A structural geometrical analysis of weakly infeasible SDPs

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Abstract

In this article, we present a geometric theoretical analysis of semidefinite feasibility problems (SDFPs). We introduce the concept of hyper feasible partitions and sub-hyper feasible directions, and show how they can be used to decompose SDFPs into smaller ones, in a way that preserves most feasibility properties of the original problem. With this technique, we develop a detailed analysis of weakly infeasible SDFPs to understand clearly and systematically how weak infeasibility arises in semidefinite programming.

1 Introduction

The problem of deciding the feasibility status of a given SDP is a fundamental issue. It is well-known that semidefinite programs have four feasibility status, i.e., (i) strongly feasible, (ii) strongly infeasible, (iii) weakly feasible and (iv) weakly infeasible. In this paper, we introduce a novel canonical form of a semidefinite program suitable for analyzing its feasibily status, and study the structure of weakly infeasible semidefinite programs. Our approach is elementary and concrete.

Among the aforementioned four feasibility status, weak infeasibility is the most difficult one to deal with, since, while the other three have apparent certificates of finite length, weak infeasibility does not. In order to detect weak infeasibility, we must show that the distance between the affine space of linear constraint and positive semidefinite cone (PSD cone) is zero but their intersection is empty. This means that we must analyze the relation of the affine space and the PSD cone at infinity. To the best of our knowledge, there are few works about the structure of weakly infeasible semidefinite programs.

Through our analysis, we will demonstrate that weak infeasibility problems have a certain common structure. This structure essentially describes how weakly infeasibility arises in semidefinite programming in a fairly general manner. As far as we know, this is the first result where the structure of weakly infeasible semidefinite programs is analyzed systematically in detail. The results might also shed some light on the boundaries that separate the four different feasibility possibilities, specially weak feasibility and weak infeasibility.

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The main tool we use is a decomposition result (Theorem 12), which implies that some semidefinite feasibility problems (SDFPs) can be decomposed into smaller subproblems in a way that the feasibility properties are mostly preserved.

Before we explain in more detail, we introduce some notation. We denote by (K, L, c) the following conic problem:

$$\begin{aligned} &\min 0 \\ &\text{s.t. } x \in (L + c) \cap K, \end{aligned}$$

where $K \subseteq \mathbb{R}^n$ is a closed convex cone, $L \subseteq \mathbb{R}^n$ is a vector subspace and $c \in \mathbb{R}^n$. By $\mathbb{S}_n$ we denote the linear space of the $n \times n$ symmetric matrices over the real field and $K_n \subseteq \mathbb{S}_n$ denotes the cone of $n \times n$ positive semidefinite matrices. When we write (K_n, L, c) it is understood that L is a subspace of $\mathbb{S}_n$ and $c \in \mathbb{S}_n$.

With this notation, let us present a simple example of weakly infeasible problem.

Example 1 (The canonical example of weakly infeasible problem)**.** *Let*

$$L + c = \left\{ \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\}. \tag{1}$$

We have that (K_2, L, c) is infeasible due to the zero in the lower right corner. It is not complicated to check that as t goes to $+\infty$, the matrices in $L + c$ approach K_2. Thus (K_2, L, c) is weakly infeasible.

This example contains many of the characteristics that are found in general weakly infeasible SDFPs. For instance, there exists a non-zero element in $L \cap K_2$. Selecting an element in $L \cap K_2$, for example, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we may pick any element in $x \in L + c$ and by making α positive and large enough, $x + \alpha A$ approaches K_2. Moreover, if we just focus on the $(2, 2)$ entry of $L + c$, which is always 0, we obtain a (trivial) feasible SDFP with no interior solution. In order to further illustrate the situation from a geometrical point of view, let us consider an embedding of K_2 in $\mathbb{R}_3$.

$$\left\{ (x, y, z) \mid \begin{pmatrix} x & y \\ y & z \end{pmatrix} \succeq 0. \right\}$$

As shown in Fig.1, this set is a second-order cone in $\mathbb{R}^3$, whose center axis is $\{(u, 0, u) \mid u \geq 0\}$ and its intersection with a plane $\{(x, y, z) \mid x + z = a,\ a > 0\}$ perpendicular to the center axis is an ellipse whose major axis is the line connecting $(a, 0, 0)$, $(0, 0, a)$ and minor axis is the line connecting $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$, $(\frac{a}{2}, -\frac{a}{2}, \frac{a}{2})$. The cone touches x-axis and z-axis on its border. Note that xy-plane is a tangent space of the second-order cone. In this setting, the affine set (1) is represented by a half line parallel to x-axis, i.e., $\{(x, y, z) = (t, 1, 0) \mid t \geq 0\}$. What happens as t goes to infinity is that the point $(t, 1, 0)$ gets arbitrary close to the boundary of the cone as the curvature of the surface of the cone becomes smaller, but they never touch.

Given this example, the following natural and simple question arises:

Does this example captures the substance of all what happens in general weakly infeasible semidefinite programs or not?

This is our main question in this paper. The answer is as follows. Example 1 captures only a part of the general case but not all aspects. Example 1 is an example of a *directionally weakly infeasible* problem,

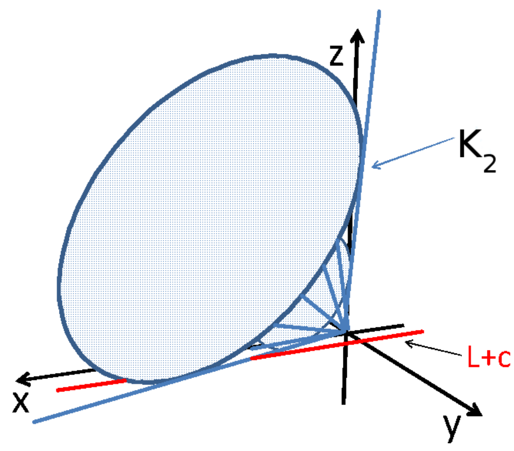

Figure 1: A prototype weakly infeasible problem

because we may approach the cone by walking along a single direction. Unfortunately, not every weakly infeasible semidefinite program is directionally weakly feasible, but the idea of directional weak infeasibility provides a good intuitive explanation about how weak infeasibility arises in semidefinite programming.

What is missing to complete the whole picture is a hierarchical structure. We will demonstrate that when dealing with a weakly infeasible SDFP (K_n, L, c) of $n \times n$ matrices, at most $(n-1)$ directions are sufficient to approach K_n (see Section 4.3). In particular, there exists a nested hierarchical structure among these directions (see Section 4.2), where each direction serves as a weak infeasibility direction to a certain subproblem. Note that this result is surprisingly strong, because, in general, if K is a closed convex cone and (K, L, c) is weakly infeasible, then the number of the directions necessary to approach the cone could be as large as the dimension of L, which is up to $(\frac{n(n+1)}{2} - 1)$ in our context.

As we mentioned before, we introduce a decomposition to tackle this problem. To this end, we define a class of conic problems called "hyper feasible". More precisely, a problem is hyper-feasible if $L \cap \mathrm{int}(K)$ is non-empty. This is equivalent to the existence of an interior recession direction. In the case of SDFPs, if (K_n, L, c) is weakly infeasible then, after performing some congruence transformation, it is possible to decompose the matrices in $L + c$ into the so called "hyper feasible part" and the "weakly feasible part". This hyper feasible part is controlled by what we call a hyper feasible partition (see Definition 15). Both the weakly feasible and the hyper feasible parts correspond to subproblems obtained by considering certain principal submatrices of the matrices in $L + c$.

Weakly feasible/infeasible problems are numerically very difficult to solve [4, 9]. A possible way to deal with such problems is to solve a strongly feasible problems obtained by adding slight perturbation. Our results could be used to find a reasonable perturbation which does not destroy the structure of the optimal solution drastically, taking in account the structure of the problem.

We review related previous works. The existence of weak infeasibility/feasibility and finite duality gap is one of the main difficulties in semidefinite programming. These situations may occur in the lack of interior-feasible solutions to the primal and/or dual. Two possible techniques to recover interior-feasibility by reducing the feasible region of the problem (without interior) or by expanding the feasible region of its dual counter-part are the facial reduction algorithm (FRA) and the conic expansion approach (CEA), respectively. FRA was developed by Borwein and Wolkowicz [1] for problems more general than conic programming, whereas CEA was developed by Luo, Sturm and Zhang [11] for conic programming. In the earlier stages of research of semidefinite programming, Ramana [12] developed an extended Lagrange-Slater dual (ELSD) to resolve the problem of finite duality gap. ELSD has a remarkable feature that the size of the extended problem is bounded by polynomial in terms of the size of the original problem. Complexity of SDFP is yet a subtle issue. This topic was studied extensively by Porkolab and Khachiyan [10]. In [13], Ramana, Tunçel and Wolkowicz demonstrated that ELSD can be interpreted as a facial reduction problem, however, we should note that in the original FRA, the size of the problem is not polynomially bounded. In [8], Pólik and Terlaky provided strong duals for conic programming over symmetric cones.

