VARIATIONAL ANALYSIS IN PSYCHOLOGICAL MODELING

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Abstract. This paper develops some mathematical models arising in psychology and some other areas of behavioral sciences that are formalized via general preferences with variable ordering structures. Our considerations are based on the recent "variational rationality approach" that unifies numerous theories in different branches of behavioral sciences by using, in particular, worthwhile change and stay dynamics and variational traps. In the mathematical framework of this approach, we derive a new variational principle, which can be viewed as an extension of the Ekeland variational principle to the case of set-valued mappings on quasimetric spaces with cone-valued ordering variable structures. Such a general setting is proved to be appropriate for broad applications to the functioning of goal systems in psychology, which are developed in the paper. In this way we give a certain answer to the following striking question: in the world, where all things change (preferences, motivations, resistances, etc.), where goal systems drive a lot of entwined course pursuits between means and ends—what can stay fixed for a while? The obtained mathematical results and new insights open the door to developing powerful models of adaptive behavior, which strongly depart from pure static general equilibrium models of the Walrasian type that are typical in economics.

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1 Introduction and Motivations

In this introductory section, we first describe the major features of some stability/stay and change dynamical models in behavioral sciences and the essence of the "variational rationality" approach to them. Then we show the need for new mathematical developments concerning variational principles and tools of variational analysis for valuable applications to such models. Finally, we discuss the main goals and contributions of this paper, from both viewpoints of mathematics and applications.

Stability/Stay and Change Dynamics in Behavioral Sciences. Recent developments on the modeling in various branches of behavioral sciences (including artificial intelligence, economics, management sciences, decision processes, philosophy, political sciences, psychology, sociology) mainly focus on the functioning/behavioral dynamics of agents, groups, and organizations. Analyzing these models, two very simple observations come to mind. First, all these disciplines, except static models in microeconomics via the classical Walrasian general equilibrium approach [1], advocate that human behaviors are driven by *adaptive processes*. Second, the vast majority of

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models in these areas (called sometimes "theories of stability/stay and change"), advocate that we live in a world where at the same time many things "stay" (e.g., habits and routines, equilibria, traps, etc.) while many other things "change" (e.g., creations, destructions, learning, innovation, attitudes as well as beliefs formation and revision, self regulation, goal setting, goal striving and revision, breaking and forming habits, etc.). As stated by Bridges [2], "we are always stuck in the middle between a current status quo position and future ends." We refer to [3] for a brief survey on stability and change theories. This may help convincing the reader that dynamical models inherent in behavior sciences are essentially different from more traditional static equilibrium models of the Walrasian type, and thus they require developing appropriate tools of analysis.

Variational Rationality in Behavioral Sciences. Since in behavioral sciences all things change (as is often said: "the only thing that does not change is change itself"), the main question in models of stability and change dynamics is: why, where, how, and when behavioral processes stop or start to change and how transitions work. To describe these issues, Soubeyran [4, 5] introduced two main variational concepts: *worthwhile changes* and *variational traps* as the end points of a succession of worthwhile single changes. The notion of variational traps includes both aspiration points and equilibria and, roughly speaking, reflects the following. Starting from somewhere and not being precisely in a trap, agents want and try approaching such traps in some feasible and acceptable ways (in the case of aspiration points), while being there, prefer to stay than to move away (in the case of equilibria). The notion is crucial in the *variational rationality approach* to modelize human behavior suggested in [4, 5]. This approach helps us to answer the aforementioned main question as well as to unify and modelize various theories of stability/stay and change. It shows how to model the course of human activities as a succession of worthwhile changes and stays, i.e., a succession of actions balancing at each step between the following:

(i) *Motivation to change* involving the utility/pleasure of advantages to change, where these advantages represent the difference between the future payoff generated by a new action and the future payoff generated by the repetition of the past action.

(ii) *Resistance to change* involving the desutility/pain of inconveniences to change, where these inconveniences are the difference between costs to be able to change and costs to be able to stay.

All these concepts, including those of actions, states, transitions, means (resources and capabilities), ends (performances, payoffs, intentions, goals, desires, preferences and values), judgments, attitudes and beliefs, require lengthy and quite intricate discussions to be fully justified in each different discipline, which have its particular points of view.

At each step we say that changes are *worthwhile* if the motivation to change is larger than a chosen fraction of resistance to change. This fraction represents an adaptive satisficing-sacrificing ratio, which helps us to choose at each step the current level of satisfaction or accepted sacrifice. As argued by Simon [6], being "bounded rational," the agent is not supposed to optimize during the transition even if he/she can or cannot reach the optimum at the end. The primary aim of the variational rationality approach is to examine the more or less worthwhile to change transition, which can lead to the desired end/goal points via a succession of worthwhile changes and stays. The major questions are as follows:

(a) When do such processes make small steps, have finite length, converge?

(b) What is the speed of convergence?

(c) Do such processes converge in finite time?

(d) For which initial points do they converge?

(e) Are such processes efficient, i.e, what are the characteristics of end points, which may be critical points, optima, equilibria, Pareto solutions, fixed points, traps, and others?

These questions become *mathematical* provided that adequate mathematical models within variational rationality approach are created and suitable tools of mathematical analysis are selected. As advocated in the aforementioned papers by Soubeyran, *variational analysis*, a relatively new mathematical discipline based on *variational principles*, potentially contains an appropriate and powerful machinery to strongly progress in these directions.

Variational Analysis. Modern variational analysis has been well recognized as a rapidly developed area of applied mathematics, which is mainly based on *variational principles.* It is much related to optimization in a broad sense (being an outgrowth of the classical calculus of variations, optimal control, and mathematical programming) while also applying variational principles and optimization techniques to a wide spectrum of problems that may not be of any variational/optimization nature. The reader can find more details on mathematical theories of variational analysis and its many applications in the now classical monograph by Rockafellar and Wets [7] as well as in more recent texts by Attouch, Buttazzo and Michaille [8], Borwein and Zhu [9], and the two-volume book by Mordukhovich [10] with the numerous references therein.

While there are powerful applications of variational analysis to important models in engineering, physics, mechanics, economics[4], etc., not much has been done on applications of variational analysis to psychology and related areas of behavioral science involving human behavior. Within the variational rationality approach, some mathematical results and applications have been recently obtained in the papers [11, 12, 13, 14, 15]. However, much more is needed to be done in this direction to capture the *dynamical nature* of human behavior reflected in the variational rationality approach. Among the most important dynamical issues, which should be adequately modelized and resolved via appropriate tools of variational analysis, we mention the following settings:

(i) *Periods of the required change* including:

• *course of motivation* (e.g., variable preferences, aspirations, hopes, moving goals, goal setting);

• *dynamics of resistance to change* (e.g., successive obstacles to overcome, goal striving), which require new concepts of distances, dissimilarity, and spaces of paths because actions can be defined as succession of operations;

• *dynamics of adaptation* concerning mainly self-regulation problems such as feedbacks, goal revision, goal pursuit, etc.

(ii) *Periods, where nothing is required to change*, namely: temporary or permanent ends as optima, stationary, equilibrium points, fixed points, traps, habits, routines, social norms, etc.

Having these dynamical issues in mind, we need to revisit available principles and techniques

[4] We particularly refer the reader to the book [10] and the more recent paper [16] with the vast bibliographies therein for applications of modern techniques of variational analysis and set-valued optimization to models of welfare economics, which are typical in microeconomics modeling.

of variational analysis and to develop new mathematical methods and results, which could be applied to solve adaptive dynamic problems arising the aforementioned goals systems of behavioral sciences. Then variational rationality and variational analysis can gain to co-evolve. Variational analysis aims to provide the main tools for the study of variational rationality, which in turn offers a variety of valuable applications for variational analysis in behavioral sciences.

Main Objectives and Contributions of the Paper. The primary objective of this paper is to study *goal systems* in *psychology* by using variational rationality approach and developing an adequate *dynamic technique* of variational analysis. To achieve this aim, we establish a new and nontrivial extension of the fundamental *Ekeland variational principle* (abbr. EVP) to a special class of *set-valued* mappings on *quasimetric* spaces with cone-valued *ordering variable structures*, which becomes the key for our applications to psychology.

The EVP, as first formulated by Ekeland [17] for extended-real-valued lower semicontinuous functions on metric spaces, is one of the most powerful results of variational analysis and its applications. It is worth mentioning that the original proof in the seminal paper by Ekeland [18] is complicated and not constructive, involving transfinite induction via Zorn's lemma. The much simplified proof of the EVP, presented in [19] as a personal communication from Michael Crandall, is remarkable for our purposes, since it is given by a *dynamical process* that itself (besides the result) contains significant information for applications to behavioral sciences. However, neither the setting and proof of the latter paper nor their subsequent numerous extensions given in the literature fully fit the main objectives of this paper required by applications to goal systems in psychology. To proceed successfully in this direction, we develop the (dynamical) approach to set-valued extensions of the EVP implemented by Bao and Mordukhovich [20, 21] for mappings in metric spaces with constant Pareto-type ordering preferences to the significantly more involved case of variable ordering structures in quasimetric spaces.

Then we establish valuable applications of the obtained mathematical results to the goal systems in psychology using and enriching the framework of variational rationality approach by Soubeyran [4, 5]. This allows us, in particular, to shed new light on the explanation, via successions of worthwhile actions and variational traps leading to the underlying dynamical relationships between means and ends in psychological goal systems.

Organization of the Paper. The rest of the paper is organized as follows. Section 2 is devoted to the qualitative description and mathematical modeling of the major goal system in psychology from the viewpoint of variational rationality. Besides these issues, we justify here the importance of an appropriate extension of the EVP and the purposes we intend to meet in this way.

Section 3 is pure mathematical containing the formulation and detailed proof of the main mathematical result of this paper, which is the variational principle discussed above. We also present here an important consequence of this result used in what follows.

