# Polynomial time algorithms for the Minimax Regret Uncapacitated Lot Sizing Model

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**Abstract**

We study the Minimax Regret Uncapacitated Lot Sizing (MRULS) model, where the production cost function and the demand are subject to uncertainty. We propose a polynomial time algorithm which solves the MRULS model in $O(n^6)$ time. We improve this running time to $O(n^5)$ when only the demand is uncertain, and to $O(n^4)$ when only the production cost function is uncertain.

*Keywords:* robust optimization, minimax regret, lot sizing, production cost and demand uncertainties

## 1. Introduction

In the Uncapacitated Lot Sizing (ULS) model [1], the goal is to find a production plan that minimizes the total production and inventory holding cost, and at the same time satisfies the demands. In an $n$-period planning horizon, we assume that the demand in period $i$ is given by $d_i > 0$ for $i = 1, \ldots, n$. For each period $i$, there is a unit production cost $p_i$, an unit inventory cost $h_i$ and a setup cost $f_i$ if production takes place. The ULS model can be formulated as follows,

$$
\begin{aligned}
ULS\ :\min\ & \sum_{i=1}^{n}(f_i y_i + p_i x_i + h_i I_i)\\
\text{s.t. }\ & I_{i-1} + x_i = I_i + d_i,\ \forall i = 1, \ldots, n\\
& x_i \le M y_i,\ \forall i = 1, \ldots, n\\
& y_i \in \{0, 1\},\ \forall i = 1, \ldots, n\\
& x_i, I_i \ge 0,\ \forall i = 1, \ldots, n,
\end{aligned}
$$

where $M$ is a large constant, $x_i$ the amount produced in period $i$, $I_i$ the inventory level at the end of period $i$, and $y_i$ equals to 1 if $x_i > 0$ and 0 otherwise. Throughout this paper, we use bold characters for vectors, e.g., $\mathbf{y} = (y_i)$. Wagner and Whitin [1] proposed an $O(n^2)$ time algorithm to solve the ULS model based on the so-called the Zero Inventory Ordering (ZIO) property, i.e., $I_{i-1}x_i = 0$. This property implies that $\mathbf{x}$ and $\mathbf{I}$ are uniquely determined by $\mathbf{y}$ and we only need to optimize over $\mathbf{y}$.

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Realizing the uncertain nature of the input data, we study a robust version of the ULS model, the Minimax Regret ULS model (MRULS). Unlike the absolute robust models, see, e.g., [2, 3] for an overview, the Minimax Regret model minimizes the worst-case regret, or opportunity loss. Under demand uncertainty, Zhang [4] proposed an $O(n^7)$ time algorithm for the MRULS model. In this paper, we show that the MRULS model with uncertain production cost function, defined by $\mathbf{f}$ and $\mathbf{p}$, and uncertain demand $\mathbf{d}$, can be solved in $O(n^6)$ time. We improve this running time to $O(n^5)$ when only the demand is uncertain and to $O(n^4)$ when only the production cost function is uncertain.

## 2. Formulation and general results

We assume that $\mathbf{f}$, $\mathbf{p}$, and $\mathbf{d}$ are subject to interval uncertainty, i.e., $\mathbf{f} \in \mathcal{U}^f = \{\mathbf{f} : f_i \in [\underline{f}_i, \overline{f}_i], i = 1, \ldots, n\}$, and similarly for $\mathbf{p}$ and $\mathbf{d}$. Let $\mathbf{u} = (\mathbf{f}, \mathbf{p}, \mathbf{d}) \in \mathcal{U} := \mathcal{U}^f \times \mathcal{U}^p \times \mathcal{U}^d$ denote a realization of the uncertain data. We denote by $\mathbf{y}$ a first-stage setup decision, i.e., before knowing the actual realization $\mathbf{u}$, and $\hat{\mathbf{y}}^\star$ an optimal setup decision once $\mathbf{u}$ has been revealed. Then, for a given first-stage decision vector $\mathbf{y}$ and a realization of uncertainty $\mathbf{u}$, the regret is defined as follows,

$$\sum_{i=1}^{n}(f_i\, y_i + p_i\, x_i + h_i\, I_i) - \sum_{i=1}^{n}(f_i\, \hat{y}_i^\star + p_i\, \hat{x}_i^\star + h_i\, \hat{I}_i^\star).$$

Hence, the MRULS model is formulated as

$$\min_{\mathbf{y}\in\{0,1\}^n} \max_{\mathbf{u}\in\mathcal{U}} [\sum_{i=1}^{n}(f_i\, y_i + p_i\, x_i + h_i\, I_i) -$$

$$-\sum_{i=1}^{n}(f_i\,\hat{y}_i^\star + p_i\,\hat{x}_i^\star + h_i\,\hat{I}_i^\star)],$$

which is equivalent to

$$\min_{\mathbf{y}\in\{0,1\}^n}\max_{\hat{\mathbf{y}}\in\{0,1\}^n}\max_{\mathbf{u}\in\mathcal{U}}[\sum_{i=1}^{n} f_i\,(y_i-\hat{y}_i) + \sum_{i=1}^{n} p_i\,(x_i-\hat{x}_i)+ + \sum_{i=1}^{n} h_i\,(I_i-\hat{I}_i)]. \tag{1}$$

