TURNPIKE THEOREMS FOR CONVEX PROBLEMS WITH UNDISCOUNTED INTEGRAL FUNCTIONALS

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Abstract. In this paper the turnpike property is established for convex optimal control problems, involving undiscounted utility function and differential inclusions defined by multi-valued mapping having convex graph. Utility function is concave but not necessarily strictly concave. The turnpike theorem is proved under the main assumption that over any given line segment, either multi-valued mapping is strictly convex or utility function is strictly concave. In this way, the strictly convexity/concavity assumption is distributed between the mapping and utility function.

1. Introduction

The following problem is considered in this paper:

$$\dot{x} \in a(x), \; x(0) = x^0, \tag{1}$$

$$\text{Maximize} : J_T(x(\cdot)) = \int_0^T u(x(t))dt. \tag{2}$$

Here multivalued mapping a is defined on a given compact set $\Omega \subset R^n$, has compact images and is continuous in the Hausdorff metric. The graph of a is denoted by

$$\operatorname{graph} a = \{(x, y) : \; x \in \Omega, \, y \in a(x)\}.$$

Utility function $u : \Omega \to R^1$ is continuous. $x^0 \in \Omega$ is a given initial point.

The main purpose of this paper is to establish the turnpike property for convex problems. Accordingly we will assume that the graph of mapping a is a convex set and function u is concave. Some strict convexity/concavity assumptions that are inevitable will be "distributed" between a and u; that is, it will be assumed that over any line segment either a is strictly convex or function u is strictly concave.

The turnpike property describes stability of all optimal trajectories when T goes to infinity. The first result in this area is obtained by John von Neumann ([33]) for discrete time systems. The phenomenon is called the turnpike property after Chapter 12, [4] by Dorfman, Samuelson and Solow. Simply saying this property states that, regardless of initial conditions, all *optimal* trajectories spend most of the time within a small neighborhood of some *optimal stationary point* when the planning period T is long enough.

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This property was further investigated by many authors (Radner [36], Makarov and Rubinov [20] , McKenzie [31] etc). Quite strong results have been obtained for discrete-time systems, in particular, for a von Neuman-Gale model. In all these studies the turnpike property was established under some convexity assumptions.

For a classification of different definitions for the turnpike property, we refer to [1, 20, 31, 44], as well as [3] for so called *exponential* turnpike property. Possible applications in Markov Games can be found in a recent study [15].

A number of powerful theoretical approaches have been suggested for both continuous and discrete systems. Some convexity assumptions are sufficient for discrete systems [20, 31]; however, rather restrictive assumptions are usually required for continuous time systems. We briefly mention here some approaches developed for continuous time systems.

A special class of terminal functionals, defined as a lower limit at infinity of utility functions, is turned out to be the most "suitable functional" enabling to prove the turnpike property for a wide range of optimal control problems in continuous time. This approach is introduced in [21] (see [28] for more references) where the turnpike property is established for a broader class of non-convex problems. In [26], [35] such functionals are used for discrete-time systems involving the notion of Statistical Convergence, where it is proved that all optimal trajectories have the same unique statistical cluster point ([6]). Recently, the turnpike property is established for a special class of time-delay systems arisen from applications in medicine and biology ([13, 27, 29]). Note that considering integral (discounted or undiscounted) functionals in the above applications will be an important advance in the field.

The majority of approaches in the literature involve optimal control problems with (discounted and undiscounted) *integral functionals* (see [1, 44] and references therein). Among the most successful approaches developed we mention here the approach developed by Rockafellar [37, 38] that applies related techniques from convex analysis, and the "direct" approach developed by Scheinkman, Brock and collaborators (see, for example, [19, 40]) that applies the Maximum principle and then reduces the main problem to the study of stability of ordinary differential equations with un-known terminal values for costate variables. We also mention approaches by Cass and Shell [2], Leizarowitz [17], Mamedov [22, 23], Montrucchio [32], Zaslavski [42, 43, 44, 45, 46, 47, 48].

There are also several approaches developed for a special class of problems (e.g. [12, 34, 41, 49]). An interesting class of control problems considered in [7, 8] involves long run average cost functions where the asymptotic behavior of optimal solutions is defined in terms of a probability measure.

When considering *discounted* integral functionals, Rockafellar's approach is the most successful one as it can be seen from a recent publication [39] where for a special class of convex problems (i.e. Ramsey's problem) the turnpike property is established without any additional restrictive assumptions. Another successful approach for *discounted* integral functionals is developed in [30, 32].

In this paper we consider *undiscounted* integral functionals that have recently attracted significant attention. [14] provides a short overview in this area together with the discounted deterministic case. In a recent paper [15] (see also [16]) the turnpike property is established Markov games involving undiscounted functionals. This property is also useful in the study of stability in Model Predictive Control. In several recent publications

[3, 9, 10, 11] strict dissipativity assumptions, similar to (3) below, are used to establish the exponential turnpike property in discrete time systems involving undiscounted integrals.

Among the most successful studies for undiscounted integral functionals, we mention the approach developed by Carlson, Haurie and Leizarowitz (see, for example, [1, 18]). In terms of problem (1),(2), when utility u is a function of x only, the main assumption employed in these studies can be formulated as follows:

There is p such that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$u(x^*) > u(x) + pz + \delta, \ \forall z \in a(x), \ \forall x \text{ with } \|x - x^*\| > \varepsilon. \tag{3}$$

A broader class of optimal control problems involving undiscounted integral functionals that possess the turnpike property have been introduced in [22, 23, 24, 25]. Since this paper builds upon the approach developed in [22, 25], more details are provided below.

The main assumption used is based on the following inequality (see Assumption H in Section 2.1), where $c(x) = \max\{pz : \ z \in a(x)\}$:

There is p such that for any x and y satisfying $px = py$, $c(x) < 0$ and $c(y) > 0$, the following inequality holds:

$$\frac{u(x) - u(x^*)}{|c(x)|} + \frac{u(y) - u(x^*)}{c(y)} < 0. \tag{4}$$

Clearly, if condition (3) holds then (4) is also satisfied. Indeed, if we rewrite (3) at two different points x and y with $c(x) < 0$, $c(y) > 0$, then for all $z \in a(x)$ we have

$$u(x) - u(x^*) < -pz - \delta < -pz, \quad \text{and} \quad u(y) - u(x^*) < -pz - \delta < -pz$$

and therefore

$$\frac{u(x) - u(x^*)}{|c(x)|} < 1, \ \frac{u(y) - u(x^*)}{c(y)} < -1.$$

This means that in this case the inequality (4) is satisfied for all x, y with $c(x) < 0$, $c(y) > 0$ (not requiring $px = py$). On the other hand it is not difficult to provide an example for which condition (3) does not hold but (4) still holds.

We provide here the geometric interpretation of condition (4). It describes a relation between u and a that forces "good" trajectories to converge to x^*. It is applied in a "worse" case only, when at two different points x and y, a trajectory under consideration may cross the hyperplane $\{z : \ pz = px \ (= py)\}$ in two different directions expressed by $c(x) < 0$ and $c(y) > 0$, where, for the sake of simplicity we let $u(x) > u(x^*) > u(y)$. Now, $\tau_x \doteq \frac{1}{|c(x)|}$ and $\tau_y \doteq \frac{1}{c(y)}$ are the "longest" and the "shortest" times that any trajectory could "spend" at the "good" state x and at the "bad" state y, respectively. Then, inequality (4) says that, over the total time period $(\tau_x + \tau_y)$, the average contribution to the functional $J_T()$; that is, $u(x)\tau_x + u(y)\tau_y$, should be strictly less than the corresponding contribution if trajectory stayed at the stationary point x^*; that is, $u(x^*)(\tau_x + \tau_y)$. In other words, condition (4) forces "good" trajectories to prefer staying closer to the stationary point x^*, rather than crossing the above hyperplane in different directions "infinitely many" times that could lead to instability (like a cyclic behavior).

For the above problem (1),(2), turnpike theorems based on condition (4), are provided in [22, 25] when optimal stationary point x^* is unique. [24] considers the case when there are a finite number of different optimal stationary points. For more references and details we refer to [28].

In this work we study the turnpike property for the convex problems. We mention here two approaches developed for convex problems.

In [37], for the convex problem (1),(2) with $u = u(x, \dot{x})$, the turnpike property is established assuming that the Hamiltonian

$$H(x, p) = \sup\{pv + u(x, v) : v \in a(x)\}$$

is strictly concave-convex in a neighborhood of a saddle point. Clearly in our case when $u = u(x)$ this assumptions does not hold; in this case, $H(x, p) = \sup\{pv : v \in a(x)\} + u(x)$ and the first term is positively homogeneous; that is, is not strictly convex.

