ON THE IRREDUCIBILITY, LYAPUNOV RANK, AND AUTOMORPHISMS OF SPECIAL BISHOP-PHELPS CONES *

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Abstract. Motivated by optimization considerations, we consider cones in $\mathbb{R}^n$ – to be called special Bishop-Phelps cones – of the form $\{(t,x) : t \geq ||x||\}$, where $||\cdot||$ is a norm on $\mathbb{R}^{n-1}$. We show that when $n \geq 3$, such cones are always irreducible. Defining the Lyapunov rank of a proper cone K as the dimension of the Lie algebra of the automorphism group of K, we show that the Lyapunov rank of any special Bishop-Phelps polyhedral cone is one. Extending an earlier known result for the l_1-cone (which is a special Bishop-Phelps cone with 1-norm), we show that any l_p-cone, for $1 \leq p \leq \infty$, $p \neq 2$, has Lyapunov rank one. We also study automorphisms of special Bishop-Phelps cones, in particular giving a complete description of the automorphisms of the l_1-cone.

Key words. Complementarity set, Lyapunov rank, Bishop-Phelps cone, Irreducible cone

1. Introduction. For a proper cone K in $\mathbb{R}^n$ with dual K^*, the *complementarity set* of K is

$$C(K) := \{(x,s) : x \in K, s \in K^*, \langle x,s\rangle = 0\}. \tag{1.1}$$

Such a set appears, for example, in complementarity problems [3], [13] and in primal and dual linear programming problems over a cone [12]. In various strategies for solving such problems, one tries to rewrite the complementarity/optimality conditions by replacing the complementarity constraints $x \in K, s \in K^*, \langle x,s\rangle = 0$ by linearly independent 'bilinear' relations. To elaborate, consider a complementarity problem corresponding to K and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, which is to find $x \in \mathbb{R}^n$ such that

$$x \in K,\ s = f(x) \in K^* \quad \text{and} \quad \langle x,s\rangle = 0.$$

Here, for the $2n$ variables $x \in K$ and $s \in K^*$, there are $n+1$ equality relations, namely, $s = f(x)$ and $\langle x,s\rangle = 0$. So, to make this a square system, it is desirable to replace the single bilinear relation $\langle x,s\rangle = 0$ by an equivalent system of n independent bilinear relations. This is clearly the case when $K = \mathbb{R}^n_+$ (the non-negative orthant in $\mathbb{R}^n$); here, the complementarity constraints are equivalently expressed as $x \geq 0, s \geq 0$, $x_i s_i = 0$ for $i = 1,2,\ldots,n$. Motivated by this, to measure the number

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of independent bilinear relations, Rudolf et al. [16], introduced the following: For a proper cone K in $\mathbb{R}^n$, an $n \times n$ real matrix Q is a *bilinear complementarity relation* if

$$(x, s) \in C(K) \Rightarrow \langle Q^T x, s \rangle = 0$$

and the *bilinearity rank of* K is

$$\beta(K) := \dim\, Q(K),$$

where $Q(K)$ is the vector space of all bilinear complementarity relations on K. While cones with $\beta(K) = n$ lead to square systems and are desirable, for cones with $\beta(K) > n$, one gets an overdetermined system of bilinear relations. In many of these overdetermined systems, such as symmetric cones [4], one can still get a square system of bilinear relations [10]. In cones with $\beta(K) < n$, the complementarity system can never be written as a square system by means of bilinear complementarity relations alone and this may indicate or cause difficulty in reformulation and solvability of the problem. In [16], Rudolf et al., initiate the study of bilinearity rank and show that isomorphic cones have the same bilinearity rank, a proper cone and its dual have the same rank, and that the rank is additive on a Cartesian product. They also compute the bilinearity rank of certain cones.

A *Lyapunov-like* matrix/transformation on a proper cone K satisfies the condition

$$(x, s) \in C(K) \Rightarrow \langle Qx, s \rangle = 0$$

and is thus the transpose of a bilinear complementarity relation. Lyapunov-like transformations were introduced in [8] as a generalization of the Lyapunov transformation $X \mapsto AX + XA^T$ that appears in linear dynamical systems theory. These are related to **Z**-matrices and have been the subject matter of several recent studies, see [8], [9], and [11]. As a consequence of a result in [17],

$$A \text{ is Lyapunov-like on } K \text{ if and only if } e^{tA} \in Aut(K) \text{ for all } t \in \mathbb{R},$$

where $Aut(K)$ denotes the automorphism group of K. Hence, Lyapunov-like transformations on K are nothing but the elements of Lie(Aut(K)), the Lie algebra of the automorphism group of the cone K [1]; thus, one may redefine the bilinearity rank of K as

$$\beta(K) = \dim Lie(Aut(K)),$$

and (henceforth) call $\beta(K)$, the *Lyapunov rank* of K.

Gowda and Tao [10], following the work of [16], established several new results on the Lyapunov rank, and in particular, described the Lyapunov rank of an arbitrary symmetric cone. It was observed in [16] (see also [10], Example (1)), that the

Lyapunov rank of the l_1-cone in $\mathbb{R}^n$ is one, where the l_1-cone is defined by

$$l^n_{1,+} := \{(t,x) : t \geq ||x||_1\},$$

with $||x||_1$ denoting the 1-norm of the vector x in $\mathbb{R}^{n-1}$. Since the Lyapunov rank is additive on a Cartesian product/sum, it follows that the l_1-cone is irreducible; see [7], Corollary 4.2.5 for an alternate proof. If the 1-norm is replaced by the 2-norm, the resulting l_2-cone

$$l^n_{2,+} = \{(t,x) : t \geq ||x||_2\}$$

is the so-called second-order cone (or the Lorentz cone or the ice-cream cone) in $\mathbb{R}^n$. This cone is irreducible and its Lyapunov rank is $\frac{n^2-n+2}{2}$, see [10], Section 5.

