residual panels					
panel (r)			item (i)		top plate			right plate		
width	height	stage	width	height	width	height	stage	width	height	stage
W_r	H_r	1	w_i	h_i	W_r	H_r - h_i	1	W_r - w_i	h_i	2
W_r	H_r	2	w_i	h_i	W_r	H_r - h_i	waste	W_r - w_i	H_r	2

Table 1: Residual panels resulting from cuts in different stages



$0 \notin R$. Finally, the parameter a^i_{kr} is equal to 1 if plate r results from cutting item i from the plate k, and equal to 0 otherwise. The FM Model reads as follows:

$$\max \sum_{i=1}^{n} \sum_{r \in R(i)} p_i x_{ir} \tag{8}$$

$$\sum_{r \in R(i)} x_{ir} \leq d_i, \qquad i = 1, \ldots, n \tag{9}$$

$$\sum_{i=1}^{n} \sum_{k \in R(i)} a^i_{kr} x_{ik} \geq \sum_{i=1}^{n} x_{ir} \qquad r \in R \tag{10}$$

$$1 \geq \sum_{i=1}^{n} x_{i0} \tag{11}$$

$$x_{ir} \in \mathbb{Z}_+ \qquad i = 1, \ldots, n; \ r \in R, \tag{12}$$

where the Objective Function (8) maximizes the profit of the cut items, Constraints (9) impose cutting at most the number of copies of each item and Constraints (10) impose that the number of used plates is not larger than the number of residual plates of the same type obtained by previous cuts. Finally, Constraint (11) imposes that the original panel is not cut more than once.

The number of cuts and plates, consequently, the number of variables and constraints of the FM model, is pseudo-polynomial, depending on panel and item dimensions. Since the set of residual panels is the same as in the case of the pseudo-polynomial size model for the 2TCS presented in [14], we refer the reader to that paper for two possible upper bounds. In particular, both these pseudo-polynomial size models require a preprocessing phase in which all the possible residual panels are generated, i.e. the set R. Again, we follow the procedures proposed in [14], where an efficient algorithm to generate all the set of residual panels is presented.

We conclude this section describing the FM model extension in order to take into consideration the rotation, i.e. the R-2TDK case. Similarly to [10], for each item i, having sizes l_i and w_i, we define its rotated counterpart i', having sizes $l_{i'} = w_i$ and $w_{i'} = l_i$, thus obtaining $2n$ items. Constraints (9) are then modified to:

$$\sum_{r \in R(i)} x_{ir} + \sum_{r \in R(i')} x_{i'r} \leq d_i, \; i = 1, \ldots, n \tag{13}$$

4 Computational Comparison

We performed an extensive computational evaluation of the performances of the different mathematical formulations presented in the previous sections, i.e. LM Model of Section 2 and FM Model of Section 3. We used `CPLEX` 12.5 [15] and `GUROBI` 5.0 [16] as MIP solvers in single thread mode, the specific parameter setting will be discussed in the table descriptions. The experiments were run on a PC with an Pentium(R) E5400 at 2.70GHz and 4 GB of RAM memory, under Linux Ubuntu (64-bit).

4.1 Instances features

For a fair comparison between the models, we consider the same set of instances used in [10]. The test set comprises 38 instances originally proposed in [8] and publicly available online[1]. These instances present a demand for each item, i.e., the C-2TDK case. The test set comprehends 24 unweighted instances and 14 weighted instances, where the weight is represented by the item area. Eight instances with the same item and panel characteristics appear in both classes thus there are globally 30 different instances whose features are summarized in Table 2. In Table 2, we report the number n of items, the total number $\tilde{n}$ of item copies ($\sum_{i=1}^{n} d_i$), the original stock panel dimension H and W, the minimum and maximum item height h_{min} and h_{max} and widths w_{min} and w_{max}. These features are particularly important since the item dimensions compared with the panel one determine the number of variables and constraints of the FM Model. On average, the presence of relatively small items leads to an high number of variables and constraints while instance with few items types and relatively large items leads to smaller number of variables and constraints for the FM Model.

4.2 Model comparison

In the following, we report the computational comparison between FM Model and LM Model, i.e., the pseudo-polynomial and the polynomial size formulations from the literature, respectively. The results are summarized in 2 tables, Tables 3 and 4. Table 3 is for the F-2TDK case and Tables 4 is for the R-2TDK case. The tables have the same structure, and the instances (column `Instance`) are divided in two groups: the first subset corresponds to the 24 unweighted instances (from *2s* to *W*) while the second subset corresponds to the 14 weighted instances (from *2* to *STS4*).