Another approach for convex problems is developed in [1], where the turnpike property is established for overtaking optimal and finitely optimal trajectories. In both cases trajectories are functions defined on $[0, +\infty)$ and the optimality criteria applies to the finite parts of trajectories related to intervals $[0, T]$ where $T \to \infty$. In this case optimal trajectory, say $x^*(t)$, is in some sense a "fixed" function defined on $[0, +\infty)$; for different values for T, say for $T_1 \neq T_2$ we are just considering different portions of the same trajectory; that is, $\{x^*(t),\ t \in [0, T_1]\}$ and $\{x^*(t),\ t \in [0, T_2]\}$.

In this paper optimality is defined in an ordinary way; as a result, we are dealing with a more complicated situation by considering a set of optimal trajectories $x^*_T(t),\ t \in [0, T]$. For example if $T_1 < T_2$ trajectories $x^*_{T_1}(t)$ and $x^*_{T_2}(t)$ do not necessarily coincide on the interval $[0, T_1]$. The main purpose is to establish the turnpike property for convex problems without involving any additional restrictive assumptions, like assumptions imposed on the Hamiltonian, dissipativity assumptions, (3) or (4). An important result here is that utility function u does need to be strictly concave if map a processes some strictly convexity properties.

2. Definitions and Main Assumptions

We call problem (1), (2) convex if, in addition, Ω is convex, the graph of mapping a is convex and function $u : \Omega \to R$ is concave.

Initial point $x^0 \in \Omega$ is fixed throughout the paper, though it will be used only for the existence of "good" trajectories starting from that point (Assumption A1 below).

Roundedness of Ω is used for the sake of simplicity; Ω is assumed to be with non-empty interior and large enough to accommodate all trajectories starting from x^0, as well as all stationary points of mapping a.

An absolutely continuous function $x(\cdot)$ is called a trajectory (solution) to the system (1) if $x(0) = x^0$ and almost everywhere on the interval $[0, T]$ the inclusion $\dot{x}(t) \in a(x(t))$ is satisfied.

We assume that given any $T > 0$ there is a trajectory to the system (1). Moreover, all trajectories starting from x^0 are bounded and the set Ω is large enough to satisfy

$$x(t) \in \Omega' \subset \text{ int } \Omega,\ \forall t \in [0, T],\ x(\cdot) \in X_T,\ T > 0, \tag{5}$$

where Ω' is a closed set. The set of trajectories defined on the interval [0,T] will be denoted by X_T and let

$$J^*_T = \sup_{x(\cdot) \in X_T} J_T(x(\cdot)).$$

Throughout the paper $||\cdot||$ stands for the Euclidean norm, $|\cdot|$ stands for the absolute value. The notation $\partial()$ denotes the boundary of the enclosed set. The scalar product of two vectors $x, y \in R^n$ will be denoted by xy.

Definition 2.1. *Trajectory $x(\cdot) \in X_T$ is called*

- *optimal if* $J_T(x(\cdot)) = J_T^*$;
- *ξ-optimal if* $J_T(x(\cdot)) \geq J_T^* - \xi$; *where* $\xi \geq 0$.

Note that if $x(\cdot) \in X_T$ is an optimal trajectory, part of this trajectory $x(t)$, $t \in [0, T']$ $(T' < T)$ may not be optimal. This means that we are dealing with a set of optimal trajectories corresponding to different values $T > 0$. The notion ξ-optimality describes "good" trajectories. Parameter ξ will be a fixed number for all T.

Stationary points play an important role in the study of asymptotical behavior of optimal trajectories. We denote the set of stationary points by M:

$$M = \{x \in \Omega : \ 0 \in a(x)\}.$$

For the sake of simplicity we assume that M is a bounded set and $M \subset \operatorname{int} \Omega$. The existence of stationary points will be guaranteed by Assumption A1 introduced below (see Lemma 4.1). It is clear that M is a closed set since mapping a is continuous. Thus, for convex problem (1), (2) M will be a compact convex set.

Definition 2.2. *$x^* \in M$ is called an optimal stationary point if*

$$u(x^*) = u^* \stackrel{\triangle}{=} \max_{x \in M} u(x).$$

If $M \neq \emptyset$, then there exists an optimal stationary point; it is also unique under some strictly convexity/concavity assumptions provided below.

Now we introduce the main assumptions.

Assumption A1: ([25]) *There exists $b < +\infty$ such that for every $T > 0$ there is a trajectory $x(\cdot) \in X_T$ satisfying the inequality*

$$J_T(x(\cdot)) \ \geq \ u^* T - b.$$

This assumption in particular means that given any $T > 0$, the set of trajectories X_T is not empty. On the other hand, the satisfaction of Assumption A1 mainly depends on initial point x^0. For example, if

(6) there exists a trajectory $x(t)$ such that $x(T_1) = x^*$ for some $T_1 < \infty$;

that is, if there exists a trajectory starting from x^0 that hits x^* in finite time, then Assumption A1 is satisfied. Thus, in some sense, it can be considered as an assumption for the existence of trajectories defined on $[0, \infty)$ that converge to (but not necessarily reach) the optimal stationary point x^*.

Assumption A2: *Given any $x_1, x_2 \in \Omega$ and any number $\alpha \in (0, 1)$, at least one of the following inequalities hold:*

$$u(\alpha\, x_1 + (1-\alpha)\, x_2) \ > \ \alpha\, u(x_1) + (1-\alpha)\, u(x_2);$$

$$\text{int } a(\alpha\, x_1 + (1-\alpha)\, x_2) \;\supset\; \alpha\, a(x_1) + (1-\alpha)\, a(x_2). \tag{7}$$

Mapping a is said to be strictly convex on the interval $[x_1, x_2]$ if the inclusion (7) is satisfied for all $\alpha \in (0,1)$. Thus, Assumption A1 means that over any line interval $[x_1, x_2]$ either function u is strictly concave or mapping a is strictly convex. We provide one example where Assumption A2 is satisfied but nether a nor u is strictly convex/concave.

Example 2.3. $\Omega = [-1, +1] \times [-1, +1]$, $a(x_1, x_2) = \{(y_1, y_2) : \; y_1^2 + y_2^2 \le 1 - x_2^2\}$, $u(x_1, x_2) = -x_1^2 - x_2$.

Clearly, for this example Assumption A2 is satisfied, however a is not strictly convex in x_1 and u is not strictly concave in x_2.

Remark 2.4. *It should be noted that strictly convexity of mapping a does not necessarily mean strictly convexity of the graph a. This can be seen from the following example.*

$$\Omega = \{x = (\xi_1, \xi_2) : \; \xi_1^2 + \xi_2^2 \le 1\}, \; a(x) = [-s(x), s(x)] \times [-s(x), s(x)],$$

where $s(x)$ is any strictly concave function satisfying $s(x) \ge 0$; for example $s(x) = 1 - \xi_1^2 - \xi_2^2$. It is not difficult to show that (7) is satisfied for all $x_1, x_2 \in \Omega$; however the images of mapping a are not strictly convex sets (they have a square shape in R^2).

Assumption A3: There exists $x' \in \Omega$ such that $u(x') > u^*$.

This assumption is natural for utility functions that are usually increasing. If it does not hold then there are two possibilities:

1. x^* is the unique optimal stationary point. In this case $u(x) < u^*$ for all $x \ne x^*$ and the turnpike property can be easily proved without any additional assumptions (see also Theorem 3.3 below and its proof). Note that this case still has some important practical applications (e.g. [34]).

2. Optimal stationary point is not unique. In this case the set of optimal stationary points is a convex set and the turnpike property should be understood in terms of convergence of optimal trajectories to the set $\{x \in \Omega : \; u(x) = u^*\}$. This set can be much larger than the set of optimal stationary points; therefore the turnpike property may easily be violated (e.g. if $u(x) \equiv constant$). Note that it is in order quite difficult to study stability of optimal trajectories when the set of optimal stationary points is not unique.

Assumption A4: There exists a stationary point $\tilde{x} \in M$ such that $0 \in \text{int } a(\tilde{x})$.

We note that in some special ceases Assumptions A3 and A4 can be eliminated (Theorems 3.3 and 3.5).

Some implications of Assumptions A1-A3 will be provided in Section 4; here we just mention that under these assumptions optimal stationary point x^* exists, is unique and $x^* \in \partial M$. In particular, $(x^*, 0)$ is not an interior point of the graph of mapping a that is the most challenging case.

This is an important issue that should be mentioned, as in many studies the assumption $(x^*, 0) \in \text{int}\,\text{graph}\, a$; that is, $0 \in \text{int}\, a(x^*)$, is essential (see for example [32]). In some studies (e.g. [1, 12]), the assumption (6) is used. In fact, such an assumption might be quite restrictive if $(x^*, 0)$ is not an interior point of graph a.