Section 4 is devoted to the major applications of the developed mathematical theory to the psychological goal system under consideration. Here we present psychological interpretations of the obtained mathematical results and show that they lead us to rather striking psychological conclusions largely discussed in this section with adding more mathematical details. Section 5 contains some concluding remarks.

2 Goal Systems in Psychology

2.1 Formalization of Goal Systems via Means-End Chain

In what follows, we define a *goal system* consisting of *four ingredients*; see [3] for more details.

(i) *Means* formalized via elements $x \in X$ belonging to the space of means X.

(ii) *Ways* formalized via elements $\omega \in \Omega(x) \subset \overline{\Omega}$ that depend on the given means $x \in X$, where $\Omega(x)$ is a subset of *feasible ways* belonging to some space $\overline{\Omega}$.

(iii) *"Means-ways of using these means"* pairs formalized as $\phi = (x, \omega) \in X \times \overline{\Omega} = \overline{\Phi}$. Their collection is denoted by $\Phi := \{\phi = (x, \omega) \in \overline{\Phi} | \, \omega \in \Omega(x)\}$.

(iv) *Ends as vectorial payoffs.* Let P be a space of payoffs. These payoffs can be gains $g \in P$ to be increased (e.g., proximal goals like performances, revenues, profits, utilities, and pleasures as well as distal goals like wishes, desires, and aspirations). These payoffs can also be costs, unsatisfied needs, desutility, or pains $f \in P$ to be decreased. For instance, $g \in P$ can be a vector of different gains $g = (g^1, \ldots, g^m) \in P = I\!R^m$, or can be a vector $f = (f^1, f^2, \ldots, f^m) \in P = I\!R^m$ of unsatisfied needs. We denote by $g : (x, \omega) \in X \times \Omega(x) \longmapsto g(x, \omega) \in P$ a vectorial *payoff function* and by $f : (x, \omega) \in X \times \Omega(x) \longmapsto f(x, \omega) \in P$ a vectorial *cost or unsatisfied need function.*

Taking the above into account, *goal systems* can be modelized as *set-valued mappings* of the following type. For *gains* we have the mapping $G(\cdot) : x \in X \longmapsto G(x) = \{g(x, \omega) | \, \omega \in \Omega(x)\} \subset P$ whose values are subsets of payoffs the agent can get given a vector of means $x \in X$. Similarly, for *unsatisfied needs* we have the mapping $F(\cdot) : x \in X \longmapsto F(x) = \{f(x, \omega) | \, \omega \in \Omega(x)\} \subset P$ whose values are subsets of unsatisfied needs.

The simplest example we can imagine for a goal system is the least interconnected one, where the unique interconnection between goals comes from the *resources constraint* **(3)** described below. To proceed, consider the following data involving $j = 1, \ldots, m$ activities:

(1) $x \in X = I\!R^d$ is a vector of means to be chosen first.

(2) $\omega = (\omega^1, \ldots, \omega^j, \ldots, \omega^m)$ is an allocation of the given means x, where $\omega^j \in I\!R^d$ for $j = 1, \ldots, m$ will be chosen later.

(3) $\omega^1 + \ldots + \omega^j + \ldots + \omega^m = x$ is a resource constraint. It defines the way in which the agent allocates the given means x to each activity, namely: the different allocations of means, which can be identified to ways of using means, $\omega^j \in X$, aim to reach the goal g^j in the activity j. It tells us that this allocation is feasible (without slack). This resource constraint can be written in the form

$$\omega^1 + \ldots + \omega^j + \ldots + \omega^m = x \iff \omega \in \Omega(x).$$

(4) $g = (g^1, \ldots, g^j, \ldots, g^m) \in P = I\!R^m$ is a vector of goals.

(5) $g^j = g^j(x^j, \omega^j) \in I\!R$ as $j = 1, \ldots, m$ represents, relative to the activity j, the goal level function $g^j(\cdot, \cdot) : (x^j, \omega^j) \in X \times \overline{\Omega} \longmapsto g^j = g^j(x^j, \omega^j) \in I\!R$. It tells us that the means $\omega^j \in X$ help to reach the goal level $g^j = g^j(x^j, \omega^j)$. Then $G(x) = \{g(x, \omega), \, \omega \in \Omega(x)\}$ defines a goal system as the set-valued "gain function" $G(\cdot) : x \in X \longmapsto G(x) \subset P$. Similarly, $F(x) = \{f(x, \omega), \, \omega \in \Omega(x)\}$ defines a goal system as the set-valued "costs or unsatisfied needs function" $F(\cdot) : x \in X \longmapsto F(x) \subset P$, where $f = (f^1, \ldots, f^j, \ldots, f^m) \in P = I\!R^m$ and $f^j = f^j(x^j, \omega^j) \in I\!R$, $j = 1, \ldots, m$.

2.2 Variational Rationality Model of Human Behavior

Simplest Adaptive Variational Rationality Model. The core of the variational rationality approach [4, 5] can be summarized by the following basic adaptive prototype, which allows a lot of variants and extensions.

(A) Adaptive processes of worthwhile changes and stays. Agent's behavior is defined as a succession $\{x_0, \ldots, x_n, \ldots\} \subset X$ of actions entwining possible stays $x_n \in X \curvearrowright x_{n+1} \in X$, $x_{n+1} = x_n$ and possible changes $x_n \in X \curvearrowright x_{n+1} \in X$, $x_{n+1} \neq x_n$. This behavior is said to be *variational rational* if at each step $n+1$ the agent chooses to change or to stay, depending on what he/she accepts to consider as the worthwhile change. Then the agent follows a succession of worthwhile stays and changes $x_{n+1} \in W_{\xi_{n+1}}(x_n)$, $\xi_{n+1} \in \Upsilon$ as $n \in I\!\!N$. Let us be more precise.

At step n, the agent performs the action x_n, given the degree of acceptability $\xi_n \in \Upsilon$ (to be defined later) he/she has chosen before. At step $n+1$, given the past action x_n done right before and the previously given degree of acceptability $\xi_n \in \Upsilon$, the agent adapts his/her behavior in the following way. He/she chooses a new degree of acceptability $\xi_{n+1} \in \Upsilon$ (which can be the same as before) of a next worthwhile change $x_{n+1} \in W_{\xi_{n+1}}(x_n)$. This degree of acceptability (satisficing with some tolerable sacrifices) represents how much worthwhile the agent considers that a change must be to accept to change this step, rather than to stay. There are two cases:

(i) A *temporary worthwhile stay* $x_n \curvearrowright x_{n+1} = x_n$. It is the case when $W_{\xi_{n+1}}(x_n) = \{x_n\}$. Then the agent will choose, in a rational variational way, to stay at $x_n = x_{n+1}$ this time. If at the next steps $n+2, n+3, \ldots$, the agent does not change the degree of acceptability, he/she will choose to stay there forever. This defines a "worthwhile to stay" trap, which is a *permanent worthwhile stay.*

(ii) *A temporary worthwhile change* $x_n \curvearrowright x_{n+1} \neq x_n$. It is the case if $W_{\xi_{n+1}}(x_n) \neq \{x_n\}$ and if the agent can find $x_{n+1} \in W_{\xi_{n+1}}(x_n)$ with $x_{n+1} \neq x_n$. Then the agent will choose to move from x_n to $x_{n+1} \in W_{\xi_{n+1}}(x_n)$, and so on.

(B) Transition phase: the definition of a worthwhile to change step. Consider step $n+1$, and let $x = x_n$ be the preceding action. At step $n+1$, the agent will choose the acceptability ratio $\xi' = \xi_{n+1} \in I\!\!R_+$ and a new action $x' = x_{n+1}$. Let $M(x, x') \in I\!\!R$ be the motivation to change from x to x', and let $R(x, x') \in I\!\!R_+$ be the resistance to change from x to x'. Then the agent will consider that, from his/her point of view, it is worthwhile to move from x to x' if the agent's motivation to change is bigger than his/her resistance to change up to the acceptability ratio ξ_{n+1}, i.e., under the validity of the condition $M(x, x') \geq \xi' R(x, x')$.

Motivation to change $M(x, x') = U[A(x, x')]$ is defined as the *pleasure* or *utility* $U[A]$ of the advantage to change $A(x, x') \in I\!\!R$ from x to x'. In the simplest (separable) case, *advantages to change* are defined as the difference $A(x, x') = g(x') - g(x)$ between a payoff to be improved (e.g., performance, revenue, profit) $g(x') \in I\!\!R$ when the agent performs a new action x' and the payoff $g(x) \in I\!\!R$ when he/she repeats a past action x supposing that repetition gives the same payoff as before. On the other hand, advantages to change $A(x, x') = f(x) - f(x')$ can also be the difference between a payoff $f(x)$ to be decreased (e.g., cost, unsatisfied need) when the agent repeats the same old action x and the payoff $f(x')$ the agent gets when he/she performs a new action x'. The *pleasure function* $U[\cdot] : A \in I\!\!R \longmapsto U[A] \in I\!\!R$ is strictly increasing with the initial condition $U[0] = 0$.

Resistance to change $R(x,x') = D[I(x,x')]$ is defined as the *pain* or *disutility* $D[I]$ of the inconveniences to change $I(x,x') = C(x,x') - C(x,x) \in I\!R_+$, which is the difference between the costs to be able to change $C(x,x') \in I\!R_+$ from x to x' and the costs $C(x,x) \in I\!R_+$ to be able to stay at x. In the simplest case, costs to be able to stay are supposed to be zero, $C(x,x) = 0$ for all $x \in X$, and costs to be able to change are defined as the *quasidistances* $C(x,x') = q(x,x') \in I\!R_+$ satisfying: **(a)** $q(x,x') \geq 0$, **(b)** $q(x,x') = 0 \Longleftrightarrow x' = x$, and **(c)** $q(x,x'') \leq q(x,x') + q(x',x'')$ for all $x, x', x'' \in X$. The *pain function* $D[\cdot] : I \in I\!R_+ \longmapsto D[I] \in I\!R_+$ is strictly increasing with the initial condition $D[0] = 0$.

