AN IMPROVISED APPROACH TO ROBUSTNESS IN LINEAR OPTIMIZATION

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Abstract. We treat uncertain linear programming problems by utilizing the notion of weighted analytic centers and notions from the area of multi-criteria decision making. In addition to many practical advantages, due to the flexibility of our approach, we are able to prove that the robust optimal solutions generated by our algorithms are at least as desirable to the decision maker as any solution generated by many other robust optimization algorithms. We then develop interactive cutting-plane algorithms for robust optimization, based on concave and quasi-concave utility functions. We present some probabilistic bounds for feasibility of robust solutions and evaluate our approach by means of computational experiments.

1. Introduction

Optimization problems are widespread in real life decision making situations. However, data perturbations as well as uncertainty in at least part of the data are very difficult to avoid in practice. Therefore, in most cases we have to deal with the reality that some aspects of the data of the optimization problem are uncertain. This uncertainty is caused by many sources such as forecasting or data approximation or noise in measurements. For example,

- factors such as change in government policies or emergence of new products can cause demand uncertainty in the market;
- the amount of rain next year is clearly uncertain which can have a profound effect on the agriculture sector among others;
- the amount of profit depends on the prices of the products which are uncertain. This uncertainty is further affected by factors such as cost of raw materials, customers' budgets and their changing preferences.

In order to handle optimization problems under uncertainty, several techniques have been proposed. The most common, widely-known approaches are

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- Sensitivity analysis,
- Chance constrained programming,
- Stochastic programming,
- Robust optimization.

In sensitivity analysis, the influence of data uncertainty is initially ignored, and then the obtained solution is justified based on the data perturbations [14]. Sensitivity analysis shows how much the optimal solution to a perturbed problem can differ from that of the nominal problem. In other words, it gives information about the local stability of a solution without any clue about improving it. This method becomes impractical for large number of uncertain parameters.

In chance constrained programming, we use some stochastic models of uncertain data to replace the deterministic constraints by their probabilistic counterparts [29, 33, 19]. It is a natural way of converting the uncertain optimization problem into a deterministic one. However, most of the time the result is a computationally intractable problem for two reasons [4]: *i)* evaluation of the probabilities with high accuracy is difficult even for simple probability distributions; *ii)* most of the time, the feasible region of the resulting problem is non-convex which makes the utilization of chance constrained programming highly problematic.

In stochastic programming the goal is to find a solution that is feasible for all (or almost all) possible instances of the data and to optimize the expectation of some function of the decisions and the random variables. The most widely used stochastic programming models are two-stage programs. At the first stage, the Decision Maker (DM) makes a decision. After that, a random event occurs and the solution from the first stage might not satisfy some of the constraints. At the second stage, a recourse decision is made which compensates for any bad effects of the first stage solution. The main assumption is that probability distributions governing the data are known or can be estimated which is a major drawback in many applications. Distributions of the random parameters are almost never known exactly, and have to be estimated which typically yields an approximate solution. Another problem with stochastic programming is that the problems can become unmanageably huge to be able to draw valid conclusions. It is discussed in [39] (for supply chain networks) that the number of scenarios might become so huge that the underlying optimization problems become ultimately unmanageable, even for a small dimensional uncertain problem.

Robust optimization is the method that is most closely related to our approach. Generally speaking, robust optimization can be applied to any optimization problem where the uncertain data can be separated from the problem's structure. This method is applicable to convex optimization problems including semidefinite programming [4]. Our focus in this paper is on uncertain linear programming problems. Uncertainty in the data means that the exact values of the data are not known, at the time when the solution has to be determined. In robust optimization framework, uncertainty in the data is described through *uncertainty sets*, which contain all (or most of) possible values that may be realized for the uncertain parameters.

Since the interest in robust formulations was revived in the 1990s, many researchers have introduced new formulations for robust optimization framework in linear programming and general convex programming [41, 7, 6, 8, 5, 12, 13, 11, 9, 32]. Ben-Tal and Nemirovski [7, 6] provided some of the first formulations for robust LP with detailed mathematical analysis. Bertsimas and Sim proposed an approach that offers control on the degree of conservatism for every constraint as well as the objective function. Bertsimas et al. [9] characterize the robust counterpart of an LP problem with uncertainty set described by an arbitrary norm. By choosing appropriate norms, they recover the formulations proposed in the above papers [7, 6, 9].

