The inexact projected gradient method for quasiconvex vector optimization problems

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Abstract

Vector optimization problems are a generalization of multiobjective optimization in which the preference order is related to an arbitrary closed and convex cone, rather than the nonnegative octant. Due to its real life applications, it is important to have practical solution approaches for computing. In this work, we consider the inexact projected gradient-like method for solving smooth constrained vector optimization problems. Basically, we prove global convergence of any sequence produced by the method to a stationary point assuming that the objective function of the problem is K-quasiconvex, instead of the stronger K-convexity assumed in the literature.

Keywords: Gradient-like method; Vector optimization; $K-$ quasiconvexity.

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1 Introduction

In many applications, it is desired to compute a point such that there exists not a preferred option. Sometimes this preference is mathematically described by means of a cone K and a function F, i.e. a point x is preferred to y if the difference of the evaluations of the function belongs to the cone. This defines the vectorial optimization problem $\min F(x)$ in which

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the order is given the cone K. In this paper we assume that F is a quasiconvex function with respect to the cone K, as in many micro-economical models devoted to maximize the utilities, which usually are quasi-concave functions.

A popular technique for solving vectorial problems is to scalarize the function F; see [15, 18] and the references therein. Many disadvantages on this scheme has been reported in, for instance, [19]. Furthermore in the quasiconvex case, the scalarizations may lead to solve non-quasiconvex models (sum of quasiconvex functions may not be quasiconvex). An appealing solution approach is to use descent directions like Gradient method for multiobjective and vector optimization problems; see, for instance, [11, 13, 14, 16]. For quasiconvex models, the convergence analysis has been obtained in [4, 6, 7, 9].

In the present paper we extend the convergence result of [4] for vector optimization and study the inexact version of the projected gradient method. Our method was inspired by the method proposed in [12] and uses similar idea exposed in [4]. Assuming existence of solutions and quasiconvexity of the vector function, we prove that every sequence generated converges to a stationary point of the vector optimization problem.

This article is organized as follows. Section 2 contains some basic definitions and preliminary material. In Section 3, we present the inexact projected gradient method for vectorial optimization. Section 4 provides the convergence analysis of the method. Finally, in Section 5 we give the final remarks.

2   Basic definitions and preliminary material

In this section, we present the vector optimization problem as well as some definitions, notations and basic properties, which are used throughout of this paper. For more details; see, [5, 12, 16].

Let $K \subset \mathbb{R}^m$ be a nonempty closed, convex and pointed cone. The partial order " $\preceq_K$ " induced by K in $\mathbb{R}^m$ is defined as $u \preceq_K v$ if $v - u \in K$. A set $Y \subset \mathbb{R}^m$ is K–bounded if there exists $z \in \mathbb{R}^m$ such that, for all $y \in Y$, $z \preceq_K y$. Assuming that $\text{int}(K)$ is nonempty, $u \prec_K v$ if $v - u \in \text{int}(K)$. As reported in [5, Remark 2.2], if $\text{int}(K)$ is nonempty, the partial order is directed, i.e, for all $y_1, y_2 \in \mathbb{R}^m$, there exists $z \in \mathbb{R}^m$ such that $y_1 \preceq_K z$ and $y_2 \preceq_K z$.

Given K, its positive polar cone is $K^\star := \{y \in \mathbb{R}^m; \langle y, x \rangle \geq 0 \text{ for all } x \in K\}$ and its generator is a compact set G such that K is the cone generated by its convex hull. As pointed out in [5, Remark 3.2], if $\text{int}(K)$ is nonempty, then $K^\star = \text{co}(\text{conv}(\text{extd}(K^\star)))$, remind that $d \in \text{extd}(K)$ is an extreme direction if $d \neq 0$ and $d = d_1 + d_2$ for some $d_1, d_2 \in K$ implies that $d_1, d_2 \in \mathbb{R}_+ d$.

Given $F : \mathbb{R}^n \to \mathbb{R}^m$, a $\mathcal{C}^1$ function and $C \subseteq \mathbb{R}^n$ a nonempty, closed and convex set, we consider the problem of finding a weakly efficient point of F in C, i.e., a point $x^* \in C$ such

that there exists no other $x \in C$ with $F(x) \prec_K F(x^*)$. We denote this constrained problem as

$$\min_K F(x), \quad \text{s.t. } x \in C. \tag{1}$$

We denote $J_F(x)$ as the Jacobian matrix of F at x and $C - x^* = \{y - x^* : y \in C\}$. We say that v is a descent direction at $x \in C$ if there is not $v \in C - x$ such that $J_F(x)v \prec_K 0$. This leads to the definition of stationarity, i.e, $x^* \in C$ is a stationary point if