For example, if $a(x) = \{-x\}$ and $x^0 \neq 0,\ x^* = 0$, there is no trajectory that reaches x^* in finite time although the graph of mapping a is convex (in this case trajectories are $x(t) = x^0 e^{-t} \to 0$). The same situation we observe for another example $a(x) = [-1, -x],\ x \in \Omega = [-1, +1]$ with $x^0 < 0,\ x^* = 0$. On the other hand, in both cases Assumption A1 is satisfied for all x^0.

2.1. **Theorem 2.5 from** [25]**.** In this section we formulate Theorem 2.5 from [25] where the turnpike property is established without involving any convexity-concavity assumptions imposed on Ω, mapping a and function u. Instead this theorem uses another assumption (called Assumption H) that is provided below.

Consider (not-necessarily convex) problem (1), (2) with the main definitions and notations introduced above. Let, in addition, the set $a(x)$ be uniformly locally connected for each x and mapping a be Lipschitz continuous in the Hausdorff metric. Assume that optimal stationary point x^* is unique. Therefore no assumptions are required about the concavity of function u and the convexity of graph a.

For a given non-zero vector $z \in R^n$, consider the following support function

$$c(z, x) = \max_{y \in a(x)} zy.$$

Then, for all x, y, for which $c(z, x) < 0$ and $c(z, y) > 0$, define function $\varphi(x, y)$ as follows

$$\varphi(x, y) = \frac{u(x) - u^*}{|c(z, x)|} + \frac{u(y) - u^*}{c(z, y)}. \tag{8}$$

The main assumption is formulated next where, as mentioned above, x^* is the unique optimal stationary point and $\mathcal{B} = \{x \in \Omega :\ u(x) \geq u(x^*)\}$

Assumption H: *There exists a vector* $z \in R^n$ *such that*

H1: $c(z, x) < 0$ *for all* $x \in \mathcal{B},\ x \neq x^*$;

H2: *there exists a point* $\bar{x} \in \Omega$ *such that* $z\bar{x} = zx^*$ *and* $c(z, \bar{x}) > 0$;

H3: *for all points* x, y*, for which*

$$zx = zy,\ c(z, x) < 0,\ c(z, y) > 0,$$

the inequality $\varphi(x, y) < 0$ *is satisfied; and, moreover, if*

$$x_k \to x^*,\ y_k \to y' \neq x^*,\ zx_k = zy_k,\ c(z, x_k) < 0,\ c(z, y_k) > 0,$$

then $\limsup_{k \to \infty} \varphi(x_k, y_k) < 0$.

Now we formulate Theorem 2.1 from [25].

Theorem 2.5. ([25]) *Consider problem (1), (2). Assume that optimal stationary point* x^* *is unique and Assumptions A1 and H are satisfied. Then*

1) there exists $C < +\infty$ *such that*

$$\int_0^T (u(x(t)) - u^*) dt \ \leq\ C$$

for all $T > 0$ *and for all trajectories* $x(\cdot) \in X_T$;

2) given any $\xi \geq 0$, for every $\varepsilon > 0$ there exists a number $K_\varepsilon < +\infty$ such that

$$\text{meas}\{t \in [0,T]: \ ||x(t) - x^*|| \geq \varepsilon\} \ \leq \ K_\varepsilon$$

for all $T > 0$ and for all ξ-optimal trajectories $x(\cdot) \in X_T$;

3) if $x(\cdot)$ is an optimal trajectory and $x(t_1) = x(t_2) = x^*$, then $x(t) \equiv x^*$ for $t \in [t_1, t_2]$.

The first assertion states that functional $J_T(x(\cdot))$ is linearly bounded; that is, $J_T(x(\cdot)) \leq Tu^* + C$ for all $T > 0$. The second assertion of the theorem is the turnpike property. It should be noted that this property is true not only for optimal but also for all "good" (that is, ξ-optimal) trajectories. The third assertion states some additional information about the behavior/structure of optimal trajectories.

3. Main results

In this section we formulate the main results of the paper. Theorem 2.5 in the previous section describes the turnpike property in general case assuming Assumption H. In this paper we concentrate on convex problems. The main goal is to show that this assumption is not necessary for convex problems.

Theorem 3.1. *Consider convex problem (1), (2). Assume that Assumptions A1-A4 hold. Then there exists a unique optimal stationary point x^* and all the assertions of Theorem 2.5 are true.*

To prove Theorem 3.1 we will mainly be verifying the assumptions of Theorem 2.5, although Theorem 3.1 is not a special case of Theorem 2.5 (see Example 3.2 below). We will show that for the convex problems with Assumptions A1-A4, Assumptions H1 and H3 hold.

In the following example all the assumptions of Theorem 3.1 are satisfied. However, Assumption H2 does not hold. This, in particular, shows that Assumption A4 does not necessarily "replace" H2. An inverse example can also be easily generated.

Example 3.2. Let $\Omega = [-1,1]$, $a(x) = [-1, -x]$, $u(x) = -x^2 + 2x$ and $x^0 = 1$.

Clearly, function u is strictly concave, the graph of mapping a is a convex set. We have

$$M = [-1,0], \ u^* = \max_{x \in M} u(x) = 0 \ \text{ and } x^* = 0.$$

Assumption A4 holds for the point $\tilde{x} = -0.5$. It is not difficult to observe that given any $T > 0$, the solution $x(t) = e^{-t}$ is optimal and Assumption A1 holds. All the other assumptions of Theorem 3.1 are also satisfied.

Consider Assumption H. We have $\mathcal{B} = [0,1]$. Then Assumption H1 is satisfied only for positive numbers $z > 0$.

Now we check Assumption H2. Take any $z > 0$. If $z\bar{x} = zx^*$ then $\bar{x} = x^* = 0$ and

$$c(z, \bar{x}) \ = \ \max_{y \in a(\bar{x})} zy \ = \ z \max_{y \in [-1,0]} y \ = \ 0.$$

Therefore, H2 does not hold for any positive z.

In what follows we consider two special cases assuming that function u is strictly concave or mapping a is strictly convex. The aim is to investigate if it is possible to eliminate one of the assumptions A3 and A4 in some special cases. We will show that Assumption A3 can be eliminated if function u is strictly concave. On the other hand, if mapping a is strictly convex then the assertions 2 and 3 (that is, the turnpike property) of Theorem 2.5 are still valid without assuming Assumption A4.

3.1. **Utility function u is strictly concave.** Clearly, in this case the optimal stationary point is unique and Assumption A2 is satisfied. Next theorem states that all the assertions 1-3 of Theorem 3.1 are valid without assuming Assumption A3. This result was first presented in [22] without proof.

Theorem 3.3. *Consider convex problem (1), (2). Assume that function u is strictly concave and Assumptions A1 and A4 hold. Then there exists a unique optimal stationary point x^* and all the assertions 1-3 of Theorem 2.5 are valid.*

This theorem shows that if the utility function u is strictly concave Assumption A3 in Theorem 3.1 can be removed. On the other hand Assumption A4 is still required in this case. The following example shows that if Assumption A4 does not hold then Theorems 3.1 and 3.3 may not be true.

Example 3.4. Let $\Omega = [-1, 1] \subset R^1$,

$$a(x) = \begin{cases} [-1, -x^4], & \text{if } x \in [0, 1] \\ [-1, 0], & \text{if } x \in [-1, 0]; \end{cases}$$

and $u(x) = -x^2 + 2x$.

It is clear that function u is strictly concave, the graph of mapping a is a convex set. We have $M = [-1, 0]$, $u^* = \max_{x \in M} u(x) = 0$ and $x^* = 0$.

It is not difficult to observe that Assumption A4 is not satisfied. We will show that, Theorem 3.1 is not true in this case.

Take an initial point $x^0 = 1$ and consider the following (obviously optimal) trajectory

$$\dot{x} = -x^4, \; x(0) = 1.$$

We have $x(t) = (3t + 1)^{-\frac{1}{3}}$. Clearly $0 \le x(t) \le 1$ and $x(t) \to 0$ as $t \to \infty$. Therefore $x(t)$ is a trajectory and

$$\int_0^T (u(x(t)) - u^*)dt = \int_0^T [-x^4(t) + 2x(t)]dt \ge \int_0^T x(t)dt = \int_0^T (3t+1)^{-\frac{1}{3}} dt \to +\infty,$$

as $T \to \infty$. Thus, the first assertion of Theorem 3.1 is not true.

3.2. **Mapping a is strictly convex.** In this case we assume that for all $x_1, x_2 \in \Omega$ and $\alpha \in (0,1)$, the following holds

$$\text{int } a(\alpha\, x_1 + (1-\alpha)\, x_2) \supset \alpha\, a(x_1) + (1-\alpha)\, a(x_2).$$

As mentioned in Remark 2.4 this assumption does not necessarily mean strictly convexity of the graph a; that is, the images $a(x)$ may not be strictly convex sets.