From the second to the fifth column, we compare the number of variables (column vars) and the number of constraints (column cons). Unsurprisingly, FM Model has an large number of variables

[1] ftp://cermsem.univ-paris1.fr/pub/CERMSEM/hifi/2Dcutting/2Dcutting.html

instance	n	$\tilde{n}$	H	W	h_{min}	h_{max}	w_{min}	w_{max}
2	10	23	40	70	9	31	7	35
3	20	62	40	70	9	33	11	43
A1	20	62	50	60	9	33	11	43
A2	20	53	60	60	12	33	14	42
A3	20	46	70	80	15	35	14	43
A4	20	35	90	70	9	33	11	43
A5	20	45	132	100	13	69	12	63
CHL1	30	63	132	100	13	69	12	63
CHL2	10	19	62	55	11	31	9	31
CHL5	10	18	20	20	1	20	2	14
CHL6	30	65	130	130	18	69	12	63
CHL7	35	75	130	130	19	57	18	54
CU1	35	90	150	175	31	96	35	112
CU2	25	82	100	125	20	58	28	80
CW1	25	67	125	105	25	78	21	66
CW2	35	63	145	165	34	93	34	104
CW3	40	96	267	207	59	170	45	130
Hchl2	35	75	130	130	19	57	18	59
Hchl3S	10	51	127	98	15	54	13	65
Hchl4S	10	32	127	98	15	54	13	65
Hchl6S	22	60	253	244	35	109	38	101
Hchl7S	40	90	263	241	33	108	38	135
Hchl8S	10	18	49	20	1	20	2	14
Hchl9	35	76	65	76	10	31	10	43
HH	5	18	127	98	18	54	13	65
OF1	10	23	70	40	9	55	4	39
OF2	10	24	70	40	13	47	4	27
STS2	30	78	55	85	10	31	10	43
STS4	20	50	99	99	14	44	16	49
W	20	62	70	40	11	43	9	33

Table 2: Instance Features

and constraints. The LM Model always has a smaller number of variables while, in some cases, it is characterized instead by a larger number of constraints (18% of the instances for the F-2TDK case and 26% for the R-2TDK case).

The columns Linear Relaxation (LR) compare the quality of the LR bound of the models (column gap$\%$). This gap has been computed comparing the LR relaxation with the optimal solution value (opt) of each instance, as $\frac{LR-opt}{opt}$. We also report the time in seconds necessary to compute the LR relaxation using CPLEX 12.5, with default parameter settings. Finally we report in bold text the winning model, i.e., the model with the tighter LR bound. As far as the LR is concerned, the FM Model is characterized by tighter values in all instances except 4 cases. For the F-2TDK case, in 1 instance the LM Model outperforms the FM model while, for the R-2TDK case, in 3 instances both models present the same LR values. As far as the computing time is concerned, the LM Model LR is systematically faster than the FM Model LR.

In order to fully compare the behavior of the two models, we solved them with two state-of-the-art commercial MIP solvers, i.e. IBM-ILOG CPLEX 12.5 and GUROBI 5.0. In this final part of the tables, we report for each configuration the time in seconds needed to solve the instance to proven optimality and the total number of branching nodes explored during the optimization. We report in bold text the winning model, i.e., the model which requires a smaller computational time (we consider a winning case if the absolute difference is greater than 0.1 seconds, otherwise we consider that case as a draw). The columns Default CPLEX and No-Reduction CPLEX reports the performances using CPLEX, respectively with default options and disabling primal-dual reductions during CPLEX preprocessing. As far as Default CPLEX and the computational time are concerned, the LM Model outperforms the FM Model. Precisely, the total time needed to solve all the instances is 13 seconds against 42 seconds for the F-2TDK case (22 winning cases against 0) and 148 seconds against 235 seconds for the R-2TDK case (25 winning cases against 0). As far as No-Reduction CPLEX is concerned, now the FM Model outperforms the LM Model. The total time needed to solve all the instances is 28 seconds against 103 seconds for the F-2TDK case (12 winning cases against 10) and 291 seconds against 2400 seconds for the R-2TDK case (17 winning cases against 12).