We introduce some notations that will be used throughout the paper. We define $d_{ij} := \sum_{k=i}^{j} d_k$, $h_{ij} := \sum_{k=i}^{j} h_k$, and $\alpha_{ij} := p_i + h_{i,j-1}$. In addition, we define $\underline{\alpha}_{ij} = \underline{p}_i + h_{i,j-1}$ and $\overline{\alpha}_{ij} = \overline{p}_i + h_{i,j-1}$. We introduce the period $n+1$ with $d_{n+1} = 0$ and $f_{n+1} = p_{n+1} = h_{n+1} = 0$. Without loss of optimality, we assume that $y_{n+1} = 1 = \hat{y}_{n+1}$. For a given first-stage decision vector $\mathbf{y}$, we define $m_i := \max\{k : y_k = 1, 1 \le k \le i\}$, i.e., the production period that satisfies the demand in period $i$. We also define $l_i := \min\{k : y_k = 1, i \le k \le n+1\}$, where if $l_i = n+1$, then there is no production in $[i, n]$. Similarly, we define $\hat{m}_i$ and $\hat{l}_i$ for $\hat{\mathbf{y}}$. Note that

$$\alpha_{ij} \le \alpha_{lj} \iff \alpha_{ir} \le \alpha_{lr} \quad \text{for } r \ge j, \tag{2}$$

while in any optimal solution of the second-stage ULS model,

$$\alpha_{\hat{m}_j,j} \le \alpha_{\hat{m}_i,j} \quad \text{for } j > i. \tag{3}$$

For given $\mathbf{y}$, $\hat{\mathbf{y}}$, $\mathbf{f}$, and $\mathbf{p}$, $m_i$ and $\hat{m}_i$ are known and we can rewrite the MRULS model as follows:

$$\max_{\mathbf{d}\in\mathcal{U}^d}[\sum_{i=1}^{n} f_i(y_i-\hat{y}_i) + \sum_{i=1}^{n}(\alpha_{m_i,i}-\alpha_{\hat{m}_i,i})d_i]. \tag{4}$$

Using (4), Zhang [4] shows that, without loss of optimality, we only need to consider the bounds of the uncertainty interval of $\mathbf{d}$ if $\mathbf{f}$ and $\mathbf{p}$ are given. We state the result here without a proof.

**Lemma 2.1.** *[4] For a given first-stage decision vector* $\mathbf{y}$*, realizations of* $\mathbf{f}$ *and* $\mathbf{p}$*, and a second-stage decision vector* $\hat{\mathbf{y}}$*, there exists an optimal scenario* $\mathbf{d}^\star$ *to (1), satisfying:*

$$d_i^\star = \begin{cases} \underline{d}_i & \text{if } \alpha_{m_i,i} < \alpha_{\hat{m}_i,i} \\ \overline{d}_i & \text{if } \alpha_{m_i,i} > \alpha_{\hat{m}_i,i} \\ \underline{d}_i \text{ or } \overline{d}_i & \text{if } \alpha_{m_i,i} = \alpha_{\hat{m}_i,i}. \end{cases}$$

We derive a similar result for the production cost function. For given $\mathbf{y}$, $\hat{\mathbf{y}}$, and $\mathbf{d}$, the MRULS model is equivalent to

$$\max_{\mathbf{f}\in\mathcal{U}^f}\max_{\mathbf{p}\in\mathcal{U}^p}[\sum_{i=1}^{n} f_i\,(y_i-\hat{y}_i) + \sum_{i=1}^{n} p_i\,(x_i-\hat{x}_i)], \tag{5}$$

in which the total inventory holding costs are eliminated.

**Lemma 2.2.** *For a given first-stage decision vector* $\mathbf{y}$*, a realization* $\mathbf{d}$*, and a second-stage decision vector* $\hat{\mathbf{y}}$*, there exist optimal scenarios* $\mathbf{f}^\star$ *and* $\mathbf{p}^\star$ *to (1), satisfying:*

$$(f_i^\star, p_i^\star) = \begin{cases} (\underline{f}_i, \underline{p}_i) & \text{if } x_i < \hat{x}_i \\ (\overline{f}_i, \overline{p}_i) & \text{if } x_i > \hat{x}_i \\ (\underline{f}_i, \underline{p}_i) \text{ or } (\overline{f}_i, \overline{p}_i) & \text{if } x_i = \hat{x}_i. \end{cases}$$

*Proof.* If $x_i < \hat{x}_i$, it is easy to see that the result holds from (5) since $(x_i - \hat{x}_i) < 0$ and $(y_i - \hat{y}_i) \le 0$. Similarly for the two other cases. □

## 3. The algorithm

In this section we propose a dynamic programming (DP) algorithm that decomposes the MRULS problem into subplans, defined by the regeneration intervals at the first level and the incoming and outgoing inventory at the second level.

