

On Blocking and Anti-Blocking Polyhedra in Infinite Dimensions

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Abstract. We consider the natural generalizations of blocking and anti-blocking polyhedra in infinite dimensions, and study issues related to duality and integrality of extreme points for these sets. Using appropriate finite truncations we give conditions under which complementary slackness holds for primal-dual pairs of the infinite linear programming problems associated with infinite blocking and anti-blocking polyhedra. We also give conditions under which the extreme points of infinite blocking and anti-blocking polyhedra are integral. We illustrate an application of our results on an infinite-horizon lot-sizing problem.

1 Introduction

Over several decades, blocking and anti-blocking polyhedra have been a useful tool in finite-dimensional combinatorial optimization. Introduced by Fulkerson [11, 12, 13], they embody duality and polarity relations in various combinatorial settings and have been applied to several fundamental problems in graph theory and discrete optimization, e.g. the study of perfect graphs and the derivation or re-derivation of duality results such as the max-flow min-cut theorem. The interested reader may consult [18, Chapter 9] for a longer treatment.

In this paper we study the generalization of blocking and anti-blocking polyhedra to problems in infinite dimensions. We first consider strong duality results for the infinite linear programming problems (LP) associated with such polyhedra. Strong duality results for infinite LPs have been studied by Romeijn, Smith and their co-authors [16, 17]. However, their results are limited to LPs with countably many variables and constraints and, more importantly, to LPs with finite-valued objective functions. Unfortunately, the LPs associated with blocking and anti-blocking polyhedra often have objective functions that are infinite-valued over some or all of the feasible region. An example where such infinite-valued objective functions naturally appear is infinite horizon planning problems. A common approach to deal with such objectives is to introduce discount factors or running averages to force the objective function to be finite-valued. To avoid problems with such approaches (e.g. strong biases towards early or later periods) we instead study the extension of a version of strong duality considered in the theory of infinite graphs by Aharoni and his co-authors [1, 3, 6, 8]. This version of strong duality considers complementary slackness conditions that are compatible with infinite-valued objective functions. We show that the existence of solutions that satisfy this version of strong duality for blocking and anti-blocking polyhedra can be proven using topological tools developed in [6]. In addition we show how the associated optimality conditions for infinite LPs with infinite-valued objectives can be interpreted as a special type of optimality for certain finite truncations of the problem. As an application of these tools, we derive properties for the optimal policies of an infinite horizon lot-sizing problem with an undiscounted and infinite-valued objective function. Our strong duality results are limited to infinite blocking and anti-blocking polyhedra defined by linear constraints with only a finite number of non-zero coefficients. Extending this result to more general blocking and anti-blocking polyhedra will likely require more advanced techniques used in [1, 3, 6, 8]. Some of these techniques are tied to the discrete structure of infinite graphs. For this reason we consider the existence of discrete or integral structures in infinite blocking and anti-blocking polyhedra as a first step towards extending these results. Specifically, we show how a topological tool developed in [10] can be used to show integrality of the extreme points of certain infinite blocking and anti-blocking polyhedra.

The remainder of this paper is organized as follows. In Section 2 we introduce notation and definitions, and review some previous work. In Section 3 we show the strong duality result, while in Section 4 we show an application of this result

to an infinite horizon lot-sizing problem. Finally, in Section 5 we give the result on integrality of extreme points.

2 Definitions and Previous Work

For arbitrary sets \mathbb{W} and L , we let \mathbb{W}^L denote all functions from L to \mathbb{W} . This set can also be thought as tuples of elements in \mathbb{W} indexed by L or as the Cartesian product $\prod_{l \in L} \mathbb{W}$. Some of the proof techniques will employ the product topology of \mathbb{W}^L , where convergence is characterized by coordinate-wise convergence. This convergence is also known as point-wise convergence. If \mathbb{W} is a compact topological space, then \mathbb{W}^L is compact when endowed with the product topology by Tychonoff's theorem. We also need the following partial converse to the Krein-Milman theorem:

Theorem 1 (Milman, quoted from [15]). *If K is a compact convex subset of a locally convex space and if $A \subseteq K$ is such that K is equal to the closure of the convex hull of A , then the extreme points of K are contained in the closure of A .*

We consider blocking anti-blocking pairs of LPs through the following definition.