The goal of classical robust optimization is to find a solution that is capable to cope best of all with *all* realizations of the data from a given (usually bounded) uncertainty set [4, 3]. By the classical definition of robustness [4, 8, 11, 21], a *robust optimal solution* is the solution of the following problem:

$$\max_{x \in \mathbb{R}^n} \left\{ \inf_{\tilde{c} \in \mathcal{C}} \langle \tilde{c}, x \rangle : \tilde{A}x \leq \tilde{b} \ , \forall \tilde{b} \in \mathcal{B}, \forall \tilde{A} \in \mathcal{A} \right\}, \tag{1}$$

where $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{B}$ are given uncertainty sets for $\tilde{c}$, $\tilde{A}$, and $\tilde{b}$, respectively. Throughout this paper, we refer to the formulation of (1) as *classical robust formulation.*

1.1. **Some drawbacks of robust optimization.** Classical robust optimization is a powerful method to deal with optimization problems with uncertain data, however, we can raise some valid criticisms. One of the assumptions for robust optimization is that the uncertainty set must be precisely specified before solving the problem. Even if the uncertainty is only in the RHS, expecting the DM to construct accurately an ellipsoid or even a hypercube for uncertainty set is not always reasonable. Another main criticism to classical robust optimization is that satisfying all of the constraints, if not make the problem infeasible, may lead to an objective value very far from the optimal value of the nominal problem. This problem is more critical for large deviations. As an example, [6, 30] considered some of the problems of NETLIB library (under reasonable assumptions on uncertainty of certain entries) and showed that classical robust counterparts of most of the problems in NETLIB become infeasible for a small perturbation. Moreover, in other problems, objective value of the classical robust optimal solution is very low and may be unsatisfactory for the decision maker.

Several modifications of classical robust optimization have been introduced to deal with this issue. One, for example, is *globalized robust conterparts* introduced in Section 3 of [4]. The idea is to consider some constraints as "soft" whose violation can be tolerated to some degree. In this method, we take care of what happens when data leaves the nominal uncertainty set. In other words, we have "controlled deterioration" of the constraint. These modified approaches have more flexibility than the classical robust methodology, but we have the problem that the modified robust counterpart of uncertain problems may become computationally intractable. Although the modified robust optimization framework rectifies this drawback to some extent, it intensifies the first criticism by putting more pressure on the DM to specify deterministic uncertainty sets before solving the problem.

Another criticism of the classical robust optimization is that it gives the same "weight" to all the constraints. In practice, this is not the case as some constraints are more important for the DM. There are some options in classical robust optimization like changing the uncertainty set which again intensifies the first criticism. We see that our approach can alleviate this difficulty.

1.2. **Contributions of this paper.** We use a utility function model rather than an uncertainty region model. Our utility function approach is at least as powerful as classical robust optimization from a theoretical point of view, but it is also advantageous in practice, since it involves the DM continuously in the optimization process in an overall less taxing way.

One of the main contributions of this paper is the development of cutting-plane algorithms for robust optimization using the notion of weighted analytic centers in a small dimensional weight-space. We also design algorithms in the slack variable space as a theoretical stepping stone towards the more applicable weight-space cutting-plane algorithms. Ultimately, we are proposing that our approach be used in practice with a small number (say somewhere in the order of 4 to 20) of *driving factors* that really matter to the DM. These driving factors are independent of the number of variables and constraints, and determine the dimension of the weight space (for interaction with the DM). Working in a low dimensional weight-space not only simplifies the interaction for the DM, but also makes the cutting-plane algorithm more efficient.