### 3.1. The dynamic programming algorithm

We construct a network with nodes $(k, \hat{m}_k, \hat{l}_k)$ in which the DP moves from node $(k, \hat{m}_k, \hat{l}_k)$ to $(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$, defining the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$, for $1 \le k \le t \le n$, $1 \le \hat{m}_k \le k \le \hat{l}_k \le \hat{l}_{t+1}$, and $\hat{m}_k \le \hat{m}_{t+1} \le t+1 \le \hat{l}_{t+1} \le n+1$. In this subplan, $[k, t]$ is a regeneration interval for the first-stage ULS model, while in the second stage, $d_{k,\hat{l}_k-1}$ units enter the subplan and $d_{t+1,\hat{l}_{t+1}-1}$ leave. This shows the correctness of the decomposition at the flow level. The incoming inventory is produced at period $\hat{m}_k$. If $\hat{m}_k$ is not a period in $[k, t]$, i.e., $\hat{m}_k < k$, then $x_{\hat{m}_k} \le d_{\hat{m}_k,k-1} \le d_{\hat{m}_k,\hat{l}_k-1} = \hat{x}_{\hat{m}_k}$, and by Lemma 2.2

$$\alpha^\star_{\hat{m}_k,i} = \underline{\alpha}_{\hat{m}_k,i} \quad \text{for } i = \hat{m}_k, \ldots, \hat{l}_k - 1. \tag{6}$$

This shows the validity of the decomposition at the uncertainty level.

We denote by $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ the minimax regret for periods $[k, t]$ within the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$. Let $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ be the minimax regret over the first $t$ periods, where $\hat{m}_{t+1}$ is the last second-stage production period in $[1, t+1]$, and $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ be the minimax regret over the first $t$ periods, where $k$ is the last first-stage production period in $[1, t+1]$ and $\hat{m}_{t+1}$ the last second-stage production period. We can derive a recursion for $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ and $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$:

$$\begin{aligned} G(1,1,1) &= 0 \\ G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1}) &= \min_{1 \le k \le t} H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1}) \\ H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1}) &= \max_{\substack{1 \le \hat{m}_k \le k \le \hat{l}_k \le \hat{l}_{t+1} \\ \hat{m}_k \le \hat{m}_{t+1}}} \{G(k, \hat{m}_k, \hat{l}_k) + \\ &\quad + \phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})\}, \end{aligned}$$

for $1 \le \hat{m}_{t+1} \le t+1 \le \hat{l}_{t+1} \le n+1$.

If all values of $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ are known, we can show that the outer DP runs in $O(n^6)$ time. Indeed, for given $t$, $\hat{m}_{t+1}$, and $\hat{l}_{t+1}$, calculating $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ can be done in $O(n)$ time due to the minimization over $k$. For each $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$, we need to consider the maximization over $\hat{m}_k$ and $\hat{l}_k$ , which takes $O(n^2)$ time. Therefore, all values of $H(k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ and $G(t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ can be calculated in $O(n^6)$ time. Next, we will show that all costs $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$ can also be obtained in $O(n^6)$ time. Therefore, the DP solves the MRULS model in $O(n^6)$ time.

### *3.2. Subplan properties*

Below, we present two useful properties of the subplans. In Proposition 3.3, we derive the structure of an optimal $\mathbf{d}^\star$, and its proof uses two lemmas in Zhang [4], while Proposition 3.4 enables us to handle $\mathbf{f}$, $\mathbf{p}$, and $\mathbf{d}$ sequentially.

**Lemma 3.1.** *[4] For a given first-stage regeneration interval $[k, t]$, a realization of $\mathbf{p}$, and a second-stage decision vector $\hat{\mathbf{y}}$, if $\alpha_{kk} > \alpha_{\hat{m}_k,k}$, there exists an optimal scenario $\mathbf{d}^\star$ such that $d_i^\star = \overline{d}_i$ for $i = k, \ldots, t$.*

**Lemma 3.2.** *[4] For a given first-stage regeneration interval $[k, t]$, a realization of $\mathbf{p}$, and a second-stage decision vector $\hat{\mathbf{y}}$, if $\alpha_{kk} \le \alpha_{\hat{m}_k,k}$ and $\hat{y}_i = 0$ for $i = k+1, \ldots, t$, there exists an optimal scenario $\mathbf{d}^\star$ such that $d_i^\star = \underline{d}_i$ for $i = k, \ldots, t$.*

**Proposition 3.3.** *Given the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$ with $\hat{m}_{t+1} \le t$, a realization of $\mathbf{p}$, and a second-stage decision vector $\hat{\mathbf{y}}$, there exists an optimal $\mathbf{d}^\star$ such that:*

*If $\hat{m}_k < \hat{m}_{t+1}$,*

$$(d_k^\star, \ldots, d_t^\star) = \begin{cases} b(i): & (\overline{d}_k, \ldots, \overline{d}_t) & \text{if } \alpha_{kk} > \alpha_{\hat{m}_k,k} \\ b(ii): & (\underline{d}_k, \ldots, \underline{d}_t) & \text{if } \alpha_{kk} \le \alpha_{\hat{m}_k,k} \text{ and } \alpha_{k,\hat{m}_{t+1}} \le \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}} \\ b(iii): & (\underline{d}_k, \ldots, \underline{d}_{\gamma-1}, \overline{d}_\gamma, \ldots, \overline{d}_t) & \text{if } \alpha_{kk} \le \alpha_{\hat{m}_k,k} \text{ and } \alpha_{k,\hat{m}_{t+1}} \le \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}, \end{cases}$$

*in which $\gamma = \min\{i : \hat{l}_k \le i \le \hat{m}_{t+1},\ \hat{y}_i = 1,\ \alpha_{ki} > \alpha_{ii}\}$.*

*and if $\hat{m}_k = \hat{m}_{t+1}$,*

$$(d_k^\star, \ldots, d_t^\star) = \begin{cases} a(i): & (\overline{d}_k, \ldots, \overline{d}_t) & \text{if } \alpha_{kk} > \alpha_{\hat{m}_k,k} \\ a(ii): & (\underline{d}_k, \ldots, \underline{d}_t) & \text{if } \alpha_{kk} \le \alpha_{\hat{m}_k,k}; \end{cases}$$

*Proof.* Note that, by construction, $m_i = k$ for $i = k, \ldots, t$.