Definition 1 (Blocking Anti-Blocking Pair). *Let I, J be arbitrary and possibly uncountable index sets, $A := (a_{ij})_{i \in I, j \in J} \in \mathbb{R}_+^{I \times J}$ be a doubly infinite non-negative "matrix", and $c := (c_j)_{j \in J} \in \mathbb{R}_+^J$ and $b := (b_i)_{i \in I} \in \mathbb{R}_+^I$ be non-negative infinite "vectors". We consider the blocking/anti-blocking pair of doubly infinite LPs given by*

$$(P) \quad z^* = \inf \sum_{j \in J} c_j x_j \tag{1a}$$

$$s.t. \quad \sum_{j \in J} a_{ij} x_j \geq b_i \quad \forall i \in I \tag{1b}$$

$$x_j \geq 0 \quad \forall j \in J \tag{1c}$$

and

$$(D) \quad w^* = \sup \sum_{i \in I} b_i y_i \tag{2a}$$

$$s.t. \quad \sum_{i \in I} a_{ij} y_i \leq c_j \quad \forall j \in J \tag{2b}$$

$$y_i \geq 0 \quad \forall i \in I. \tag{2c}$$

The blocking polyhedron defined by A and b is the feasible region of (P) given by $\mathcal{P}_\uparrow(A, b) := \{x \in \mathbb{R}^J : (1b)-(1c)\}$ and the anti-blocking polyhedron defined by A and c is the feasible region of (D) given by $\mathcal{P}_\downarrow(A, c) := \{y \in \mathbb{R}^I : (2b)-(2c)\}$. When A, b and c are clear from the context, we use the notation \mathcal{P}_\uparrow and \mathcal{P}_\downarrow .

If I and J are finite this pair of problems reduces to the traditional finite blocking/anti-blocking pair introduced by Fulkerson. In addition, however, special infinite versions of this problem have been studied by several authors. In [10], Dalang studies the extreme points of the anti-blocking polyhedron associated with the stable sets of an infinite perfect graph and in [14] Grzkaślewicz does the same for the anti-blocking polyhedron associated with matchings and b -matchings of a bipartite infinite graph. In a series of papers, Aharoni and his co-authors study duality for several problems in infinite graphs and hypergraphs, including integer and fractional matching [1, 3, 6, 8, 7], connectivity [2, 5] and flows [4]. Another stream of related work is a series of papers by Romeijn, Smith and their co-authors. These papers study extreme points [9] and duality [16, 17] for problems with more general structure, but only with countably infinite I and J ; i.e. for the case in which both \mathcal{P}_\uparrow and \mathcal{P}_\downarrow are contained in $\mathbb{R}^\mathbb{N} = \prod_{i=0}^\infty \mathbb{R}$. When applied to countably infinite blocking/anti-blocking pairs the results in [9] imply that the extreme points of the finite projections of \mathcal{P}_\downarrow converge in the product topology of $\mathbb{R}^\mathbb{N}$ to the closure of the extreme points of \mathcal{P}_\downarrow and hence, if the extreme points of the finite projections are all integral, then the extreme points of \mathcal{P}_\downarrow are also integral. The results in [16, 17] imply that, under some technical conditions, strong duality holds for (P)/(D) for countable I, J .

3 LP Duality

The following straightforward lemma shows that weak duality holds for (P)/(D).