$$-\mathrm{int}(K) \cap J_F(x^*)(C - x^*) = \varnothing. \tag{2}$$

For characterizing stationary points, we consider $\varphi : \mathbb{R}^m \to \mathbb{R}$,

$$\varphi(y) := \max\langle y, \omega\rangle, \quad \text{s.t. } \omega \in G.$$

As reported in [16], $-K = \{y \in \mathbb{R}^m : \varphi(y) \leq 0\}$ and $-\mathrm{int}(K) = \{y \in \mathbb{R}^m : \varphi(y) < 0\}$. Furthermore, in [14, Proposition 2], it is shown that φ is positively homogeneous of degree 1, subadditive and a Lipschitz continuous with Lipschitz constant $L = 1$. If $y \prec_K z$ ($y \preceq_K z$, respectively), $\varphi(y) < \varphi(z)$ ($\varphi(y) \leq \varphi(z)$). Now we define $h_x : C - x \to \mathbb{R}$ by

$$h_x(v) := \hat{\beta}\varphi(J_F(x)v) + \|v\|^2/2,$$

where $\hat{\beta} > 0$ and the following constrained parametric optimization problem:

$$\min h_x(v), \quad \text{s.t. } v \in C - x. \tag{3}$$

This problem has only one solution, namely $\bar{v}$, and it fulfills that $\bar{v} = P_{C-x}(-\hat{\beta}J_F(x)^T\omega)$ for some $\omega \in \mathrm{conv}(G)$; see [14]. So, we have the following:

Definition 2.1. *The projected gradient direction function of F is defined as $v : C \to \mathbb{R}^n$, where $v(x)$ is the unique solution of Problem (3). The optimal value function associated to (3) is $\theta : C \to \mathbb{R}$, where $\theta(x) := h_x(v(x))$.*

Lemma 2.1. *x is a stationary point of F, if and only if $\theta(x) = 0$ and $v(x) = 0$. $v(x)$ and $\theta(x)$ are continuous functions.*

Proof. For the first part see [14, Proposition 3]. For the continuity of $v(x)$; see [13, Proposition 3.4]. The second part is a direct consequence of this fact. □

Now we consider the inexact case. Let us present the concept of approximate directions.

Definition 2.2. *Let $x \in C$ and $\sigma \in [0, 1)$. A vector $v \in C - x$ is a σ−approximate projected gradient direction at x if $h_x(v) \leq (1 - \sigma)\theta(x)$.*

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A particular class of σ–approximate directions for F at x is given by the so called *scalarization compatible* (or simply s–compatible) directions, i.e., those $v \in \mathbb{R}^n$ such that

$$v = P_{C-x}(-\hat{\beta}J_F(x)^T\omega), \text{ for some } \omega \in \text{conv}(G). \tag{4}$$

Relations between s–compatible and σ–approximate directions can be found in [12]. The convergence of the method is obtained using the following definition and results.

Definition 2.3 (Definition 4.1 of [17])**.** *Let S be a nonempty subset of $\mathbb{R}^n$. A sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ is said to be quasi-Fejér convergent to S if and only if for all $x \in S$ there exist $k_0 \geq 0$ and a sequence $(\delta_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\sum_{k=0}^{\infty}\delta_k < \infty$ and*

$$\|x^{k+1} - x\|^2 \leq \|x^k - x\|^2 + \delta_k,$$

for all $k \geq k_0$.

Lemma 2.2 (Theorem 4.1 of [17])**.** *If $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ is* quasi-Fejér *convergent to some set S, then:*

  i) The sequence $(x^k)_{k\in\mathbb{N}}$ is bounded;

  ii) If there exists an accumulation point, x, of the sequence $(x^k)_{k\in\mathbb{N}}$ belongs to S, then $\{x^k\}$ is convergent to x.

We end with a brief introduction to quasiconvexity in the vectorial framework.

Definition 2.4. *The vector function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be K–quasiconvex if for all $y \in \mathbb{R}^m$ the level set $L_F(y) = \{x \in \mathbb{R}^n : F(x) \preceq_K y\}$ is convex.*

The following characterization will be useful.

Theorem 2.1. *Assume that $(\mathbb{R}^m, \preceq_K)$ is partially ordered. Then F is K–quasiconvex if and only if $\langle d, F(.)\rangle : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex for every extreme direction $d \in K^\star$.*

Proof. As already remarked, $\text{int}(K)$ is a nonempty set, $K^\star$ is the conic hull of the closed convex hull of $\text{extd}(K^\star)$ and $(\mathbb{R}^m, \preceq_K)$ is directed. Combining these two facts, the desired result follows from [5, Theorem 3.1]. $\qquad\square$

3 Inexact projected gradient algorithm

This part is devoted to present the method proposed in [12] and some properties of it. Fix $\hat{\beta} > 0$, $\delta \in (0,1)$, $\tau > 1$ and $\sigma \in [0,1)$. The inexact projected gradient method is defined as follows.