Motivated by the above results, we consider cones in $\mathbb{R}^n$ of the form

$$K = \{(t,x) : t \geq ||x||\}, \tag{1.2}$$

where $||\cdot||$ is a norm on $\mathbb{R}^{n-1}$, $n > 1$. We will call these *special Bishop-Phelps cones* (abbreviated as special BP cones) as they are particular instances of the so-called Bishop-Phelps cones [5] given by

$$\{z \in \mathbb{R}^n : ||z|| \leq \phi(z)\}, \tag{1.3}$$

where $||\cdot||$ is a norm on $\mathbb{R}^n$ and ϕ is a continuous linear functional on $\mathbb{R}^n$.

The above results on l_1 and l_2 cones motivate a number of interesting questions:

- Is every special BP cone irreducible?
- What is the Lyapunov rank of such a cone? What if this cone is polyhedral? What if the norm is the p-norm?
- Can one describe the automorphism group of such a cone?

Answering these, in this paper, we prove the following results for $n \geq 3$:

(i) Every special BP cone is irreducible.
(ii) Every polyhedral special BP cone has Lyapunov rank one.
(iii) The Lyapunov rank of the l_p-cone, for $1 \leq p \leq \infty$, $p \neq 2$, is one.
(iv) Every automorphism of the l_1-cone on $\mathbb{R}^n$ is of the form

$$\theta \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix},$$

where $\theta > 0$ and D is a generalized permutation matrix (that is, it is a product of a permutation matrix and a diagonal matrix with diagonal entries ± 1).

We remark that the above results (i) and (ii) do not extend to arbitrary Bishop-Phelps cones as every closed and pointed cone in $\mathbb{R}^n$ (in particular, $\mathbb{R}^n_+$) is a Bishop-Phelps cone and conversely [14]. However, Bishop-Phelps cones with strictly convex norm and $||\phi|| > 1$ are irreducible, see [7], Example 4.1. We also note that the above results fail for special BP cones when $n = 2$.

The organization of the paper is as follows. In Section 2, we cover some basic material. Section 3 deals with the irreduciblity issue. In Section 4, we consider the Lyapunov ranks of polyhedral special BP cones and l_p-cones. Our final section deals with automorphisms of special BP cones.

2. Preliminaries. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean n-space where the vectors are written as row vectors or column vectors depending on the context. The usual inner product is written as $\langle x, y\rangle$ or as x^Ty. The standard unit vectors in $\mathbb{R}^n$ are denoted by $e_1, e_2, \ldots, e_n$; thus, e_i has one in the ith slot and zeros elsewhere.

For a set K in $\mathbb{R}^n$, $\mathrm{int}(K)$ and $\overline{K}$ denote, respectively, the interior and closure of K. The subspace generated by K is denoted by $\mathrm{span}(K)$. We let

$$\mathrm{cone}\,(K) = \{\lambda x : \lambda \geq 0,\, x \in K\}.$$

The dual of K is given by

$$K^* := \{y \in \mathbb{R}^n : \langle y, x\rangle \geq 0\ \forall x \in K\}.$$

A nonempty set K is a *cone* if $K = \mathrm{cone}\,(K)$. A *closed convex cone* K in $\mathbb{R}^n$ is said to be, see [2],

(a) *pointed* if $K \cap -K = \{0\}$;
(b) *proper* if K is pointed and has nonempty interior.

For a closed convex set S, a vector x in S is an *extreme vector* if $x = ty + (1-t)z$ with $0 < t < 1$, $y, z \in S$ holds only when $y = z = x$; we denote the set of all extreme vectors of S by $\mathrm{ext}(S)$. Note that when S is also compact, by the (finite dimensional) Krein-Milman theorem, see Theorems 3.21 and 3.25 in [15], S is the convex hull of $\mathrm{ext}(S)$:

$$S = \mathrm{conv}\,(\mathrm{ext}(S)).$$

For a convex cone K, we say that a nonzero vector x in K is an *extreme direction* if the equality $x = y + z$ with $y, z \in K$ holds only when y and z are nonnegative multiples of x.

Given any norm $||\cdot||$ on $\mathbb{R}^{n-1}$, $n > 1$, consider the cone in (1.2). That this is a special case of a Bishop-Phelps cone (1.3) is seen by defining, on $\mathbb{R}^n$, the norm

$||(t,x)|| := |t|+||x||$ and the continuous linear functional $\phi : (t,x) \mapsto 2t$. Bishop-Phelps cones are always closed and pointed, and proper when $||\phi|| > 1$ (see Proposition 2.2 and Theorem 2.5 in [5]). Thus, *any cone of the form (1.2) is proper.* If S denotes the closed unit ball in $\mathbb{R}^{n-1}$ with respect to a norm $||\cdot||$, we see that the cone K in (1.2) is also given by

$$K = \text{cone}\,(\{1\} \times S)$$

and, as a consequence, every extreme direction of K is a positive multiple of $(1,x)$ for some $x \in \text{ext}(S)$. In this setting, given $x \in \text{ext}(S)$, we note that $-x \in \text{ext}(S)$; We say that $(1,-x)$ is the *conjugate* of $(1,x)$ and say that $(1,x)$ and $(1,-x)$ form a *conjugate pair.* Corresponding to a norm $||\cdot||$ on $\mathbb{R}^{n-1}$, we define the *dual norm* $||\cdot||_D$ on $\mathbb{R}^{n-1}$ by