- Cases a(i) and b(i): The result follows directly from Lemma 3.1. Please note that the condition $\hat{m}_{t+1} \le t$ is not required.

- Case a(ii): Since $\hat{m}_k = \hat{m}_{t+1}$, $\hat{y}_i = 0$ for $i = k+1, \ldots, t$, and the result follows from Lemma 3.2. Please note that the condition $\hat{m}_{t+1} \le t$ is not required.

- Case b(ii): From Lemma 2.1, it will be enough to show that $\alpha_{ki} \le \alpha_{\hat{m}_i,i}$ for $i = k, \ldots, t$. From the statement of the proposition, the inequality holds for $i = k$. For $i = \hat{m}_{t+1}, \ldots, t$, the inequality holds by observing that $\alpha_{ki} \le \alpha_{\hat{m}_{t+1},i}$ from (2) and the fact that $\hat{m}_i = \hat{m}_{t+1}$. It remains to analyze $i = k+1, \ldots, \hat{m}_{t+1} - 1$. By (3), $\alpha_{\hat{m}_{t+1},\hat{m}_{t+1}} \le \alpha_{\hat{m}_i,\hat{m}_{t+1}}$. Combining this inequality with $\alpha_{k,\hat{m}_{t+1}} \le \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}$, we obtain $\alpha_{k,\hat{m}_{t+1}} \le \alpha_{\hat{m}_i,\hat{m}_{t+1}}$ and the desired inequality follows again using (2).

- Case b(iii): Using a similar argument as in Case b(ii), the result holds for periods $i = k$ and $i = \hat{m}_{t+1}, \ldots, t$. For $k+1, \ldots, \hat{m}_{t+1} - 1$, we need to consider whether there exists a period $i$, $k+1 \le i \le \hat{m}_{t+1} - 1$, such that $\hat{y}_i = 1$ and $\alpha_{ki} > \alpha_{ii}$. If such a period does not exists, then $\alpha_{ki} \le \alpha_{\hat{m}_i,i}$ for $i = k+1, \ldots, \hat{m}_{t+1} - 1$, and the result follows by letting $\gamma = \hat{m}_{t+1}$. Otherwise, from the definition of $\gamma$, $\alpha_{k,\hat{m}_i} \le \alpha_{\hat{m}_i,\hat{m}_i}$, and using (2), $\alpha_{ki} \le \alpha_{\hat{m}_i,i}$ for $i = k+1, \ldots, \gamma - 1$. For $i = \gamma, \ldots, \hat{m}_{t+1} - 1$, we have $\alpha_{k\gamma} > \alpha_{\gamma\gamma}$, which implies $\alpha_{ki} > \alpha_{\gamma i}$ by (2), while $\alpha_{\gamma i} > \alpha_{\hat{m}_i,i}$ by (3). Combining the two inequalities, the result follows by Lemma 2.1. □

**Proposition 3.4.** *Given the subplan $[(k, \hat{m}_k, \hat{l}_k), (t+1, \hat{m}_{t+1}, \hat{l}_{t+1})]$, a realization of $\mathbf{d}$, and a second-stage decision vector $\hat{\mathbf{y}}$, there exists an optimal $\mathbf{p}^\star$ such that:*

$$(p_k^\star, \ldots, p_t^\star) = \begin{cases} I: & (\overline{p}_k, \ldots, \overline{p}_t) & \text{if } \hat{l}_k = \hat{l}_{t+1} \\ II: & (\overline{p}_k, \ldots, \overline{p}_{\hat{l}_k - 1}, \underline{p}_{\hat{l}_k}, \ldots, \underline{p}_t) & \text{if } k < \hat{l}_k < \hat{l}_{t+1} \\ III: & (\overline{p}_k, \ldots, \overline{p}_{\theta-1}, \underline{p}_\theta, \ldots, \underline{p}_t) & \text{if } k = \hat{l}_k < \hat{l}_{t+1}, \end{cases}$$

*where*

$$\theta = \begin{cases} k & \text{if } \hat{y}_i = 0 \text{ for } i = k+1, \ldots, t, \\ \min\{i : k < i \le t,\ \hat{y}_i = 1\} & \text{otherwise.} \end{cases}$$

*Proof.* We consider each case separately.

• Case I: In this case $\hat{l}_k \geq t+1$ and therefore $\hat{x}_i = 0 \leq x_i$ for $i = k, \ldots, t$. We can assume without loss of optimality that $p^\star_i = \overline{p}_i$ for $i = k, \ldots, t$ by Lemma 2.2.

• Case II: In this case $\hat{l}_k \leq t$. Since $k < \hat{l}_k$, we can use the same argument as in Case I to show that the result holds for $i = k, \ldots, \hat{l}_k - 1$. For $i = \hat{l}_k, \ldots, t$, the result follows by using $x_i = 0 \leq \hat{x}_i$.