In this case Assumption A2 holds. In addition if Assumptions A1, A3 and A4 hold, then, according to Theorem 3.1, there exists a unique optimal stationary point x^* and all the assertions 1-3 are valid. The aim here is to investigate if it is possible to eliminate Assumption A4 if mapping a is strictly convex.

It is not difficult to show that Assumption A4 holds if, for example, the set of stationary points M is not a singleton; that is, there are $x_1 \neq x_2$ with $0 \in a(x_i),\ i = 1, 2$. Thus, if Assumption A4 does not hold then x^* must be the only stationary point.

Example 3.4 shows that the first assertion of Theorem 3.1 may not be true in this case. However, the following theorem states that the turnpike property (that is, the assertions 2 and 3 of Theorem 3.1) is still valid without assuming Assumption A4.

Theorem 3.5. *Consider convex problem (1), (2). Assume that mapping a is strictly convex, Assumptions A1 and A3 hold. Then there exists a unique optimal stationary point x^* and the assertions 2 and 3 of Theorem 2.5 are valid.*

4. Preliminary results

In this section we consider the convex problem (1), (2). For a given set $A \subset R^n$, we denote $a(A) = \cup_{x \in A} a(x)$. First we show that there exists a stationary point.

Lemma 4.1. *Assume that Assumption A1 holds. Then the set of stationary points M is not empty.*

Proof: Let $M = \emptyset$; that is, $0 \notin a(x)$ for all $x \in \Omega$. This means that $0 \notin a(\Omega)$. Since the graph of mapping a is convex, the set $a(\Omega)$ is also convex. Therefore there is a non-zero vector $v \in R^n$ and a number $\varepsilon > 0$ such that

$$v\, y \ \geq \ \varepsilon \text{ for all } y \in a(\Omega).$$

Consider any solution $x(\cdot)$ to the system (1). The last inequality shows that for all t we have

$$v\, x(t) \ = \ v x^0 + \int_0^t v\dot{x}(s)\, ds \ \geq \ v\, x^0 + t\,\varepsilon.$$

Thus $\|x(t)\| \to \infty$ as $t \to \infty$. Therefore, for large T system (1) does not have any bounded solutions remaining in Ω. This contradicts Assumption A1. Lemma is proved.

Now consider the following set

$$\mathcal{B} = \{x \in \Omega : \ u(x) \geq u^*\}. \tag{9}$$

Since u is concave, $\mathcal{B}$ is a convex compact set. Then, the set $a(\mathcal{B})$ is also convex and compact.

Lemma 4.2. *Assume that Assumption A3 holds. Then* $\operatorname{int}\mathcal{B} \neq \emptyset$ *and*

$$x \in \partial\mathcal{B}, \text{ for all } x \in \operatorname{int}\Omega \text{ satisfying } u(x) = u^*. \tag{10}$$

Proof: By Assumption A3 there is x' such that $u(x') > u^*$. Since u is continuous it follows that this inequality is satisfied in a small neighborhood of x'; that is, $\operatorname{int}\mathcal{B} \neq \emptyset$.

Now consider any point $x \in \operatorname{int}\Omega$ satisfying $u(x) = u^*$. Take an arbitrary small number $\varepsilon > 0$ and denote $x_\varepsilon = x - \varepsilon(x' - x)$. We have $x = \frac{1}{1+\varepsilon}\, x_\varepsilon + \frac{\varepsilon}{1+\varepsilon}\, x'$ and since u is concave

$$u^* = u(x) \geq \frac{1}{1+\varepsilon}\, u(x_\varepsilon) + \frac{\varepsilon}{1+\varepsilon}\, u(x') > \frac{1}{1+\varepsilon}\, u(x_\varepsilon) + \frac{\varepsilon}{1+\varepsilon}\, u^*$$

that yields $u(x_2^\varepsilon) < u^*$. Thus for all sufficiently small $\varepsilon > 0$ the relation $x_2^\varepsilon \notin \mathcal{B}$ holds; that is, $x \in \partial\mathcal{B}$.

Lemma is proved.

Lemma 4.3. *Assume that Assumptions A1 and A3 hold. Then there exist a non-zero vector* $q \in R^n$ *and a number* γ *such that*

$$q\,x \;\geq\; \gamma, \text{ if } x \in \mathcal{B}, \tag{11}$$

$$q\,x \;\leq\; \gamma, \text{ if } x \in M. \tag{12}$$

Proof: Assumption A1 ensures that the set of stationary points M is not empty (Lemma 4.1), on the other hand, Assumption A3 ensures that set $\mathcal{B}$ has a non-empty interior (Lemma 4.2). Since these sets are convex compact, it is sufficient to show that

$$\operatorname{int}\mathcal{B} \cap M = \emptyset.$$

This relation is straightforward; if $x \in \operatorname{int}\mathcal{B}$ then from Lemma 4.2 it follows $u(x) > u^*$ that means $x \notin M$. Therefore, the prof of the lemma follows from the linear separability of disjoint convex sets $\operatorname{int}\mathcal{B}$ and M.

Lemma is proved.

From Lemma 4.2 it follows that all all optimal stationary points are on the boundary of the set of stationary points; that is,

$$\text{if } u(x^*) = u^* \text{ and } 0 \in a(x^*) \text{ then } x^* \in \partial M. \tag{13}$$

We fix the vector q, for which relations (11), (12) hold, and then consider the set

$$B^* \;=\; \{x \in \Omega :\; q\,x \;\geq\; \gamma\}. \tag{14}$$

It is clear that $\mathcal{B} \subset B^*$ and B^* is convex. Then, the set $a(B^*) = \bigcup_{x \in B^*} a(x)$ is also convex.

Lemma 4.4. *Assume that Assumptions A1 and A3 hold. Then there exists a non-zero vector* $p \in R^n$ *such that*

$$p\,y \;\leq\; 0, \text{ for all } y \in a(x),\, x \in B^*. \tag{15}$$

Proof: We note that $0 \in a(B^*)$, because optimal stationary points x^* exist (Lemma 4.1) and $x^* \in \mathcal{B} \subset B^*$. Since the set $a(B^*)$ is convex, to prove the lemma it is sufficient to show that $0 \in \partial a(B^*)$.

Take any sequence $\varepsilon_n \to 0$ ($\varepsilon_n > 0$) and consider the sequence of sets

$$B_{\varepsilon_n} = \{x \in B^* : \, q\,x \geq \gamma + \varepsilon_n\}.$$

It is clear that B_{ε_n} is convex compact and $0 \notin a(x)$ for all $x \in B_{\varepsilon_n}$ (see (12)). Therefore $0 \notin a(B_{\varepsilon_n})$. Moreover, $a(B_{\varepsilon_n})$ are convex compact and $a(B_{\varepsilon_m}) \subset a(B_{\varepsilon_k})$ for $\varepsilon_k > \varepsilon_m$.

Denote $A = \mathrm{cl}\left(\cup_{n\geq 1} a(B_{\varepsilon_n}\right)$. Clearly $0 \notin \mathrm{int}\, A$. We show that $A = a(B^*)$.

Since $a(B_\varepsilon) \subset a(B^*)$ for all $\varepsilon > 0$ we have $A \subset a(B^*)$.

Let $y' \in a(B^*)$; that is, $y' \in a(x')$ for some $x' \in B^*$. If $q\,x' > \gamma$ then for sufficiently large numbers n the inclusion $x' \in B_{\varepsilon_n}$ holds. Therefore in this case $y' \in A$. Now consider the case when $q\,x' = \gamma$. In this case there is a sequence $x_n \in B_{\varepsilon_n}$ such that $x_n \to x'$ as $n \to \infty$. Since mapping a is continuous there is a sequence $y_n \in a(x_n) \subset a(B_{\varepsilon_n})$ such that $y_n \to y'$. As the set A is closed $y' \in A$. Thus $a(B^*) \subset A$.

Therefore, $A = a(B^*)$. Since $0 \in a(B^*)$ and $0 \notin \mathrm{int}\, A$ we obtain $0 \in \partial\left(a(B^*)\right)$. Lemma is proved.

Now we fix non-zero vector p for which relation (15) holds and define the following support function

$$c(x) = \max_{y \in a(x)} py. \tag{16}$$

For the sake of simplicity we will take $||p|| = 1$. It is clear that c is a concave function and

$$c(x) \leq 0 \text{ for all } x \in B^*. \tag{17}$$

Note that if $0 \in \mathrm{int}\, a(\tilde{x})$ for some $\tilde{x} \in \Omega$, then $c(\tilde{x}) > 0$.

The following lemma directly follows from the definition of $c(x)$ in (16).