Lemma 1 (Weak Duality). $w^* \leq z^*$

Proof. Direct from non-negativity of the data. □

While it is possible to show strong duality under certain settings, e.g. [6, 16, 17], there are many cases in which the objective values of every reasonable feasible solution for (P) and (D) are infinite. A common alternative for these cases is to study the existence of primal/dual solutions that satisfy the following definition of complementary slackness [1, 3, 6, 7, 8].

Definition 2 (Complementary Slackness and Optimality). *Feasible solutions $x \in \mathcal{P}_\uparrow$ and $y \in \mathcal{P}_\downarrow$ satisfy complementary slackness if and only if*

$$\left(\sum_{i \in I} a_{ij} y_i - c_j \right) x_j = 0 \quad \forall j \in J \tag{3a}$$

$$\left(b_i - \sum_{j \in J} a_{ij} x_j \right) y_i = 0 \quad \forall i \in I. \tag{3b}$$

We say $x \in \mathbb{R}^J$ is CS Optimal if and only if $x \in \mathcal{P}_\uparrow$ and there exists $y \in \mathcal{P}_\downarrow$ such that (x, y) satisfies (3).

For the countably infinite case, [16, 17] show that the limits of solutions that satisfy complementary slackness for certain finite truncations of (P)/(D) also satisfy complementary slackness for (P)/(D). A similar technique is used in [6] to show that complementary solutions exist for certain fractional matching/covering pairs in hypergraphs, even in the uncountable setting. This proof directly extends to blocking/anti-blocking problems so we use it to show that complementary solutions for (P)/(D) exist under fairly general conditions. For this we use the following finite truncations of (P)/(D).

Definition 3. Let $U \subseteq J$ be such that $|U| < \infty$ and $I(U) = \{i \in I : a_{i,j} = 0 \forall j \in J \setminus U\}$. We define the finite truncation of P associated with U as

$$(P(U)) \quad w^* = \inf \sum_{j \in U} c_j x_j \tag{4a}$$

$$s.t. \quad \sum_{j \in U} a_{ij} x_j \geq b_i \quad \forall i \in I(U) \tag{4b}$$

$$x_j \geq 0 \quad \forall j \in U \tag{4c}$$

and its feasible region as $\mathcal{P}_\uparrow(A, b, U) := \{x \in \mathbb{R}^U : (4b)-(4c)\}$. Similarly, we define the truncation of D associated with U as

$$(D(U)) \quad z^* = \sup \sum_{i \in I(U)} b_i y_i \tag{5a}$$

$$s.t. \quad \sum_{i \in I(U)} a_{ij} y_i \leq c_j \quad \forall j \in U \tag{5b}$$

$$y_i \geq 0 \quad \forall i \in I(U) \tag{5c}$$

and its feasible region as $\mathcal{P}_\downarrow(A, c, U) := \{y \in \mathbb{R}^{I(U)} : (5b)-(5c)\}$. Again, when A, b and c are clear from context, we use the abbreviated notations $\mathcal{P}_\uparrow(U)$ and $\mathcal{P}_\downarrow(U)$.

With this definition $(P(U))$ is the truncation of (P) that only considers variables in U and constraints that only involve variables in U . To show the existence of CS optimal solutions we need the following conditions.

Assumption 1. The following conditions hold:

- i) **Row Finiteness:** For all $i \in I$, $|j \in J : a_{ij} > 0| < \infty$.
- ii) **Column Finiteness:** For all $j \in J$, $|i \in I : a_{ij} > 0| < \infty$.

Before we prove the existence of CS Optimal solutions we give the following pseudo-finite interpretation of CS Optimality. Here, for given vectors $x \in \mathbb{R}^J$ and $y \in \mathbb{R}^I$ and sets $U \subseteq J$ and $V \subseteq I$ we let $x_U := (x_j)_{j \in U}$ and $y_V := (y_i)_{i \in V}$. We then refer to x_U and y_V as truncations of x and y .