Initialization: Take $x^0 \in C$.

Iterative step: Given x^k, compute a σ-approximate direction v^k at x^k. If $h_{x^k}(v^k) = 0$, then stop. Otherwise compute

$$j(k) := \min \left\{ j \in \mathbb{Z}_+ : \ F(x^k + \tau^{-j} v^k) \preceq_K F(x^k) + \delta \tau^{-j} J_F(x^k) v^k \right\}. \tag{5}$$

Set $t_k = \tau^{-j(k)}$ and $x^{k+1} = x^k + t_k v^k$.

If $m = 1$ and $\sigma = 0$, the method becomes the classical exact projected gradient method. In the inexact unconstrained case, we retrieve the method introduced in [16]. The following holds.

Proposition 3.1. $h_{x^k}(v^k) = 0$, *implies the stationarity of* x^k. *Let* $\delta \in (0,1)$, $x^k \in C$ *and let* v^k *be a descent direction. Then, there exits* $\bar{\gamma} > 0$ *such that (5) holds for all* $\gamma \in [0, \bar{\gamma}]$, *i.e.,*

$$F(x^k + \gamma v^k) \preceq_K F(x^k) + \delta \gamma J_F(x^k) v^k.$$

So, the Armijo rule is well defined.

Proof. For the first part note that if $h_{x^k}(v^k) = 0$, then, by the definition of σ-approximation, $\theta(x^k) \geq 0$, but as $\theta(x^k) \leq 0$, so $\theta(x^k) = 0$, concluding that x^k is a stationary point. On the other hand, if x^k is stationary, then $\theta(x^k) = 0$, and therefore $h_{x^k}(v^k) = 0$. The last part follows from [14, Proposition 1]. $\square$

4 Convergence analysis

In this section, we show the global convergence of the inexact projected gradient method. If the method stops after a finite number of iterations, it computes a stationary point as desired. So, we will assume that $(x^k)_{k\in\mathbb{N}}$, $(v^k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}$ are the infinite sequences generated by the inexact projected gradient method. From [12, Lemma 3.6], we recall that if $(F(x^k))_{k\in\mathbb{N}}$ is K–bounded, then

$$\sum_{k=0}^{\infty} t_k |\langle \omega, J_F(x^k) v^k \rangle| < +\infty, \ \text{for all } \omega \in \text{conv}(G). \tag{6}$$

Proposition 4.1. *Sequence $(x^k)_{k \in \mathbb{N}}$ is feasible and $F(x^k) - F(x^{k+1}) \in K$ for all k.*

Proof. The feasibility is a consequence of the definition of the method and the K–decreasing property follows from (5). □

Under differentiability, the convergence is obtained in [12, Theorem 3.5] as follows.

Proposition 4.2. *Every accumulation point, if any, of $(x^k)_{k \in \mathbb{N}}$ is a stationary point of Problem (1).*

In what follows we present the main novelty of this paper. For the convergence of the method we need the following hypotheses.
Assumption 1. $T \neq \varnothing$, where $T := \left\{ x \in C \colon F(x) \preceq_K F(x^k), \ k = 0, 1, \ldots, \right\}$.
Assumption 2. Each v^k of the sequence $(v^k)_{k \in \mathbb{N}}$ is scalarization compatible, i.e., exists a sequence $(\omega^k)_{k \in \mathbb{N}} \subset \mathrm{conv}(G)$ such that

$$v^k = P_{C - x^k}(-\hat{\beta} J_F(x^k)^T \omega^k), \qquad k = 0, 1, \ldots.$$

The convergence of several methods for solving vector optimization problems is usually obtained under Assumption 1; see [4, 6, 9, 11, 13, 14, 16]. Although the existence of a weakly efficient solution does not imply that T is nonempty, it is closely related with the completeness of the $\mathrm{Im}(F)$, which ensures the existence of efficient points; see [19]. Moreover, if the sequence $(x^k)_{k \in \mathbb{N}}$ has an accumulation point, then T is nonempty; see [6, 9].