Lemma 4.5. *Given any points $x_1, x_2 \in \Omega$ and any number $\alpha \in (0,1)$, if*

$$\mathrm{int}\, a(\alpha\, x_1 + (1-\alpha)\, x_2) \supset \alpha\, a(x_1) + (1-\alpha)\, a(x_2)$$

then

$$c(\alpha\, x_1 + (1-\alpha)\, x_2) > \alpha\, c(x_1) + (1-\alpha)\, c(x_2).$$

In the next Lemma the uniqueness of the optimal stationary point x^* is established.

Lemma 4.6. *Assume that Assumptions A1, A2 and A3 hold. Then there exists a unique optimal stationary point x^*.*

Proof: Note that the existence of stationary points follows from Lemma 4.1.

Assume that, together with x^*, there exists another optimal stationary point $x_1^* \neq x^*$, with $u(x_1^*) = u(x^*) = u^*$. Since $0 \in a(x^*)$ and $0 \in a(x_1^*)$, the relations $c(x_1^*) \geq 0$ and $c(x^*) \geq 0$ hold. On the other hand, $x_1^*, x^* \in \mathcal{B} \subset B^*$ and, therefore, from (17) we have

$$c(x_1^*) = c(x^*) = 0. \tag{18}$$

Consider convex combinations of these points:

$$x_\alpha \;=\; \alpha\, x_1^* + (1-\alpha)\, x^*,\;\; \alpha \in (0,1).$$

Since M is convex we have $x_\alpha \in M$. Therefore,

$$u^* \;\geq\; u(x_\alpha) \;\geq\; \alpha\, u(x_1^*) + (1-\alpha)\, u(x^*) \;=\; u^*;$$

that is, $u(x_\alpha) = u^*$ for all $\alpha \in (0,1)$.

The last equality means that $x_\alpha \in \mathcal{B} \subset B^*$ or $c(x_\alpha) \leq 0$ for all $\alpha \in (0,1)$ (see (17)). On the other hand function c is concave and therefore from (18) we have

$$c(x_\alpha) \;\geq\; \alpha\, c(x_1^*) + (1-\alpha)\, c(x^*) \;=\; 0,$$

for all $\alpha \in (0,1)$. Thus $c(x_\alpha) \;=\; 0$ for all $\alpha \in (0,1)$.

Therefore, functions u and c are not strictly concave over the interval $[x_1^*, x^*]$. This contradicts Assumption A2 and Lemma 4.5.

Lemma is proved.

4.1. **Verifying Assumption H.** In this section we assume that Assumptions A1-A4 hold. In this case we show that Assumptions H1 and H3 are satisfied for $z = p$, where p is a nonzero vector defined in Lemma 4.4. Therefore, we deal with the support function $c(z,x) = c(x)$.

1. First we show that Assumption H1 holds; that is,

$$c(x) < 0 \text{ for all } x \in \mathcal{B},\;\; x \neq x^*. \tag{19}$$

According to Lemma 4.2 the set B^* has a non-empty interior. Take any point $x' \in \operatorname{int} B^*$. From (17) it follows $c(x') \leq 0$. We show that in fact $c(x') < 0$.

By the contrary assume that $c(x') = 0$. From Assumption A4 there is a point $\tilde{x}$ for which the inclusion $0 \in \operatorname{int} a(\tilde{x})$ holds. Clearly $c(\tilde{x}) > 0$. Consider the points $x_\alpha = \alpha\, \tilde{x} + (1-\alpha)\, x'$, $0 < \alpha \leq 1$. We have

$$c(x_\alpha) \geq \alpha\, c(\tilde{x}) + (1-\alpha)\, c(x').$$

From this inequality we obtain $c(x_\alpha) > 0$ for all $\alpha > 0$. Clearly, for sufficiently small numbers $\alpha > 0$, points x_α belong to the interior of the set B^* and, therefore, from (17) it follows that $c(x_\alpha) \leq 0$. This is a contradiction.

Therefore, the following is true

$$c(x) < 0 \text{ for all } x \in \operatorname{int} B^*. \tag{20}$$

In particular this inequality holds for all points $x \in \operatorname{int} \mathcal{B} \subset \operatorname{int} B^*$.

Now we show that the relation (20) is valid for all boundary points $x' \neq x^*$ of the set $\mathcal{B}$. For such points x' the relation $u(x') = u^*$ holds.

Take any $x' \in \partial\mathcal{B}$, with $x' \neq x^*$, and assume by the contrary that $c(x') = 0$. Note that $c(x^*) = 0$ and

$$c(x_\alpha) \;\geq\; \alpha\, c(x') + (1-\alpha)\, c(x^*) = 0 \tag{21}$$

for all $x_\alpha = \alpha\, x' + (1-\alpha)\, x^*$, $0 \leq \alpha \leq 1$.

Since function u is concave we have

$$u(x_\alpha) \;\geq\; \alpha\, u(x') + (1-\alpha)\, u(x^*) = u^* \text{ for all } \alpha \in (0,1). \tag{22}$$

This means that $x_\alpha \in \mathcal{B} \subset B^*$. Then according to Assumption A2, the inequality in (22) should be strong; that is, $u(x_\alpha) > u^*$. This means that $x_\alpha \in \operatorname{int}\mathcal{B} \subset \operatorname{int} B^*$, which leads to a contradiction thanks to $c(x_\alpha) = 0$ and (20).

Therefore, $c(x) < 0$ for all $x \in \mathcal{B}$, $x \neq x^*$; that is, Assumption H1 holds.

2. Now we show that Assumption H3 holds.

Take any two points x, y for which $c(x) < 0$, $c(y) > 0$. Consider numbers α, β defined as follows:

$$\alpha = \frac{c(y)}{|c(x)| + c(y)} > 0, \ \ \beta = \frac{|c(x)|}{|c(x)| + c(y)} > 0.$$

Clearly $\alpha + \beta = 1$. Let $x' = \alpha\, x + \beta\, y$. Since function u is concave we have

$$\alpha\, u(x) + \beta\, u(y) \ \leq \ u(x'). \tag{23}$$

On the other hand, c is also concave and, therefore, we obtain

$$c(x') \ \geq \ \alpha\, c(x) + \beta\, c(y) \ = \ 0. \tag{24}$$

This inequality shows that $x' \notin \operatorname{int}\mathcal{B}$; that is, $u(x') \leq u^*$ (see (20)).

We show that

$$\alpha\, u(x) + \beta\, u(y) \ < \ u^*. \tag{25}$$

Consider two cases according to Assumption A2.

a). Let function u is strictly concave on $[x, y]$. In this case the inequality in (23) is strong and therefore inequality (25) follows from $u(x') \leq u^*$.

b). Let function c is strictly concave. In this case the inequality in (24) is strong: $c(x') > 0$. This means that $x' \notin \mathcal{B}$; that is, $u(x') < u^*$. Then from (23) we have (25).

Therefore (25) is true. Then

$$\alpha\,(u(x) - u^*) + \beta\,(u(y) - u^*) \ < \ 0,$$

or

$$\varphi(x, y) \ = \ \frac{u(x) - u^*}{|c(x)|} + \frac{u(y) - u^*}{c(y)} \ < \ 0. \tag{26}$$

Thus, for all points x, y for which $c(x) < 0$, $c(y) > 0$ the inequality $\varphi(x, y) < 0$ is satisfied; i.e. the first part of Assumption H3 holds even for a larger set of points (x, y) (the additional condition $px = py$ is not required in this case).

Now we check the second part of Assumption H3. By Assumption A4 the relation $0 \in \operatorname{int} a(\tilde{x})$ holds for some $\tilde{x}$ and, therefore, $c(\tilde{x}) > 0$. We set $y = \tilde{x}$ in (26) and obtain

$$\frac{u(x) - u^*}{|c(x)|} \ < \ -\frac{u(\tilde{x}) - u^*}{c(\tilde{x})} \ < \ +\infty \text{ for all } \ x, \ c(x) < 0.$$

Therefore

$$\frac{u(x) - u^*}{|c(x)|} \ \leq \ \lambda \text{ for all } \ x, \ c(x) < 0, \tag{27}$$

where

$$\lambda = \sup_{x,\, c(x) < 0} \frac{u(x) - u^*}{|c(x)|} < \infty.$$

Consider sequences x_k, y_k such that

$$c(x_k) < 0,\ x_k \to x^* \text{ and } c(y_k) > 0,\ y_k \to y' \neq x^*. \tag{28}$$

First we note that $x_k \neq x^*$ since $c(x^*) = 0$. Moreover from Assumption H1 and inequality $c(y_k) > 0$ it follows that $y_k \notin \mathcal{B}$; that is $u(y_k) < u^*$ for all k.

Let $c(y') = 0$. In this case $u(y') < u^*$ holds. Indeed, otherwise $u(y') \geq u^*$ and we obtain

$$u(x^\mu) \geq \mu\, u(x^*) + (1-\mu)u(y') \geq u^*,\ \forall \mu \in (0,1).$$

This means $x^\mu \neq x^*$ and $x^\mu \in \mathcal{B} \subset B^*$ that contradicts (19). Therefore, $u(y') < u^*$ and

$$\frac{u(y_k) - u^*}{c(y_k)} \to -\infty \text{ as } k \to \infty;$$

that is, Assumption H3 holds thanks to (27).