Waki and Muramatsu [16] considered a FRA for conic programming and showed that FRA can be regarded as a dual version of CEA. See an excellent review by Pataki [7] for FRA, where he points out the relation between facial reduction and extended duals. In particular, he indicates the necessary steps to derive Ramana's ELSD from FRA (see [7, Section 5]). Pataki also found recently that all ill-conditioned semidefinite programs contains a common 2×2 semidefinite programming structure [6]. When applied to an infeasible problem, FRA and CEA detect infeasibility, but no information is given about whether it is weakly infeasible or strongly infeasible. Recently, Klep and Schweighofer developed a polynomially bounded certificate of feasibility and infeasibility based on real algebraic geometry without any constraint qualification [3]. They succeeded in obtaining a certificate for weakly infeasibility as well, but geometric intuition behind it is not clear yet. Finally, we mention that Waki showed weakly infeasible instances can be obtained from semidefinite relaxation of polynomial optimization problems [15].

This paper is organized as follows. In Section 2, we prove a few results that will be used in later sections. In Section 3, we prove Theorems 11 and 12 that concern the inheritance of feasibility between problems and subproblems. In Section 4, we introduce *hyper feasible partitions* and *sub-hyper feasible directions* in order to analyze weakly infeasible problems. We also define directionally weakly infeasible programs and show that not all weakly infeasible SDFP belong to that category. Moving on to Section 5, we present a procedure that aims to distinguish between weakly feasible and weakly infeasible SDFPs. Finally, in Section 6 we summarize and conclude this work.

We use the following notation. The linear space of $m \times n$ matrices is denoted by $M_{m\times n}$. For a matrix $A \in \mathbb{S}_n$, we denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimum and maximum eigenvalue of A, respectively. If $\mathcal{C}, \mathcal{D}$ are subsets of some real space, we write $\mathrm{dist}(\mathcal{C}, \mathcal{D}) = \inf\{\|x - y\| \mid x \in \mathcal{C}, y \in \mathcal{D}\}$, where $\|\cdot\|$ is the Euclidean norm or the Frobenius norm, in the case of subsets of $\mathbb{S}_n$. By $\mathrm{int}(\mathcal{C})$ and $\mathrm{ri}(\mathcal{C})$ we denote the interior and the relative interior of $\mathcal{C}$, respectively. We use I_n to denote the $n \times n$ identity matrix and $0_{m\times n}$ to represent the $m \times n$ matrix with all entries equal to 0.

2 Feasibility in Semidefinite Programming

In this section, we develop our main theoretical tools, namely Theorems 11 and 12. These results will be extensively used in Sections 3 and 4. First, in Subsection 2.1, we make a few remarks concerning feasibility in general conic linear programming. The most important result in that section is that if (K, L, c) is weakly infeasible, there is always a non-zero vector in $K \cap L$. In Subsection 2.2, we turn our focus to SDFPs and develop the concepts needed to state and prove Theorems 11 and 12.

2.1 Feasibility in General Conic Linear Programs

First, let us briefly discuss general conic linear programs (K, L, c), where K is an arbitrary closed convex full-dimensional cone in a finite dimensional space. With that in mind, we have the following classical definitions.

Definition 2. *We say that* (K, L, c) *is:*

- Strongly feasible, *if* $(L + c) \cap \operatorname{int}(K) \neq \emptyset$.
- Weakly feasible, *if* $(L + c) \cap \operatorname{int}(K) = \emptyset$, *but* $(L + c) \cap K \neq \emptyset$.
- Weakly infeasible*: if* $(L + c) \cap K = \emptyset$ *and* $\operatorname{dist}(K, L + c) = 0$.
- Strongly infeasible*: if* $(L + c) \cap K = \emptyset$ *and* $\operatorname{dist}(K, L + c) > 0$.

In this work, we felt the need to consider two additional definitions. We say that (K, L, c) is *hyper feasible* if $L \cap \operatorname{int}(K) \neq \emptyset$. It is clear that if (K, L, c) is hyper feasible, then it must be strongly feasible. In fact, if $x \in L \cap \operatorname{int}(K)$, there exists $\lambda > 0$ such that $x + \lambda c \in \operatorname{int}(K)$, so that $\frac{1}{\lambda}x + c \in (L + c) \cap \operatorname{int}(K)$. When (K, L, c) is either weakly feasible or weakly infeasible, we say that the problem has *weak status*.

We now prove some properties of weakly infeasible problems. Suppose (K, L, c) is weakly infeasible. The fact that $\operatorname{dist}(K, L + c) = 0$, implies the existence of sequences $\{x_i\} \subset L + c$ and $\{y_i\} \subset K$, such that $\lim_{i\to\infty} \|x_i - y_i\| = 0$. In this context, we call $\{x_i\}$ and $\{y_i\}$ *weakly infeasible sequences*. What is special about weakly infeasible sequences is that they cannot have any convergent subsequences, because the limit would be a feasible solution, which is impossible. Therefore, we have the following propositions.

Proposition 3. *If* (K, L, c) *is weakly infeasible and* $\{y_i\} \subseteq K$ *and* $\{x_i\} \subseteq L + c$ *are sequences such that* $\|x_i - y_i\| \to 0$*, then both* $\{x_i\}$ *and* $\{y_i\}$ *do not have convergent subsequences. Therefore,* $\|x_i\| \to \infty$ *and* $\|y_i\| \to \infty$.

Proposition 4. *If* (K, L, c) *is weakly infeasible, there are non-zero vectors in* $L \cap K$.

Proof. Since (K, L, c) is weakly infeasible, there is a sequence $\{x_i\} \subseteq L + c$, such that

$$\lim_{i\to\infty} \operatorname{dist}(x_i, K) = 0$$
$$\lim_{i\to\infty} \|x_i\| = \infty. \tag{2}$$

It is no loss of generality to assume that the elements of the sequence are non-zero, so we assume so. Now, $\left\{\frac{x_i}{\|x_i\|}\right\}$ has a convergent subsequence which we also denote by $\left\{\frac{x_i}{\|x_i\|}\right\}$ and let $\tilde{x}$ be its limit. If we write $x_i = l_i + c$, with $l_i \in L$, then it becomes clear that since Equation 2 holds, $\tilde{x} \in L$. Moreover, $\operatorname{dist}\left(\frac{x_i}{\|x_i\|}, K\right) = \frac{1}{\|x_i\|}\operatorname{dist}(x_i, K)$. Hence, $\tilde{x} \in K$. □

2.2 Preliminaries

In this subsection, we review known results for later development. We also introduce some notations.

Proposition 5. *Let* $x, y \in \mathbb{S}_n$*, then:*

i. as functions from $\mathbb{S}_n$ *to* $\mathbb{R}$*,* $\lambda_{\min}$ *is a concave function and* $\lambda_{\max}$ *is a convex function.*

ii. $\lambda_{\min}(x+y) \geq \lambda_{\min}(x) + \lambda_{\min}(y)$.

iii. $\lambda_{\max}(x+y) \leq \lambda_{\max}(x) + \lambda_{\max}(y)$.

iv. $\lambda_{\min}(x) + \lambda_{\max}(y) \geq \lambda_{\min}(x+y)$.

v.

$$\mathrm{dist}(x, K_n) = \left[\sum_{\lambda} \lambda^2\right]^{\frac{1}{2}},$$

where the summation runs through all negative eigenvalues of x, including algebraic multiplicities. If there are no negative eigenvalues, then $\mathrm{dist}(x, K_n) = 0$.

Notice that item *v.* implies that in order for a sequence $\{x_i\} \subseteq \mathbb{S}_n$ to satisfy $\mathrm{dist}(x_i, K_n) \to 0$, it is sufficient that $\lambda_{\min}(x_i) \to 0$. Moreover, if the elements of the sequence are not in K_n, then this is also a necessary condition.

Theorem 6 (Ostrowski [5])**.** *Let A be a Hermitian matrix and S a non-singular matrix over the complex field, both with the same dimensions. Then there exists n positive scalars θ_i such that each θ_i satisfy $\lambda_{\min}(SS^*) \leq \theta_i \leq \lambda_{\max}(SS^*)$ and $\lambda_i(SAS^*) = \theta_i \lambda_i(A)$, where λ_i is the i-th smallest eigenvalue and S^* is the Hermitian transpose of S.*

Proof. See [2][Theorem 4.5.9] for a proof of this result. ☐

Proposition 7 (Schur Complement)**.** *Suppose $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is a square matrix divided in blocks in a way that A is positive definite, then:*

- *M is positive definite if and only if $C - B^T A^{-1} B$ is.*
- *M is positive semidefinite if and only if $C - B^T A^{-1} B$ is.*

The next proposition is easily proved with the aid of previous propositions.

Proposition 8. *Let (K, L, c) be a semidefinite program and let $\lambda^* = \sup_{x \in L+c} \lambda_{\min}(x)$. If:*

- *$\lambda^* < 0$, then (K, L, c) is strongly infeasible.*
- *$\lambda^* = 0$, but the supremum is not attained in $L + c$, then (K, L, c) is weakly infeasible.*
- *$\lambda^* = 0$, but the supremum is attained in $L + c$, then (K, L, c) is weakly feasible.*
- *$0 < \lambda^* \leq +\infty$ then (K, L, c) is strongly feasible.*
- *$\lambda^* = +\infty$ then (K, L, c) is hyper feasible.*

Proposition 8 implies that the value of λ^* is enough to distinguish the different feasibility possibilities of (K, L, c), except when $\lambda^* = 0$. In this case, (K, L, c) has weak status and can be either weakly infeasible or weakly feasible.