The results of the analysis show that primal-dual reductions embedded within the CPLEX solver are very beneficial for the LM Model, while they impair the solution of the FM Model. This observation tells us that the influence of the reductions on different formulations can be misleading and it can lead to incorrect conclusions about the performances of different approaches. We are not suggesting systematically removing primal-dual reductions that, on average, are very effective. We want to stress that reductions are not always useful, and that the solution time of a specific integer program can largely depend on the set-up of the solver. In other words, computing time is not enough to conclude that one formulation is better than another, and other information like the LR value and the number of explored branching nodes must be taken into account. Computing time is enough to conclude that one formulation is better than another when a *specific solver* with a *specific configuration* is used.

Concerning the number of branching nodes, on average the FM Model explores a smaller number of branching nodes, due to having a tighter LR. Clearly this stronger information is not "free", since solving the LR of the FM Model is computationally more expensive.

For the sake of completeness, the Default GUROBI columns report the performances using GUROBI, with default options. In this case the differences between the performances are less noticeable with respect to Default CPLEX, but still the LM Model slightly outperforms the FM Model. The total time needed to solve all the instances is 24 seconds against 29 seconds for the F-2TDK case (17 winning cases against 4) and 88 seconds against 145 seconds for the R-2TDK case (25 winning cases against 6). The behavior of the branching nodes explored is in line with the previous cases.

To summarize, the principal points of strength of the FM Model are the following:

- *Stronger linear relaxation*
- *Less branching nodes*
- *Less dependency from MIP-Solver reductions*

We conclude this section stressing the fact that the FM Model can be considered as a valid alternative to the standard LM Model, even if in some cases the LM outperforms it. The principal cause is the computational difficulty of the LR of the FM Model, which is systematically tighter but slower than the LM Model LR. In the majority of the considered instances, this trade-off between speed and quality of the LR plays in favor of the LM Model.

5 Conclusions

In this paper, we considered the *2-stage two-dimensional knapsack* (2TDK) problem which requires finding the maximum profit subset of rectangular items obtainable through 2-stage guillotine cuts in a rectangular panel. We introduced and tested, through extensive computational experiments, a pseudo-polynomial size formulation. The experiments showed that the proposed formulation is characterized by a strong linear programming relaxation and can be effectively solved with a general-purpose MIP solver. We considered all the classical problem variations, i.e. we perform experiments considering *weighted* and *unweighted* instances from the literature and allowing and non-allowing item rotation.

Further developments can be the implementation of a simultaneous row-and-column generation approach instead of a complete enumeration of the variable and the constraints of the FM model. Finally another interesting stream of research could be the application of the pseudo-polynomial size model to different combinatorial optimization problems, exploiting the strong linear relaxations and the fact that state-of-the-art MIP solvers are now able to deal effectively with large models.