$$||x||_D = \max\{\langle x,u\rangle : ||u|| = 1\}.$$

It is easily seen that the dual cone of $K = \{(t,x) : t \geq ||x||\}$ is

$$K^* = \{(t,x) : t \geq ||x||_D\}.$$

For $1 \leq p \leq \infty$ and $x \in \mathbb{R}^{n-1}$, the p-norm is $||x||_p := [\sum_1^{n-1} |x_i|^p]^{\frac{1}{p}}$ when $p < \infty$ and $||x||_\infty = \max\, |x_i|$. The dual norm of $||\cdot||_p$ is $||\cdot||_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. We define the l_p-cone as

$$l^n_{p,+} := \{(t,x) : t \geq ||x||_p\}.$$

3. Irreducibility. Given a closed convex cone K in $\mathbb{R}^n$, we say that it is *reducible* if there exist nonempty sets $K_1 \neq \{0\}$ and $K_2 \neq \{0\}$ such that

$$K = K_1 + K_2, \quad \text{span}(K_1) \cap \text{span}(K_2) = \{0\}.$$

(As in [7], it can be shown that K_1 and K_2 are then closed convex cones in $\mathbb{R}^n$.) In this case, we say that K is a *direct sum* of K_1 and K_2. A closed convex cone that is not reducible is said to be *irreducible.*

THEOREM 3.1. *In $\mathbb{R}^n$, for $n \geq 3$, every special BP cone is proper and irreducible.*

Proof. The properness of K has already been noted. Let S denote the closed unit ball in $(\mathbb{R}^{n-1}, ||\cdot||)$ so that $K = \text{cone}\,(\{1\} \times S)$. As all norms are equivalent on $\mathbb{R}^{n-1}$, we see that the compact convex set S has nonempty interior. Since $\text{conv}\,(\text{ext}(S)) = S$, $\text{ext}(S)$ must contain $n-1$ linearly independent vectors, say, $z_1, z_2, \ldots, z_{n-1}$. Now let $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a matrix/linear transformation with $T(z_i) = e_i$ for all $i = 1,2,\ldots,n-1$, where (we recall that) $e_1, e_2 \ldots, e_{n-1}$ are the standard unit vectors in $\mathbb{R}^{n-1}$. Clearly, T is invertible. Define a new norm $||\cdot||_*$ on $\mathbb{R}^{n-1}$ by

$$||x||_* = ||T^{-1}x|| \quad (x \in \mathbb{R}^{n-1}).$$

Then the closed unit ball corresponding to $||\cdot||_*$ is $S_* = T(S)$ and the corresponding norm induced cone is

$$\text{cone}\,(\{1\} \times S_*) = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} (\text{cone}\,(\{1\} \times S)).$$

Note that the cones induced by $||\cdot||$ and $||\cdot||_*$ are isomorphic and irreducibility of one implies that of the other. So, we may assume without loss of generality that the closed unit ball S of the given norm $||\cdot||$ on $\mathbb{R}^{n-1}$ contains $e_1, e_2, \ldots, e_{n-1}$ as extreme vectors and, as $x \in \text{ext}(S) \Rightarrow -x \in \text{ext}(S)$, write

$$E := \{\pm e_1, \pm e_2, \ldots, \pm e_{n-1}\} \subseteq \text{ext}(S). \tag{3.1}$$

Now suppose, if possible, that K is reducible: $K = K_1 + K_2$, where K_1 and K_2 are closed convex cones with $K_1 \neq \{0\}, K_2 \neq \{0\}, \text{span}(K_1) \cap \text{span}(K_2) = \{0\}$. Define, for $i = 1, 2$,

$$S_i = \{x \in S : (1, x) \in K_i\}.$$

Clearly, these sets are compact, convex, disjoint, and $S_1 \cup S_2 \subseteq S$. We claim that

$$\text{ext}(S) \subseteq \text{ext}(S_1) \cup \text{ext}(S_2). \tag{3.2}$$

To see this, let $x \in \text{ext}(S)$ so that $||x|| = 1$. Then $(1, x) \in K_1 + K_2$ and we may write

$$(1, x) = (\lambda_1, x_1) + (\lambda_2, x_2),$$

where $(\lambda_i, x_i) \in K_i$ for $i = 1, 2$. Then $\lambda_i \geq ||x_i||$ for $i = 1, 2$, $1 = \lambda_1 + \lambda_2$, and $x = x_1 + x_2$. Now,

$$1 = ||x|| \leq ||x_1|| + ||x_2|| \leq \lambda_1 + \lambda_2 = 1$$

implies that $||x_i|| = \lambda_i$ for $i = 1, 2$. If one λ_i is zero, say $\lambda_1 = 0$, then $x_1 = 0$ and so $(1, x) = (1, x_2) \in K_2$, $x \in S_2$. As $x \in \text{ext}(S)$ and $S_1 \cup S_2 \subseteq S$, we must have $x \in \text{ext}(S_2)$. If both λ_1 and λ_2 are nonzero (that is, positive), then the equality

$$x = (\frac{x_1}{\lambda_1})\lambda_1 + (\frac{x_2}{\lambda_2})\lambda_2$$

says that x is a convex combination of two unit vectors. Since $x \in \text{ext}(S)$, we must have $x = \frac{x_1}{\lambda_1} = \frac{x_2}{\lambda_2}$ which further implies that for $i = 1, 2$,

$$(1, x) = (1, \frac{x_i}{\lambda_i}) = \frac{1}{\lambda_i}(\lambda_i, x_i) \in K_i.$$