The properties of a semidefinite problem are not changed when we perform a congruence, in particular, for any non-singular matrix P, we have that (K_n, L, c) and (K_n, PLP^T, PcP^T) have the same feasibility properties, where $PLP^T = \{PlP^T \mid l \in L\}$. This is useful, because we may change a problem in a way to give emphasis to a certain block structure. For convenience, let us state this fact as a proposition.

Proposition 9. *If P is non-singular then:*

i. (K_n, L, c) *is hyper feasible if and only* (K_n, PLP^T, PcP^T) *is.*

ii. (K_n, L, c) *is strongly feasible if and only* (K_n, PLP^T, PcP^T) *is.*

iii. (K_n, L, c) *is weakly feasible if and only* (K_n, PLP^T, PcP^T) *is.*

iv. (K_n, L, c) *is weakly infeasible if and only* (K_n, PLP^T, PcP^T) *is.*

v. (K_n, L, c) *is strongly infeasible if and only* (K_n, PLP^T, PcP^T) *is.*

Proof. These facts all follow from the nature of K_n, i.e., $A \in K_n$ if and only if $PAP^T \in K_n$. We just remark that if we have a sequence $\{l_i + c\} \subset L + c$ that approaches K_n, then $\{P(l_i + c)P^T\}$ must also approach K_n. The reason is that Theorem 6 states that the eigenvalue of a matrix changed by a congruence differs by a positive constant from the corresponding eigenvalue of the original matrix and this constant is bounded by the eigenvalues of a certain positive definite matrix. Thus it becomes clear that if $\lambda_{\min}(l_k + c) \to 0$, then $\lambda_{\min}(P(l_k + c)P^T) \to 0$. □

We now introduce a notation that will help us define precisely what is meant by "subproblems". For a $n \times n$ matrix x, we define $U_k(x)$ to be the $k \times k$ submatrix of x consisting of the first k columns and the first k rows. We also define $L_k(x)$ to be the $(n-k) \times (n-k)$ submatrix of x consisting of the last $n-k$ rows and the last $n-k$ columns of x. Finally, we define $R_k(x)$ to be the $k \times (n-k)$ submatrix of x consisting of the first k-rows and the last $n-k$ columns. It is clear that all these functions are linear mappings. The following relation holds:

$$x = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} = \begin{pmatrix} U_k(x) & R_k(x) \\ R_k(x)^T & L_k(x) \end{pmatrix},$$
$$\underbrace{\phantom{X_{11}}}_{k} \quad \underbrace{\phantom{X_{22}}}_{n-k} \qquad \underbrace{}_{k} \quad \underbrace{}_{n-k}$$

where X_{11} denotes a $k \times k$ block, X_{22} a $(n-k) \times (n-k)$ block and X_{12} a $n \times (n-k)$ block. In addition, we adopt the convention that L_0 and U_n are both equal to the identity operator on $\mathbb{S}_n$. Notice that if $U_k(x)$ is positive definite, we may calculate the Schur complement of x. More precisely, we say that $L_k(x) - R_k(x)^T U_k(x)^{-1} R_k(x)$ is the *Schur complement of x with respect to k*.

It is clear that both $U_k(L + c)$ and $L_k(L + c)$ are affine spaces on their own rights, as a result of being images of affine spaces by linear maps. Also, $U_k(K_n)$ is precisely the cone of $k \times k$ positive semidefinite matrices and $L_k(K_n)$ is the cone of $(n-k) \times (n-k)$ positive semidefinite matrices. Therefore, $(U_k(K_n), U_k(L), U_k(c))$ and $(L_k(K_n), L_k(L), L_k(c))$ are well-defined semidefinite feasibility problems. The feasibility problem $(L_k(K_n), L_k(L), L_k(c))$ is interpreted as trying to find a positive semidefinite matrix that is a lower right principal submatrix of a matrix in $L+c$. That is, we only focus on the smaller $(n-k)\times(n-k)$ block disregarding the other parts of the matrix. The interpretation for $(U_k(K_n), U_k(L), U_k(c))$ is analogous. We say that $(L_k(K_n), L_k(L), L_k(c))$ is the *lower $(n-k)$-subproblem* of (K_n, L, c) and $(U_k(K_n), U_k(L), U_k(c))$ is the *upper k-subproblem* of (K_n, L, c). For the sake of notational convenience, we define

$$L_k(K_n, L, c) = (L_k(K_n), L_k(L), L_k(c))$$
$$U_k(K_n, L, c) = (U_k(K_n), U_k(L), U_k(c)).$$

Let us give an example of this notation.

Example 10. *Let*

$$L + c = \left\{ \begin{pmatrix} t & 1 & s \\ 1 & t & s+1 \\ s & s+1 & 0 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}.$$

We have that $U_1(K_3, L, c)$ is an SDFP of 1×1 matrices and

$$U_1(L + c) = \{t \mid t \in \mathbb{R}\}.$$

In addition, $L_1(K_3)$ is the cone of 2×2 positive semidefinite matrices and

$$L_1(L + c) = \left\{ \begin{pmatrix} t & s+1 \\ s+1 & 0 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}.$$

Thus $L_1(K_3, L, c)$ is an SDFP of 2×2 matrices. We also have that

$$R_1(L + c) = \left\{ \begin{pmatrix} 1 & s \end{pmatrix} \mid s \in \mathbb{R} \right\}.$$

Notice that we understand L_1, R_1 and U_1 as polymorphic functions depending on n. So it makes sense to consider iterates such as $L_1(L_1)$ and $U_1(L_1)$. For instance:

$$\begin{aligned} L_1(L_1(L + c)) &= \{0\} \\ R_1(L_1(L + c)) &= \{s + 1 \mid s \in \mathbb{R}\} \\ U_1(L_1(L + c)) &= \{t \mid t \in \mathbb{R}\}. \end{aligned}$$

3 A Decomposition Result

Now we are ready to derive our main results. It is reasonable to expect that subproblems of (K_n, L, c) give *some* information about the full problem. The next theorem suggests the proper way of finding a subproblem that preserves most of the feasibility properties of the original problem. The key idea is to perform an appropriate transformation, isolate a direction in $K_n \cap L$ and decompose (K_n, L, c) accordingly. By doing so, arguments using the Schur complement ensure that the feasibility properties are inherited. Repeating this process, we may reduce systematically the dimension of a given SDFP.

Theorem 11. *Let (K_n, L, c) be an SDFP and suppose that $L \cap K_n$ admits an element of the form*

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is a $k \times k$ positive definite matrix with $k < n$. Then the following relations hold between (K_n, L, c) and its $(n-k)$-lower subproblem $L_k(K_n, L, c)$:

i. (K_n, L, c) is hyper feasible if and only if $L_k(K_n, L, c)$ is,

ii. (K_n, L, c) is strongly feasible if and only if $L_k(K_n, L, c)$ is,

iii. (K_n, L, c) has weak status if and only if $L_k(K_n, L, c)$ has,

iv. (K_n, L, c) is strongly infeasible if and only if $L_k(K_n, L, c)$ is.

Proof. Let

$$\lambda^* = \sup_{x \in L+c} \lambda_{\min}(x)$$
$$\lambda_k^* = \sup_{x \in L_k(L+c)} \lambda_{\min}(x).$$

By using Proposition 8, we can prove items *i.* to *iv.* by checking the values of λ^* and λ_k^*. Notice that the relation

$$\lambda^* \leq \lambda_k^*, \tag{3}$$

always holds, because the matrices in $L_k(L+c)$ are principal submatrices of the matrices in $L+c$. Also, it is only necessary to prove the first three items. If this is done, then *iv.* must necessarily follow.

($i. \Rightarrow$),($ii. \Rightarrow$) These two implications follow easily from (3).

($i. \Leftarrow$),($ii. \Leftarrow$) Let $x \in L+c$ be such that $L_k(x) \succ 0$. If α is sufficiently large then $U_k(x+\alpha A) \succ 0$. The Schur complement of $x+\alpha A$ with respect to k is

$$L_k(x) - R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x).$$

Since $L_k(x) \succ 0$, if we make α large, the Schur complement will be positive definite and $x+\alpha A$ will be a positive definite matrix in $L+c$. This proves that $\lambda_k^* > 0$ implies $\lambda^* > 0$. Now, if there exists $y \in L$ such that $L_k(y) \succ 0$, the same argument shows that $y+\alpha A \succ 0$, for large α. Therefore, $\lambda_k^* = +\infty$ implies $\lambda^* = +\infty$.

($iii. \Rightarrow$) If $\lambda^* = 0$, then the relation given by (3), implies that $\lambda_k^* \geq 0$. Hence $\lambda_k^* = 0$, otherwise we would have that $\lambda^* > 0$ and this would contradict items (i) and (ii).

($iii. \Leftarrow$) Suppose that $\lambda_k^* = 0$, we need to show that $\lambda^* = 0$ also. From (3), we already have that $\lambda^* \leq 0$, we will now show that $\lambda^* = 0$ by showing that there are elements in $L+c$ such that the minimum eigenvalue function assume values arbitrarily close to 0.