Thus, in this simplest case, the worthwhile to change and stay process satisfies the conditions $g(x_{n+1}) - g(x_n) \geq \xi_{n+1} q(x_n, x_{n+1})$ or $f(x_n) - f(x_{n+1}) \geq \xi_{n+1} q(x_n, x_{n+1})$ for each n. This yields

$$W_{\xi'}(x) = \left\{x' \in X \big|\ g(x') - g(x) \geq \xi' q(x,x')\right\} \ \text{ or } \ W_{\xi'}(x) = \left\{x' \in X \big|\ f(x) - f(x') \geq \xi' q(x,x')\right\}.$$

(C) End points as traps. Given the final worthwhile to change rate $\xi_* > 0$, we say that the end point $x_* \in X$ of the process under consideration is a *stationary trap* if $W_{\xi_*}(x_*) = \{x_*\}$. In the simplest case above, $x_* \in X$ is a stationary trap if $g(x') - g(x_*) < \xi_* q(x_*, x')$ or $f(x_*) - f(x') < \xi_* q(x_*, x')$ for all $x' \neq x_*$. The definition of a *variational trap* requires more. It involves the given initial state x_0 and requires that the final stationary state (trap) can be reachable from this initial state via a worthwhile and feasible transition of single worthwhile changes and temporary stays.

(D) Variational rationality problems include the following major components.

Starting from any given $x_0 \in X$ and depending on the motivation and resistance to change functions, we want to find a *path of worthwhile changes* so that:

(i) the steps go to *zero* and have *finite length*;

(ii) the corresponding iterations *converge to a variational trap*;

(iii) the *convergence rate* and *stoping criteria* are investigated;

(iv) the *efficiency* or *inefficiency* of such worthwhile to change processes are studied to clarify whether the worthwhile to change process ends at a *critical point*, a *local or global optimum*, a *local or global equilibrium*, an *epsilon-equilibrium*, a *Pareto solution*, etc.

2.3 Ekeland's Variational Principle (EVP) in the Simplest Nonadaptive Model of Variational Rationality

Let us discuss here how the variational rationality approach of [4, 5] interprets the classical EVP [18] in the case of the simplest nonadaptive model. First recall the seminal Ekeland's result.

The classical EVP. *Let* (X,d) *be a complete metric space, and let* $f(\cdot) : x \in X \longmapsto f(x) \in I\!R \cup \{\infty\}$ *be a lower semicontinuous (l.s.c.) function not identically to* ∞ *and bounded from below. Denote* $\underline{f} := \inf\{f(x)|\ x \in X\} > -\infty$. *Then for every* $\varepsilon > 0$, $\lambda > 0$, *and* $x_0 \in X$ *with* $f(x_0) < \underline{f} + \varepsilon$ *there exists* $x_* \in X$ *satisfying the conditions:*

(a) $f(x_0) - f(x_*) \geq (\varepsilon/\lambda) d(x_0, x_*)$,

(b) $f(x_*) - f(x') < (\varepsilon/\lambda) d(x_*, x')$ *for all* $x' \neq x_*$,

(c) $d(x_0, x_*) \leq \lambda$.

By taking $f(x) := -g(x)$, we can immediately reformulate the EVP for the case of *maximization* of $g(.) : x \in X \longmapsto g(x) \in \mathbb{R} \cup \{-\infty\}$. Let us next present a *variational rationality* interpretation of the latter result by using terminology and notation of Sections 2.1 and 2.2.

Variational rationality interpretation of the EVP. Consider the maximization formulation for a payoff to be improved. Then the EVP variational rationality framework tells us the following. Impose the **assumptions**:

- the worthwhile to change process $x_{n+1} \in W_{\xi_{n+1}}(x_n)$ is *nonadaptive*, which means that the "satisficing-sacrificing" ratio ξ_{n+1} is constant along the process $\xi_{n+1} \equiv \xi = \varepsilon/\lambda > 0$ for all $n \in N$;
- advantages to change are *separable*, i.e., $A = A(x, x') = g(x') - g(x)$, where $g(\cdot) : x \in X \longmapsto g(x) \in \mathbb{R}$ is a *payoff function* to be improved (in the sense of maximization);
- costs to be able to change $C = C(x, x') \in \mathbb{R}_+$ represent a *distance* $C(x, x') = d(x, x')$, which implies that costs to be able to change are *symmetric* $C(y, x) = C(x, y)$, costs to be able to stay are *zero* $C(x, x) = 0$ for all $x \in X$, and costs to be able to change satisfy the *triangular inequality*;
- pleasure and pain are identified with *advantages to change* and *inconveniences to change*, respectively, i.e., $U[A] = A$ for all $A \in \mathbb{R}$, and $D[I] = I$ for all $I \geq 0$.

Then we have the **conclusions:**

(a) There exists an acceptable *one step transition* from any initial position x_0 to the end $x_* \in W_\xi(x_0)$. This means that it is *worthwhile* to move directly from x_0 to x_*.

(b) The end is a *stable position*, which means that $W_\xi(x_*) = \{x_*\}$. In other words, it is *not worthwhile* to move from x_* to any different action $x' \neq x_*$.

(c) The end can be reached in a *feasible way* $C(x_0, x_*) \leq \lambda$. This means that if the agent cannot spend more than the $\lambda > 0$ amount in terms of costs, then the move from x_0 to x_* is feasible in the model under consideration.

2.4 Variational Traps and Behavioral Essence of the Ekeland Principle

As stated by Alber and Heward [22], the essence of a trap, given in behavioral terms, is that only "a relatively simple response is necessary to enter the trap, yet once entered, the trap cannot be resisted in creating general behavior changes." Let us give (among many others) a short list of traps we can find in different disciplines.

(A) Psychology. Baer and Wolf [23] seem to be the first to use the term of behavioral trap in describing "how natural contingencies of reinforcement operate to promote and maintain generalized behavior changes." Plous [24] lists five behavioral traps defined as more or less easy to fall into and more or less difficult to get out: investment, deterioration, ignorance, and collective traps. Behavioral traps have been shown to end reinforcement processes [25]. Ego-depletion can generate behavioral traps due to fatigue costs, in the context of self regulation failures [26, 27]. Among several cognitive and emotional traps we can list all-or-nothing thinking, labeling, overgeneralization, mental filtering, discounting the positive, jumping to conclusions, magnification, emotional reasoning, should and shouldn't statements, personalizing the blame, etc.

(B) Economics and decision sciences. Making traps in decision represents hidden biases, heuristics, and routines; e.g., anchoring, status quo, sunk costs, confirming evidence, framing, estimation, and forecasting traps; see [28] and the references therein.

(C) Management sciences. The importance of success and failure traps within organizations due to the so-called "myopia of learning" is emphasized in [29, 30].

(D) Development theory. To explain the formation of poverty traps, Appadurai [31] defines *aspiration traps*, which describe the inability to aspire of the poor; see also [32, 33].

Variational approach of [4, 5] shows that, from the viewpoint of behavioral sciences dealing with essentially *dynamic models* of human behaviors (contrary to pure static developments in general equilibrium theory of economics), the very *essence of the EVP* concerns *variational traps*. More precisely, conditions **(a)** and **(c)** of the Ekeland theorem presented above define a variational trap, which is rather easy to reach in an acceptable and feasible way, while is difficult to leave due to condition **(b)**. This corresponds to the intuitive sense of variational traps in behavioral sciences given in [24]. From this viewpoint, the EVP not only ensures the *existence* of variational traps, but also indicates (particularly in its proof) the *dynamics* of how to reach a variational trap.

It is worth mentioning that the usage and understanding of the EVP in the variational rationality approach to behavioral sciences is different from those in mathematics. Indeed, in *behavioral sciences* (where inertia, frictions, and learning play a major role), natural solutions are variational traps that are reachable in a worthwhile way as *maximal elements* of certain *dynamic* relationships for *worthwhile changes*. In this way, the exact solutions become variational traps, since they include costs to be able to change in their definition. The approximate solution becomes optimum, since they ignore costs to be able to change in their definition.

In *mathematics*, the treatment of the EVP is actually *opposite*. Variational traps resulting from the EVP are seen as *approximate solutions* to the original problem while providing the *exact optimum* to another optimization problem, with a small *perturbation* term.

2.5 Variationally Rational Model of Goal Systems

Variational Rationality Concepts: Worthwhile to Change Payoffs. In the context of goal systems, we define the following variational concepts following [4, 5].

1. Changes. We say that $\phi = (x, \omega) \in \Phi \curvearrowright \phi' = (x', \omega') \in \Phi$ signifies a *change* from the old feasible "means-way of using these means" pair $\phi \in \Phi$ to the new feasible pair $\phi' \in \Phi$, where

$$\Phi := \left\{ \phi = (x, \omega) \in \overline{\Phi} \text{ such that } \omega \in \Omega(x) \right\}$$

stands for the set of all the feasible pairs.

2. Advantages to change. Consider now a change from the present feasible means-end pair (x, g) with $g \in G(x)$ to the next one (x', g') with $g' \in G(x')$. Then $A := A(\phi, \phi') = g(\phi') - g(\phi) \in P$ is the *advantage* to change from the old feasible pair $\phi \in \Phi$ to the new feasible pair $\phi' \in \Phi$.