• Case III: We need to consider whether $\hat{\mathbf{y}}$ has a production period in $[k+1, t]$. If such a period does not exist, $x_k = d_{kt} \leq d_{k,\hat{l}_{t+1}} = \hat{x}_k$ and $x_i = 0 = \hat{x}_i$ for $i = k+1, \ldots, t$. So, we can assume $p^\star_i = \underline{p}_i$ for $i = k, \ldots, t$, and the desired result follows by choosing $\theta = k$.

Now suppose that such a production period exists and $\theta = \min\{i : k < i \leq t, \hat{y}_i = 1\}$. By construction we have $x_k = d_{kt} \geq d_{k,\theta-1} = \hat{x}_k$ and $x_i = 0 = \hat{x}_i$ for $i = k+1, \ldots, \theta - 1$, which implies that $p^\star_i = \overline{p}_i$ for $i = k, \ldots, \theta - 1$. For $i = \theta, \ldots, t$, $x_i = 0 \leq \hat{x}_i$ implies that $p^\star_i = \underline{p}_i$ and the desired result follows. □

### *3.3. Minimax regret of a subplan*

In this section we calculate $\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, \hat{l}_{t+1})$. We start with diagonal subplans, i.e., $\hat{m}_{t+1} \leq t$. For convenience, we let $\underline{\mathcal{L}}_i^j$ be the optimal objective value of the ULS model defined between periods $i$ and $j$ with all uncertain parameters at lower bounds. We define $\overline{\mathcal{L}}_i^j$ in a similar fashion, but where the demands are at their upper bounds.

#### *3.3.1. Case I: $\hat{l}_k = \hat{l}_{t+1}$, $\hat{m}_{t+1} \leq t$*

Using Proposition 3.4, $(p^\star_k, \ldots, p^\star_t) = (\overline{p}_k, \ldots, \overline{p}_t)$, while $\alpha^\star_{\hat{m}_k, i} = \underline{\alpha}_{\hat{m}_k, i}$, $i = k, \ldots, t$, by (6). In addition, according to Proposition 3.3, we need to consider demand scenarios in Cases a(i)-a(ii). Therefore,

$$\phi = \begin{cases} \overline{f}_k + \sum_{i=k}^t \overline{\alpha}_{ki}\overline{d}_i - \sum_{i=k}^t \underline{\alpha}_{\hat{m}_k, i}\overline{d}_i \text{ if } \overline{\alpha}_{kk} > \underline{\alpha}_{\hat{m}_k, k} \\ \overline{f}_k + \sum_{i=k}^t \overline{\alpha}_{ki}\underline{d}_i - \sum_{i=k}^t \underline{\alpha}_{\hat{m}_k, i}\underline{d}_i \text{ if } \overline{\alpha}_{kk} \leq \underline{\alpha}_{\hat{m}_k, k}. \end{cases} \quad (7)$$

#### *3.3.2. Case II: $k < \hat{l}_k < \hat{l}_{t+1}$, $\hat{m}_{t+1} \leq t$*

We have $(p^\star_k, \ldots, p^\star_t) = (\overline{p}_k, \ldots, \overline{p}_{\hat{l}_k - 1}, \underline{p}_{\hat{l}_k}, \ldots, \underline{p}_t)$ by Proposition 3.4, while again $\alpha^\star_{\hat{m}_k, i} = \underline{\alpha}_{\hat{m}_k, i}$, $i = k, \ldots, \hat{l}_k - 1$. In addition, according to Proposition 3.3, we need to consider demand scenarios in Cases (b)(i)-(b)(iii).

• Case II(b)(i): $\overline{\alpha}_{kk} > \underline{\alpha}_{\hat{m}_k, k}$

In this case, $(d^\star_k, \ldots, d^\star_t) = (\overline{d}_k, \ldots, \overline{d}_t)$. In the second stage, there is no production in $[k, \hat{l}_k - 1]$ or $[\hat{m}_{t+1} + 1, t]$. Therefore,

$$\begin{aligned} \phi =& \overline{f}_k + \sum_{i=k}^{t} \overline{\alpha}_{ki}\overline{d}_i - \sum_{i=k}^{\hat{l}_k - 1} \underline{\alpha}_{\hat{m}_k, i}\overline{d}_i - \overline{\mathcal{L}}_{\hat{l}_k}^{\hat{m}_{t+1}-1} - \\ & - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1}, i}\overline{d}_i. \end{aligned} \quad (8)$$

• Case II(b)(ii): $\overline{\alpha}_{kk} \leq \underline{\alpha}_{\hat{m}_k, k}$ and $\overline{\alpha}_{k,\hat{m}_{t+1}} \leq \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In this case, $\phi$ is identical to (8) with $(d^\star_k, \ldots, d^\star_t) = (\underline{d}_k, \ldots, \underline{d}_t)$, i.e.,

$$\begin{aligned} \phi =& \overline{f}_k + \sum_{i=k}^{t} \overline{\alpha}_{ki}\underline{d}_i - \sum_{i=k}^{\hat{l}_k - 1} \underline{\alpha}_{\hat{m}_k, i}\underline{d}_i - \underline{\mathcal{L}}_{\hat{l}_k}^{\hat{m}_{t+1}-1} - \\ & - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1}, i}\underline{d}_i. \end{aligned}$$

• Case II(b)(iii): $\overline{\alpha}_{kk} \leq \underline{\alpha}_{\hat{m}_k, k}$ and $\overline{\alpha}_{k,\hat{m}_{t+1}} > \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$