The fact that $\lambda_k^* = 0$, implies that for every $\epsilon > 0$ there exists $x \in L+c$ such that $-\epsilon < \lambda_{\min}(L_k(x)) \leq 0$. For large α, we have that $U_k(x+\alpha A)$ is positive definite and the Schur Complement with respect to k exists. We have the following relation:

$$V(x+\alpha A)V^T = \begin{pmatrix} U_k(x+\alpha A) & 0 \\ 0 & L_k(x) - R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x) \end{pmatrix},$$

where $V = \begin{pmatrix} I_k & 0 \\ -R_k(x)^T U_k(x+\alpha A)^{-1} & I_{n-k} \end{pmatrix}$. By Theorem 6, we have that $\lambda_{\min}(V(x+\alpha A)V^T) = \theta_1 \lambda_{\min}(x+\alpha A)$, where θ_1 is bounded by the eigenvalues of

$$VV^T = \begin{pmatrix} I_k & (-U_k(x+\alpha A))^{-1} R_k(x) \\ -R_k(x)^T (U_k(x+\alpha A))^{-1} & I_{n-k} + R_k(x)^T (U_k(x+\alpha A))^{-1} (U_k(x+\alpha A))^{-1} R_k(x) \end{pmatrix}.$$

However, since $\lambda_{\min}(x+\alpha A) \leq 0$ and $\lambda_{\min}(V(x+\alpha A)V^T) \leq 0$, we have that the minimum eigenvalue of $V(x+\alpha A)V^T$ is precisely the minimum eigenvalue of the Schur Complement, i.e,

$$\lambda_{\min}(L_k(x) - R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x)) = \theta_1 \lambda_{\min}(x+\alpha A).$$

After all, $V(x+\alpha A)V^T$ is divided in two blocks. Since the first block is positive definite, all the non-positive eigenvalues must be concentrated in the Schur Complement.

Now the proof goes as follows: as α increases, VV^T goes to the identity matrix and θ goes to 1. Moreover, the Schur complement goes to $L_k(x)$, so for $\tilde{\epsilon} > 0$, it should be possible to take α large enough in order to have $\lambda_{\min}(x+\alpha A) > -\epsilon - \tilde{\epsilon}$. We now describe in detail this argument.

Notice that $U_k(x+\alpha\tilde{A})^{-1}$ is just $[U_k(x)+\alpha U_k(\tilde{A})]^{-1}$. Since $U_k(\tilde{A}) \succ 0$, we have that $[U_k(x)+\alpha U_k(\tilde{A})]^{-1}$ goes to 0 as α increases. Hence, VV^T converges to I, as α goes to infinity.

We have that $\lambda_{\min}$ and $\lambda_{\max}$ are continuous functions, thus $\lambda_{\min}(VV^T)$ and $\lambda_{\max}(VV^T)$ are continuous functions of α. Thus for every $\tilde{\epsilon} > 0$, there exists M such that $\alpha > M$, implies $\theta_1 \in [1-\tilde{\epsilon}, 1+\tilde{\epsilon}]$. So for $\alpha > M$ we have

$$\begin{aligned}
\lambda_{\min}(x+\alpha A) &= \frac{1}{\theta_1}\lambda_{\min}(L_k(x) - R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x)) \\
&\geq \frac{1}{1+\tilde{\epsilon}}\left[\lambda_{\min}(L_k(x)) - \lambda_{\max}(R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x))\right] \\
&\geq \frac{1}{1+\tilde{\epsilon}}\left[-\epsilon - \lambda_{\max}(R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x))\right].
\end{aligned}$$

We have to control the term $R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x)$ to ensure that its maximum eigenvalue does not get too positive. Since it goes to 0 as α increases, for every $\epsilon' > 0$, there exists M' such that $\alpha > M'$ implies that

$$-\lambda_{\max}(R_k(x)^T U_k(x+\alpha A)^{-1} R_k(x)) \geq -\epsilon'.$$

Assuming that $\alpha > \max\{M, M'\}$, we may plug this bound in the previous inequality:

$$\lambda_{\min}(x+\alpha A) \geq \frac{-1}{1+\tilde{\epsilon}}(\epsilon + \epsilon').$$

Therefore, it is possible to make $\lambda_{\min}(x+\alpha A) > -\hat{\epsilon}$ for any $\hat{\epsilon} > 0$. □

Theorem 12. *Let (K_n, L, c) be a SDP and suppose that there is a matrix $\tilde{A}$ with rank k, such that $\tilde{A} \neq 0$, and $\tilde{A} \in K_n \cap L$. Let P be any non-singular matrix such that $U_k(P\tilde{A}P^T) > 0$ and all the other entries of $A = P\tilde{A}P^T$ are zero. Then $U_k(K, PLP^T, PcP^T)$ is hyper feasible. In addition, if $k < n$, we have that:*

i. *(K_n, L, c) is hyper feasible if and only if $L_k(K_n, PLP^T, PcP^T)$ is,*

ii. *(K_n, L, c) is strongly feasible if and only if $L_k(K_n, PLP^T, PcP^T)$ is,*

iii. *(K_n, L, c) has weak status if and only if $L_k(K_n, PLP^T, PcP^T)$ has,*

iv. *(K_n, L, c) is strongly infeasible if and only if $L_k(K_n, PLP^T, PcP^T)$ is.*

Proof. First it is clear that $U_k(K_n, PLP^T, PcP^T)$ is hyper feasible, since $A \in PLP^T \cap K_n$. Now suppose that $k < n$. The theorem follows from Proposition 9 and Theorem 11 applied to $L_k(K_n, PLP^T, PcP^T)$ and $P\tilde{A}P^T$.

□

Theorem 12 implies that when $K_n \cap L$ is non-trivial and $k < n$, we have a subproblem that has almost the same feasibility properties. This subproblem itself might satisfy the conditions of Theorem 12, so it might be possible to apply Theorem 12 several times. This idea will be explored in the next section.

4 Hyper Feasible Partitions and Sub-hyper Feasible Directions

In this section we discuss the concepts of *hyper feasible partition* and *sub-hyper feasible directions* and prove many of its properties. This is mostly done in Subsection 4.1. Then, in Subsection 4.2, we try to convey some of the geometrical intuition behind that concept and we explore the hierarchical structure that weakly infeasible SDFPs have. In particular, we show how weakly infeasible sequences can be constructed from sub-hyper feasible directions associated to hyper feasible partitions and vice-versa. Finally, in Subsection 4.3, we define directional weak infeasibility and show that not all weakly infeasible SDFPs have this property.

4.1 Definition and properties

If $(\mathcal{K}_n, \mathcal{L}, c)$ is feasible, we have that $\mathcal{L} \cap \mathcal{K}_n$ is the recession cone of the feasible region. Thus if there are non-zero vectors in $\mathcal{L} \cap \mathcal{K}_n$, the feasible region must be unbounded. This condition, however, plays an important role even when the problem is infeasible. For instance, if the problem is not hyper feasible but $\mathcal{L} \cap \mathcal{K}_n$ is non-trivial, by applying Theorem 12 we may determine many properties of $(\mathcal{K}_n, \mathcal{L}, c)$ by analyzing a smaller subproblem $\mathcal{L}_k(\mathcal{K}_n, P\mathcal{L}P^T, PcP^T)$.

In case it happens that $\mathcal{L}_k(P\mathcal{L}P^T) \cap \mathcal{L}_k(\mathcal{K}_n) \neq \{0\}$, we may apply Theorem 12 again. Unless the subproblem is hyper feasible, every time we apply Theorem 12 we succeed in reducing the dimension of the problem.

It is interesting to notice what would happen if we apply this idea to a weakly infeasible problem. Clearly, each subproblem obtained must have weak status, so it is either weakly feasible or weakly infeasible. Also, each time we apply Theorem 12, we are reducing the dimension of the problem by at least 1. This, together with the fact that there are no 1×1 weakly infeasible SDFPs, implies that the last subproblem must be *weakly feasible*. Before we progress further and develop this idea in more detail, we need the following definitions.

Definition 13 (Hyper feasible direction)**.** *Let $(\mathcal{K}_n, \mathcal{L}, c)$ be an SDFP. Let $A \in \mathbb{S}_n$ be non-zero. We say that A is a* hyper feasible direction of size k for $(\mathcal{K}_n, \mathcal{L}, c)$ *if $A \in \mathcal{K}_n \cap \mathcal{L}$, $\mathcal{U}_k(A) \succ 0$ and all the other entries of A are 0. In other words, we have*

$$A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{A}$ is a $k \times k$ positive definite matrix.

Definition 14 (Sub-hyper feasible direction)**.** *Let $(\mathcal{K}_n, \mathcal{L}, c)$ be an SDFP. Let $A \in \mathcal{L}$ be non-zero. We say that A is a* sub-hyper feasible direction of size k associated with the subproblem $\mathcal{L}_s(\mathcal{K}_n, \mathcal{L}, c)$ *if $\mathcal{L}_s(A)$ is a hyper feasible direction of size k for $\mathcal{L}_s(\mathcal{K}_n, \mathcal{L}, c)$. In other words, we have*

$$\left(\begin{array}{c|cc} * & * & \\ \hline \multirow{2}{*}{$*$} & \tilde{A} & 0 \\ & 0 & 0 \end{array}\right),$$

where A is divided into blocks with $\mathcal{L}_s(A) = \left(\begin{smallmatrix} \tilde{A} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\tilde{A}$ is $k \times k$ positive definite matrix and $*$ represents arbitrary entries.