Clearly this cannot happen as $\text{span}(K_1) \cap \text{span}(K_2) = \{0\}$. We thus have our claim. Recalling the definition of E from (3.1) let, for $i = 1, 2$, $E_i := E \cap S_i$. We claim that

E_1 and E_2 are nonempty. To see this, assume the contrary and suppose (without loss of generality) $E_2 = \emptyset$ so that, by (3.2), $E \subseteq S_1$. Then, $\{(1, \pm e_i) : i = 1, 2, \ldots, n-1\} \subseteq K_1$. As the set $\{(1, e_1), (1, e_2), \ldots, (1, e_{n-1}), (1, -e_1)\}$ forms a basis of $\mathbb{R}^n$, we see that

$$\mathbb{R}^n = \operatorname{span}(\{(1, \pm e_i) : i = 1, 2, \ldots, n-1\}) \subseteq \operatorname{span}(K_1).$$

This means that $\operatorname{span}(K_2) = \{0\}$, leading to a contradiction.

Thus, E_1 and E_2 are nonempty and $E = E_1 \cup E_2$. Let

$$E_1 = \{u_1, u_2, \ldots, u_k\} \quad \text{and} \quad E_2 = \{v_1, v_2, \ldots, v_l\} \text{ so that } k + l = 2(n-1).$$

Let $C_1 := \{(1, u_1), (1, u_2), \ldots, (1, u_k)\}$ and $C_2 = \{(1, v_1), (1, v_2), \ldots, (1, v_l)\}$; we note that $C_i \subset K_i$ so that

$$\operatorname{span}(C_1) \cap \operatorname{span}(C_2) = \{0\}.$$

Now for any given element $(1, x)$ in $\{1\} \times E$, we recall that $(1, -x)$ is the *conjugate* of $(1, x)$ and $(1, x)$ and $(1, -x)$ form a conjugate pair. As every element of E is of the form $\pm e_i$ for some i, the conjugate of any element in C_1 (likewise C_2) is either in C_1 or in C_2. We now consider the following cases:

(1) Both C_1 and C_2 contain some conjugate pairs.
(2) Both C_1 and C_2 are without conjugate pairs.
(3) Only C_1 (say) contains conjugate pairs.

We show that each case leads to a contradiction.

Case 1: Suppose that $(1, e_i), (1, -e_i) \in C_1$ and $(1, e_j), (1, -e_j) \in C_2$ for some $i \neq j$. In this case, $(1, e_i) + (1, -e_i) = (2, 0) = (1, e_j) + (1, -e_j) \in \operatorname{span}(C_1) \cap \operatorname{span}(C_2) = \{0\}$ which is not possible.
Case 2: In this case, the conjugate of any element of C_1 (of C_2) is found in C_2 (respectively, in C_1). This sets up a one-to-one correspondence between elements of C_1 and C_2 showing that the cardinalities of C_1 and C_2 are equal, that is, $k = l$. Since these cardinalities add up to $2(n-1)$, we must have $k = l = n-1$. As there are no conjugate pairs in C_1 and in C_2, both C_1 and C_2 are linearly independent sets in $\mathbb{R}^n$. Thus, $\dim(\operatorname{span}(C_i)) = n-1$ for $i = 1, 2$. Since $\operatorname{span}(C_1) \cap \operatorname{span}(C_2) = \{0\}$, we must have $n \geq (n-1) + (n-1)$, that is, $n \leq 2$. This cannot happen, as we have assumed that $n \geq 3$.
Case 3: In this case, we write C_1 and C_2 in terms of distinct elements:
$C_1 = \{(1, w_1), \ldots, (1, w_m), (1, -w_1), \ldots, (1, -w_m), (1, z_1), \ldots, (1, z_r)\}$ and
$C_2 = \{(1, -z_1), \ldots, (1, -z_r)\}$. (Note that $(1, z_1), \ldots, (1, z_r)$ are elements in C_1 whose conjugates are not in C_1 but in C_2.) It follows that $r = l$ and $k = 2m + r = 2m + l$. Since $k + l = 2(n-1)$, we must have $m + l = n - 1$ or $m + l + 1 = n$. Since the subset

$\{(1,w_1),\dots,(1,w_m),(1,-w_1),(1,z_1),\dots,(1,z_r)\}$ of C_1 is linearly independent and its cardinality is n, $\text{span}(C_1)=\mathbb{R}^n$. This leads to $K_2=\{0\}$ and to a contradiction.

We have thus proved that the reducibility of K leads to a contradiction. Hence the theorem. $\blacksquare$.

Remark (1). The following examples show that for general BP cones or for special BP cones with $n=2$, the above theorem may not hold.
For $n\geq 2$, consider the BP cone

$$\{x\in\mathbb{R}^n : ||x||_1\leq \phi(x)\},$$

where $||x||_1$ is the 1-norm of x and $\phi(x)=\langle x,e\rangle$, with e denoting the vector of ones. This cone, being $\mathbb{R}^n_+$, is reducible.
For $n=2$, consider the special BP cone

$$K=\{(t,x): t\geq |x|\}.$$

This is isomorphic to the nonnegative orthant in $\mathbb{R}^2$ and hence reducible.