Consider the case $c(y') > 0$. From (26), (27) we have

$$\frac{u(y) - u^*}{c(y)} \ < \ -\lambda \text{ for all } \ y,\ c(y) > 0. \tag{29}$$

This inequality, in particular, is satisfied for all $y = y_k$. Consider the limit point $y = y'$.

First we show that the equality

$$\frac{u(y') - u^*}{c(y')} \ = \ -\lambda. \tag{30}$$

is not possible.

Denote $y^\mu = \mu\, x^* + (1-\mu)\, y'$, $\mu \in [0,1]$. As function $c(y)$ is concave, $c(x^*) = 0$ and $c(y') > 0$, for any given $\mu' \in (0,1)$ the following holds

$$c(y^\mu) \ \geq \ \mu\, c(x^*) + (1-\mu)\, c(y') \ = \ (1-\mu)\, c(y') \ > \ 0, \text{ for all } \ \mu \in [0, \mu']. \tag{31}$$

This in particular, means that (see Assumption H1) $y^\mu \notin \mathcal{B}$ or $u(y^\mu) - u^* < 0$. Moreover, from (29) we obtain

$$\frac{u(y^\mu) - u^*}{c(y^\mu)} \ < \ -\lambda \text{ for all } \ \mu \in [0, \mu']. \tag{32}$$

On the other hand function $u(y)$ is concave. Then

$$u(y^\mu) \ \geq \ \mu\, u(x^*) + (1-\mu)\, u(y'),$$

or

$$(1-\mu)\, [u(y') - u^*] \leq u(y^\mu) - u^* < 0, \text{ for all } \ \mu \in [0, \mu']. \tag{33}$$

Thus, if (30) is true, then from (31), (33) we have (note that $u(y^\mu) - u^*$ is negative)

$$\frac{u(y^\mu) - u^*}{c(y^\mu)} \ \geq \ \frac{(1-\mu)\, [u(y') - u^*]}{(1-\mu)\, c(y')} \ = \ -\lambda.$$

This contradicts (32); i.e. the case (30) is not possible.

Therefore, at the limit point $y = y'$ the inequality

$$\frac{u(y') - u^*}{c(y')} \ < \ -\lambda$$

holds. In this case, as $y_k \to y'$ and functions u and c are continuous, there is a small number $\varepsilon > 0$ such that for sufficiently large numbers k the following holds

$$\frac{u(y_k) - u^*}{c(y_k)} < -\lambda - \varepsilon.$$

Therefore, for sufficiently large numbers k from (27) we have

$$\varphi(x_k, y_k) = \frac{u(x_k) - u^*}{|c(x_k)|} + \frac{u(y_k) - u^*}{c(y_k)} \leq -\varepsilon < 0.$$

Then $\limsup_{k\to\infty} \varphi(x_k, y_k) \leq -\varepsilon < 0$; that is, Assumption H3 holds.

5. Proof of Theorems

5.1. **Proof of Theorem 3.1.** In the previous section we have shown that all the assumptions of Theorem 2.5 are satisfied, except H2. For each x, the set $a(x)$ is uniformly locally connected as it is a convex set. Lipschitz continuity of mapping a is guaranteed with relation (5) in a closed subset Ω' that contains all trajectories starting from x^0. The proof of Theorem 2.5 is based on the following lemmas, namely Lemmas 5.1, 5.2, 5.5 and 5.6 provided below. To prove Theorem 3.1 it sufficient to show that all these lemmas are true under the assumptions of this theorem, without involving H2.

We start with some notations. Recall that $\mathcal{B} = \{x \in \Omega : u(x) \geq u^*\}$. Denote

$$\mathcal{M}^* = \{x \in \Omega : c(x) \geq 0\}.$$

Clearly the set of stationary points $M \subset \mathcal{M}^*$. Denote by $\mathcal{D} \subset \Omega$ a compact set for which the following conditions hold:

a) $x \in \text{ int } \mathcal{D}$ for all $x \in \mathcal{B},\ x \neq x^*,\ x \neq \partial\Omega$;
b) $c(x) < 0$ for all $x \in \mathcal{D},\ x \neq x^*$;
c) $\mathcal{D} \cap \mathcal{M}^* = \{x^*\}$ and $\mathcal{B} \subset \mathcal{D}$.

It is not difficult to construct such a set $\mathcal{D}$ with properties a), b), c). For example, it can be constructed as follows.

Let $x \in \mathcal{B},\ x \neq x^*$. Then $c(x) < 0$. Since mapping a is continuous in the Hausdorff metric function $c(x)$ is also continuous. Therefore there exists $\varepsilon_x > 0$ such that $c(x') < 0$ for all $x' \in V_{\varepsilon_x}(x) \cap \Omega$. Here the notion $V_\varepsilon(x)$ stands for the open ε-neighborhood of the point x. In this case the set

$$\mathcal{D} = \text{ cl } \{ \bigcup_{x\in\mathcal{B}, x\neq x^*} V_{\frac{1}{2}\varepsilon_x}(x)\} \cap \Omega$$

satisfies conditions a) - c).

The following lemma follows from the continuity of functions u and c.

Lemma 5.1. (Lemmas 1 and 2 in [25]) *For every $\varepsilon > 0$ there exist $\nu_\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that*

$$u(x) \leq u^* - \nu_\varepsilon,\ \forall x \in \Omega,\ x \notin \text{int } \mathcal{D},\ ||x - x^*|| \geq \varepsilon;$$
$$c(x) < -\eta_\varepsilon,\ \forall x \in \mathcal{D},\ ||x - x^*|| \geq \varepsilon.$$

This lemma is used in the proof of the following lemma that is dealing with the transformation of Assumption H3.

Lemma 5.2. (Lemma 3 in [25]) *Assume that at the point $x' \in \mathcal{D}$, $y \in \mathcal{M}^*$ we have $px' = py$, $c(x') < 0$, $c(y) > 0$. Then for every point x and numbers δ_1, δ_2, satisfying*

$$x \in \mathrm{cl}\left(V_{\eta(x',y)}(x')\right),\ c(x) < 0,\ 0 \le \delta_1 \le \bar{\delta}_1,\ 0 \le \delta_2 \le \delta_2(y),$$

the following inequality holds

$$\frac{u(x)}{|c(x)| + \delta_1} + \frac{u(y)}{c(y) + \delta_2} \le u^* \left(\frac{1}{|c(x)| + \delta_1} + \frac{1}{c(y) + \delta_2} \right) - \delta(y).$$

Here functions $\eta(x', y)$ and $\delta(y)$ such that $\delta(y)$ is continuous and for every $\varepsilon > 0$, $\tilde{\varepsilon} > 0$ there exist $\hat{\delta}_\varepsilon > 0$ and $\hat{\eta}_{\varepsilon,\tilde{\varepsilon}} > 0$ such that

$$\delta(y) \ge \hat{\delta}_\varepsilon \text{ and } \eta(x', y) \ge \hat{\eta}_{\varepsilon,\tilde{\varepsilon}}, \quad \text{for all } (x', y) \text{ satisfying } ||x' - x^*|| \ge \tilde{\varepsilon},\ ||y - x^*|| \ge \varepsilon.$$

In the proof of this lemma, Assumption H2 is only used to show the existence of a finite number $b \in (0, +\infty)$ such that the following inequality holds (see (5.6) in [25])

$$\frac{u(x) - u^*}{|c(x)|} \le b,\ \forall x \in \mathcal{D},\ x \ne x^*. \tag{34}$$

For convex problems under Assumptions A1-A4, a similar inequality (27) is proved in the previous section, where the required finite number is $b = \lambda$ and the inequality holds for all x satisfying $c(x) < 0$. From definition of set $\mathcal{D}$ (part b) we know that for all $x \in \mathcal{D}$, $x \ne x^*$, the inequality $c(x) < 0$ holds and therefore (34) is also true. The rest of the proof remains the same as the proof of Lemma 3 in [25].

We define two types of sets.

Definition 5.3. *$\pi \subset [0, T]$ is called a set of 1-st type on the interval $[p_2, p_1]$ if the following conditions hold:*

a). *The set π consists of two sets π_1 and π_2, that is $\pi = \pi_1 \cup \pi_2$, such that $x(t) \in \mathrm{int}\ \mathcal{D}$, $\forall t \in \pi_1$ and $x(t) \notin \mathrm{int}\ \mathcal{D}$, $\forall t \in \pi_2$.*

b). *The set π_1 consists at most countable number of intervals Δ_k, with end-points $t_1^k < t_2^k$, and the intervals $(px(t_2^k), px(t_1^k))$, $(k = 1, 2, ...)$ are disjoint.*
Clearly, in this case intervals $\Delta_k^0 \doteq \mathrm{int}\ \Delta_k = (t_1^k, t_2^k)$ are also disjoint.

c). $p_1 \ge \sup_k px(t_1^k)$, $p_2 \le \inf_k px(t_2^k)$.