We need to go through all candidates to $\gamma$, i.e., $\hat{l}_k \leq \gamma \leq \hat{m}_{t+1}$ where $\overline{\alpha}_{k\gamma} > \underline{\alpha}_{\gamma\gamma}$, and force $\hat{y}_\gamma = 1$ and $(d_k, \ldots, d_t) = (\underline{d}_k, \ldots, \underline{d}_{\gamma-1}, \overline{d}_\gamma, \ldots, \overline{d}_t)$. By optimizing over all these candidates we get

$$\begin{aligned} \phi =& \overline{f}_k + \max_{\substack{\hat{l}_k \leq \gamma \leq \hat{m}_{t+1} \\ \overline{\alpha}_{k\gamma} > \underline{\alpha}_{\gamma\gamma}}} [\sum_{i=k}^{\gamma-1} \overline{\alpha}_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \overline{\alpha}_{ki}\overline{d}_i - \sum_{i=k}^{\hat{l}_k - 1} \underline{\alpha}_{\hat{m}_k, i}\underline{d}_i - \\ & - \underline{\mathcal{L}}_{\hat{l}_k}^{\gamma-1} - \overline{\mathcal{L}}_{\gamma}^{\hat{m}_{t+1}-1}] - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1}, i}\overline{d}_i. \end{aligned}$$

#### *3.3.3. Case III: $k = \hat{l}_k < \hat{l}_{t+1}$, $\hat{m}_{t+1} \leq t$*

Since $\hat{m}_k = k$, we cannot have demand scenarios a(i) or b(i). If $\hat{m}_{t+1} = k$, the demands in both stages are satisfied at period $k$ and $\phi = 0$. Otherwise, $k < \hat{m}_{t+1} \leq t$ and $\alpha^*_{k,\hat{m}_{t+1}} = \overline{\alpha}_{k,\hat{m}_{t+1}}$ while $\alpha^*_{\hat{m}_{t+1},\hat{m}_{t+1}} = \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$.

• Case III(a)(ii): $\hat{m}_k = \hat{m}_{t+1}$

As argued above, $\phi = 0$.

• Case III(b)(ii): $\hat{m}_k < \hat{m}_{t+1}$ and $\overline{\alpha}_{k,\hat{m}_{t+1}} \leq \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In this case $(d^\star_k, \ldots, d^\star_t) = (\underline{d}_k, \ldots, \underline{d}_t)$. In the second stage, we need to go through all candidates for $\theta$, $k < \theta \leq$

$\hat{m}_{t+1}$. We force $\hat{y}_\theta = 1$, $\hat{y}_i = 0$ for $i = k, \dots, \theta - 1$, and $(p_k, \dots, p_t) = (\overline{p}_k, \dots, \overline{p}_{\theta-1}, \underline{p}_\theta, \dots, \underline{p}_t)$, and optimise over all these candidates. Therefore,

$$\phi = \max_{k<\theta\leq\hat{m}_{t+1}} [\sum_{i=\theta}^{t} \overline{\alpha}_{ki}\underline{d}_i - \underline{\mathcal{L}}_\theta^{\hat{m}_{t+1}-1}] - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1},i}\underline{d}_i.$$

- Case III(b)(iii): $\hat{m}_k < \hat{m}_{t+1}$ and $\overline{\alpha}_{k,\hat{m}_{t+1}} > \underline{\alpha}_{\hat{m}_{t+1},\hat{m}_{t+1}}$

Besides $\theta$, we need to optimize over all the candidates for $\gamma$ as in Case II(b)(iii):

$$\phi = \max_{\substack{k<\theta\leq\gamma \\ \theta\leq\gamma\leq\hat{m}_{t+1} \\ \overline{\alpha}_{k\gamma}>\underline{\alpha}_{\gamma\gamma}}} [\sum_{i=\theta}^{\gamma-1} \overline{\alpha}_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \overline{\alpha}_{ki}\overline{d}_i - \underline{\mathcal{L}}_\theta^{\gamma-1} - \overline{\mathcal{L}}_\gamma^{\hat{m}_{t+1}-1}] - \underline{f}_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \underline{\alpha}_{\hat{m}_{t+1},i}\overline{d}_i.$$

#### 3.3.4. Non-diagonal subplans

In this section, we focus on non-diagonal subplans, i.e., $\hat{m}_{t+1} = \hat{l}_{t+1} = t+1$. Note that if $\hat{l}_k = \hat{l}_{t+1}$, then (7) still holds. Therefore, we will assume that $\hat{l}_k \leq t$, i.e., there is at least one second-stage production period in $[k, t]$.