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					Linear Relaxation				Default CPLEX				No-Reduction CPLEX				Default GUROBI			
	FM Model		LM Model		FM Model		LM Model		FM Model		LM Model		FM Model		LM Model		FM Model		LM Model	
Instance	vars	cons	vars	cons	gap%	time	gap%	time	time	nodes	time	nodes	time	nodes	time	nodes	time	nodes	time	nodes
2s	541	95	144	83	**8.35**	0.01	15.23	0.00	0.45	720	**0.05**	141	0.52	605	**0.11**	97	0.43	259	**0.04**	68
3s	626	90	393	149	**1.73**	0.01	7.73	0.01	0.08	26	0.02	28	**0.01**	0	0.14	0	0.04	0	0.04	5
A1s	559	106	378	141	**0.00**	0.00	1.69	0.00	0.04	0	0.02	0	0.01	0	0.04	34	0.02	0	0.03	0
A2s	717	146	464	161	**0.00**	0.01	5.17	0.00	0.05	0	0.05	11	**0.00**	0	0.11	0	0.04	0	0.08	7
A3	1332	211	532	165	**0.58**	0.03	4.09	0.00	0.10	0	0.10	14	**0.03**	0	0.33	728	0.14	0	0.12	17
A4	3042	536	400	121	**4.88**	0.10	7.05	0.00	0.54	87	**0.08**	88	0.45	21	**0.22**	49	0.55	3	**0.18**	49
A5	5157	736	484	161	**1.92**	0.26	5.15	0.00	0.80	37	**0.10**	111	0.33	0	0.25	412	0.50	0	**0.15**	23
CHL1s	14448	1342	974	223	**0.68**	0.72	1.26	0.00	2.95	28	**0.14**	327	1.63	0	**0.76**	1286	1.26	0	**0.29**	401
CHL2s	409	98	116	67	**5.99**	0.01	7.84	0.00	0.09	84	0.03	17	0.08	44	0.05	17	0.06	1	0.02	4
CHL5	267	81	118	63	**4.49**	0.00	10.19	0.00	0.03	0	0.01	7	0.02	0	0.02	0	0.04	0	0.01	0
CHL6	10806	1018	1000	231	**1.33**	0.47	1.98	0.01	1.83	0	**1.58**	2088	**1.45**	3	7.95	11232	**1.31**	3	1.81	11243
CHL7	13282	1026	1421	266	**0.58**	0.71	1.03	0.01	1.73	67	**1.36**	1634	**1.48**	0	15.78	11697	**0.99**	0	5.13	29883
CU1	2563	258	1327	326	**0.00**	0.07	0.57	0.01	0.27	0	**0.13**	139	**0.06**	0	0.19	129	0.23	0	0.13	12
CU2	2470	365	994	304	**0.00**	0.06	1.53	0.00	0.25	0	**0.08**	80	**0.06**	0	0.20	259	0.25	0	**0.14**	17
Hchl3S	1739	396	319	195	**2.13**	0.06	4.05	0.00	0.32	197	**0.17**	151	0.48	99	**0.16**	73	1.05	41	**0.20**	440
Hchl4S	1739	396	200	119	**6.76**	0.07	9.10	0.00	3.93	2132	**0.28**	823	1.44	608	**0.74**	2273	1.61	304	**1.02**	2357
Hchl6S	7366	890	724	219	**0.93**	0.46	2.60	0.00	1.30	33	**0.26**	349	0.70	0	**0.24**	25	1.00	0	**0.25**	83
Hchl7S	40833	2914	2053	321	**0.60**	2.22	1.48	0.01	6.76	139	**5.39**	2873	**8.79**	9	68.25	40062	**6.49**	0	9.20	37821
Hchl8S	1028	254	118	63	25.93	0.02	**24.63**	0.01	0.72	235	**0.06**	30	0.74	295	**0.10**	47	0.70	117	**0.06**	90
OF1	457	123	166	83	**0.00**	0.00	3.21	0.01	0.04	0	0.01	0	0.00	0	0.02	0	0.04	0	0.01	0
OF2	362	97	156	79	**4.10**	0.00	11.33	0.00	0.05	7	0.02	83	0.02	0	0.04	0	0.04	0	0.04	12
STS2s	2798	306	1178	255	**0.52**	0.09	2.32	0.01	0.23	0	0.24	149	**0.11**	0	2.35	2603	0.29	0	0.27	119
STS4s	2974	446	583	181	**1.57**	0.11	3.38	0.01	0.74	178	0.68	1848	**0.50**	50	2.34	5882	0.48	5	**0.34**	139
W	799	209	469	149	**1.25**	0.01	6.75	0.00	0.09	0	0.03	55	0.02	0	0.05	104	0.10	0	0.09	0
2	541	95	144	83	**8.13**	0.00	13.53	0.00	0.20	275	**0.03**	64	0.18	63	0.10	4	0.43	156	**0.10**	92
3	626	90	393	149	**5.93**	0.01	16.59	0.00	0.15	122	**0.03**	36	0.13	98	0.16	8	0.17	33	**0.04**	6
A1	559	106	378	141	**2.56**	0.01	17.97	0.00	0.04	0	0.03	34	0.01	0	0.06	0	0.04	0	0.04	0
A2	717	146	464	161	**4.32**	0.01	17.08	0.00	0.08	20	0.06	188	**0.03**	0	0.13	2	0.09	1	0.08	19
CHL1	14448	1342	974	223	**5.31**	0.66	9.42	0.01	5.00	304	**0.45**	80	3.62	23	**0.36**	64	2.25	3	**0.26**	119
CHL2	409	98	116	67	**6.70**	0.01	10.68	0.00	0.06	22	0.02	17	0.02	0	0.03	0	0.03	0	0.02	0
CW1	1922	270	816	244	**2.66**	0.06	11.67	0.00	0.32	0	**0.07**	33	0.06	0	0.07	0	0.27	0	**0.08**	5
CW2	2439	266	1144	218	**1.22**	0.06	12.26	0.01	0.22	0	**0.07**	12	0.06	0	0.15	2	0.27	0	**0.08**	4
CW3	3944	394	1746	345	**0.89**	0.14	12.82	0.01	0.59	0	**0.15**	72	0.17	0	0.16	0	0.36	0	0.27	6
Hchl2	14025	1083	1416	266	**2.71**	0.78	5.63	0.01	10.86	1997	**0.40**	285	3.85	67	**0.61**	22	5.62	11	**2.32**	1613
Hchl9	5528	535	1478	266	**2.08**	0.25	4.37	0.01	0.60	35	**0.23**	127	0.66	42	**0.29**	16	1.38	1	**0.50**	20
HH	286	105	71	68	**12.18**	0.00	16.44	0.00	0.05	16	0.01	0	0.00	0	0.01	0	0.04	0	0.01	0
STS2	2798	306	1178	255	**2.28**	0.09	7.48	0.00	**0.28**	3	0.42	594	**0.19**	0	0.65	109	**0.48**	0	0.84	140
STS4	2974	446	583	181	**0.70**	0.10	5.69	0.01	0.37	107	**0.24**	89	0.19	3	0.21	9	0.44	0	0.35	71
tot									42.21		13.14		28.10		103.47		29.53		24.64	