4. The Lyapunov rank. Recall that given a proper cone K in $\mathbb{R}^n$, the Lyapunov rank of K is the dimension of the space of all Lyapunov-like matrices on K. It has been shown in [10], Theorem 3, that the Lyapunov rank of a polyhedral cone in $\mathbb{R}^n$ can be any natural number m with $1\leq m\leq n$, $m\neq n-1$. In particular, the Lyapunov rank of the nonnegative orthant in $\mathbb{R}^n$ is n. In this section, we consider cones of the form (1.2).

THEOREM 4.1. *In $\mathbb{R}^n$, for $n\geq 3$, every polyhedral special BP cone has Lyapunov rank one.*

The result follows immediately from Theorem 3.1 (of the previous section) and Corollary 5 of [10] that says that *for any polyhedral proper cone, the Lyapunov rank is one if and only if it is irreducible.* Below, we offer a direct and elementary proof.

Proof. Let $n\geq 3$ and K given by (1.2) be polyhedral. We show that every Lyapunov-like matrix on K is a multiple of the identity matrix, thus proving the result. As done in the proof of Theorem 3.1, we may assume that $\pm e_i$, $i=1,2,\dots,n-1$ are extreme vectors of the closed unit ball of $\mathbb{R}^{n-1}$ under the given norm. Then $(1,\pm e_i)$, $i=1,2,\dots,n-1$, are extreme directions of K. Assuming that vectors in $\mathbb{R}^n$ are now written as column vectors, consider a Lyapunov-like matrix given by

$$A=\begin{bmatrix} a & b^T\\ c & D\end{bmatrix},$$

where $a\in\mathbb{R}$, $b,c\in\mathbb{R}^{n-1}$, and D is an $(n-1)\times(n-1)$ matrix. As K is a polyhedral cone, by Theorem 2 in [10], every (column) vector $[1\ e_i]^T$, $i=1,2,\dots,n-1$, is an eigenvector of A. Thus, there exist real numbers λ_i and μ_i, $i = 1, 2, \ldots, n-1$, such that

$$A \begin{bmatrix} 1 \\ e_i \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ e_i \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ -e_i \end{bmatrix} = \mu_i \begin{bmatrix} 1 \\ -e_i \end{bmatrix},$$

for all $i = 1, 2, \ldots, n-1$. From these, we get

$$a + \langle b, e_i \rangle = \lambda_i,\ a - \langle b, e_i \rangle = \mu_i,\ \ c + De_i = \lambda_i e_i,\ \ c - De_i = -\mu_i e_i$$

for all $i = 1, 2, \ldots, n-1$. These lead to $a = \frac{\lambda_i + \mu_i}{2}$, $De_i = \frac{\lambda_i + \mu_i}{2} e_i = a\, e_i$, $2c = (\lambda_i - \mu_i) e_i$, and $2\langle b, e_i \rangle = \lambda_i - \mu_i$ for all $i = 1, 2, \ldots, n-1$. As $n \geq 3$, the conditions $2c = (\lambda_i - \mu_i) e_i$ for all $i = 1, 2, \ldots, n-1$ imply that $c = 0$ and $\lambda_i = \mu_i$ for all i. We see that $D = a\, I_{n-1}$, where I_{n-1} is the identity matrix of size $n-1$ and $b = 0$. From these we see that $A = a\, I_n$. Thus, multiples of identity are the only Lyapunov-like matrices on K. Hence the Lyapunov rank of K is one. ∎

Corollary 4.2. *Suppose, $n \geq 3$ and S is a compact polyhedral set in $\mathbb{R}^{n-1}$ with nonempty interior. Further suppose that S is symmetric about the origin. Let $K = \text{cone}\,(\{1\} \times S)$ in $\mathbb{R}^n$. Then, K is irreducible and the Lyapunov rank of K is one.*

Proof. The Minkowski functional of S is a norm whose closed unit ball is S [15]. The corresponding cone induced by this norm is K. Thus, K is a polyhedral special BP cone and the result follows from the above theorem. ∎

Theorem 4.3. *Let $n \geq 3$. For any p with $1 \leq p \leq \infty$, $p \neq 2$, the Lyapunov rank of $l^n_{p,+}$ is one.*

Proof. For $p = 1, \infty$, the cone $l^n_{p,+}$ is polyhedral; hence the result follows from the previous theorem. We assume $1 < p < \infty$, $p \neq 2$, and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Consider a matrix

$$A = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix},$$

which is Lyapunov-like on $l^n_{p,+}$, where $a \in \mathbb{R}$, D is an $(n-1) \times (n-1)$ matrix, etc. Our goal is to show that $A = aI$. Now, for each $x \in \mathbb{R}^{n-1}$ with $||x||_p = 1$, define $s \in \mathbb{R}^{n-1}$ by

$$s = sgn(x) * |x|^{\frac{p}{q}},$$

whose ith component is $s_i = sgn(x_i)\, |x_i|^{\frac{p}{q}}$, where $sgn(\alpha)$ is $1, 0, -1$ according as whether the number α is positive, zero, or negative. Then, $||s||_q = 1$ and $\langle x, s \rangle = 1$. Now viewing vectors in $\mathbb{R}^n$ as column vectors, we see that $u = [1\ x]^T \in l^n_{p,+}$, $v =$