Definition 5.4. *$\omega \subset [0, T]$ is called a set of 2-nd type on the interval $[p_2, p_1]$ if the following conditions hold:*

a). $x(t) \notin \mathrm{int}\ \mathcal{D}$, $\forall t \in \omega$.

b). *The set ω consists at most countable number of intervals $[s_2^k, s_1^k]$, such that the intervals $(px(s_2^k), px(s_1^k))$, $(k = 1, 2, ...)$ are nonempty and disjoint, and*

$$p_1 - p_2 = \sum_k [px(s_1^k) - px(s_2^k)].$$

Lemma 5.5. (Lemma 7 in [25]) *Assume that trajectory $x(t) \in X_T$ is a continuously differentiable function, the sets π $(= \pi_1 \cup \pi_2)$ and ω are of 1-st type and 2-nd type on the same interval $[p_2, p_1]$, respectively. Then*

$$\int\limits_{\pi\cup\omega} u(x(t))dt \;\leq\; u^* \text{ meas}\,(\pi \cup \omega) - \int\limits_{Q} [u^* - u(x(t))]dt - \int\limits_{E} \delta^2(x(t))dt;$$

Where

a). $Q \cup E = \omega \cup \pi_2 = \{t \in \pi \cup \omega : \; x(t) \notin \text{ int } \mathcal{D}\}$.

b). For every $\varepsilon > 0$ there exists a number $\delta_\varepsilon > 0$ such that

$$\delta^2(x) \geq \delta_\varepsilon, \; \forall x, \; ||x - x^*|| \geq \varepsilon.$$

c). For every $\delta > 0$ there exists a number $K(\delta) < \infty$ such that

$$\text{meas}[(\pi \cup \omega) \cap Z_\delta] \;\leq\; K(\delta) \text{ meas}\,[(Q \cup E) \cap Z_\delta],$$

here $Z_\delta = \{t \in [0, T] : \; |px(t) - p^*| \geq \delta\}$.

The proof of this lemma is based on Lemmas 5.1, 5.2 and Assumptions H1, H3. Since they are true in our case Lemma 5.5 is also true.

Lemma 5.6. (Lemmas 13 and 14 in [25]) *Assume that trajectory $x(t) \in X_T$ is a continuously differentiable function. Then, interval $[0, T]$ can be divided into at most countable number of disjoint subsets such that*

$$[0, T] = \cup_n(\pi_n \cup \omega_n) \cup F_1 \cup F_2 \cup E, \tag{35}$$

$$\int\limits_0^T u(x(t))\,dt = \sum_n \int\limits_{\pi_n\cup\omega_n} u(x(t))\,dt + \int\limits_{F_1\cup F_2} u(x(t))\,dt + \int\limits_{E} u(x(t))\,dt. \tag{36}$$

Here

1. π_n and ω_n are the sets of 1-st type and 2-nd types, respectively, on the intervals $[p_n^2, p_n^1]$, $n = 1, 2, ...$

2. F_1 and F_2 are the sets of 1-st type on the intervals $[p_1^2, p_1^1]$ and $[p_2^2, p_2^1]$, respectively, and

$$x(t) \in \text{ int } \mathcal{D}, \; \textit{for all } t \in F_1 \cup F_2, \tag{37}$$

$$p_i^1 - p_i^2 \leq C < +\infty, \; i = 1, 2. \tag{38}$$

3.

$$x(t) \notin \text{ int } \mathcal{D}, \; \textit{for all } t \in E. \tag{39}$$

4. For every $\delta > 0$ there is a number $C(\delta)$ such that

$$\text{meas}\,[(F_1 \cup F_2) \cap Z_\delta] \leq C(\delta), \tag{40}$$

where the number $C(\delta) < +\infty$ does not depend on trajectory $x(t)$, on T and on the intervals in (35).

5. There is a number $L < +\infty$ such that

$$\int\limits_{F_1 \cup F_2} [u(x(t)) - u^*] \, dt < L \tag{41}$$

and L does not depend on trajectory $x(t)$ and on T.

Proof: The proof of all the assertions, except inequality (41), do not require Assumption H2 (see the proof of Lemma 13 in [25]). This assumption is used only for the proof of inequality (41). We show that this inequality is valid if Assumption A4, instead of Assumption H2, is satisfied.

For the sake of simplicity consider just one of the sets in (41); say $F = F_1$ and show that

$$\int\limits_F [u(x(t)) - u^*] \, dt \ < L_1 \tag{42}$$

where L_1 does not depend on trajectory $x(t)$ and on T.

F is a set of 1-st type on some bounded interval. According to Definition 5.3 and (37) this means that $p\dot{x}(t) < 0$ for all $t \in F$. Moreover $F = \cup_{k \geq 1} \Delta_k$, every set Δ_k is an interval with endpoints t_1^k, t_2^k, $\operatorname{int} \Delta_k \cap \operatorname{int} \Delta_m = \emptyset$ if $k \neq m$, the intervals $(px(t_2^k), px(t_1^k))$, $(k = 1, 2, ...)$ are disjoint. Since trajectories are bounded the inequality

$$\sum_k [px(t_1^k) - px(t_2^k)] < \hat{L}. \tag{43}$$

holds where $\hat{L}$ does not depend on trajectory $x(t)$ and on T.

We show that the inequality (42) is true. Take any k and denote $s = px(t)$. Let $s_i^k = px(t_i^k), i = 1, 2$. Since $p\dot{x}(t) < 0$ for all $t \in (t_1^k, t_2^k) \subset \Delta_k$, there exists an inverse function $t = t(s)$. We have $dt = ds/p\dot{x}(t)$ and

$$\int\limits_{\Delta_k} [u(x(t)) - u^*] \, dt \ = \int\limits_{s_1^k}^{s_2^k} \frac{u(x(t(s))) - u^*}{p\, x(t(s))} \, ds \ = \int\limits_{s_2^k}^{s_1^k} \frac{u(x(t(s))) - u^*}{-p\, x(t(s))} \, ds. \tag{44}$$

On the other hand, for all $s \in (s_1^k, s_2^k)$ we have $x(t(s)) \in \operatorname{int} \mathcal{D}$ that means $c(x(t(s))) < 0$. Then from $p\, x(t(s)) \leq c(x(t(s)))$ we have

$$-p\, x(t(s)) \geq -c(x(t(s))) = |c(x(t(s)))| > 0.$$

Thus if $u(x(t(s))) - u^* > 0$ for $s \in (s_1^k, s_2^k)$, then from relation (27) it follows that

$$\frac{u(x(t(s))) - u^*}{-p\, x(t(s))} \leq \frac{u(x(t(s))) - u^*}{|c(x(t(s)))|} \leq \lambda. \tag{45}$$

Now, if $u(x(t(s))) - u^* \leq 0$ for some $s \in (s_1^k, s_2^k)$ then

$$\frac{u(x(t(s))) - u^*}{-p\, x(t(s))} \leq 0.$$

Thus the relation (45) is true for all $s \in (s_1^k, s_2^k)$ (note that $\lambda > 0$) and therefore from (44) it follows

$$\int_{\Delta_k} [u(x(t)) - u^*]\, dt \;\le\; \lambda[s_1^k - s_2^k].$$

Summing over k and taking into account (43) we obtain

$$\int_F [u(x(t)) - u^*]\, dt \;=\; \sum_k \int_{\Delta_k} [u(x(t)) - u^*]\, dt \;\le\; \lambda \sum_k [s_1^k - s_2^k] \;<\; \lambda \hat{L}.$$

This means that (42) holds for $L_1 = \lambda \hat{L}$, and therefore (41) also holds.

We have shown that under the assumptions of Theorem 3.1 all the required preliminaries for the proof of Theorem 2.5 (that is, Lemmas 5.1, 5.2, 5.5 and 5.6) are true without involving H2. This completes the proof of Theorem 3.1.

We note that Lemmas 5.5 and 5.6 are for continuously differentiable trajectories. It is known that (see, for example, Theorem 6 in [5], all conditions of this theorem hold in our case) given any $\varepsilon > 0$ and any trajectory $x(t)$ defined on $[0, T]$, there exists a continuously differentiable trajectory $\hat{x}(t)$ such that $\hat{x}(0) = x(0)$ and

$$\|x(t) - \hat{x}(t)\| < \varepsilon, \;\; \forall t \in [0, T].$$

The proof of Theorem 2.5 in [25] is first performed for continuously differentiable trajectories. Then by using the above property the theorem is extended to any absolutely continuous trajectory of the system (1).