Due to our convention that $\mathcal{L}_0 = \mathcal{U}_n$, we have that a hyper feasible direction is also a sub-hyper feasible direction.

$$tA_1 + sA_2 = t \left(\begin{array}{c|cc} \tilde{A}_1 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 0 \\ & 0 & 0 \end{array} \right) + s \left(\begin{array}{c|cc} * & \multicolumn{2}{c}{*} \\ \hline * & \tilde{A}_2 & 0 \\ & 0 & 0 \end{array} \right)$$

Figure 2: A depiction of a linear combination of sub-hyper feasible directions, where $*$ denotes arbitrary non-specified entries. We have $\tilde{A}_1 \succ 0$ and $\tilde{A}_2 \succ 0$. Although we have that $A_1 \succeq 0$, we do not necessarily have that A_2 is positive semidefinite.

Definition 15 (Hyper feasible partition)**.** *Let (K_n, L, c) be an SDFP and suppose that there exist m positive integers with $0 = k_0 < k_1 < k_2 < \ldots < k_m \leq n$ satisfying the following condition:*

for each subproblem $L_{k_{j-1}}(K_n, L, c)$ $j = 1, \ldots, m$, there exists a sub-hyper feasible direction A_j of size $k_j - k_{j-1}$,

then $\{k_i \mid 1 \leq i \leq m\}$ is called a hyper feasible partition *of (K_n, L, c). A hyper feasible partition is* maximal *if $k_m = n$ or $L_{k_m}(K_n) \cap L_{k_m}(L) = \{0\}$.*

The sub-hyper feasible directions appearing in Definition 15 are said to be *associated* to the hyper feasible partition. See Figure 2 for a depiction of directions associated to a hyper feasible partition. Given a hyper feasible partition for (K_n, L, c), as a convention, we say that the *bottom subproblem of* (K_n, L, c) is the lower $(n - k_m)$ subproblem of (K_n, L, c), in other words, the SDFP $L_{k_m}(K, L, c)$.

Example 16. *Let*

$$L + c = \left\{ \begin{pmatrix} t & 1 & v & u \\ 1 & u-1 & v+1 & s \\ v & v+1 & z+2 & z+1 \\ u & s & z+1 & 0 \end{pmatrix} \mid t, u, s, v, z \in \mathbb{R} \right\}.$$

Then (K_4, L, c) is infeasible due to the presence of 0 in the lower corner forcing $u = s = 0$ and making -1 appear in the diagonal. A hyper feasible partition is given by taking $k_1 = 1$, $k_2 = 2$ and $m = 2$. A possible choice of sub-hyper feasible directions is given by

$$A_1 = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \qquad A_2 = \left(\begin{array}{c|c|cc} 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right).$$

A important observation is that even if $K_n \cap L \neq \{0\}$, it does not follow that (K_n, L, c) has a hyper feasible partition. However, in that case, we may "rotate" the problem by a non-singular matrix P thus ensuring that (K_n, PLP^T, PcP^T) has a hyper feasible partition. Let us register this fact as a proposition.

Proposition 17. *If (K_n, L, c) is an SDFP with $K_n \cap L \neq \{0\}$, then there exists an orthogonal matrix P such that (K_n, PLP^T, PcP^T) has a maximal hyper feasible partition.*

Proof. The proof is constructive and we provide an algorithm that constructs the partition and gives the corresponding P. See Algorithm 1.

At Line 12, we have that $\tilde{P}\tilde{A}\tilde{P}^T \in \tilde{P}L_{k_{i-1}}(P_{i-1}LP_{i-1}^T)\tilde{P}^T$. However, by the choice of P_i at Line 11, the following relation holds:

$$\tilde{P}L_{k_{i-1}}(P_{i-1}LP_{i-1}^T)\tilde{P}^T = L_{k_{i-1}}(P_iLP_i^T).$$

Therefore, it is indeed possible to find $A_i \in P_i L P_i^T$ such that

$$L_{k_{i-1}}(A_i) = \tilde{P}\tilde{A}\tilde{P}^T.$$

At Line 13, we update the previous directions to ensure that $\{A_1, A_2, \dots, A_{i-1}\} \subseteq \{P_i L P_i^T\}$. Notice that these directions remain sub-hyper feasible directions.

It is also clear that the Algorithm 1 must halt in a finite number of steps, for if the condition in Line 6 is satisfied, we for sure reduce the dimension of the problem by at least one. The fact that we exit the loop when the condition in Line 6 fails ensures that the hyper feasible partition obtained is indeed maximal.

□

Algorithm 1: Constructing a maximal hyper feasible partition

Input : (K_n, L, c) with $K_n \cap L \neq \{0\}$

Output: A maximal hyper feasible partition, a set of sub-hyper feasible directions and an orthogonal matrix P

1 $i \leftarrow 1$
2 $\tilde{A} \leftarrow$ a non-zero matrix in $K_n \cap L$
3 $k_1 \leftarrow \operatorname{rank}(\tilde{A})$
4 Feed P_1, $\tilde{A}$ and (K_n, L, c) to Theorem 12, with P_1 satisfying the hypothesis of Theorem 12.
5 $A_1 \leftarrow P_1 \tilde{A} P_1^T$
6 **while** $k_i < n$ *and* $L_{k_i}(K_n) \cap L_{k_i}(P_i L P_i^T) \neq \{0\}$ **do**
7     $\tilde{A} \leftarrow$ a non-zero matrix in $L_{k_i}(P_i L P_i^T) \cap L_{k_i}(K_n)$
8   Feed $\tilde{P}$, $\tilde{A}$ and $L_{k_i}(K_n, P_i L P_i^T, P_i c P_i^T)$ to Theorem 12, with $\tilde{P}$ satisfying the hypothesis of Theorem 12
9     $i \leftarrow i + 1$
10     $M \leftarrow \begin{pmatrix} I_{k_{i-1}} & 0 \\ 0 & \tilde{P} \end{pmatrix}$
11     $P_i \leftarrow M P_{i-1}$
12     $A_i \leftarrow$ any matrix in $P_i L P_i^T$ such that $L_{k_{i-1}}(A_i) = \tilde{P}\tilde{A}\tilde{P}^T$
13   For each $1 \leq j < i$ exchange A_j for $M A_j M^T$
14     $s_i \leftarrow \operatorname{rank}(\tilde{A})$
15     $k_i \leftarrow k_{i-1} + s_i$
16 $m \leftarrow i$
17 $P \leftarrow P_i$
18 **return** $\{k_i \mid 1 \leq i \leq m\}, \{A_i \mid 1 \leq i \leq m\}$ *and* P

We remind again that from the point of view of feasibility there is no difference in dealing with (K_n, L, c) or (K_n, PLP^T, PcP^T), so performing this kind of transformation is harmless, when P is non-singular. Our next results concern the relation between a SDFP equipped with a hyper feasible partition and its bottom subproblem.

Proposition 18. *If (K_n, L, c) is an SDFP with a hyper feasible partition $\{k_i \mid 1 \leq i \leq m\}$ and $\{A_i \mid 1 \leq i \leq m\}$ a set of sub-hyper feasible directions then:*

i. if $k_m = n$, (K_n, L, c) is hyper feasible.

ii. if $k_m < n$, then (K_n, L, c):

- *is hyper feasible if and only if $L_{k_m}(K_n, L, c)$ is.*

- *is strongly feasible if and only if* $L_{k_m}(K_n, L, c)$ *is.*
- *has weak status if and only if* $L_{k_m}(K_n, L, c)$ *has.*
- *is strongly infeasible if and only if* $L_{k_m}(K_n, L, c)$ *is.*

Proof. Notice that we can use Theorem 12 to analyze a hyper feasible partition. First, feed A_1 and (K_n, L, c) to Theorem 12. Next, feed A_i and $L_{k_i}(K_n, L, c)$ successively, for $2 \leq i \leq m$. By doing that, we ensure that each subproblem inherits the feasibility properties in the sense of Theorem 12. Therefore, it is clear that item *ii*. holds.

If $m = 1$ we must have $k_1 = n$, thus $A_1 \in L \cap \text{int}(K_n)$ and it is clear that (K_n, L, c) is hyper feasible. If $m > 1$ then A_m satisfies $U_{n-k_{m-1}}(L_{k_{m-1}}(A_m)) \succ 0$. Therefore

$$L_{k_{m-1}}(K_n, L, c)$$

is hyper feasible, so the same must be true of (K_n, L, c). □

Proposition 19. *Let* (K_n, L, c) *be an SDFP equipped with a maximal hyper feasible partition* $\{k_i \mid 1 \leq i \leq m\}$ *then*

i. $k_m = n$ *if and only if* (K, L, c) *is hyper feasible.*

ii. if $k_m < n$, *then* (K, L, c)*:*

- *is strongly feasible if and only if* $L_{k_m}(K_n, L, c)$ *is.*
- *has weak status if and only if* $L_{k_m}(K_n, L, c)$ *is weakly feasible.*
- *is strongly infeasible if and only if* $L_{k_m}(K_n, L, c)$ *is.*

Proof. This is a direct consequence of Proposition 18. A maximal hyper feasible partition $\{k_i \mid 1 \leq i \leq m\}$ satisfies either $k_m = n$ or

$$L_{k_m}(K_n) \cap L_{k_m}(L) \neq \{0\}.$$

So if (K_n, L, c) is hyper feasible, we must necessary have $k_m = n$, because for every $k < n$, we have that $\text{int}(L_k(K_n)) \cap L_k(L) \neq \{0\}$. This proves item *i*.