$[1\ \ -s]^T \in l^n_{q,+}$, and $\langle u, v\rangle = 0$. Since A is Lyapunov-like, we have $\langle Au, v\rangle = 0$. This leads to

$$a + \langle b, x\rangle - \langle c, s\rangle - \langle Dx, s\rangle = 0.$$

Since this equation is valid if we replace x by $-x$ and s by $-s$, we must have $\langle b, x\rangle - \langle c, s\rangle = 0$ and $\langle (D - aI)x, s\rangle = 0$. We specialize x and s to show that $b = c = 0$ and $D = aI$.
(i) By taking $x = s = e_i$, $i = 1, 2, \ldots, n-1$, we see that $b = c$ and that any diagonal element of $D - aI$ is zero.
(ii) Recalling that $n \geq 3$, for any $\varepsilon_i = \pm 1$, we let $x = (\frac{1}{n-1})^{\frac{1}{p}} \sum_1^{n-1} \varepsilon_i e_i$ and $s = (\frac{1}{n-1})^{\frac{1}{q}} \sum_1^{n-1} \varepsilon_i e_i$. Then with $b = c$ and $p \neq q$, $\langle b, x\rangle - \langle c, s\rangle = 0$ leads to $\sum_1^{n-1} b_i \varepsilon_i = 0$. Since $\varepsilon_i = \pm 1$ are arbitrary, we deduce that $b = 0$.
(iii) For any t, $0 < t < 1$, we let $x_1 = t^{\frac{1}{p}}$, $x_2 = (1-t)^{\frac{1}{p}}$, $x_3 = x_4 = \cdots = x_{n-1} = 0$, and $s_1 = t^{\frac{1}{q}}$, $s_2 = (1-t)^{\frac{1}{q}}$, $s_3 = s_4 = \cdots = s_{n-1} = 0$. Putting these in $\langle (D - aI)x, s\rangle = 0$ and simplifying, we deduce that the leading 2×2 principal submatrix of $D - aI$ is zero. By a similar argument, we show that any 2×2 principal submatrix of $D - aI$ is also zero. We conclude that $D - aI = 0$.
Thus we have proved that $A = aI$. Hence, the Lyapunov rank of $l^n_{p,+}$ is one. ∎

Remark (2). For $n = 2$, consider the special BP cone $K = \{(t, x) : t \geq |x|\}$. This, being isomorphic to the nonnegative orthant in $\mathbb{R}^2$, has Lyapunov rank 2.

5. Automorphisms. Given a proper cone K in $\mathbb{R}^n$, we say that an $n \times n$ matrix A is an *automorphism* of K and write $A \in Aut(K)$ if A is nonsingular and $A(K) = K$. As noted in the Introduction, if A is Lyapunov-like on K, then $e^{tA} \in Aut(K)$ for all $t \in \mathbb{R}$. When $\beta(K) = 1$, multiples of the identity matrix are the only Lyapunov-like matrices. Motivated by these, we raise the question of describing $Aut(K)$, when K is a special BP cone. While this remains an open problem, we describe some special automorphisms that are induced by isometries of the given norm on $\mathbb{R}^{n-1}$. As a special case, we completely describe the automorphisms of the l_1-cone.

Given a norm on $\mathbb{R}^{n-1}$, $n > 1$, with the corresponding closed unit ball S, we consider the special BP cone K defined by (1.2). Relative to this K, we say that an $n \times n$ real matrix A is *conjugate-pair-preserving* if for any $x \in ext(S)$ and $\lambda > 0$

$$A \begin{bmatrix} 1 \\ x \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ y \end{bmatrix} \Rightarrow A \begin{bmatrix} 1 \\ -x \end{bmatrix} = \mu \begin{bmatrix} 1 \\ -y \end{bmatrix}.$$

Recall that a matrix D on $\mathbb{R}^{n-1}$ is an *isometry* of $||\cdot||$ if $||Dx|| = ||x||$ for all $x \in \mathbb{R}^{n-1}$.

THEOREM 5.1. *For $n \geq 3$, consider a special BP cone given by (1.2). Then for* *any $\theta > 0$ and an isometry D of $||\cdot||$, the matrix*

$$\theta \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \tag{5.1}$$

is a conjugate-pair-preserving automorphism of K. Conversely, every conjugate-pair-preserving automorphism of K arises this way.

Proof. The first part of the theorem is easily verified. For the second part, we take a conjugate-pair-preserving automorphism A of K and show that it is of the specified form. We write A in the form

$$A = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix},$$

where $a \in \mathbb{R}$, D is an $(n-1) \times (n-1)$-matrix, etc. Since the vector $u = [1\ 0]^T$ in $\mathbb{R}^n$ is in the interior of K, the first column of A, namely Au, is also in the interior of K. This means that $a > ||c||$. Thus, by scaling A if necessary (which results in $\theta = a$), we may assume that

$$A = \begin{bmatrix} 1 & b^T \\ c & D \end{bmatrix}.$$

Our immediate goal is to show that $c = 0 = b$.

Let u_i, $i = 1, 2, \ldots, n-1$, be linearly independent vectors in $\text{ext}(S)$, where S is the closed unit ball in $\mathbb{R}^{n-1}$. As $n \geq 3$, we have at least two (different) vectors u_1 and u_2. Now, A is nonsingular and maps extreme directions of K to extreme directions of K; so, we have

$$A \begin{bmatrix} 1 \\ u_1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ u_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ w \end{bmatrix}, \tag{5.2}$$

where $\lambda, \alpha > 0$ and $||x|| = 1 = ||w||$. Since A is conjugate-pair-preserving, we must have

$$A \begin{bmatrix} 1 \\ -u_1 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ -x \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -u_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -w \end{bmatrix}, \tag{5.3}$$

where $\mu, \beta > 0$. Expanding these we get

$$1 + \langle b, u_1 \rangle = \lambda,\ c + Du_1 = \lambda x,\ 1 - \langle b, u_1 \rangle = \mu,\ c - Du_1 = -\mu x$$

with similar statements for u_2 in place of u_1. These yield

$$\lambda + \mu = 2, 2c = (\lambda - \mu)x,\ \alpha + \beta = 2,\ 2c = (\alpha - \beta)w.$$

Now suppose, to get a contradiction, that $c \neq 0$. As $||x|| = 1 = ||w||$, the equality $(\lambda - \mu)x = (\alpha - \beta)w$ implies that $|\lambda - \mu| = |\alpha - \beta|$ and $x = \pm w$. From these and the equality $\lambda + \mu = 2 = \alpha + \beta$, we get the following two cases:

(i) $\lambda = \alpha, \mu = \beta, x = w$.
(ii) $\lambda = \beta, \mu = \alpha, x = -w$.