3. Costs to be able to change and costs to be able to stay. Denote by $C(\cdot, \cdot) : (\phi, \phi') \in \Phi \longmapsto C[\phi, \phi'] \in P$ the *costs to be able to change* from the old feasible pair $\phi \in \Phi$ to the new feasible pair $\phi' \in \Phi$. It is worth mentioning here that, in the context of our new version of the EVP for *variable ordering structures* developed in Section 3, the costs to be able to change exhibit the following two specific properties:

(i) They do *not depend* on the ways of using means $C[\phi, \phi'] = C(x, x') \in P$. This signifies that they actually behave as if the *ways of using means are free.*

(ii) They have a *directional shape* $C(x, x') = q(x, x')\xi$, where $\xi \in P$ and $q(x, x') \in I\!R_+$ is a scalar *quasidistance*, which represents the total cost to be able to change from the old means x to the new means x'. In the case where $P = I\!R^J$, the vector $\xi = (\xi^1, \ldots, \xi^J) \in P$ with $\xi^j \in I\!R_+$, $j = 1, \ldots, J$, and $\|\xi\| = 1$ represents the *internal shares* of this scalar total cost $q(x, x')$ among the different activities. In general these shares $\xi^j = \xi^j(x, x') > 0$ can change along the process.

The detailed justification that the total costs to be able to change can be modelized as a quasidistance $q(x, x') \in I\!R_+$ is given in [4]. To save space, let us just mention that this comes from the definition of the costs to be able to change as the infimum of the costs to be able to perform a succession of operations of deletions, conservations, and acquisitions. The fact that the costs to be able to stay satisfy $C(x, x) = 0$ for all $x \in X$ must also be carefully justified. In the general case, the costs to be able to change modelize inertia, i.e., the resistance to change. There are two extreme cases. *Strong resistance to change* is modelized by the costs to be able to change as scalars or cone quasidistances. This is the case of *variational principles* of Ekeland's type. On the other hands, *weak resistance* to change is modelized by the costs to be able to change via *convex increasing functions* of scalar or cone quasidistances. This is the case of *proximal algorithms*; see [34, 15] for more details and discussions.

4. Inconveniences to change. They represent the difference $I(\phi, \phi') = C(x, x') - C(x, x)$ between the costs to be able to change $C(x, x')$ and the costs to be able to stay $C(x, x)$.

5. Worthwhile to change payoffs. Consider the difference between the advantages to change and the costs to be able to change given by

$$\Delta := \Delta\left[(x, \omega), (x', \omega')\right] = \Delta\left[\phi, \phi'\right] = A(\phi, \phi') - \xi I(\phi, \phi') = \big(g(\phi') - g(\phi)\big) - \xi q(x, x') \in P.$$

This defines the worthwhile to change payoff for the change $\phi = (x, \omega) \curvearrowright \phi' = (x', \omega')$, where $\phi, \phi' \in \Phi$. Then the *change* $\phi := (x, \omega) \curvearrowright \phi' = (x', \omega')$ *is worthwhile* if $\Delta[\phi, \phi'] \geq_{K[f(\phi)]} \mathbf{0}$.

6. Pleasure and pain. To simplify our model of goal systems in this paper, we will not consider the pleasure and pain functions in full generality, i.e., defined as the utilities $U[A(\phi, \phi')] \in \mathbf{U}$ of the advantages to change and the pains $D[I(\phi, \phi')] \in \mathbf{D}$ as the disutilities of inconveniences to be able to change. We simply identify the pleasures with the advantages to change $U[A] = A$ and the pains with the inconveniences to be able to change $D[I] = I$. However, the variable cones $K[f(\phi)]$ or $K[g(\phi)]$ represent these variable pleasures and pains feelings. They define *variable preferences* in the payoff space P. Then the change $\phi = (x, \omega) \curvearrowright \phi' = (x', \omega')$ is *worthwhile* if $\Delta[\phi, \phi'] \geq_{K[f(\phi)]} \mathbf{0}$. This defines the corresponding *variable preference on feasible "means-way of using these means" pairs* in the following way, respectively:

$$\phi'' \geq_\phi \phi' \Longleftrightarrow \Delta\left[\phi, \phi''\right] \geq_{K[f(\phi)]} \Delta\left[\phi, \phi'\right],$$

$$\phi'' \geq_\phi \phi' \Longleftrightarrow \Delta\left[\phi, \phi''\right] \geq_{K[g(\phi)]} \Delta\left[\phi, \phi'\right],$$

where the reference point is the current feasible pair $\phi = (x, \omega) \in \Phi$.

3 Variational Principle for Variable Ordering Structures

The preceding section describes in detail the primary adaptive psychological model of this paper and also discusses the importance of *variational analysis* (particularly an appropriate variational principle of the Ekeland type) as the main mathematical tool of our study and applications. In comparison with the original version of the EVP presented above, the following *three requirements* are absolutely mandatory for an appropriate extension of the EVP for its possible application to the psychological model under consideration:

(a) *vectorial* (actually *set-valued*) nature of the cost function;

(b) *quasimetric* (instead of metric) structure of the topological space of arguments;

(c) *variable preference* structure of ordering on the space of values.

By now, a great many of numerous versions and extensions of the EVP are known in the literature; see, e.g., [9, 10, 20, 21, 35, 36, 37, 38, 39] and the references therein for more recent publications. Some of them address the above issues **(a)** and **(b)** while *none* of them, to the best of our knowledge, deals with *variable structures* in **(c)**, which is the main issue required for applications to *adaptive* models in psychology and other branches of behavioral sciences.

Note that problems with variable preferences have drawn some attention in recent publications (see, e.g., [14, 40, 41, 42, 43, 44], but not from the viewpoint of variational principles.

In this section, we derive a general variational principle that addresses all the three issues **(a)**–**(c)** listed above. Furthermore, we consider a general *parametric* setting when the mapping in the variational principle depends on a *control* parameter, which allows us to take into account the "ways of using means" providing in this way a kind of *feedback* in adaptive psychological models; see Section 4 for more discussions. Our approach and main result extend those (even in nonparametric settings of finite-dimensional spaces) from the papers by Bao and Mordukhovich [20, 21], which dealt with nonparametric mappings between Banach spaces in the standard (not variable) preference framework. Addressing the new challenges in this paper requires a significant improvement of the previous techniques, which is done below.

To describe the class of variable preferences invoking in our main result, take vectors $p_1, p_2 \in P$ from some linear topological *decision space* P, denote $d := p_1 - p_2$, and say that p_2 is *preferred* by the decision maker to p_1 with the *domination factor* d for p_1. The set of all the domination factors for p_1 together with the zero vector $\mathbf{0} \in P$ is denoted by $K[p_1]$. Then the set-valued mapping $K : P \rightrightarrows P$ is called a *variable ordering structure*. We define the *ordering relation* induced by the variable ordering structure K by

$$p_2 \ \leq_{K[p_1]} p_1 \ \text{ if and only if } \ p_2 \in p_1 - K[p_1]$$

and say that $p_* \in \Xi$ is *Pareto efficient/minimal* to the set Ξ in P with respect to the ordering structure K if there is no other vector $p \in \Xi \setminus \{p_*\}$ such that $p \leq_{K[p_*]} p_*$, i.e.,

$$\big(p_* - K[p_*]\big) \cap \Xi = \{p_*\}.$$

It follows from the definition that $p_* \in \operatorname{Min}(\Xi; K[p_*])$ in the sense of set optimization with the ordering cone $K[p_*]$; see, e.g., [35, 45]. This order reduces to the one in set optimization when

$K[p] \equiv \Theta$ for some convex ordering cone $\Theta \subset P$.

Fixing a *direction* $\xi \in P$ and a *threshold/accuracy* $\varepsilon > 0$, we say that p_* is an *approximate* $\varepsilon\xi$*-minimal point* of Ξ with respect to K if

$$\big(p_* - K[p_*] - \varepsilon\xi\big) \cap \Xi = \emptyset.$$

Next we recall the definition of quasimetric spaces and the corresponding notions of closedness, compactness, and completeness in such topological spaces.

Definition 3.1 (quasimetric spaces). *A pair (X, q) with the collection of elements X and the function $q : X \times X \longmapsto I\!R$ on $X \times X$ is said to be a* QUASIMETRIC SPACE *if the following hold:*

(i) $q(x, x') \geq 0$ *for all* $x, x' \in X$*;*

(ii) $q(x, x') = 0$ *if and only if* $x' = x$ *for all* $x, x' \in X$*;*

(iii) $q(x, x'') \leq q(x, x') + q(x', x'')$ *for all* $x, x', x'' \in X$.

Definition 3.2 (left-sequential closedness). *A subset $S \subset X$ is said to be* LEFT-SEQUENTIALLY CLOSED *if for any sequence $\{x_n\} \subset X$ converging to $x_* \in X$ in the sense that the numerical sequence $\{q(x_n, x_*)\}$ converges to zero, the limit x_* belongs to S.*

Definition 3.3 (left-sequential completeness). *A sequence $\{x_n\} \subset X$ is said to be* LEFT-SEQUENTIAL CAUCHY *if for each $k \in I\!N$ there exists N_k such that*

$$q(x_n, x_m) < 1/k \ \text{ for all } \ m \geq n \geq N_k.$$

A quasimetric space (X, q) is said to be LEFT-SEQUENTIALLY COMPLETE *if each left-sequential Cauchy sequence is convergent.*

Let $f : T \to P$ be a mapping from a quasimetric space (T, q) to an ordered vector space P equipped with a variable ordering structure $K : P \rightrightarrows P$, and let $S \subset T$. Then:

• f is (left-sequentially) *level-closed with respect to* K if for any $p \in P$ the p-level set of f with respect to K defined by

$$\text{lev}_p(f, K) := \big\{t \in X \big|\ f(t) \leq_{K[p]} p\big\} = \big\{t \in X \big|\ f(t) \in p - K\,[p]\big\}$$

is left-sequentially closed in (T, q).

• f is *quasibounded from below* on $S \subset \text{dom}\, f := \{t \in T |\ f(t) \in P\}$ *with respect to a cone* Θ, or it is Θ-*quasibounded from below* for short, if there is a bounded subset $M \subset P$ such that $f(S) \subset M + \Theta$ for the image set $f(S) := \cup\{f(t) \in P |\ t \in S\}$.