The only difference between item *ii*. of Proposition 19 and item *ii*. of Proposition 18 is when (K_n, L, c) has weak status. In that case, we have a slightly better result because $L_{k_m}(K_n) \cap L_{k_m}(L) = \{0\}$, so it cannot be weakly infeasible.

□

Thus when there is a hyper feasible partition readily available such that $k_m < n$, we may analyze the bottom subproblem and except when there is weak status involved, we may determine the feasibility status of (K_n, L, c) by analyzing this smaller subproblem.

When (K_n, L, c) is weakly infeasible, we may transform it into an equivalent problem that admits a maximal hyper feasible partition. Then, this problem can be decomposed in a way that there are several intermediate blocks have weak status and a weakly feasible bottom subproblem that has compact feasible region. Now, suppose that (K_n, L, c) has a maximal hyper feasible partition such that the bottom subproblem, (i.e, $L_{k_m}(K_n, L, c)$) is weakly feasible. Unfortunately, this is not enough to ensure that (K_n, L, c) is weakly infeasible, however it ensures that (K_n, L, c) must at least have weak status. In order to distinguish between weak infeasibility and weak feasibility, it is enough to be able to distinguish feasibility from infeasibility. After all, the other feasibility possibilities are excluded. This theme will be explored in Section 5.

Example 20 (Example 16 revisited)**.** *Notice that the partition given in Example 16 is maximal and* $L_2(L+c) = \left\{ \begin{pmatrix} z+2 & z+1 \\ z+1 & 0 \end{pmatrix} \mid z \in \mathbb{R} \right\}$*, so the bottom subproblem* $(L_2(K), L_2(K), L_2(c))$ *is weakly feasible. Therefore, by Proposition 19,* (K_4, L, c) *must be in weakly status. Since it is infeasible, it must be weakly infeasible.*

4.2 Geometric Interpretation

In this subsection, we will see a hierarchical structure associated with weakly infeasible problems. A weakly infeasible problem always has a weakly infeasible sequence, i.e., there exists a sequence $\{x_i\} \subseteq L + c$ such that $\text{dist}(x_i, K_n)$ approaches K_n as i goes to infinity. However, we will show that such a sequence can always be constructed by linear combinations of sub-hyper feasible directions associated to a hyper feasible partition added to an appropriate displacement. Before we see the general case, let us see an example.

Example 21. *Let*

$$L + c = \left\{ \begin{pmatrix} s & 1 & t \\ 1 & t & 1 \\ t & 1 & 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}. \tag{4}$$

Thus (K_3, L, c) *is an SDFP. A hyper feasible partition for this problem is given by* $\{1, 2\}$ *and a set of associated sub-hyper feasible directions is given by*

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Owing to the fact that $L_2(K_3, L, c) = \{0\}$*, we have that* (K_3, L, c) *has weak status. However, it is infeasible due to the 0 at position* $(3, 3)$ *and the* 1 *at position* $(2, 3)$ *in Equation* (4)*, thus* (K_3, L, c) *is weakly infeasible. Now, we describe how to construct a weakly infeasible sequence.*

First, observe that the matrix

$$c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

has a negative eigenvalue, because of the $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *block at the bottom. This is bad, but we can control the minimum eigenvalue of the bottom block by adding* $\alpha_1 A_2$*. Then we have*

$$\alpha_1 A_2 + c = \begin{pmatrix} 0 & 1 & \alpha_1 \\ 1 & \alpha_1 & 1 \\ \alpha_1 & 1 & 0 \end{pmatrix}. \tag{5}$$

For given $\epsilon > 0$*, by taking sufficiently large* α_1*, we have that* $\lambda_{\min}(L_1(\alpha_1 A_2 + c)) > -\epsilon$*. But (5) may have a large negative eigenvalue because of* α_1 *at positions (1,3) and (3,1). This problem is fixed by further adding "a higher level" sub-hyper feasible direction* A_1*. Now, we consider the element* $\alpha_2 A_1 + \alpha_1 A_2 + c$*. For* $\alpha_2 > 0$*, its Schur complement with respect to* 1 *is*

$$\begin{aligned} & L_1(\alpha_2 A_1 + \alpha_1 A_2 + x) - \frac{1}{\alpha_2} R_1(\alpha_2 A_1 + \alpha_1 A_2 + x)^T R_1(\alpha_2 A_1 + \alpha_1 A_2 + x) \\ &= L_1(\alpha_1 A_2 + x) - \frac{1}{\alpha_2} R_1(\alpha_1 A_2 + x)^T R_1(\alpha_1 A_2 + x). \end{aligned}$$

An argument using Theorem 6 then shows that if α_2 is large, the minimum eigenvalue of $\alpha_2 A_1 + \alpha_1 A_2 + x$ becomes very close to the minimum eigenvalue of $L_1(\alpha_1 + A_2 + x)$. So, for instance, we may choose α_2 such that $\alpha_2 > \alpha_1$ and

$$\lambda_{\min}(\alpha_2 A_1 + \alpha_1 A_2 + x) > -2\epsilon.$$

Note that, at this stage, we need to control the minimum eigenvalue of the submatrix

$$\begin{pmatrix} \alpha_2 & 0 & \alpha_1 \\ 0 & \alpha_1 & 0 \\ \alpha_1 & 0 & 0 \end{pmatrix}.$$

This implies that α_2 is much larger than α_1 to keep the minimum eigenvalue close to zero.

This example has a hierarchical structure in the following sense. In order approach K_3, we ensure first that we are sufficiently close to K_2. We will see that given an arbitrary weakly infeasible problem, we may always construct weakly infeasible sequences in this hierarchical fashion. We first approach the "bottom" cones and then we work our way to the "top".

Proposition 22. *Let (K_n, L, c) be equipped with a hyper feasible partition $\mathcal{V} = \{k_i \mid 1 \leq i \leq m\}$ satisfying $k_m < n$. If $x \in L+c$ is such that $L_{k_m}(x) \succeq 0$ and we consider the subspace $L' \subseteq L$ generated by $\{A_1, \ldots, A_m\}$, where the A_i are sub-hyper feasible directions for $\mathcal{V}$ then* $\operatorname{dist}(L' + x, K) = 0$.

Proof. We have that $\mathcal{V}$ is also a hyper feasible partition for (K_n, L', x). Since $L_{k_m}(K_n, L', x)$ is at least weakly feasible due to the fact that $L_{k_m}(x) \succeq 0$, it follows that (K_n, L', x) must at least have weak status. □

Theorem 23. *If (K_n, L, c) is weakly infeasible then there exists an affine space of dimension at most $n-1$ such that $L' + c' \subseteq L + c$ and (K_n, L', c') is weakly infeasible.*

Proof. From Proposition 17, there exists an orthogonal matrix P such that (K_n, PLP^T, PcP^T) has a hyper feasible partition. This hyper feasible partition must satisfy $k_m < n$, otherwise the problem would be hyper feasible. Using Proposition 22, we ensure the existence of an affine subspace $P(L' + c')P \subseteq P(L + c)P^T$ such that $(K, PL'P^T, Pc'P^T)$ is weakly infeasible and $P(L' + c')P$ has dimension at most $n - 1$. Hence $L' + c'$ is an affine subspace of $L + c$ having the desired properties. □

Notice that $\{A_1, \ldots, A_m\}$ is a basis for L' and it may well happen that $L' \subsetneq L$. In any case, what the previous proposition states is that we can build a weakly infeasible sequence by picking any $x \in L + c$ satisfying $L_{k_m}(x) \succeq 0$ and adding a linear combination of the sub-hyper feasible directions associated to a partition. A hyper feasible partition that guarantees the existence of such an x always exist, for instance, any that is maximal will suffice. But, in reality, we do not need maximality here, as the weak feasibility of the $(n - k_m)$-subproblem delimited by the partition is enough. Another important fact is that although the dimension of $L + c$ is of order $O(n^2)$, Theorem 23 asserts that in reality we only need at most $n - 1$, hence $O(n)$, dimensions to approach K_n.

It was shown in Theorem 23 that if (K_n, L, c) is weakly infeasible then there exists an affine subspace of dimension at most $n - 1$ that also corresponds to a weakly infeasible subproblem. From this discussion, it is natural to ask what is the minimal dimension that such a space could have. A first step in this direction is to study under which conditions a single dimension is enough. If a single dimension is enough, we say that (K_n, L, c) is *directionally weakly infeasible*. We postpone this discussion to Subsection 4.3.