Table 3: F-2TDK case results

					Linear Relaxation				Default CPLEX				No-Reduction CPLEX				Default GUROBI			
	FM Model		LM Model		FM Model		LM Model		FM Model		LM Model		FM Model		LM Model		FM Model		LM Model	
Instance	vars	cons	vars	cons	gap%	time	gap%	time	time	nodes	time	nodes	time	nodes	time	nodes	time	nodes	time	nodes
2s	1879	215	494	155	**1.08**	0.04	3.02	0.00	0.34	222	**0.07**	16	0.30	7	**0.15**	0	0.57	0	**0.08**	5
3s	1574	147	1566	285	**0.00**	0.02	2.83	0.01	0.16	0	**0.03**	0	0.02	0	0.10	0	0.27	0	**0.10**	0
A1s	1794	227	1611	281	**0.00**	0.03	1.15	0.01	0.17	0	0.12	118	**0.03**	0	0.28	139	0.35	0	0.28	7
A2s	2230	272	1748	301	**0.61**	0.04	2.04	0.02	0.18	7	0.16	147	**0.07**	0	1.29	1565	0.31	0	**0.19**	143
A3	3915	381	1970	297	**0.46**	0.10	1.28	0.01	0.33	0	0.29	380	**0.24**	23	3.42	2564	0.53	0	0.57	472
A4	11467	1041	1505	221	**2.22**	0.34	2.87	0.01	2.18	259	**1.68**	1935	**3.19**	128	7.13	7876	**1.82**	4	3.08	8492
A5	21353	1504	1869	301	1.21	1.11	1.21	0.01	5.64	1045	**0.77**	1298	11.39	612	**2.87**	2026	3.04	8	**1.00**	3107
CHL1s	52769	2472	3871	415	0.27	4.04	0.27	0.02	17.21	125	**2.15**	1709	26.26	160	**3.18**	1472	7.24	0	**1.53**	1700
CHL2s	2005	312	419	123	**3.79**	0.04	4.03	0.00	0.32	101	**0.08**	212	0.22	29	0.24	544	0.65	35	**0.07**	76
CHL5	1039	182	407	115	**0.00**	0.01	0.25	0.00	0.07	0	0.01	0	0.01	0	0.02	0	0.06	0	0.02	0
CHL6	38219	1953	4070	431	0.42	2.01	0.42	0.02	10.65	0	**2.21**	1188	**10.11**	10	20.12	8848	6.38	1	**1.12**	2015
CHL7	45457	1793	5680	496	**0.52**	4.33	0.59	0.04	**14.20**	1948	25.06	13442	**15.83**	100	937.00	194367	**8.50**	53	34.15	141796
CU1	8790	526	5143	612	**0.39**	0.39	0.57	0.04	**0.48**	0	0.68	498	**0.40**	0	2.87	1038	0.49	0	**0.37**	104
CU2	5996	480	3655	558	0.00	0.21	0.00	0.03	0.54	0	**0.18**	0	0.21	0	0.21	0	0.34	0	0.43	11
Hchl3S	8358	1007	1179	375	**0.74**	0.59	1.77	0.02	3.43	608	**1.08**	1064	**1.88**	23	35.87	49030	1.33	1	**0.55**	82
Hchl4S	8358	1007	715	223	**2.19**	0.73	3.42	0.01	8.69	1516	**0.35**	254	5.31	240	**1.33**	1428	4.85	18	**0.55**	391
Hchl6S	31273	2029	2768	415	**0.01**	2.65	0.46	0.01	5.40	0	**1.80**	1114	**7.72**	0	27.06	11365	2.75	0	2.71	11431
Hchl7S	147490	4850	7587	601	**0.07**	32.68	0.38	0.04	**78.02**	17	94.67	47639	**123.43**	0	1285.85	140701	55.87	0	**17.69**	38948