The notion of weight has been widely used in the area of multi-criteria decision making: when we have several objective values to optimize, a natural way is to optimize a weighted sum of them [27], [26]. Authors in [27] presented an algorithm for evaluating and ranking items with multiple attributes. [27] is related to our work as the proposed algorithm is a cutting-plane one. However, our algorithm uses the concept of weighted analytic center which is completely different. Authors in [26] proposed a family of models (denoted my McRow) for multi-expert multi-criteria decision making. Their work is close to ours as they derived compact formulations of the McRow model by assuming some structure for the weight region, such as polyhedral or conic descriptions. Our work has fundamental differences with [26]: cutting-plane algorithms in the weight-space find a weight vector w in a fixed weight region (the unit simplex) such that the weighted analytic center of w, say $x(w)$, is the desired solution for the decision maker. The algorithms we design in this paper make it possible to implement the ideas we mentioned above to help overcome some of the difficulties for robust optimization to reach a broader, practicing user base. For some further details and related discussion, also see Moazeni [30] and Karimi [28].

1.3. **Notations and assumptions.** Before introducing our approach in the next section, let us first explain some of the assumptions and notations we are going to use. Much of the prior work on robust linear programming addresses the uncertainty through the coefficient matrix. Bertsimas and Sim [13] considered linear programming problems in which all data except the right-hand-side (RHS) vector is uncertain. In [8, 7, 11], it is assumed that the uncertainty affects the coefficient matrix and the RHS vector. Some papers deal with uncertainty only in the coefficient matrix [6, 12, 9]. Optimization problems in which all of the data in the objective function, RHS vector and the coefficient matrix are subject to uncertainty, have been considered in [5]. As we explain in Section 2, the nominal data and a rough approximation of the uncertainty set are enough for our approach. However, the structure of uncertainty region is useful for the probability analysis. In this paper, we assume that the coefficient matrix is deterministic, where the coefficients of the objective function and the RHS vector are subject to uncertainty. Some of the reasons we may assume uncertainty in the objective value and the RHS (at least in some applications) are:

(1) Instead of specifying uncertainty for each local variable, we handle the whole uncertainty with some global variables. These global variables can be, for example, the whole budget, human resources, availability of certain critical raw materials, government quotas, etc. It is easier for the DM to specify the uncertainty set for these global variables. Then, we can approximate the uncertainty in the coefficient matrix with the uncertainty in the RHS and the objective function. In other words, we may fix the coefficient matrix on one of the samples from the

uncertainty set and then handle the uncertainty by introducing uncertainty to the RHS vector as in [10].

(2) A certain coefficient matrix is typical for many real world problems. In many applications of planning and network design problems such as scheduling, manufacturing, electric utilities, telecommunications, inventory management and transportation, uncertainty might only affect costs (coefficients of the objective function) and demands (the RHS vector)[31, 37]:
  - **Transportation system:** In many problems in this domain, we can assume that in a road network, the nodes and the arcs are fixed. However, the cost associated to each arc, i.e. the vehicle travel time, and/or the capacity associated to each arc are not known precisely.
  - **Traffic assignment problem:** In some problems, we assume that the drivers have perfect information about the arcs and nodes, which are the structure of the road network and the existing streets. However, their route choice behavior makes the travelling time uncertain.
  - **Distribution system:** In some applications, the locations of warehouses and their capacities (in inventory planning and distribution problems) are well-known and fixed for the DM. However, the size of orders and the demand rate of an item could translate to an uncertain RHS vector. Holding costs, set up costs and shortage costs, which affect the optimal inventory cost, are also typically uncertain. These affect at least the objective function.
  - **Medical/health applications:** In these applications (see for instance, [18, 15, 40, 17]) the DM may be a group of people (including medical doctors and a patient who are more comfortable with a few, say 4-20, driving factors which may be more easily handled by the mathematical model, if these factors could be represented as uncertain RHS values.

  In all of the aforementioned applications, well-understood existing resources, reliable structures (well-established street and road networks, warehouses, and machines which are not going to change), and logical components of the formulation are translated into a certain coefficient matrix. The data in the objective function and the RHS vector are usually estimated by statistical techniques by the DM, or affected by uncertain elements such as institutional, social, or economical market conditions. Therefore, determining these coefficients with precision is often difficult or practically impossible. Hence, considering uncertainty in the objective function and the RHS vector seems to be very applicable, and motivates us to consider such formulation in LP problems separately.