For given $k$, $\hat{m}_k$ and $\hat{l}_k$, let us define $\Phi(t) = \phi(k, \hat{m}_k, \hat{l}_k, t, t, t)$. If $\hat{m}_k < k$, we have

$$\begin{aligned}
\Phi(\hat{l}_k) =& \overline{f}_k + \sum_{i=k}^{\hat{l}_k-1} \overline{\alpha}_{ki}d_i^\star - \sum_{i=k}^{\hat{l}_k-1} \underline{\alpha}_{\hat{m}_k,i}d_i^\star \\
\Phi(t+1) =& \max\{\max_{\hat{l}_k\leq\hat{m}_t<t} [\phi(k, \hat{m}_k, \hat{l}_k, t, \hat{m}_t, t+1) + \\
&+ (\overline{\alpha}_{kt} - \underline{\alpha}_{\hat{m}_t,t})d_t^\star],\ \Phi(t) - \overline{f}_t + \\
&+ (\overline{\alpha}_{kt} - \underline{\alpha}_{tt})d_t^\star\},
\end{aligned}$$

for $t = \hat{l}_k, \dots, n$, where the direction of bounds of $\alpha$ follows from $k < \hat{m}_t \leq t$, and $d_i^\star$ is determined using Lemma 2.1. If $\hat{m}_k = k$, the first period to be considered by the recursion is $\hat{l}_k + 1$, and the initial condition should be replaced by $\Phi(\hat{l}_k + 1) = 0$.

### 3.4. The running time

In this section we show that all values of $\phi$ can be calculated in $O(n^6)$ time. Note that $\overline{\mathcal{L}}_i^j$, $\underline{\mathcal{L}}_i^j$ and $\sum_{s=i}^{j} \alpha_{rs}^\star d_s^\star$ can be calculated in $O(n^3)$ time for all values of $r \leq i \leq j$ in a preprocessing procedure. For diagonal subplans, $\phi$ can be calculated in constant time, except for Cases II(b)(iii) and III(b)(ii), in which it takes $O(n)$ time, and for Case III(b)(iii), in which it takes $O(n^2)$ time. The desired running time follows by observing that there are $O(n^4)$ Case III subplans, while for Case II(b)(iii) there are only $O(n^5)$ relevant subplans since

$$\phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, a+1) = \phi(k, \hat{m}_k, \hat{l}_k, t+1, \hat{m}_{t+1}, a), \quad (9)$$

with $a > t+1$ Once the values of $\phi$ for all diagonal subplans are available, it is straightforward to see from Section 3.3.4 that all non-diagonal subplans can be handled in $O(n^5)$ time.

## 4. Single type uncertainty

In this section, an $O(n^5)$ time algorithm is proposed for demand uncertainty only (the MRULS$^d$ model), and an $O(n^4)$ time one for production cost function uncertainty only (the MRULS$^p$ model).

### 4.1. Demand uncertainty

When only the demand is uncertain, the nodes of the network proposed in Section 3 can be simplified to $(k, \hat{m}_k)$ and the number of subplans is reduced to $O(n^4)$. Let $\phi^d(k, \hat{m}_k, t+1, \hat{m}_{t+1})$ denote the minimax regret for periods $[k, t]$ within the subplan $[(k, \hat{m}_k), (t+1, \hat{m}_{t+1})]$. By including the $O(n^3)$ time preprocessing procedure in Section 3, we show below that any diagonal subplan can be computed in at most $O(n^2)$ time. Noting that

$$\phi^d(k, \hat{m}_k, t+1, b) = \phi^d(k, \hat{m}_k, t, b) + (\alpha_{kt} - \alpha_{bt})d_t^\star,$$

for $k \leq b \leq t$, all diagonal subplans can be handled in $O(n^5)$ time, and the desired running time for the MRULS$^d$ model follows. To calculate $\phi^d(k, \hat{m}_k, t+1, \hat{m}_{t+1})$, $\hat{m}_{t+1} \leq t$, we use Proposition 3.3.

#### 4.1.1. Case a: $\hat{m}_k = \hat{m}_{t+1}$

Please note that $\hat{m}_k = \hat{m}_{t+1}$ implies $\hat{m}_{t+1} \leq t$. We have

$$\phi^d = \begin{cases} f_k + \sum_{i=k}^{t} \alpha_{ki}\overline{d}_i - \sum_{i=k}^{t} \alpha_{\hat{m}_k,i}\overline{d}_i & \text{if } \alpha_{kk} > \alpha_{\hat{m}_k,k} \\ f_k + \sum_{i=k}^{t} \alpha_{ki}\underline{d}_i - \sum_{i=k}^{t} \alpha_{\hat{m}_k,i}\underline{d}_i & \text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k} \text{ and } \hat{m}_k < k \\ 0 & \text{if } \alpha_{kk} \leq \alpha_{\hat{m}_k,k} \text{ and } \hat{m}_k = k. \end{cases}$$

#### 4.1.2. Case b: $\hat{m}_k < \hat{m}_{t+1} \leq t$

- Case b(i): $\alpha_{kk} > \alpha_{\hat{m}_k,k}$

In this case, $\hat{m}_k < k$. If $\eta$ is the first second-stage production period in $[k+1, \hat{m}_{t+1}]$, then

$$\phi^d = f_k + \sum_{i=k}^{t} \alpha_{ki}\overline{d}_i - \min_{k<\eta\le\hat{m}_{t+1}} [\sum_{i=k}^{\eta-1} \alpha_{\hat{m}_k,i}\overline{d}_i + \overline{\mathcal{L}}_{\eta}^{\hat{m}_{t+1}-1}] - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\overline{d}_i.$$

- Case b(ii): $\alpha_{kk} \le \alpha_{\hat{m}_k,k}$ and $\alpha_{k,\hat{m}_{t+1}} \le \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In this case, $\phi^d$ is similar to that in Case b(i), where $(d_k^\star, \dots, d_t^\star) = (\underline{d}_k, \dots, \underline{d}_t)$. Thus,

$$\phi^d = f_k + \sum_{i=k}^{t} \alpha_{ki}\underline{d}_i - \min_{k<\eta\le\hat{m}_{t+1}} [\sum_{i=k}^{\eta-1} \alpha_{\hat{m}_k,i}\underline{d}_i + \underline{\mathcal{L}}_{\eta}^{\hat{m}_{t+1}-1}] - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\underline{d}_i$$

if $\hat{m}_k < k$, and

$$\phi^d = f_k + \sum_{i=k}^{t} \alpha_{ki}\underline{d}_i - \underline{\mathcal{L}}_{k}^{\hat{m}_{t+1}-1} - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\underline{d}_i$$

if $\hat{m}_k = k$.