Theorem 24 (A hierarchical construction of weakly infeasible sequences)**.** *Let (K_n, L, c) be weakly infeasible and equipped with a hyper feasible partition $\{k_i \mid 1 \leq i \leq m\}$ such that the lower $(n-k_m)$-subproblem is weakly feasible. Let $\{A_1, \ldots, A_m\}$ be sub-hyper feasible directions associated to the partition and $x \in L + c$ such that $L_{k_m}(x) \succeq 0$. Then $\forall \epsilon > 0$, there exists $\alpha_1 < \ldots < \alpha_m$ such that*

$$\lambda_{\min}(\alpha_m A_1 + \ldots + \alpha_2 A_{m-1} + \alpha_1 A_m + x) > -\epsilon.$$

Proof. If we want to construct an element in $L + c$ that has small eigenvalue, we first start with $x \in L + c$, since it satisfies $L_{k_m}(x) \succeq 0$. Now, of course, $\lambda_{\min}(x)$ might be very negative, so we shall fix that by adding sub-hyper feasible directions. Since A_m satisfies $U_{k_m - k_{m-1}}(L_{k_{m-1}}(A_m)) \succ 0$ and all the other entries of $L_{k_{m-1}}(A_m)$ are 0, by using arguments such as the ones we used to prove item ($iii.\Leftarrow$) of Theorem 11, it is possible to show that for a given $\epsilon > 0$ and α_1 large enough, the minimum eigenvalue of $L_{k_{m-1}}(\alpha_1 A_m + x)$ is greater than $-\frac{\epsilon}{m}$. Using the same idea, we select α_2 big enough in way that

$$\lambda_{min}(L_{k_{m-2}}(\alpha_2 A_{m-1} + \alpha_1 A_m + x)) > -\frac{\epsilon}{m} - \frac{\epsilon}{m}.$$

Of course, the larger we select α_2 the better, so we may select it in a way that $\alpha_1 < \alpha_2$. This way, working in a bottom-up fashion we ensure $\alpha_1 < \ldots < \alpha_m$ holds and that

$$\lambda_{\min}(\alpha_m A_1 + \ldots + \alpha_2 A_{m-1} + \alpha_1 A_m + x) > -\epsilon.$$

□

Informally, we may say that a hyper feasible partition reveals how $L + c$ approaches K_n. Notice the hierarchical relation between the sub-hyper feasible directions. We first ensure that the bottom part is sufficient close to the corresponding positive semidefinite cone and then we work our way until the top. This is a feature that only becomes apparent when $n > 2$, so it is in this sense that Example 1 does not fully capture the intricateness of weakly infeasible SDFPs.

To conclude this subsection, we show how a weakly infeasible sequence can be used to generate a hyper feasible partition. It is a weak converse for Theorem 24.

Theorem 25. *Let (K_n, L, c) be a weakly infeasible and $\{x_s\} \subseteq (L + c)$ be a weakly infeasible sequence. Then, for some subsequence of $\{x_s\}$ (which we also denote by $\{x_s\}$), there exists a non-singular $n \times n$ matrix P such that (K_n, PLP^T, PcP^T) has a hyper feasible partition $\{k_i \mid 1 \leq i \leq m\}$ and an associated set of sub-hyper feasible directions $\{A_1, \ldots, A_m\}$ satisfying*

$$L_{k_i}(A_{i+1}) = \lim_{s \to \infty} \frac{L_{k_i}(y_s)}{\|L_{k_i}(y_s)\|}, \tag{6}$$

where $k_0 = 0$, $y_s = P x_s P^T$ and $0 \leq i \leq m-1$. In addition, the hyper feasible partition can be chosen in a way that the lower $(n - k_m)$-subproblem is weakly feasible.

Proof. We need some preparation for this proof. If $\{x_s\}$ is a weakly infeasible sequence (WIS) and $L_{k_i}(K_n, L, c)$ is weakly infeasible for some i, then $\{L_{k_i}(x_s)\}$ is a weakly infeasible sequence too. This is a consequence of the following inequality:

$$\lambda_{\min}(x_s) \leq \lambda_{\min}(L_{k_i}(x_s)) \leq 0.$$

Also, since $\{x_s\}$ is WIS, $\|x_s\|$ goes to infinity as s increases. Hence, after removing any zeroes, $\{x_s\}$ has a subsequence such that $\left\{\frac{x_s}{\|x_s\|}\right\}$ converges to a point in $L \cap K$. We substitute $\{x_s\}$ by this subsequence. After appropriate transformation, this point will serve as our first hyper feasible direction of size k_1. Denoting the matrix that performs the transformation by P_1, we have that $L_{k_1}(K_n, P_1^T L P_1, P_1^T c P_1)$ has weak status. If it is weakly feasible, we are done. Otherwise, $\{L_{k_1}(P_1^T x_s P_1)\}$ is a WIS and we repeat the process. This is the essence of this proof.

The actual proof is by modifying Algorithm 1 and running it with (K_n, L, c) and $\{x_s\}$. It is no loss of generality to assume that all the elements in the sequence are non-zero. We perform the following changes:

1. Input: (K_n, L, c) and $\{x_s\}$;
2. Line 2: substitute $\{x_s\}$ by a subsequence such that $\lim_{s\to\infty} \frac{x_s}{\|x_s\|}$ exists and take $\tilde{A}$ to be the limit;
3. Line 6: change the condition to

$$k_i < n, L_{k_i}(K_n) \cap L_{k_i}(P_i L P_i^T) \neq \{0\} \text{ and } L_{k_i}(K_n, P_i L P_i^T, P_i c P_i^T) \text{ is weakly infeasible;}$$

4. Line 7: substitute $\{x_s\}$ by a subsequence such that

$$\lim_{s\to\infty} \frac{L_{k_i}(P_i x_s P_i^T)}{\|L_{k_i}(P_i x_s P_i^T)\|}$$

   exists and take $\tilde{A}$ to be the limit.

With these changes, the output of Algorithm 1 produces the required directions and the matrix P. Algorithm 1 correctly produces a hyper feasible partition together with a set of sub-hyper feasible directions, see Proposition 17. And by our choice, the directions satisfy Equation (6).

□

4.3 Directional Weak Infeasibility

In this section, our objective is to give a precise definition of directional weak infeasibility and show that not all weakly infeasible SDFPs have this property. Before that, we briefly return to the situation where (K, L, c) is a general conic linear program.

Definition 26. *Let (K, L, c) be a weakly infeasible conic linear program. We say that (K, L, c) is* directionally weakly infeasible (DWI) *if there exists an affine subspace $L' + c' \subseteq L + c$ such that $L' + c'$ has dimension 1 and (K, L', c') is weakly infeasible.*

Notice that (K, L, c) is DWI if and only if there exists $x \in L$ and $c' \in L + c$ such that $\mathrm{dist}(tx + c', K) \to 0$ as $t \to +\infty$. For such a x, we must necessarily have that $x \in K$. Our main objective in this subsection is to study directional weak infeasibility and give an example of an SDFP that is weakly infeasible but is not DWI. By doing so, it becomes clear why Example 1 does not fully capture the essence of weak infeasibility since it is easy to check that Example 1 is DWI.

Proposition 27. *Let $x \in L \cap K$ and $c' \in L + c$ be such that $\lim_{t\to+\infty} \mathrm{dist}(tx + c', K) = 0$, where (K, L, c) is a conic linear program. Then, for every $y \in \mathrm{ri}(L \cap K)$, we have*

$$\lim_{t\to+\infty} \mathrm{dist}(ty + c', K) = 0.$$

Proof. For $y \in \mathrm{ri}(L \cap K)$ there exists $u > 1$ such that $(1-u)x + uy \in L \cap K$. Let $l = (1-u)x + uy$, then $y = \frac{l}{u} + \frac{(u-1)}{u}x$. It follows that for $t \geq 0$, we have

$$\begin{aligned}\mathrm{dist}(ty + c', K) &= \mathrm{dist}\left(t\left[\frac{l}{u} + \frac{(u-1)}{u}x\right] + c', K\right)\\ &\leq \mathrm{dist}\left(\frac{t(u-1)}{u}x + c', K\right).\end{aligned}$$

Since $\lim_{t\to+\infty} \mathrm{dist}(tx + c', K) = 0$, it is clear that the same must be true for $\mathrm{dist}\left(\frac{t(u-1)}{u}x + c', K\right)$, since $\frac{u-1}{u} > 0$. □

Example 28 (A weakly infeasible problem that is not directionally weakly infeasible)**.** *Let (K_3, L, c) be as in Example 21. Notice that $A_1 \in \mathrm{ri}(K_3 \cap L)$, so if (K_3, L, c) were DWI, we would have*

$$\lim_{t\to+\infty} \mathrm{dist}(tA_1 + c', K) = 0, \tag{7}$$

for some $c' \in L + c$, by Proposition 27. Now, there are many ways to show that Equation (7) does not hold. For instance, let L' be the vector space space spanned by A_1. Then (K_3, L', c') has a hyper feasible partition $\{1\}$ and a hyper feasible direction A_1. However, $L_1(K_3, L', c')$ is strongly infeasible no matter which c' we choose, thus (K_3, L', c') itself is strongly infeasible and Equation (7) cannot hold.

5 A Backwards Procedure

In this section we present a procedure that directly explores the structure of the problem in order to distinguish between weak infeasibility and weak feasibility. We begin this section by revisiting Example 16 again.