• $t_* \in S$ is a *Pareto minimizer* (resp. *approximate* $\varepsilon\xi$*-minimizer*) of f over S with respect to K if $f(t_*)$ is the corresponding Pareto minimal point (resp. approximate $\varepsilon\xi$-minimal point) of the image set $f(S) \subset P$.

Note that our applications in Section 4, $x \in X$ represents actions, states, or some means; $g \in P$ represents vectors of ends to be increased (e.g., performances, payoffs, revenues, profits, utilities, pleasures) while $f \in P$ represents vectors of ends to be decreased (a vector of costs, unsatisfied needs, disutilities, pains, etc.). Until arriving at applications, in the mathematical framework here we consider for definiteness the "minimization" setting (instead of the "maximization" one), which is more appropriate for certain applications. Correspondingly, $f \in P$ as a vector of ends to be *decreased* and $K[f] \subset P$ is the cone of vectorial costs *lower* than the given cost vector f. We say that the vectorial payoff f_2 (a vector of payoffs to be decreased) is *smaller* than f_1 with respect to K and write $f_2 \leq_{K[f_1]} f_1$ if $f_2 \in f_1 - K[f_1]$.

Consider now a set-valued mapping $\Omega : X \rightrightarrows \overline{\Omega}$ from a *quasimetric* space (X, q) to a *compact* subset $\overline{\Omega} \subset Y$ of a *Banach* space Y. Let P be a *linear topological* space (of *payoffs*) equipped with some variable *cone-valued ordering* structure $K : P \rightrightarrows P$ (called *variable preference on payoffs*), and let $\{\mathbf{0}\} \neq \Theta \subset P$ be a nontrivial cone. Our standing assumptions are as follows.

Now let us formulate our *standing assumptions* on the initial data of the problem under consideration needed for the proof of the new variational principle in Theorem 3.4. Recall that a cone $K \subset P$ is *proper* if we have $K \neq \{\mathbf{0}\}$ and $K \neq P$.

(H1) The quasimetric space (X, q) is *left-sequentially complete.* Furthermore, the quasimetric $q(x, \cdot)$ is (left-sequentially) *l.s.c.* with respect to the second variable for all $x \in X$.

(H2) All the values of $K \colon P \rightrightarrows P$ are *closed, convex*, and *pointed* subcones of P. Furthermore, the *common ordering cone* of K, denoted by $\Theta_K := \cap_{f \in P} K[f]$, also has these properties.

(H3) The mapping $K \colon P \rightrightarrows P$ enjoys the *transitivity property* in the sense that

$$\Big(f_1 \in f_0 - K[f_0],\ f_2 \in f_1 - K[f_1]\Big) \Longrightarrow \Big(f_2 \in f_0 - K[f_0]\Big).$$

(H4) The mapping $\Omega : X \rightrightarrows \overline{\Omega}$ is (left-sequentially) *closed-graph.*

(H5) The cone $\Theta \subset P$ is *closed* and *convex.*

It is easy to check that the relation $f_1 \leq_{K[f_0]} f_0$ implies that $K[f_1] \subset K[f_0]$ with the equality $K[f_1] + K[f_0] = K[f_0]$ under assumption **(H2)**.

Theorem 3.4 (parametric variational principle for mappings with variable ordering structures). *Let $f : X \times \overline{\Omega} \longmapsto P$ be a mapping with* $\operatorname{dom} f = \operatorname{gph} \Omega$ *in the setting described above. In addition to the standing assumptions* **(H1)**–**(H5)**, *suppose that:*

(A1) *f is quasibounded from below on* $\operatorname{gph} \Omega$ *with respect to the cone* Θ.

(A2) *f is (left-sequentially) level-closed with respect to K on* $\operatorname{gph} \Omega$.

(A3) *$f(x, \cdot)$ is continuous for each* $x \in \operatorname{dom} \Omega$.

Then for any $\varepsilon > 0$, $\lambda > 0$, $(x_0, \omega_0) \in \operatorname{gph} \Omega$, and $\xi \in \Theta_K \setminus (-\Theta - K[f_0])$ with $\|\xi\| = 1$ and $f_0 := f(x_0, \omega_0)$, there is a pair $(x_*, \omega_*) \in \operatorname{gph} \Omega$ with $f_* := f(x_*, \omega_*) \in \operatorname{Min}\big(F(x_*); K[f_*]\big)$ and $F(x_*) := \cup\{f(x_*, \omega) |\ \omega \in \Omega(x_*)\}$ satisfying the relationships

$$f_* + (\varepsilon/\lambda) q(x_0, x_*) \xi \leq_{K[f_0]} f_0, \tag{3.1}$$

$$f + (\varepsilon/\lambda) q(x_*, x) \xi \not\leq_{K[f_*]} f_* \ \text{ for all } (x, \omega) \in \operatorname{gph} \Omega \ \text{ with } f := f(x, \omega) \neq f_*. \tag{3.2}$$

If furthermore (x_0, ω_0) is an approximate $\varepsilon\xi$–minimizer of f over $\operatorname{gph} \Omega$ *with respect to $K[f_0]$, then x_* can be chosen so that in addition to* (3.1) *and* (3.2) *we have*

$$q(x_0, x_*) \leq \lambda. \tag{3.3}$$

Proof. Without loss of generality we assume that $\varepsilon = \lambda = 1$. Indeed, the general case can be easily reduced to this special one by applying the latter to the mapping $\widetilde{f}(x,\omega) := \varepsilon^{-1} f(x,\omega)$ and the left-sequentially complete quasimetric space $(X,\widetilde{q})$ with $\widetilde{q}(x,x') := \lambda^{-1} q(x,x')$.

Define now a set-valued mapping $W : X \times \overline{\Omega} \rightrightarrows X$ by

$$W(x,\omega) := \left\{x' \in X \middle|\, \exists\, \omega' \in \Omega(x') \;\text{ with }\; f(x',\omega') + q(x,x')\xi \;\leq_{K\,[f]} f(x,\omega)\right\}, \tag{3.4}$$

where $f := f(x,\omega)$. It is easy to check that for such a pair (x',ω') satisfying the inequality in (3.4) we get, by denoting $f' := f(x',\omega')$, that

$$f(x',\omega') \;\leq_{K[f]} f(x,\omega) \;\text{ and }\; K\left[f'\right] \subset K\,[f] \tag{3.5}$$

under the imposed assumptions for K. Indeed, the inclusion in (3.5) follows directly from the inequality therein while the latter is valid by

$$\begin{aligned}
& f(x',\omega') + q(x,x')\xi \;\leq_{K[f]} f(x,\omega) \\
\Longleftrightarrow\;\; & f(x',\omega') + q(x,x')\xi \in f(x,\omega) - K\,[f] \\
\Longleftrightarrow\;\; & f(x',\omega') \in f(x,\omega) - q(x,x')\xi - K\,[f] \\
\Longrightarrow\;\; & f(x',\omega') \in f(x,\omega) - K\,[f]\,,
\end{aligned}$$

where the implication holds due to the choice of $\xi \in \Theta_K \subset K\,[f]$ and the convexity of the cone $K\,[f]$. In fact $K\,[f] + q(x,x')\xi \subset K\,[f] + K\,[f] = K\,[f]$.

Next we list some important properties of the set-valued mapping W used in what follows.

• The sets $W(x,\omega)$ are *nonempty* for all $(x,\omega) \in \operatorname{gph}\Omega$ due to $(x,\omega) \in W(x,\omega)$.

• The sets $W(x,\omega)$ are *left-sequentially closed* in (X,q) for all $(x,\omega) \in \operatorname{gph}\Omega$. To verify this property, it is sufficient to show that the limit of any convergent sequence $\{x_k\} \subset W(x,\omega)$ with $x_k \to x_*$ as $k \to \infty$ belongs to the set $W(x,\omega)$. By definition of W, find a sequence $\{\omega_k\} \subset \overline{\Omega}$ with $\omega_k \in \Omega(x_k)$ for all $k \in I\!N$ satisfying

$$f(x_k,\omega_k) + q(x,x_k)\xi \in f(x,\omega) - K\,[f]\,.$$

Since $\overline{\Omega}$ is a compact set, extract (without relabeling) a convergent subsequence from $\{\omega_k\}$ that converges to some $\omega_* \in \overline{\Omega}$. This gives us $(x_*,\omega_*) \in \operatorname{gph}\Omega$ by the (left-sequential) closedness assumption on $\operatorname{gph}\Omega$. Then passing to limit with taking into account the level-closedness and lower semicontinuity assumptions imposed on f and q tells us that

$$f(x_*,\omega_*) + q(x,x_*)\xi \in f(x,\omega) - K\,[f]\,, \;\text{ i.e., }\; x_* \in W(x,\omega).$$

• The sets $W(x,\omega)$ are *bounded from below* with respect to $\Theta + K\,[f]$ for all $(x,\omega) \in \operatorname{gph}\Omega$, where $f = f(x,\omega)$. Indeed, it follows from

$$W(x,\omega) \subset \left\{x' \in X \;\text{ such that }\; q(x,x')\xi \in f(x,\omega) - M - \Theta - K\,[f]\right\},$$

where the bounded set M is taken from the definition of the assumed quasiboundedness from below of the mapping f with respect to the cone Θ.