Hchl8S	4836	563	407	115	**17.04**	0.16	17.08	0.00	12.01	3037	**0.22**	62	16.17	3933	**0.38**	92	4.73	480	**0.29**	383
OF1	2723	450	518	153	**0.11**	0.07	3.21	0.00	0.23	0	**0.04**	63	0.09	0	0.05	0	0.28	0	**0.06**	6
OF2	2328	403	535	153	**1.65**	0.05	5.07	0.00	0.35	0	**0.04**	59	0.09	0	0.09	52	0.41	0	**0.17**	11
STS2s	7443	464	4595	479	**0.03**	0.32	0.34	0.02	0.66	0	0.63	417	**0.31**	0	2.14	867	**0.50**	0	0.62	24
STS4s	10593	848	2244	341	**0.21**	0.40	1.22	0.01	0.95	11	0.92	499	**0.73**	0	46.45	32145	**0.82**	0	3.77	7151
W	4168	548	1644	285	**0.38**	0.12	1.67	0.01	0.57	0	**0.13**	93	0.14	0	0.18	96	0.41	0	**0.16**	14
2	1879	215	494	155	**1.59**	0.04	4.26	0.01	0.53	424	**0.06**	18	0.50	48	**0.20**	59	0.58	4	**0.13**	42
3	1574	147	1566	285	**3.19**	0.02	6.76	0.00	0.22	36	**0.07**	61	**0.13**	11	0.30	28	0.28	0	**0.12**	8
A1	1794	227	1611	281	**2.19**	0.04	9.35	0.01	0.20	0	0.17	317	**0.04**	0	0.21	0	0.24	0	0.21	9
A2	2230	272	1748	301	**4.74**	0.04	7.86	0.01	0.34	61	**0.20**	266	**0.29**	43	0.51	39	0.81	22	**0.42**	44
CHL1	52769	2472	3871	415	**2.66**	5.79	5.23	0.02	23.87	208	**2.94**	1276	26.57	4	**1.71**	321	10.77	1	**2.08**	601
CHL2	2005	312	419	123	**2.65**	0.04	6.50	0.00	0.23	72	**0.05**	51	0.11	0	0.13	0	0.30	1	**0.07**	29
CW1	12762	1044	2971	454	**2.38**	0.40	6.59	0.02	1.75	0	**0.33**	416	1.08	0	**0.55**	13	0.78	0	**0.39**	163
CW2	8318	596	4210	384	**4.36**	0.39	9.80	0.03	0.91	161	**0.44**	415	**1.04**	41	1.59	166	1.22	12	**0.63**	175
CW3	43719	2818	6732	625	**2.75**	2.05	11.86	0.04	12.01	11	**0.61**	511	8.25	3	**1.80**	58	3.56	0	**0.89**	58
Hchl2	46163	1835	5670	496	**1.39**	4.38	3.76	0.03	28.25	1214	**6.20**	1546	25.28	354	**10.70**	2441	21.88	93	**9.71**	6109
Hchl9	15148	747	5502	484	**0.81**	0.48	2.75	0.03	**1.85**	61	2.31	506	2.51	0	**1.76**	259	**1.21**	0	1.97	224
HH	1802	490	219	126	**6.50**	0.05	10.13	0.00	0.28	21	**0.03**	1	0.09	0	0.04	0	0.35	0	**0.03**	0
STS2	7443	464	4595	479	**0.78**	0.30	4.45	0.03	**0.34**	0	1.16	485	**0.42**	0	1.40	140	**0.41**	0	1.05	44
STS4	10593	848	2244	341	**0.59**	0.40	3.12	0.01	1.66	17	**0.62**	272	1.04	4	**0.88**	260	0.86	0	0.93	336
tot									235.22		148.60		291.50		2400.10		145.75		88.19	

Table 4: R-2TDK case results

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