Assumption 2 holds if v^k is the exact gradient projected direction at x^k. It was also used in [12] for proving the full convergence of the sequence generated by the method in the case that F is K–convex.
From now on, we will assume that Assumptions 1-2 hold. We start with the following result

Lemma 4.1. *For each $\hat{x} \in T$ and $k \in \mathbb{N}$, it holds that*

$$\langle v^k, \hat{x} - x^k \rangle \geq \hat{\beta} \langle J_F(x^k)^T \omega^k, v^k \rangle + \|v^k\|^2.$$

Proof. Take $k \in \mathbb{N}$ and $\hat{x} \in T$. As v^k is s-compatible at x^k, then $v^k = P_{C - x^k}(-\hat{\beta} J_F(x^k)^T \omega^k)$, for some $\omega^k \in \mathrm{conv}(G)$. As v^k is a projection, $\langle -\hat{\beta} J_F(x^k)^T \omega^k - v^k, v - v^k \rangle \leq 0$, for all $v \in C - x^k$. In particular, for $v = \hat{x} - x^k$, we obtain $\langle -\hat{\beta} J_F(x^k)^T \omega^k - v^k, \hat{x} - x^k - v^k \rangle \leq 0$. So, from the last inequality, we get

$$\langle v^k, \hat{x} - x^k \rangle \geq -\hat{\beta} \langle J_F(x^k)^T \omega^k, \hat{x} - x^k \rangle + \hat{\beta} \langle J_F(x^k)^T \omega^k, v^k \rangle + \|v^k\|^2. \tag{7}$$

Since F is K–quasiconvex, by Theorem 2.1, for each $d \in \text{extd}(K^\star)$, $\langle d, F \rangle : \mathbb{R}^n \to \mathbb{R}$ is a quasiconvex function. As $\text{co}(\text{conv}(\text{extd}(K^\star))) = K^\star$ and $\omega^k \in \text{conv}(G) \subset K^\star$, $\omega^k = \sum_{\ell=1}^p \gamma_\ell^k d_\ell$, where $\gamma_\ell^k \in \mathbb{R}_+$ and $d_\ell \in \text{extd}(K^\star)$, for all $1 \leq \ell \leq p$. Therefore,

$$\hat{\beta}\langle J_F(x^k)^T \omega^k, \hat{x} - x^k \rangle = \hat{\beta}\langle J_F(x^k)^T \sum_{\ell=1}^p \gamma_\ell^k d_\ell, \hat{x} - x^k \rangle = \hat{\beta} \sum_{\ell=1}^p \gamma_\ell^k \langle J_F(x^k)^T d_\ell, \hat{x} - x^k \rangle.$$

As $\hat{x} \in T$, we have $F(x^k) - F(\hat{x}) \in K$. So, $\langle d_\ell, F(x^k) - F(\hat{x}) \rangle \geq 0$, for all $d_\ell \in \text{extd}(K^\star)$. But as $\langle d_\ell, F \rangle$ is a real-valued, quasiconvex differentiable function, it follows that $\langle J_F(x^k)^T d_\ell, \hat{x} - x^k \rangle \leq 0$. This implies that $\hat{\beta}\langle J_F(x^k)^T \omega^k, \hat{x} - x^k \rangle \leq 0$. Now, the result follows from combining of the last inequality with (7). $\qquad \square$

Next lemma presents the quasi-Fejér convergence.

Lemma 4.2. *Suppose that F is K–quasiconvex. Then, the sequence $(x^k)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to the set T.*

Proof. Since T is nonempty, take $\hat{x} \in T$ and fix $k \in \mathbb{N}$. Using the definition of x^{k+1}, after some algebraic work, we are lead to

$$\|x^{k+1} - \hat{x}\|^2 = \|x^k - \hat{x}\|^2 + \|x^{k+1} - x^k\|^2 - 2t_k\langle v^k, \hat{x} - x^k \rangle. \tag{8}$$

Using Lemma 4.1, recall that $t_k \in (0, 1)$, we get

$$\|x^{k+1} - x^k\|^2 - 2t_k\langle v^k, \hat{x} - x^k \rangle \leq t_k\|v^k\|^2 - 2t_k(\hat{\beta}\langle J_F(x^k)^T \omega^k, v^k \rangle + \|v^k\|^2). \tag{9}$$

On the other hand, $t_k\|v^k\|^2 - 2t_k(\hat{\beta}\langle J_F(x^k)^T \omega^k, v^k \rangle + \|v^k\|^2) \leq -2t_k\hat{\beta}\langle J_F(x^k)^T \omega^k, v^k \rangle$, Recalling that $\alpha \leq |\alpha|$, from (9), we obtain $\|x^{k+1} - x^k\|^2 - 2t_k\langle v^k, \hat{x} - x^k \rangle \leq 2t_k|\hat{\beta}\langle J_F(x^k)^T \omega^k, v^k \rangle|$. Combining last inequality with (8), we get

$$\|x^{k+1} - \hat{x}\|^2 \leq \|x^k - \hat{x}\|^2 + 2t_k\hat{\beta}|\langle J_F(x^k)^T \omega^k, v^k \rangle|. \tag{10}$$