From (5.2) and (5.3), along with the invertibility of A, the first case leads to $[1\ u_1]^T = [1\ u_2]^T$ and the second case leads to $[1\ u_1]^T = [1\ -u_2]^T$. Clearly, these cannot happen. Hence $c = 0$. From $2c = (\lambda - \mu)x$, we get $\lambda = \mu$ or $x = 0$. Now, $x \neq 0$ as the vector $[1\ u_1]^T$, which is on the boundary of K, cannot map to $\lambda[1,\ 0]^T$, which is in the interior of K. Thus, we must have $\lambda = \mu$. But then,

$$1 + \langle b, u_1 \rangle = \lambda, \quad 1 - \langle b, u_1 \rangle = \mu \Rightarrow \langle b, u_1 \rangle = 0.$$

Likewise, $\langle b, u_2 \rangle = 0$. By similar considerations, we arrive at $\langle b, u_i \rangle = 0$ for all $i = 1, 2, \ldots, n-1$, yielding $b = 0$. Thus,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}.$$

We now claim that D is an isometry. Let u be any unit vector in $\mathbb{R}^{n-1}$. Then, the vector $[1\ u]^T$ is on the boundary of K. Hence $A[1\ u]^T$ is a positive multiple of a vector of the form $[1\ v]^T$, where $||v|| = 1$. This leads to $Du = v$ and to $||Du|| = ||v|| = 1$. Thus, D is an isometry. This completes the proof. ■

In the result below, we say that a square matrix is a *generalized permutation matrix* if it is the product of a permutation matrix and a diagonal matrix with diagonal entries ± 1.

THEOREM 5.2. *For $n \geq 3$, every matrix in $Aut(l^n_{1,+})$ is of the form*

$$\theta \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \tag{5.4}$$

where $\theta > 0$ and D is a generalized permutation matrix.

Proof. It is clear that every matrix of the form (5.4) is an automorphism of the l_1-cone. Now we prove the converse. Let $A \in Aut(l^n_{1,+})$. We first claim that A is conjugate-pair-preserving. If S denotes the closed unit ball of l_1-norm on $\mathbb{R}^{n-1}$, then $\text{ext}(S) = \{\pm e_i : i = 1, 2, \ldots, n-1\}$. As the l_1-cone is cone $(\{1\} \times S)$, we note that the extreme directions of the l_1-cone are given by

$$\left\{ \begin{bmatrix} 1 \\ \pm e_i \end{bmatrix} : i = 1, \ldots, n-1 \right\}. \tag{5.5}$$

Now, let $A \in Aut(l^n_{1,+})$. As in the proof of the previous theorem, we see that the $(1,1)$ entry of A is positive; thus, we can scale A and assume without loss of generality that A is in the form

$$A = \begin{bmatrix} 1 & b^T \\ c & D \end{bmatrix}.$$

Now, A is nonsingular and maps extreme directions to extreme directions; so, we have

$$A \begin{bmatrix} 1 \\ e_1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -e_1 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ y \end{bmatrix}, \tag{5.6}$$

where $\lambda, \mu > 0$ and $x, y \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$. Since $2u = [1\ e_1]^T + [1\ -e_1]^T$ is in the interior of $l^n_{1,+}$, $A(2u)$ is in the interior of $l^n_{1,+}$. From the above relations, we see that $\lambda + \mu = |\lambda + \mu| > ||\lambda x + \mu y||_1$. Since $x, y \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$, using the definition of l_1-norm, we see that $y = -x$. This proves that A is conjugate-pair-preserving. By the previous result,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}, \tag{5.7}$$

where D is an isometry of the l_1-norm. Since the isometries of the l_1-norm are generalized permutations, see [6], we have the stated result. ■

Remark (3). That D is a generalized permutation in (5.7) can be shown in a different way (without using a result of [6]): Using (5.7) in (5.6), we get

$$1 = \lambda,\ De_1 = \lambda x.$$

As $x \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$, we see that $De_1 \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$. More generally, $De_j \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$ for any j. Note that such an inclusion is valid for D^{-1} in place of D as A^{-1} is also an automorphism. Thus,

$$D(\{\pm e_i : i = 1, 2, \ldots, n-1\}) = \{\pm e_i : i = 1, 2, \ldots, n-1\}.$$

This shows that D is a generalized permutation.

Remark (4). For $n = 2$, let $K = \{(t, x) : t \geq |x|\}$. Then the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is an automorphism of K which is clearly not of the form given in the above theorem.

Remark (5). For any proper cone K, $A \in Aut(K)$ if and only if $A^T \in Aut(K^*)$. Thus, knowing the automorphisms of the l_1-cone, one can describe the automorphisms of the l_∞-cone.

6. Concluding Remarks.. In this paper, we have studied the so-called special Bishop-Phelps cones and described some results pertaining to irreducibility, Lyapunov rank, and automorphisms. We end this paper by noting a characterization result on self-dual special Bishop-Phelps cones and raising a question on the homogeneity property. The following result provides a simple answer for the self-duality property (which is likely to be known).