• We have the inclusion $W(x',\omega') \subset W(x,\omega)$ for all $x' \in W(x,\omega)$ and $\omega' \in \overline{\Omega}$ with

$$f(x',\omega') + q(x,x')\xi \leq_{K[f]} f(x,\omega)$$

To verify it, pick $x'' \in W(x',\omega')$ and by construction of $W(x,\omega)$ in (3.4) find $\omega'' \in \Omega(x")$ satisfying the inequality $f(x'',\omega'')+q(x',x'')\xi \leq_{K[f']} f(x',\omega')$. Summing up the last two inequalities and taking into account the triangle inequality for the quasimetric $q(x,x'') \leq q(x,x')+q(x',x'')$, the choice of $\xi \in \Theta_K$ as well as the transitivity and convexity properties of K ensuring that $K[f]+K[f']+\Theta_K = K[f]$, we get the relationships

$$\begin{aligned} & f(x'',\omega'')+q(x,x'')\xi \\ = \; & \big(f(x',\omega')+q(x,x')\xi\big)+\big(f(x'',\omega'')+q(x',x'')\xi\big) \\ & +\big(q(x,x'')-q(x,x')-q(x',x'')\big)\xi - f(x',\omega') \\ \in \; & f(x,\omega)-K[f]+f(x',\omega')-K[f']-\Theta_K-f(x',\omega') \\ \subset \; & f(x,\omega)-K[f]. \end{aligned}$$

This clearly implies that $f(x'',\omega'')+q(x,x'')\xi \leq_{K[f]} f(x,\omega)$, i.e., $x'' \in W(x,\omega)$. Since x'' was chosen arbitrarily in $W(x',\omega')$, we conclude that $W(x',\omega') \subset W(x,\omega)$.

To proceed further, let us inductively construct a sequence of pairs $\{(x_n,\omega_n)\} \subset \operatorname{gph}\Omega$ and denote $f_n := f(x_n,\omega_n)$ for all $n \in I\!N \cup \{0\}$ by the following *iterative procedure*: starting with (x_0,ω_0) given in the theorem and having the n-iteration (x_n,ω_n), we select the next one (x_{n+1},ω_{n+1}) by

$$\begin{cases} x_{n+1} \in W(x_n,\omega_n), \\ q(x_n,x_{n+1}) \geq \sup\limits_{x \in W(x_n,\omega_n)} q(x_n,x) - \dfrac{1}{n+1}, \\ \omega_{n+1} \in \Omega(x_{n+1}), \quad f(x_{n+1},\omega_{n+1})+q(x_n,x_{n+1})\xi \; \leq_{K[f_n]} f(x_n,\omega_n) \end{cases} \tag{3.6}$$

It follows from construction (3.4) of the sets $W(x,\omega)$ and their properties listed above that this iterative procedure is *well defined*. By (3.5) the sequence $\{f_n\}$ with $f_n := f(x_n,\omega_n)$ is *nonincreasing* with respect to the ordering structure K in the sense that $f_{n+1} \leq_{K[f_n]} f_n$ for all $n \in I\!N \cup \{0\}$. Furthermore, the cone sequence $\{K(f_n)\}$ is *nonexpansive*, i.e.,

$$K[f_{n+1}] \subset K[f_n] \quad \text{for all} \;\; n \in I\!N \cup \{0\},$$

which implies together with the convex-valuedness of K that

$$\sum_{n=0}^{m} K[f_n] = K[f_0] \quad \text{for all} \;\; m \in I\!N \cup \{0\}.$$

Summing up the last inequality in (3.6) from $n=0$ to m, we get with $t_m := \sum_{n=0}^{m} q(x_n,x_{n+1})$ that

$$t_m\xi \in f_0 - f_{m+1} - K[f_0] \subset f_0 - M - \Theta - K[f_0] \quad \text{for all} \;\; m \in I\!N \cup \{0\}. \tag{3.7}$$

Let us next prove by passing to the limit in (3.7) as $m \to \infty$ that

$$\sum_{n=0}^{\infty} q(x_n,x_{n+1}) < \infty. \tag{3.8}$$

Arguing by contradiction, suppose that (3.8) does not hold, i.e., the increasing sequence $\{t_m\}$ tends to ∞ as $m \longrightarrow \infty$. By the first inclusion in (3.7) and the boundedness of the set M taken from the quasiboundedness of f from below, find a bounded sequence $\{w_m\} \subset f_0 - M$ satisfying

$$t_m\xi - w_m \in -\Theta - K[f_0], \;\; \text{i.e.,} \;\; \xi - w_m/t_m \in -\Theta - K[f_0], \quad m \in I\!N.$$

Passing now to the limit as $m \longrightarrow \infty$ and taking into account the closedness of Θ, the boundedness of $\{w_m\}$, and that $t_m \to \infty$, we arrive at $\xi \in -\Theta - K[f_0]$ in contradiction to the choice of $\xi \in \Theta_K \setminus (-\Theta - K[f_0])$. Thus (3.8) holds and allows us for any $\varepsilon > 0$ find a natural number $N_\varepsilon \in I\!N$ so that $t_m - t_n = \sum_{k=n}^{m-1} q(x_k, x_{k+1}) \le \varepsilon$ whenever $m \ge n \ge N_\varepsilon$. Hence

$$q(x_n, x_m) \le \sum_{k=n}^{m-1} q(x_k, x_{k+1}) \le \varepsilon \ \text{ for all } \ m \ge n \ge N_\varepsilon,$$

which means that $\{x_k\}$ is a (left-sequential) *Cauchy sequence* in the quasimetric space (X, q). Since X is left-sequentially complete, there is $x_* \in X$ such that $x_k \longrightarrow x_*$ as $k \longrightarrow \infty$. Taking into account that $W(x_{k+1}, \omega_{k+1}) \subset W(x_k, \omega_k)$ and the choice of x_{k+1}, we get the estimate

$$\operatorname{radius} W(x_k, \omega_k) := \sup_{x \in W(x_k, \omega_k)} q(x_k, x) \le q(x_k, x_{k+1}) + \frac{1}{k+1}$$

ensuring that $\operatorname{radius} W(x_k, \omega_k) \downarrow 0$ as $k \to \infty$. It follows from the left-sequential completeness of X and the left-sequential closedness of $W(x_k, \omega_k)$ that

$$\bigcap_{k=0}^{\infty} W(x_k, \omega_k) = \{x_*\} \quad \text{for some } x_* \in X. \tag{3.9}$$

Now we justify the existence of $\omega_* \in \Omega(x_*)$ such that $f_* := f(x_*, \omega_*) \in \operatorname{Min}\big(F(x_*), K[f_*]\big)$ satisfies the relationships in (3.1) and (3.2). For each pair $(x_k, \omega_k) \in \operatorname{gph} \Omega$, define a subset of $\overline{\Omega}$ by

$$R(x_k, \omega_k) := \big\{\omega \in \Omega(x_*) \big|\ f(x_*, \omega) + q(x_k, x_*)\xi \le_{K[f_k]} f(x_k, \omega_k)\big\}, \quad k \in I\!N. \tag{3.10}$$

Then we have the following properties:

- The set $R(x_k, \omega_k)$ is *nonempty* and *closed* for any $k \in I\!N \cup \{0\}$ under the assumptions made. Indeed, the nonemptiness follows directly from $x_* \in W(x_k, \omega_k)$ and the definition of W in (3.4). The closedness property holds since $R(x_k, \omega_k)$ is the $(f(x_k, \omega_k) - q(x_k, x_*)\xi)$-level set of the mapping $f(x_*, \cdot)$ with respect to the closed and convex cone $K[f_k]$ and since $f(x_*, \cdot)$ is assumed to be continuous. Furthermore, it follows from the inclusion $R(x_k, \omega_k) \subset \Omega(x_*) \subset \overline{\Omega}$ and the compactness of $\overline{\Omega}$ that $R(x_k, \omega_k)$ is a compact subset as well.

- The sequence $\{R(x_k, \omega_k)\}$ is *nonexpansive.* To verify it, pick any $w \in R(x_{k+1}, \omega_{k+1})$ and get

$$f(x_*, w) + q(x_{k+1}, x_*)\xi \in f(x_{k+1}, \omega_{k+1}) + K[f_{k+1}].$$

Combining this with (3.6) and then using the quasimetric triangle inequality together with the equality $K[f_{k+1}] + K[f_k] = K[f_k]$ tell us that

$$f(x_*, w) + q(x_k, x_*)\xi \in f(x_k, \omega_k) + K[f_k], \ \text{ i.e., } \ w \in R(x_k, \omega_k),$$

which therefore justifies that $w \in R(x_{k+1}, \omega_{k+1}) \subset R(x_k, \omega_k)$.

It follows from the properties of $R(\cdot, \cdot)$ established above and the compactness of $\overline{\Omega}$ that there exists $\overline{\omega} \in \overline{\Omega}$ satisfying the inclusion

$$\bar{\omega} \in \bigcap_{k=0}^{\infty} R(x_k, \omega_k). \tag{3.11}$$

Denoting $f_{\bar{\omega}} := f(x_*, \bar{\omega})$ and forming the $f_{\bar{\omega}}$–level set of $f(x_*, \cdot)$ over $\Omega(x_*)$ by

$$\Xi := \left\{ \omega \in \Omega(x_*) \middle| \; f_\omega := f(x_*, \omega) \leq_{K[f_{\bar{\omega}}]} f(x_*, \bar{\omega}) =: f_{\bar{\omega}} \right\}, \tag{3.12}$$

we obviously have that Ξ is compact with $\bar{\omega} \in \Xi$. Employ now [45, Corollary 5.10], which ensures in our setting the existence of $\omega_* \in \Xi$ such that

$$f_* = f(x_*, \omega_*) \in \mathrm{Min}\left(f(x_*, \Xi), K[f_{\bar{\omega}}]\right) \;\text{ with }\; f(x_*, \Xi) := \bigcup_{\omega \in \Xi} \{f_\omega := f(x_*, \omega) \in P\}.$$

This reads by the definition of Pareto efficiency that

$$\left(f_* - K[f_{\bar{\omega}}]\right) \cap f(x_*, \Xi) = \{f_*\}.$$

Since $f_* \leq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$, we have $K[f_*] \subset K[f_{\bar{\omega}}]$; cf. the justifications for (3.5). Thus we get

$$(f_* - K[f_*]) \cap f(x_*, \Xi) = \{f_*\}, \;\text{ i.e., }\; f_* \in \mathrm{Min}\,(f(x_*, \Xi), K[f_*]).$$

Actually the following stronger conclusion holds:

$$f_* \in \mathrm{Min}\,(F(x_*), K[f_*]) \;\text{ with }\; F(x_*) = f\big(x_*, \Omega(x_*)\big) \supset f(x_*, \Xi). \tag{3.13}$$

Arguing by contradiction, suppose that (3.13) does not hold and find $\omega \in \Omega(x_*) \setminus \Xi$ such that $f_\omega \leq_{K[f_*]} f_*$. Since $\omega_* \in \Xi$, we have $f_* \leq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$. Then the transitivity assumption **(H3)** ensures that $f_\omega \leq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}}$, and so $\omega \in \Xi$ contradicting the choice of $\omega \in \Omega(x_*) \setminus \Xi$. This justifies (3.13).