Since K is a pointed, closed and convex cone, $\text{int}(K^\star)$ is a nonempty set; see [20, Propositions 2.1.4, 2.1.7(i)]. Therefore, $K^\star$ contains a basis of $\mathbb{R}^m$. Without loss of generality, we assume that the basis $\{\widetilde{\omega}^1, ..., \widetilde{\omega}^m\} \subset \text{conv}(G)$. Thus, for each k, there exist $\eta_i^k \in \mathbb{R}$, $i = 1, \ldots, m$, such that $\omega^k = \sum_{i=1}^m \eta_i^k \widetilde{\omega}^i$. By the compactness of $\text{conv}(G)$, there exists $L > 0$, such that $|\eta_i^k| \leq L$ for all i and k. Thus, inequality (10) becomes $\|x^{k+1} - \hat{x}\|^2 \leq \|x^k - \hat{x}\|^2 + 2t_k\hat{\beta}L \sum_{i=1}^m |\langle \widetilde{\omega}^i, J_F(x^k)v^k \rangle|$.

Defining $\delta_k := 2t_k\hat{\beta}L \sum_{i=1}^m |\langle \widetilde{\omega}^i, J_F(x^k)v^k \rangle|$, it follows that $\delta_k > 0$. Since $(F(x^k))_{k \in \mathbb{N}}$ is K–bounded and using (6), we have $\sum_{k=0}^\infty \delta_k < \infty$. Therefore, since $\hat{x}$ is an arbitrary element of T, the desired result follows from Definition 2.3. $\qquad \square$

Next theorem establishes a sufficient condition for the convergence of the sequence $(x^k)_{k\in\mathbb{N}}$.

Theorem 4.1. *Assume that F is a K–quasiconvex function. Then, $(x^k)_{k\in\mathbb{N}}$ converges to a stationary point.*

Proof. Since F is K–quasiconvex, from Lemma 4.2 it follows that $(x^k)_{k\in\mathbb{N}}$ is quasi-Fejér convergent and, hence, bounded; see Lemma 2.2(i). Therefore $(x^k)_{k\in\mathbb{N}}$ has at least one accumulation point, say x^*. From Proposition 4.2, x^* is a stationary point. Moreover, since C is closed and the sequence is feasible, $x^* \in C$.

We proceed to prove $x^* \in T$. Since F is continuous $(F(x^k))_{k\in\mathbb{N}}$ has $F(x^*)$ as accumulation point. By Proposition 4.1, $(F(x^k))_{k\in\mathbb{N}}$ is a K–decreasing sequence. Hence, the whole sequence $(F(x^k))_{k\in\mathbb{N}}$ converges to $F(x^*)$ and holds $F(x^*) \preceq_K F(x^k)$ with $k \in \mathbb{N}$, which implies that $x^* \in T$. Therefore, the desired result follows from Lemma 2.2(ii) and Proposition 4.2. $\qquad\square$

This theorem extends the full convergence obtained under K–convexity in [12] to the K–quasiconvex case. This class is actually larger than K–convex problems as next example shows.

Example 4.1. *Let $F : \mathbb{R} \to \mathbb{R}^2$, $F(t) = (4t^2, t^4 - 4t^2 + 2)$, and $K = co(conv(\{(1,0),(1,1)\}))$. The function F is not K–convex because $(F(0) + F(1))/2 - F(1/2) = (1, -9/16) \notin K$, but, as $\langle(1,0), F(t)\rangle = 4t^2$ and $\langle(1,1), F(t)\rangle = t^4 + 2$, are quasiconvex, by Theorem 2.1, F is K–quasiconvex.*

5   Final Remarks

In this paper we considered the inexact projected gradient method presented in [12]. We explored strongly the structure of problem (1), mainly the quasiconvexity of the function F, and obtained that the sequence generated by the approach converges to a stationary point. So, the method will be successful for a class which is larger than the cases studied so far.

Future research is focused into two directions: the practical implementation of the method and its generalization to other cases. In particular, we are looking for the convergence of a subgradient method for solving the non-differentiable and K–quasiconvex problem without to use scalarizations.

Recently were published in [2, 3, 8] the subgradient approaches for solving vectorial and feasibility problems; see also [10] using strongly scalarizations technics. We also pretend to extend these methods to the variable ordering case; see, for instance, [1].

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