- Case b(iii): $\alpha_{kk} \le \alpha_{\hat{m}_k,k}$ and $\alpha_{k,\hat{m}_{t+1}} > \alpha_{\hat{m}_{t+1},\hat{m}_{t+1}}$

In addition to $\eta$, and similarly to Cases IIb(iii) and IIIb(iii) in Section 3.3, we need to optimize over candidates for $\gamma$. Therefore,

$$\phi^d = f_k + \max_{\substack{k<\eta\le\gamma \\ \eta\le\gamma\le\hat{m}_{t+1} \\ \alpha_{k\gamma}>\alpha_{\gamma\gamma}}} [\sum_{i=k}^{\gamma-1} \alpha_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \alpha_{ki}\overline{d}_i - \sum_{i=k}^{\eta-1} \alpha_{\hat{m}_k,i}\underline{d}_i - \underline{\mathcal{L}}_{\eta}^{\gamma-1} - \overline{\mathcal{L}}_{\gamma}^{\hat{m}_{t+1}-1}] - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\overline{d}_i$$

if $\hat{m}_k < k$, and

$$\phi^d = f_k + \max_{\substack{k\le\gamma\le\hat{m}_{t+1} \\ \alpha_{k\gamma}>\alpha_{\gamma\gamma}}} [\sum_{i=k}^{\gamma-1} \alpha_{ki}\underline{d}_i + \sum_{i=\gamma}^{t} \alpha_{ki}\overline{d}_i - \underline{\mathcal{L}}_{k}^{\gamma-1} - \overline{\mathcal{L}}_{\gamma}^{\hat{m}_{t+1}-1}] - f_{\hat{m}_{t+1}} - \sum_{i=\hat{m}_{t+1}}^{t} \alpha_{\hat{m}_{t+1},i}\overline{d}_i$$

if $\hat{m}_k = k$.

### *4.2. Production cost function uncertainty*

The nodes of the network for the MRULS$^p$ model can be simplified to $(k, \hat{l}_k)$ and again the number of subplans is reduced to $O(n^4)$. Let $\phi^p(k, \hat{l}_k, t+1, \hat{l}_{t+1})$ denote the minimax regret for periods $[k, t]$ within the subplan $[(k, \hat{l}_k), (t+1, \hat{l}_{t+1})]$, and $\mathcal{L}^p(i, t, j)$, $i \le t \le j$, denote the optimal objective value of a ULS model from periods $i$ to $t$ with demands $(d_i, \dots, d_{t-1}, d_{t,j-1})$ and setup and unit production costs at their lower bounds. We have

$$\phi^p = \begin{cases} \overline{f}_k + \overline{p}_k d_{kt} & \text{if } \hat{l}_k = \hat{l}_{t+1} \\ \overline{f}_k + \overline{p}_k d_{kt} - \mathcal{L}^p(\hat{l}_k, t, \hat{l}_{t+1}) & \text{if } k < \hat{l}_k < \hat{l}_{t+1} \\ \max\{\underline{p}_k d_{t+1,\hat{l}_{t+1}-1}, \max_{k<\theta\le t}[\overline{p}_k d_{\theta t} - \mathcal{L}^p(\theta, t, \hat{l}_{t+1})]\} & \text{if } k = \hat{l}_k < \hat{l}_{t+1}. \end{cases}$$

For a given value of $j$, $\mathcal{L}^p(i, t, j)$ can be calculated in $O(n^3)$ time for all values of $i$ and $t$. Hence we can calculate all $\mathcal{L}^p(i, t, j)$ in $O(n^4)$ time in a preprocessing procedure. It is then straightforward to see that $\phi^p(k, \hat{l}_k, t+1, \hat{l}_{t+1})$ can be calculated in constant time for $\hat{l}_k = \hat{l}_{t+1}$ and $k < \hat{l}_k < \hat{l}_{t+1}$. For $k = \hat{l}_k < \hat{l}_{t+1}$, the running time is $O(n)$, but there are only $O(n^3)$ of these subplans. Therefore, the overall running time remains at $O(n^4)$, and the desired running time for the MRULS$^d$ model follows.

## 5. Conclusions

In this paper, we have formulated the minimax regret ULS model with both production cost function and demand uncertainty. We gave a theoretical analysis to reduce the number of optimal uncertainty scenarios and proposed an algorithm that solves the problem in $O(n^6)$ time. Lower running times were found for the case in which only one single type of uncertainty is present. In particular, an $O(n^4)$ time algorithm is proposed for production cost function uncertainty only, and an $O(n^5)$ time one for demand uncertainty only. The latter reduces the running time of the $O(n^7)$ time algorithm proposed in [4].

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