5.2. **Proof of Theorem 3.3.** We assume that the function u is strictly concave. In this case optimal stationary point x^* is unique and Assumption A2 holds. If Assumption A3 holds then Theorem 3.3 follows from Theorem 3.1.

Assume that Assumption A3 does not hold. Since u is strictly concave the set $\mathcal{B} = \{x \in \Omega : \; u(x) \ge u(x^*)\}$, defined in (9), consists of one point; that is $\mathcal{B} = \{x^*\}$, and

$$u(y) \;<\; u^* \text{ for all } y \in \Omega, \; y \neq x^*. \tag{46}$$

Moreover, given any $\varepsilon > 0$ there is $\delta > 0$ such that

$$u(y) - u^* \le -\delta \text{ for all } y \in \Omega, \; \|y - x^*\| \ge \varepsilon. \tag{47}$$

In this case, for optimal and ξ-optimal trajectories meas$\{t \in [0, T] : \; ||x(t) - x^*|| \ge \varepsilon\}$ can not growth infinitely; that is the turnpike property is true. Assertions 1 and 3 are also trivially satisfied in this case. Therefore Theorem 3.3 is true.

In what follows we provide another proof; we show that Assumption H holds in this case and then Theorem 3.3 follows from Theorem 2.5.

Take any $z \in R^n$ and consider a support function

$$c(z, x) = \max_{y \in a(x)} zy.$$

Verifying H1. Clearly, Assumption H1 holds for all $z \in R^n$, since $\mathcal{B} \setminus x^* = \emptyset$.

Verifying H2. By the assumption A4 there is a point $\tilde{x} \in M$ such that $0 \in \operatorname{int} a(\tilde{x})$. Since mapping a is continuous, the relation $0 \in \operatorname{int} a(x)$ holds for all points x in a small neighborhood of $\tilde{x}$. Thus there is a point $x' \in M$ such that $0 \in \operatorname{int} a(x')$ and $x' \neq x^*$.

Take any non-zero z such that the scalar product $z(x^*-x')=0$; that is, $zx^*=zx'$. Since $0\in\ \mathrm{int}\,a(x')$ we have

$$c(z,x')=\max_{y\in a(x')} zy>0.$$

Thus Assumption H2 is satisfied.

Verifying H3. Take any z satisfying Assumption H2. Consider points $x,y\in\Omega$ such that $c(z,x)<0$ and $c(z,y)>0$. Clearly $x\neq x^*$ and from (46) we have $u(x)<u^*$, $u(y)\le u^*$. Thus the inequality

$$\varphi(x,y)=\frac{u(x)-u^*}{|c(z,x)|}+\frac{u(y)-u^*}{c(z,y)}<0 \tag{48}$$

holds (see (8)). Now let

$$x_k\to x^*,\ y_k\to y'\neq x^*,\ zx_k=zy_k,\ c(z,x_k)<0,\ c(z,y_k)>0.$$

We have $u(x_k)<u^*$, $u(y_k)\le u^*$ and $u(y')<u^*$. Since function u is continuous there is a number $\varepsilon>0$ such that for all sufficiently large numbers k the inequality $u(y_k)\le u^*-\varepsilon$ is satisfied. Then from (48) for sufficiently large k we obtain

$$\varphi(x_k,y_k)=\frac{u(x_k)-u^*}{|c(z,x_k)|}+\frac{u(y_k)-u^*}{c(z,y_k)}<-\frac{\varepsilon}{c(z,y_k)}.$$

Note that z is a fixed point and therefore $c(z,y_k)$ is bounded above. Thus, $\limsup_{k\to\infty}\varphi(x_k,y_k)<0$; that is, Assumption H3 is also satisfied.

Therefore Assumption H holds. Theorem is proved.

5.3. **Proof of Theorem 3.5.** Since mapping a is assumed to be strictly convex, Assumptions A1-A3 are satisfied. Then according to Lemma 4.6, there exists a unique optimal stationary point x^*. If Assumption A4 is also satisfied then Theorem 3.5 follows from Theorem 3.1.

Consider the case when Assumption A4 is not satisfied; that is,

$$0\notin \mathrm{int}\,a(x),\ \forall x\in\Omega. \tag{49}$$

Clearly, in this case the set of stationary points M consists of one point x^*. Indeed, if $0\in a(x')$ for some $x'\neq x^*$, then as mapping a is strictly convex, $0\in\ \mathrm{int}\,a(\frac{x'+x^*}{2})$ that contradicts (49). Consider the set

$$a(\Omega)=\bigcup_{x\in\Omega} a(x).$$

Note that $a(\Omega)\subset R^n$ is a convex compact set. Indeed, if $y_1,y_2\in a(\Omega)$; that is, $y_i\in a(x_i), i=1,2$, then from strictly convexity of mapping a we have

$$\lambda y_1+(1-\lambda)y_2\in \mathrm{int}\,a(\lambda x_1+(1-\lambda)x_2)\subset a(\Omega),\ \text{for all } \lambda\in(0,1).$$

Clearly, $0\in a(\Omega)$ since $0\in a(x^*)$. Now we show that 0 is on boundary of the set $a(\Omega)$. By the contrary assume that $0\in \mathrm{int}\,a(\Omega)$. Then, there are points $y_i\in a(x_i)$, $i=1,...,m$, $(m\le n+1)$ and nonnegative numbers λ_i, satisfying $\lambda_1+\cdots+\lambda_m=1$, such that

$$\lambda_1 y_1+\cdots+\lambda_m y_m=0.$$

In this case again, since mapping a is strictly convex,

$$0 = \lambda_1 y_1 + \cdots + \lambda_m y_m \in \operatorname{int} a(\lambda_1 x_1 + \cdots + \lambda_m x_m)$$

that contradicts (49).

Therefore, $a(\Omega)$ is a convex compact set and $0 \in \partial a(\Omega)$. Then there is a non-zero vector p such that

$$p\,y \leq 0,\ \forall y \in a(x),\ x \in \Omega. \tag{50}$$

Now we show that given any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$p\,y \leq -\delta_\varepsilon,\ \forall y \in a(x), x \in \Omega, \|x - x^*\| \geq \varepsilon. \tag{51}$$

If (51) is not true, then there is $\varepsilon > 0$ for which $p\,y_k \to 0$ for some sequence $y_k \in a(x_k)$, $\|x_k - x^*\| \geq \varepsilon$. Without loss of generality we can assume that $y_k \to y'$ and $x_k \to x'$, as $k \to \infty$. Thus, taking into account the fact that mapping a is continuous,

$$p\,y' = 0,\ y' \in a(x'),\ x' \neq x^*.$$

In this case we obtain $\frac{y'+0}{2} \in \operatorname{int} a(\frac{x'+x^*}{2})$. Then there is $\tilde{y} \in a(\frac{x'+x^*}{2})$ such that $p\,\tilde{y} > p\,\frac{y'+0}{2} = 0$. This contradicts (50). Thus (51) is true.

Take any trajectory $x(\cdot) \in X_T$ and denote

$$\tau_\varepsilon \ = \ \{t \in [0,T]:\ ||x(t) - x^*|| \geq \varepsilon\}.$$

From (50) we have $p\,\dot{x}(t) \leq 0$ for all t where $\dot{x}(t)$ exists. Thus function $p\,x(t)$ is decreasing and

$$p\,x(T) - p\,x^0 = \int_0^T p\,\dot{x}(t)\,dt \leq \int_{\tau_\varepsilon} p\,\dot{x}(t)\,dt \leq -\delta_\varepsilon \operatorname{meas} \tau_\varepsilon.$$

From this inequality we obtain

$$\operatorname{meas} \tau_\varepsilon \leq \frac{p\,x(T) - p\,x^0}{\delta_\varepsilon}.$$

Since $x(T) \in \Omega$ is bounded there exists $K_\varepsilon < \infty$ such that $\frac{p\,x(T) - p\,x^0}{\delta_\varepsilon} \leq K_\varepsilon$ for all T. Therefore the relation

$$\operatorname{meas} \{t \in [0,T]:\ ||x(t) - x^*|| \geq \varepsilon\} \leq K_\varepsilon$$

holds for all trajectories, and in particular for all ξ-optimal trajectories. That is, the assertion 2 of Theorem 3.1 is valid.

Now we show the third assertion. Let $x(t_1) = x(t_2) = x^*$. By the contrary, assume that $x(t') \neq x^*$ for some $t' \in (t_1, t_2)$. Then, thanks to (50), (51) the following holds

$$p\,x(t_2) < p\,x(t') < p\,x(t_1)$$

that leads to a contradiction. Thus, the third assertion is also true for all trajectories, and in particular for all optimal and ξ-optimal trajectories.

Theorem is proved.

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