THEOREM 6.1. *For $n \geq 2$, the special BP cone K in $\mathbb{R}^n$ given by (1.2) is is self-dual in $\mathbb{R}^n$, that is, $K = K^*$ if and only if the norm $||\cdot||$ on $\mathbb{R}^{n-1}$ is the 2-norm.*

Proof. When the norm is the 2-norm, the corresponding special BP-cone is either the second-order cone $l^n_{2,+}$ (see Section 1) or the cone $K = \{(t, x) : t \geq |x|\}$ in $\mathbb{R}^2$. These cones are self-dual. Now suppose that K is self-dual so that $K = K^*$. We recall that

$$K^* = \{(s, y) : s \geq ||y||_D\},$$

where $||y||_D$ denotes the dual norm of y. Now for any $x \in \mathbb{R}^{n-1}$,

$$(||x||, x) \in K = K^*$$

implies that $||x|| \geq ||x||_D$. Similarly, the inclusion $(||x||_D, x) \in K^* = K$ implies that $||x||_D \geq ||x||$. Hence, $||x|| = ||x||_D$ for all $x \in \mathbb{R}^{n-1}$. Now,

$$||x||_2^2 = \langle x, x\rangle \leq ||x||\,||x||_D = ||x||^2.$$

Thus, $||x||_2 \leq ||x||$ for all $x \in \mathbb{R}^{n-1}$. Finally, by definition of the dual norm, for any $x \in \mathbb{R}^{n-1}$, there exists a vector u with $||u|| = 1$ such that $||x||_D = |\langle x, u\rangle|$. Thus,

$$||x|| = ||x||_D = |\langle x, u\rangle| \leq ||x||_2\,||u||_2 \leq ||x||_2\,||u|| \leq ||x||_2.$$

We conclude that $||x|| = ||x||_2$ for all $x \in \mathbb{R}^{n-1}$. This completes the proof. ∎

We say that a proper cone K is *homogeneous* [18] if for any two elements $x, y \in \operatorname{int} K$, there exists $A \in Aut(K)$ such that $A(x) = y$. A self-dual homogeneous cone is said to be a *symmetric cone* [4]. It is known that every symmetric cone is the cone of squares in some Euclidean Jordan algebra (and conversely). The second order cone $l^n_{2,+}$ is a symmetric cone. It is easily seen, from Theorem 5.2, that the cone $l^n_{1,+}$ ($n \geq 3$) is not homogeneous. (If not, any element of the open unit ball of $(\mathbb{R}^{n-1}, ||\cdot||_1)$ can be mapped onto any another in the open unit ball by a generalized permutation.) These two examples motivate the following problem:
Which special Bishop-Phelps cones are homogeneous? In particular, is $l^n_{p,+}$ non-homogeneous for $p \neq 2$?

REFERENCES

[1] A. Baker. *Matrix Groups.* Springer, London 2002.

[2] A. Berman and R. J. Plemmons. *Nonnegative Matrices in Mathematical Sciences.* SIAM, Philadelphia, 1994.

[3] R.W. Cottle, J.-S. Pang, and R. Stone. The Linear Complementarity Problem. Academic Press, Boston, 1992.

[4] J. Faraut and A. Korányi. *Analysis on Symmetric Cones.* Clarendon Press, Oxford, 1994.

[5] J. Jahn. Bishop-Phelps cones in optimization. International Journal of Optimization: Theory, Methods, and Applications, 1: 123-130, 2009.

[6] C.-K. Li and W. So. Isometries of l_p-norm. The American Mathematical Monthly, 101: 452-453, 1994.

[7] R. Loewy and H. Schneider. Indecomposable cones. Linear Algebra and Its applications, 11: 235-245, 1975.

[8] M.S. Gowda and R. Sznajder. Some global uniqueness and solvability results for linear complementarity problems over symmetric cones. SIAM Journal on Optimization, 18: 461-481, 2007.

[9] M.S. Gowda and J. Tao. **Z**-transformations on proper and symmetric cones. Mathematical Programming, Series B, 117: 195-221, 2009.

[10] M.S. Gowda and J. Tao. On the bilinearity rank of a proper cone and Lyapunov-like transformations. Mathematical Programming, Series A, DOI 10.1007/s10107-013-0715-3, 2013.

[11] M.S. Gowda, J. Tao and G. Ravindran. On the **P** property of **Z** and Lyapunov-like transformations on Euclidean Jordan algebras. Linear Algebra and its Applications, 436: 2201-2209, 2012.

[12] Y. Nesterov and A. Nemirovskii. *Interior-point Polynomial Algorithms in Convex Programming.* SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[13] F. Facchinei and J.-S. Pang. *Finite Dimensional Variational Inequalities.* Springer- Verlag, New York, 2003.

[14] M. Petschke. On a theorem of Arrow, Barankin, and Blackwell. SIAM Journal of Control Optimization, 28: 395-401, 1980.

[15] W. Rudin. *Functional Analysis.* McGraw-Hill, New York 1973.

[16] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh. Bilinearity optimality constrains for the cone of positive polynomials Mathematical Programming, Series B, 129: 5–31, 2011.

[17] H. Schneider and M. Vidyasagar. Cross-positive matrices. SIAM J. Numerical Analysis, 7: 508-519, 1970.

[18] E.B. Vinberg. The theory of homogeneous convex cones. Transactions of Moscow Mathematical Society, 12: 340–403, 1963.