Example 29 (Example 16 revisited again)**.** *We already know that (K_4, L, c) is weakly infeasible. However, using a maximal hyper feasible partition, we were only able to prove that (K_4, L, c) has weak status. Weak infeasibility came as a result of showing the infeasibility of (K_4, L, c). Let us do this in a more systematic way. We had that $L_2(L + c) = \left\{\begin{pmatrix} z+2 & z+1 \\ z+1 & 0\end{pmatrix} \mid z \in \mathbb{R}\right\}$, which corresponds to a weakly feasible problem.*

Now, we try to consider a few necessary conditions for feasibility. If $x \in L + c$ is feasible, we must have that $L_2(x)$ should also be feasible, so at this point we must have $z = -1$. Moreover, the zero in the lower right corner $L_2(L + c)$ forces $R_3(x) = 0$ and we must have $u = s = 0$. The introduction of these constraints suggests that we may consider an auxiliary problem $(K_4, L^{(2)}, c^{(2)})$ in a way that

$$L^{(2)} + c^{(2)} = \left\{\begin{pmatrix} t & 1 & v & 0 \\ 1 & -1 & v+1 & 0 \\ v & v+1 & 1 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} \mid t, v \in \mathbb{R}\right\}.$$

But $L^{(2)} + c^{(2)}$ can be seen as an affine space in $\mathbb{S}_3$ in a natural way, so instead, we write

$$L^{(2)} + c^{(2)} = \left\{\begin{pmatrix} t & 1 & v \\ 1 & -1 & v+1 \\ v & v+1 & 1\end{pmatrix} \mid t, v \in \mathbb{R}\right\},$$

and consider the problem $(K_3, L^{(2)}, c^{(2)})$. Notice that $(K_3, L^{(2)}, c^{(2)})$ admits a hyper feasible partition $\{1\}$ with $U_3(A_1)$ serving as a hyper feasible direction. So, again, we may consider the problem $(L_1(K_3), L_1(L^{(2)}), L_1(c^{(2)}))$

and we have that $L_2(L^{(2)} + c^{(2)}) = \left\{ \left(\begin{smallmatrix} -1 & v+1 \\ v+1 & 1 \end{smallmatrix} \right) \mid v \in \mathbb{R} \right\}$*, which is a strongly infeasible SDFP. This implies that* $(K_3, L^{(2)}, c^{(2)})$ *is strongly infeasible (by Theorem 12) and* (K_n, L, c) *must be infeasible. After all, we introduced a few necessary conditions as constraints and it turned out that all the matrices in* $L + c$ *that satisfy these constraints have a leading principal submatrix that is not positive semidefinite.*

We now discuss what was done in the previous example in more detail and greater generality. Knowing that a given SDFP has weak status, i.e., it is either weakly feasible or weakly infeasible, in order to distinguish between these two possibilities it is enough to prove or disprove that the problem is feasible. The following theorem shows that the task of checking weak feasibility of (K_n, L, c) is reduced to checking feasibility of a smaller dimensional semidefinite program. Based on this result, we are able to develop a procedure to completely distinguish whether a given SDFP is strongly feasible, weakly feasible, weakly infeasible and strongly infeasible.

Theorem 30. *Let* (K_n, L, c) *be a given SDFP. Assume* $n > k + m$*, and suppose that the* k*-lower subproblem* $L_k(K_n, L, c)$ *is weakly feasible and that any feasible solution to* $L_k(K_n, L, c)$ *is written as*

$$\begin{aligned} &\begin{pmatrix} X_{11} & 0_{(n-k)-m,m} \\ 0_{m,(n-k)-m} & 0_{m,m} \end{pmatrix} \in L_k(L + c), \\ &X_{11} \succeq 0. \end{aligned} \tag{8}$$

Then, (K_n, L, c) *is weakly feasible if and only if the following SDFP is feasible:*

$$\begin{pmatrix} \widetilde{X}_{11} & 0_{n-m,m} \\ 0_{m,n-m} & 0_{m,m} \end{pmatrix} \in L + c, \quad \widetilde{X}_{11} \succeq 0.$$

In other word, if we represent the affine space

$$\mathcal{U} = \left\{ \widetilde{X}_{11} \mid \begin{pmatrix} \widetilde{X}_{11} & 0_{n-m,m} \\ 0_{m,n-m} & 0_{m,m} \end{pmatrix} \in L + c \right\},$$

as $\mathcal{U} = L' + c'$*, where* L' *is a subspace and* $c' \in \mathbb{S}^{n-m}$*, then,* (K_n, L, c) *is weakly feasible if and only if* $\mathcal{U} \neq \emptyset$ *and* (K_{n-m}, L', c') *is feasible.*

Proof. Since "if part" is obvious, we focus on the "only if" part. Let

$$\mathcal{W} = \{X \mid L_k(X) \succeq 0, X \in L + c\}. \tag{9}$$

The set $\mathcal{W}$ is the set of X such that $L_k(X)$ is a feasible solution to the k-lower subproblem $L_k(K, L, c)$. Therefore, by (8), for any $X \in \mathcal{W}$, its bottom $m \times m$ principal submatrix is zero. We denote by $\mathcal{V}$ the feasible set of (K_n, L, c). Then, $\mathcal{V}$ is written as follows:

$$\mathcal{V} = \{X \mid X \in L + c \text{ and } X \succeq 0\} = \{X \mid L_k(X) \succeq 0,\ X \in L + c \text{ and } X \succeq 0.\} \tag{10}$$

Comparing the right hand sides of (10) and (9), we see that $\mathcal{V} \subseteq \mathcal{W}$ and, as we have already observed, the $m \times m$ bottom principal submatrix of any element of $\mathcal{W}$ is zero. This implies that the $m \times m$ bottom principal submatrix of any $X \in \mathcal{V}$ is zero. Since X is a positive semidefinite matrix with $m \times m$ bottom principal submatrix being zero, the last m rows and m columns of X are zero. This completes the proof. □

Algorithm 2: A Backwards Procedure

Input : (K_n, L, c)
Output: The feasibility status of (K_n, L, c)
1 $(K', L', c') \leftarrow (K_n, L, c)$
2 Apply a congruence transformation to (K', L', c') in order to obtain an equivalent SDFP which admits a hyper feasible partition. Denote this new SDFP by (K', L', c')
3 Apply Algorithm 1 to (K', L', c')
4 Let (K'', L'', c'') be the bottom block of (K', L', c')
5 *firstTime* $\leftarrow$**True**
6 **if** (K'', L'', c'') *is Strongly Feasible* **then**
7   **if** *firstTime* **then**
8   **return** *"Strongly Feasible"*
9   **else**
10   **return** *"Weakly Feasible"*
11 **if** (K'', L'', c'') *is Strongly Infeasible* **then**
12   **if** *firstTime* **then**
13   **return** *"Strongly Infeasible"*
14   **else**
15   **return** *"Weakly Infeasible"*
`/* If we reach this part, then the bottom block must be weakly feasible. */`
16 Apply a congruence transformation to (K', L', c') which puts the feasible region of (K'', L'', c'') in the shape

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

as in Equation 8. Denote the transformed problem by (K', L', c')
17 Apply Theorem 30 to (K', L', c') to obtain a smaller subproblem as described in the theorem. If the smaller subproblem is empty, return *"Weakly Infeasible"*
18 Replace (K', L', c') by the resulting subproblem, set *firstTime* to **False** and go back to Line 2

Notice that if $L_k(K_n, L, c)$ is weakly feasible, there is a non-singular matrix P such that $L_k(K_n, PLP^T, PcP^T)$ has the shape described by Equation (8)[1]. Theorem 30 enables us to develop a procedure to determine the feasibility status of (K_n, L, c), see Algorithm 2. It halts in a finite number of steps, because the size of the problem is reduced every time Line 18 is reached.

Associated to (K, L, c) we consider the problem of finding a dual improving direction (D_D), which comes in hand to prove generalizations of Farkas' Lemma:

$$\begin{aligned} &\min 0 \\ &\text{s.t. } s \in L^\perp \cap K, c^T s \le -1. \end{aligned}$$

Then, we have that (K, L, c) is strongly infeasible if and only if (D_D) is feasible and it is weakly infeasible if and only if (D_D) is weakly infeasible (see [11, Theorem 1]). Thus, to certificate strong infeasibility it is enough to present a solution to (D_D). In addition, an interior solution to (K, L, c) provides a certificate of strong feasibility. Finally, to pinpoint weak feasibility, a feasible solution together with a hyperplane that separates $L + c$ and K properly gives an "if and only if" condition (See [14, Theorem 11.3]).

[1] This is due to the facial structure of K_n. If F is a proper face of K_n, then there is a non-singular matrix P, such that all the matrices in PFP^T have the shape $\left(\begin{smallmatrix} X_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for some $X_{11} \succeq 0$.

We would like to remark that with Algorithm 2 we can exactly pinpoint weak infeasibility, at least in theory. Notice that the conditions in Lines 6 and 11 of Algorithm 2 can be attested by the above-mentioned certificates. The condition on Line 17 is just a matter of checking linear equations. We conclude that to certify weak infeasibility, we may first transform the problem into an equivalent problem that admits a maximal hyper feasible partition, then we use a certificate to attest the weak feasibility of the bottom subproblem. Finally, we run Algorithm 2 using the aforementioned certificates to check the conditions on Lines 6, 11 and 17.

6 Final Remarks

In this work we presented a new approach for analyzing SDFPs that have a non-zero vector in $L \cap K$, in particular, we can use our techniques to study weakly infeasible problems. By isolating a hyper feasible direction, we obtain a subproblem that inherits most of the feasibility properties of the original problem. As far as we know, this is the first work which systematically analyzes the structure of weak infeasible semidefinite programs.

Our approach does not use duality theory and we work exclusively in the primal space. Of course, duality theory is a very important tool and we were not deliberately trying to avoid its use. As a future work we plan to understand how our decomposition technique works on the dual side. Notice that by Theorem 1 in [11], we have that (D_D) is weaky infeasible if and only if (K_n, L, c) is. This means that in that case, (D_D) must also posses a hyper feasible partition. At this moment, it is not clear how to generate it in a transparent way from the corresponding partition in the primal space.

Our approach appears different from FRA and CAE though their purpose is similar. It would be interesting to study relation among them and to develop something corresponding to Ramana's ELSD.

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