Now we are ready to show that the pair (x_*, ω_*) satisfies the conclusions (3.1) and (3.2) of our variational principle. The inequality in (3.1) immediately follows from $\omega_* \in R(x_0, \omega_0)$. To verify (3.2), suppose the contrary and find a pair $(x, \omega) \in \mathrm{gph}\,\Omega$ with $f(x, \omega) \neq f(x_*, \omega_*)$ satisfying

$$f(x, \omega) + q(x_*, x)\xi \in f(x_*, \omega_*) + K[f_*]. \tag{3.14}$$

Fix $k \in I\!N \cup \{0\}$ and sum up the three inequalities: (3.14), (3.12) with $\omega = \omega_*$, and (3.10) with $\omega = \bar{\omega}$. This gives us, by taking into account the triangle inequality as well as the relationships $f_* \leq_{K[f_{\bar{\omega}}]} f_{\bar{\omega}} \leq_{K[f_k]} f_k$, $K[f_*] \subset K[f_{\bar{\omega}}] \subset K[f_k]$, and $K[f_*] + K[f_\omega] + K[f_k] = K[f_k]$, that

$$f(x, \omega) + q(x_k, x)\xi \in f(x_k, \omega_k) - K[f_k], \;\text{ i.e., }\; x \in W(x_k, \omega_k), \quad k \in I\!N.$$

This means that x belongs to the set intersection in (3.9), and thus $x = x_*$. Substituting it into (3.14), we obviously get $f(x_*, \omega) + q(x_*, x_*)\xi \in f(x_*, \omega_*) + K[f_*]$ and reduce it to

$$f(x_*, \omega) \in f(x_*, \omega_*) - K[f_*], \;\text{ i.e., }\; f(x_*, \omega) \leq_{K[f_*]} f(x_*, \omega_*).$$

The latter shows that $f(x_*, \omega) = f(x_*, \omega_*) \in \mathrm{Min}\,(F(x_*), K[f_*])$, which contradicts the assumption of $f(x, \omega) \neq f(x_*, \omega_*)$ and hence justifies (3.2).

To complete the proof of the theorem, it remains to estimate the distance $q(x_0, x_*)$ in (3.3) when (x_0, ω_0) is an approximate $\varepsilon\xi$–minimizer of f over $\mathrm{gph}\,\Omega$. Arguing by contradiction, suppose that (3.3) does not hold, i.e., $q(x_0, x_*) > \lambda$. Since $x_* \in W(x_0, \omega_0)$, we have

$$f(x_*, \omega_*) + (\varepsilon/\lambda)q(x_*, x_0)\xi \in f(x_*, \omega_*) - K[f_*],$$

which together with $f(x_*, \omega_*) \leq_{K[f_0]} f(x_0, \omega_0)$ yields $f(x_*, \omega_*) + \varepsilon\xi \in f(x_*, \omega_*) - K[f_0]$. This contradicts the approximate minimality of (x_0, ω_0) and thus ends the proof. △

Finally in this section, we present a direct consequence of Theorem 3.4 for the case when the mapping f does not depend on the control variable ω, which also provides a new variational principle for systems with variable ordering structures and is used in what follows.

Corollary 3.5 (variational principle for parameter-independent mappings with respect to variable ordering). *Let $f = f(x)$ be a mapping from X to P with $\mathrm{dom}\, f \neq \emptyset$ in the setting of Theorem 3.4 under the assumptions made therein. Then for any $\varepsilon > 0$, $\lambda > 0$, $x_0 \in \mathrm{gph}\,\Omega$, and $\xi \in \Theta_K \setminus (-\Theta - K[f(x_0)])$ with $\|\xi\| = 1$ there is a point $x_* \in \mathrm{dom}\, f$ satisfying the relationships*

$$f(x_*) + (\varepsilon/\lambda) q(x_0, x_*)\xi \leq_{K[f(x_0)]} f(x_0), \tag{3.15}$$

$$f(x) + (\varepsilon/\lambda) q(x_*, x)\xi \not\leq_{K[f(x_*)]} f(x_*) \quad \text{for all } x \in \mathrm{dom}\, f \quad \text{with } f(x) \neq f(x_*). \tag{3.16}$$

If furthermore x_0 is an approximate $\varepsilon\xi$–minimizer of f with respect to $K[f(x_0)]$, then x_* can be chosen so that in addition to (3.15) *and* (3.16) *we have the estimate* (3.3).

Proof. It follows from Theorem 3.4 applied to the mapping $\widetilde{f} : X \times \overline{\Omega} \to P$ with $\widetilde{f}(x, \omega) = f(x)$ and a (compact) set $\overline{\Omega}$ consisting of just one point, say $\{\omega_*\}$. △

4 Applications to Goal Systems in Psychology

4.1 What Our Variational Principle Add to Goal System Theory

(A) Variational rationality via variational analysis. In the context of the variational rationality framework of [4, 5], the new variational principle in Theorem 3.4 shows that, considering an *adaptive goal system* endowed with variable cone-valued preferences in the payoff space and a *quasimetric* on the space of means under (fairly natural) hypotheses of the theorem and starting from any feasible "means-way of using these means" pair $\phi_0 = (x_0, \omega_0) \in \Phi$, there exists a *succession of worthwhile changes* $\phi_{n+1} \in W(\phi_n)$ with $n \in I\!N$, which ends at some *variational trap* $\phi_* = (x_*, \omega_*) \in \Phi$, where the agent *prefers to stay than to move.* The meaning of this is as follows; see the notation and psychological description in Section 2.

(i) Reachability and acceptability aspects along the transition: we have $\phi_* \in W(\phi_0)$, i.e., it *is worthwhile* to move directly from the starting means-end pair ϕ_0 to the ending one ϕ_*.

(ii) Stability aspect at the end: we have $W(\phi_*) = \{\phi_*\} \iff \phi \notin W(\phi_*)$ for any $\phi \in \Phi$, $\phi \neq \phi_*$, meaning that it is *not worthwhile* to move from the means-end pair ϕ_* to a different one.

(iii) Feasibility aspect along the transition: if $\phi_0 = (x_0, \omega_0) \in \Phi$ is any $\varepsilon\xi$–*approximate minimizer* of $G(\cdot)$, then x_* can be chosen such that in addition to **(i)** and **(ii)** we have $C(x_0, x_*) \leq \lambda$.

(iv) The end is efficient as a Pareto optimal solution. This is shown in the proof of Theorem 3.4 and discussed above.

(B) When proof says more than statement. Analyzing the statement and the proof of our variational principle in Theorem 3.4, we can observe that–besides the variational trap interpretations, which are discussed above in **(A)** and follow from the *statement* of the theorem–the *proof* itself offers much more from the psychological point of view. Indeed, the statement of Theorem 3.4 is an *existence result* while the proof provides a constructive *dynamical process*, which leads us to a solution. From the mathematical viewpoint, the situation is similar to the classical Ekeland principle with the proof given in [19]. From the psychological viewpoint, this is in accordance with the message popularized by Simon [6]: *decision and making process matters and can determine the end.* It is also a major point of the variational rationality approach [4, 5]: *to explain human desirable ends requires to exhibit human behavioral processes that can lead to them.* It senses that desirable ends must be reachable in an acceptable way by using feasible means. In other words, if the agent starts from any "means-way of using these means" pair, pursues his/her goals by exploring enough each step and performing a succession of worthwhile changes or stays, then he/she will end in a strong behavioral trap, i.e., a Pareto solution more preferable to stay than to move even without any resistance to change. The given proof of Theorem 3.4 reveals at least *four* very important points discussed below in the rest of this subsection.

(C) Worthwhile to change processes. Parallel to [4] with using [19] in the case of nonadaptive models, the proof of Theorem 3.4 for the *adaptive* psychological models under consideration shows how the agent explicitly forms at each step a "consideration set" to evaluate and balance his/her current motivation and resistance to change "exploring enough" within the current worthwhile to change set trying to "improve it enough" by inductively constructing a sequence of feasible pairs $\{(x_n,\omega_n)\}\subset \mathrm{gph}\,\Omega$. This nicely fits the famous concept of "consideration sets" in marketing sciences defined first as "evoked sets" by Howard and Sheth [46]. The idea is that, at any given consumption occasion, consumers do not consider all the brands available while the current consideration/relevant set represents "those brands that the consumer considers seriously when making a purchase and/or consumption decision" as discussed, e.g., in [47, 48]. The size of the consideration set is usually *small* relative to the total number of brands, which the consumer could evaluate. Then, using various heuristics, the consumer tries to simplify his/her decision environment.

(D) Variational traps as desirable ends. The worthwhile to change dynamical process given in the proof of Theorem 3.4 allows the agent to reach a *variational trap* by a succession of worthwhile changes. In this model, it is a *Pareto "means-way of using these means" pair.* Such a variational trap is related to two important concepts, *aspiration points* and *efficient points*, at the individual and collective levels discussed as follows.