Proposition 1 (Pseudo-Finite Optimality). Under Assumption 1 the following conditions are equivalent:

i) x is CS optimal.

ii) There exists $y \in \mathbb{R}^I$ such that for every finite $U \subseteq J$ we have

$$x_U \in \mathcal{P}_\uparrow(U) \quad (6a)$$

$$y_{I(U)} \in \mathcal{P}_\downarrow(U) \quad (6b)$$

$$\left(\sum_{i \in I} a_{ij} y_i - c_j \right) x_j = 0 \quad \forall j \in U \quad (6c)$$

$$\left(b_j - \sum_{j \in U} a_{ij} x_j \right) y_i = 0 \quad \forall i \in I(U). \quad (6d)$$

iii) There exists $y \in \mathbb{R}^I$ such that for every finite $U \subseteq J$ we have

$$x_U \in \mathcal{P}_\uparrow(U) \quad (7a)$$

$$y_{I(U)} \in \mathcal{P}_\downarrow(U) \quad (7b)$$

$$\left(\sum_{i \in I(U)} a_{ij} y_i - c'_j \right) x_j = 0 \quad \forall j \in U \quad (7c)$$

$$\left(b_i - \sum_{j \in U} a_{ij} x_j \right) y_i = 0 \quad \forall i \in I(U) \quad (7d)$$

where $c'_i = c_i - \sum_{i \in I \setminus I(U)} a_{ij} y_i$.

Proof. First note that by the row finiteness assumption and the definition of $I(U)$ we have that $x \in \mathcal{P}_\uparrow$ is equivalent to $x_U \in \mathcal{P}_\uparrow(U)$ for all finite U . Similarly, by column finiteness and non-negativity of the a_{ij} we have that $y \in \mathcal{P}_\downarrow$ is equivalent to $y_{I(U)} \in \mathcal{P}_\downarrow(U)$ for all finite U . The result then follows by noting that, because of the definition of $I(U)$, we have $\sum_{j \in U} a_{ij} x_j = \sum_{j \in J} a_{ij} x_j$ for all $i \in I(U)$. \square

Conditions (6) and (7) are almost equivalent to optimality of x for the truncations $P(U)$ of P . For instance, if c' is replaced by c in (7c), condition (7) implies that for every finite U there exists a dual solution $y_{I(U)}$ such that $(x_U, y_{I(U)})$ is a primal/dual pair of feasible solutions for $(P(U))/(D(U))$ that satisfy complementary slackness, which implies x_U is optimal for $P(U)$. However, the requirements of CS optimality are slightly stronger. First, finite dual solutions $y_{I(U)}$ are required to come from a common infinite dual solution y . Second, optimality of x_U for $P(U)$ needs to be augmented to consider neighboring truncations. For example, in (6) this augmentation requires the complementary condition of x_j to consider dual variables y_i for all constraints of P in which x_j appears. This can be different from the analog requirement for $(P(U))/(D(U))$, which only considers dual variables y_i for constraints that exclusively contain x_j variables for $j \in U$. In turn, condition (7) implicitly considers the additional dual variables by changing the objective coefficients to c' . An unfortunate consequence of this difference is that CS optimality of x cannot be directly guaranteed through optimality of x_U for $P(U)$. Fortunately, we can extend Theorem 3.4 of [6] to show that existence of optimal solutions for $P(U)$ for every finite U does indeed

imply the existence of at least one infinite CS optimal solution x . Furthermore, certain properties of the finite optimal solutions of $P(U)$ are guaranteed to be inherited by this infinite solution x . To describe such properties we consider sets $F, G \subseteq \mathbb{R}$ representing a restriction on the domain of the solutions. For example, $F = G = \mathbb{N}$ represents the constraint that solutions are integral. Using this we can show that, if all truncations $(P(U))/(D(U))$ have optimal solutions taking values in F, G , then $(P)/(D)$ has a CS Optimal primal-dual pair that also takes values in F, G . However, to prove this result we need the following assumption, guaranteeing certain compactness requirements in the proof.