(a) Aspiration points and Pareto points. The Pareto point achieved in Theorem 3.4 is an aspiration point as defined in [4, 5] and then further studied and applied in [13, 14]. An *aspiration point* is such that, starting from any point of the worthwhile to change process, it is worthwhile to move directly (in an acceptable way) to the given point of aspiration. It represents the "*rather easy to reach*" aspect of a variational trap while the other one ("difficult to leave") is more traditional as an equilibrium or stability condition.

(b) Optimal solutions. The proof developed in Theorem 3.4 allows to study other types of *optimal solutions/minimal points*; compare, e.g., [20, 21, 35, 39] for various notions of this kind and employ iterative procedures to derive for them variational principles of the Ekeland type.

(c) Individual or collective aspects: agents versus organizations. In this paper we focus on the individual aspects of goal systems. The case of organizations requires to consider *bilevel optimization problems* with leaders and followers. This will be a subject of our future research.

(E) Variable preferences and efficiency for course pursuit processes. Variable preferences can take different forms; see [3] for more discussions. In this paper we pay the main attention to ordering structures defined by *variable cone-valued preference*. Among other types of variable preferences we mention *attention based preferences* discussed, e.g., in [49, 50]. Such variable preferences can also be modelized and resolved by the approach developed in the proof of Theorem 3.4.

(F) Habituation processes as ends of course pursuit problems. Our results help to modelize the emergence of *multiobjective habituation processes* with variable preferences. Such a formulation can represent agents who follow a habituation process with *multiple goals* as well as an organization, where each agent can have different goals. Then the procedure developed in the proof of Theorem 3.4 ends in a variational trap, which is a goal system habit for agents or a bundle of routines for organizations. This represents a habituation process in various areas of life, which is characterized by several properties such as repetitions, automaticity, control and economizing, etc.; see [4, 5, 51] for more details and discussions.

4.2 When Costs to Be Able to Change "Ways of Using Means" Do Matter

Consider a more general problem to change from a "means-way of using these means" feasible pair $\phi = (x, \omega)$ with $\omega \in \Omega(x) \subset \overline{\Omega}$ to a new pair $\phi' = (x', \omega')$ with $\omega' \in \Omega(x') \subset \overline{\Omega}$. In this general behavioral case, the full costs $C\left[(x,\omega),(x',\omega')\right] = C\left[\phi,\phi'\right]$ to be able to change from a feasible pair $\phi = (x,\omega)$ with $\omega \in \Omega(x)$ to another feasible pair $\phi' = (x', \omega')$ with $\omega' \in \Omega(x')$ must include the *two kinds of costs* in the sum: $C\left[(x,\omega),(x',\omega')\right] = C_X(x,x') + C_\Omega(\omega,\omega') \in P$.

Suppose now in the line of Theorem 3.4 that such vectorial costs are *proportional* to a vector $\xi \in P$, i.e., $C\left[(x,\omega),(x',\omega')\right] = q\left[\phi,\phi'\right]\xi$, where $q\left[\phi,\phi'\right] \in I\!R_+$ is a quasidistance on $\overline{\Phi} := X \times \overline{\Omega}$. This quasidistance modelizes the *total costs* to be able to change from one pair to another.

Let $\Phi := \{(x,\omega), \omega \in \Omega(x)\} \subset \overline{\Phi}$ be the subset of feasible "means-way of using these means" pairs. Then the *worthwhile to changes preference* over all the "means-way of using these means" pairs $\phi = (x,\omega) \in \Phi$ is

$$\begin{aligned}
\phi' \geq_{K[f(\phi)]} \phi \iff& (x',\omega') \geq_{K[f(x,\omega)]} (x,\omega)\\
\iff& f(x',\omega') + C\left[(x,\omega),(x',\omega')\right] \leq_{K[f(x,\omega)]} f(x,\omega)\\
\iff& f(\phi') + C\left[\phi,\phi'\right] \leq_{K[f(\phi)]} f(\phi),
\end{aligned}$$

where $C\left[(x,\omega),(x',\omega')\right] = q\left[(x,\omega),(x',\omega')\right]\xi$ while the pairs $\phi = (x,\omega) \in \Phi \subset \overline{\Phi}$ and $\phi' = (x',\omega') \in \Phi \subset \overline{\Phi}$ are feasible, i.e., $\omega \in \Omega(x) \subset \overline{\Omega}$ and $\omega' \in \Omega(x') \subset \overline{\Omega}$. In this general case, the previous worthwhile to change sets read as follows:

$$\begin{aligned}
W(x,\omega) &= \left\{x' \in X \,\middle|\, \exists\, \omega' \in \Omega(x') \text{ with } (x',\omega') \geq_{K[f(x,\omega)]} (x,\omega)\right\}\\
&= \left\{x' \in X \,\middle|\, \exists\, \omega' \in \Omega(x') \text{ with } f(x',\omega') + C\left[(x,\omega),(x',\omega')\right] \leq_{K[f(x,\omega)]} f(x,\omega)\right\}.
\end{aligned}$$

Instead, we consider now the *new worthwhile to change set* defined by

$$\begin{aligned} W(\phi): &= \left\{\phi' \in \Phi \middle|\ \phi' \geq_{K[f(\phi)]} \phi\right\} = \left\{\phi' \in \Phi \middle|\ f(\phi') + q\left[\phi, \phi'\right]\xi \leq_{K[f(\phi)]} f(\phi)\right\} \\ &= \left\{(x', \omega') \in \Phi \middle|\ f(x', \omega') + q\left[(x, \omega), (x', \omega')\right]\xi \leq_{K[f(x,\omega)]} f(x, \omega)\right\}. \end{aligned}$$

In this setting, we can apply Corollary 3.5, where we replace the means $x \in X$ by the "means-way of using these means" pairs $\phi = (x, \omega) \in \overline{\Phi} \subset X \times \overline{\Omega}$ to modelize such a situation. This variant has the following *two advantages*: **(i)** it helps to modelize goal systems, where changing the ways of using means is costly; **(ii)** it allows us to *drop the compactness* assumption on the set $\overline{\Omega}$ of ways as in Theorem 3.4. Now the state space is that of pairs $\phi = (x, \omega) \in \overline{\Phi}$, the vectorial payoff mapping is that of unsatisfied needs $f : \phi \in \overline{\Phi} \longmapsto f(\phi) \in P$, and the real function $q(\phi, \phi') \in I\!R_+$ denoted the quasidistance between two pairs of "means-way of using these means." Then, in this context of "means-way of using means" pairs, we reformulate Corollary 3.5 as follows.

Corollary 4.1 (variational principle in "means-way of using these means" setting). *Let $(\overline{\Phi}, q)$ be a left-sequentially complete quasimetric space, and let $K : P \rightrightarrows P$ be a cone-valued ordering structure satisfying assumptions* **(H2)** *and* **(H3)**. *Consider a mapping $f : \overline{\Phi} \to P$ with* $\operatorname{dom} f \neq \emptyset$ *being a left-sequentially closed subset of $\overline{\Phi}$. Assume also that:*

(A1) *f is quasibounded from below on* $\operatorname{dom} f$ *with respect to a convex cone Θ.*

(A2) *$f(\cdot) + (\varepsilon/\lambda)q(\phi, \cdot)$ is (left-sequentially) level-closed with respect to K for all pairs $\phi \in \overline{\Phi}$ and positive numbers $\varepsilon, \lambda > 0$.*

Then for any ε, $\lambda > 0$, $\phi_0 \in \operatorname{dom} f$, and $\xi \in \Theta_K \setminus (-\Theta - K[f_0]))$ with $\|\xi\| = 1$ and $f_0 := f(\phi_0)$ there is a point $\phi_* \in \operatorname{dom} f$, satisfying the relationships

$$f(\phi_*) + (\varepsilon/\lambda)q(\phi_0, \phi_*)\xi \leq_{K[f_0]} f(\phi_0),$$

$$f(\phi) + (\varepsilon/\lambda)q(\phi_*, \phi)\xi \not\leq_{K[f_*]} f(\phi_*) \quad \textit{for all} \quad \phi \in \operatorname{dom} f \quad \textit{with} \quad f(\phi) \neq f(\phi_*),$$

where $f_* := f(\phi_*)$. If furthermore ϕ_0 is an approximate $\varepsilon\xi$–minimizer of f with respect to $K[f_0]$, then ϕ_* can be chosen so that in addition to (3.15) *and* (3.16) *we have the estimate* (3.3).

Comments. From the psychological point of view, Corollary 4.1 can be interpreted as follows. Starting from any "means-way of using these means" pair $\phi_0 \in \overline{\Phi}$, the agent who manages several goals by enduring costs to be able to change both the means used and the way of using them and whose next preference over the relative importance of each goal changes with the current pair ϕ, can reach, in *only one worthwhile to change step*, a certain *variational trap* ϕ_*, where it is not worthwhile to move. Moreover, given the desirability level $\varepsilon > 0$ of the initial pair $\phi_0 \in \overline{\Phi}$ and the size $\lambda > 0$ of the limited resource, the agent accomplishes this worthwhile change in a *feasible way*, since the costs to be able to change $q(\phi_*, \phi)$ are lower than the resource constraint λ.

5 Conclusion

The main mathematical result of this paper, Theorem 3.4, as well as its consequences provide a far-going extension of the Ekeland variational principle aimed, first of all, to cover multiobjective problems with variable ordering structures. This major feature allows us to obtain new applications to adaptive psychological models within the variational rationality approach. Following this way, we plan to develop in our future research further applications of variational analysis to qualitative and algorithmic aspects of adaptive modeling in behavior sciences. One of our major attention in this respect is to extend the variational rationality approach and the corresponding tools variational analysis to decision making problems, where "all things can be changed", i.e., with changeable decision sets, payoffs, goals, preferences, and contexts/parameters. Note that in a different setting, where decision sets and parameters can change along some Markov chain, another approach to similar issues has been developed in the context of *habitual domain theory*; see [52, 53, 54, 55, 56]. A detailed comparison between the variational rationality approach and that of habitual domain theory has been recently given in [34].

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