Assumption 2. We assume $\sup_{i \in I: a_{ij} > 0} \left\{ \frac{b_i}{a_{ij}} \right\} < \infty$ for all $j \in J$ and $\sup_{j \in J: a_{ij} > 0} \left\{ \frac{c_j}{a_{ij}} \right\} < \infty$ for all $i \in I$.

Theorem 2. Let $F, G \subseteq \mathbb{R}$ be closed subsets such that $0 \in F, G$. Suppose for any $U \subseteq J$ with $|U| < \infty$ there exist $x^U \in F^U \cap \mathcal{P}_\uparrow(U)$ and $y^U \in G^{I(U)} \cap \mathcal{P}_\downarrow(U)$ such that

$$\left(\sum_{i \in I(U)} a_{ij} y_i^U - c_j \right) x_j^U = 0 \quad \forall j \in U \tag{8a}$$

$$\left(b_i - \sum_{j \in U} a_{ij} x_j^U \right) y_j^U = 0 \quad \forall i \in I(U), \tag{8b}$$

where the superscript indicates the solutions' dependence on the set U . Then, under Assumptions 1 and 2, there exists a primal dual pair $(x, y) \in F^J \cap \mathcal{P}_\uparrow \times G^I \cap \mathcal{P}_\downarrow$ of CS optimal solutions for $(P)/(D)$.

Proof. We follow and extend the proof of a similar result in [6] for fractional matchings in infinite graphs and hypergraphs.

Let $X := \prod_{j \in J} [0, g_j]$ and $Y := \prod_{i \in I} [0, h_i]$, where $g_j := \sup_{i \in I: a_{ij} > 0} \left\{ \frac{b_i}{a_{ij}} \right\}$ and $h_i := \sup_{j \in J: a_{ij} > 0} \left\{ \frac{c_j}{a_{ij}} \right\}$. By Assumption 2 and Tychonoff's Theorem, $X \times Y$ is compact. Furthermore, $\mathcal{P}_\downarrow \subseteq Y$ and every $x^U \in F^U \cap \mathcal{P}_\uparrow(U)$ satisfying (8) is contained in $\prod_{j \in U} [0, g_j]$ (because such x^U is optimal for $P(U)$).

Now, for any finite $U \subseteq J$, let

$$C(U) = \left\{ (x, y) \in X \times Y : \begin{array}{ll} x_U \in \mathcal{P}_\uparrow(U), & y \in \mathcal{P}_\downarrow, \\ (x, y) \in F^J \times G^I, & (x, y) \text{ satisfies (3)} \end{array} \right\}$$

By the assumptions, $C(U)$ is closed. To see that it is non-empty take $x \in F^U \cap \mathcal{P}_\uparrow(U)$ and $y \in G^{I(U)} \cap \mathcal{P}_\downarrow(U)$ satisfying (8) and extend them to $X \times Y$ by appending zeros. Family $C(U)$ has the finite intersection property because for any finite subfamily $(U_i)_i$ we have $C(\bigcup_i U_i) \subseteq \bigcap_i C(U_i)$. Then, by compactness we have that $\bigcap_{U \subseteq J: |U| < \infty} C(U) \neq \emptyset$. The result then follows by row finiteness (condition (i) of Assumption 1). \square

4 Application to Lot Sizing

We next discuss our results' application to an infinite-horizon version of the single-item lot-sizing problem [16]. For this we assume there is an infinite sequence of demands $b_t > 0$, $t = 1, \dots$, that must be met either with product produced in the same period or with product held over in inventory from previous periods; backlogging is not allowed. There are time-dependent unit production and holding costs, $c_t > 0$ and $h_t > 0$ respectively. Instead of the traditional inventory balance formulation, we employ assignment variables x_{st} that indicate how much of period t 's demand is produced in period $s \leq t$; these variables have a composite unit cost of $c_{st} = c_s + \sum_{\tau=s}^{t-1} h_\tau$. This naturally leads to the formulation

$$\inf \sum_{s=1}^{\infty} \sum_{t=s}^{\infty} c_{st} x_{st} \quad (9a)$$

$$\text{s.t. } \sum_{s=1}^t x_{st} \geq b_t, \quad t = 1, \dots \quad (9b)$$

$$x_{st} \geq 0 \quad s = 1, \dots, t, \quad t = 1, \dots \quad (9c)$$

The model clearly satisfies Assumptions 1 and 2, and the dual is

$$\begin{aligned} \sup \quad & \sum_{t=1}^{\infty} b_t y_t \\ \text{s.t. } \quad & y_t \leq c_{st}, \quad s = 1, \dots, t, \quad t = 1, \dots \\ & y_t \geq 0, \quad t = 1, \dots \end{aligned}$$

Applying Theorem 2 and our definition of CS optimality, we can derive the following conclusions. Let x^* and y^* be CS optimal; since $b_t > 0$, some x_{st}^* must be positive, and therefore

$$y_t^* = \min_{s \leq t} c_{st} \quad t = 1, \dots \quad (10a)$$

$$x_{st}^* > 0 \Rightarrow y_t^* = c_{st} \quad s = 1, \dots, t \quad t = 1, \dots \quad (10b)$$

That is, production to meet demand in period t should only occur in the period(s) $s \leq t$ that affords the cheapest overall unit cost. Since $c_t > 0$, (10a) in turn implies $y_t^* > 0$ and hence

$$\sum_{s=1}^t x_{st}^* = b_t \quad t = 1, \dots; \quad (10c)$$

i.e. each period's demand is met exactly. Finally, a simple calculation shows

$$c_{st} = \min_{\sigma \leq t} c_{\sigma t} \Rightarrow c_{s\tau} = \min_{\sigma \leq \tau} c_{\sigma\tau}, \quad \tau = s, \dots, t, \quad (10d)$$

and therefore if it is optimal to produce in period s for period t , it is also optimal to produce in s for all intervening periods. This reasoning gives the following result, an extension of [16, Theorem 5.1] for the case when (9) does not necessarily have a finite optimal value.

Theorem 3. *For problem (9), we can choose a CS optimal production plan x^* with a production epoch or regeneration interval structure:*

$$x_{st}^* > 0 \Rightarrow x_{s\tau}^* = b_\tau, x_{\sigma\tau}^* = 0, \quad \sigma \in \{1, \dots, \tau\} \setminus \{s\}, \quad \tau = s, \dots, t. \quad (11)$$

5 Extreme Point Structure

The topological techniques used in Section 3 are strongly dependent on the finite row/column conditions of Assumption 1. While duality results have been proven in the context of infinite graphs [1, 3, 6, 8] without these assumptions, the proofs require more elaborate techniques that are usually connected to the discrete structure of these problems. This suggests that extending duality results in the absence of Assumption 1 might be possible for infinite LPs with discrete structures or solutions. In this section we study the existence of such discrete structures in infinite LPs as a first step towards this extension. Theorem 2 already shows that CS optimal solutions can inherit discrete structures of the optimal solutions of the truncations. For instance, if all truncations have integral optimal solutions ($F = G = \mathbb{N}$), then there exists an integral CS optimal solution. We show that a similar inheritance holds even in the absence of Assumption 1. More precisely, we give a generalization of a result in [10] to show that the extreme points of \mathcal{P}_\uparrow and \mathcal{P}_\downarrow inherit the properties of the following finite truncations.

Definition 4. *For $U \subseteq J$ and $V \subseteq I$, let*

$$\begin{aligned} i) \quad \widetilde{\mathcal{P}}_\uparrow(U) &:= \left\{ x \in \mathbb{R}^U : \sum_{j \in U} a_{ij} x_j \geq b_i \quad \forall i \in I, \quad x_j \geq 0 \quad \forall j \in U \right\}, \\ ii) \quad \widetilde{\mathcal{P}}_\downarrow(V) &:= \left\{ y \in \mathbb{R}^V : \sum_{i \in V} a_{ij} y_i \leq c_j \quad \forall j \in J, \quad y_i \geq 0 \quad \forall i \in V \right\}. \end{aligned}$$

5.1 Anti-Blocking Polyhedra

Theorem 4. *Let $F \subseteq \mathbb{R}$ be closed and such that $0 \in F$. If for any finite $V \subseteq I$ the extreme points of $\widetilde{\mathcal{P}}_\downarrow(V)$ are in F^V , then under Assumption 2 the extreme points of \mathcal{P}_\downarrow are in F^I .*

Proof. We follow and extend the proof of a similar result in [10] for stable set polyhedra of perfect graphs⁴. Equip \mathbb{R}^I with the product topology; this ensures it is a locally convex, Hausdorff space. Assume $(a_{ij})_{j \in J}$ contains at least one non-zero for each i , implying the boundedness (and compactness) of \mathcal{P}_\downarrow . The general case is a simple extension.

The conclusion follows from Theorem 1 with $K = \mathcal{P}_\downarrow$ and $A = F^I \cap \mathcal{P}_\downarrow$, a closed set. It is enough to show that the closure of the convex hull of A contains

⁴ An infinite graph is perfect if every one of its finite induced graphs is perfect.

\mathcal{P}_\downarrow , and we show this now. Let $\hat{y} \in \mathcal{P}_\downarrow$, and consider any open set containing \hat{y} . Let $V \subseteq I$ be the finite index set of coordinates for which this open set's component is not equal to \mathbb{R} . Define $\bar{y} \in \mathcal{P}_\downarrow$ as $\bar{y}_j = \hat{y}_j$ for $i \in V$ and 0 otherwise. Then $\bar{y}_V \in \mathcal{P}_\downarrow(V)$, and hence it is a convex combination of points in $F^V \cap \mathcal{P}_\downarrow(V)$. By extending with zeros in all other coordinates, we can take this combination also in \mathbb{R}^I ; thus \hat{y} is in the closure of the convex hull of points in A . \square

Theorem 4 has the following direct corollaries.

Corollary 1 ([10]). *The stable set polyhedron of a perfect graph has integer extreme points.*

Corollary 2. *The b -matching polyhedron of a bipartite graph has integer extreme points.*

Corollary 3 ([3, 8]). *The extreme points of the fractional matching polyhedron of a graph are half-integral.*

5.2 Blocking Polyhedra

Similarly to Theorem 4, we may consider solutions with values in certain closed sets containing 0. However, for simplicity we restrict our attention to integral solutions.

Theorem 5. *If for any finite $U \subseteq J$ the extreme points of $\widetilde{\mathcal{P}}_\uparrow(U)$ are integral, then, under Assumption 2, the extreme points of \mathcal{P}_\uparrow are integral.*

Proof. Let $d_j := \lceil \sup_{i \in I: a_{ij} > 0} \{ \frac{b_i}{a_{ij}} \} \rceil + 1$. The proof is similar to the proof of Theorem 4. Equip \mathbb{R}^J with the product topology, and consider the set $\mathcal{P}_\uparrow \cap \{x \in \mathbb{R}^J : x_j \leq d_j, \forall j \in J\}$. Using the same argument from the previous proof with $\bar{x}_j = d_j, \forall j \in J \setminus J'$, it follows that this set has integer extreme points, since the upper bound constraints cannot add fractional extreme points. Moreover, any $x \in \mathcal{P}_\uparrow$ having some $j \in J$ with $x_j > d_j$ cannot be extreme, so the extreme points of this truncated set contain the extreme points of \mathcal{P}_\uparrow . \square

Theorem 5 has the following direct corollary.

Corollary 4 ([3, 8]). *The extreme points of the fractional vertex cover polyhedron of a graph are half-integral.*

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