(3) In our approach, we need the uncertainty sets for probabilistic analysis. Uncertainty in the RHS and the objective value is easier to handle mathematically.

By the above explanation, in this paper, we fix the coefficient matrix A. It is clear that changing each entry of A could change the geometry of the feasible region. On the other hand, neither we nor the DM know how each coefficient may affect the optimal solution before starting to solve the problem. Therefore, to fix matrix A, we rely on the nominal values (expected values) of the coefficients estimated by a method agreed by the DM. An uncertain linear programming problem with deterministic coefficient matrix $A \in \mathbb{R}^{m\times n}$, is of the form:

$$\begin{aligned} \max \quad & \langle \tilde{c}, x\rangle \\ \text{s.t.} \quad & Ax \leq \tilde{b}, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{2}$$

where $\tilde{c} \in \mathcal{C}$ and $\tilde{b} \in \mathcal{B}$ are an n-vector and an m-vector respectively, whose entries are subject to uncertainty. $\mathcal{C}$ and $\mathcal{B}$ are called *uncertainty sets*. In this paper, we deal with problem (2) and suppose that the data uncertainty affects only the elements of the vectors $\tilde{b}$ and $\tilde{c}$. We assume entries of $\tilde{c}$ and $\tilde{b}$ are random variables with unknown distributions, as it is impractical to assume that the exact distribution is explicitly known. By classical view of robust optimization, *classical robust counterpart* of problem (2) is defined in (1) with a certain A. Feasible/Optimal solutions of problem (1) are called *classical robust feasible/classical robust optimal solutions* of problem (2) [4]. Without loss of generality, we make the following assumptions on $\tilde{b}$ and $\tilde{c}$:

- For every $i \in \{1, 2, \cdots, m\}$, $\tilde{b}_i$ can be written as $\tilde{b}_i = b_i^{(0)} + \sum_{l=1}^{N_i} \Delta b_i^l \tilde{z}_i^l$ where $\{\tilde{z}_i^l\}_{l=1}^{N_i}$ are independent random variables for every $i \in \{1, \cdots, m\}$.
- For each $\tilde{c}_i$, $i \in \{1, \cdots, n\}$, we have $\tilde{c}_i = c_i^{(0)} + \sum_{l=1}^{N_{ic}} \Delta c_i^l \tilde{z}_{ic}^l$ where $\{\tilde{z}_{ic}^l\}_{l=1}^{N_{ic}}$ are independent random variables.

As can be seen above, each variable $\tilde{b}_i$ is the summation of a nominal value $b_i^{(0)}$ with scaled random variables $\{\tilde{z}_i^l\}_{l=1}^{N_i}$. In practice, the number of these random variables N_i is small compared to the dimension of A as we explained above: each random variable $\tilde{z}_i^l$ represents a major source of uncertainty in the system.

Here, we impose the following restrictions on the problem (2) [30]:

- The matrix A has full column rank, i.e., $\text{rank}(A) = n \leq m$.
- The set $\{x \in \mathbb{R}^n : Ax \leq b^{(0)}\}$ is bounded.
- The set $\{x \in \mathbb{R}^n : Ax \leq b^{(0)}\}$ has nonempty interior.

If A does not have full column rank, then a problem of the form

$$\begin{aligned} \max \quad & \langle c^{(0)}, x \rangle \\ \text{s.t.} \quad & Ax \leq b^{(0)}, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{3}$$

either is unbounded, or can be projected to a smaller dimensional space, with the same optimal value. Let μ be a $(n-1)$-vector such that $\sum_{i=1}^{n-1} \mu_i A_i = A_n$, where A_i is the i-th column of A. By some simple linear algebra, we can eliminate x_n and represent problem (2) in the $(n-1)$-dimensional space. If the problem is not unbounded, the optimal objective function value of the problem in the $(n-1)$-dimensional space equals the optimal objective function value of the original problem. If the LP represent a practical problem such as a combinatorial optimization problem, typically non-negativity constraints imply that A has a full column rank. The assumption on the boundedness of $\{x \in \mathbb{R}^n : Ax \leq b^{(0)}\}$ is not very restrictive as it is satisfied in many practical problems representing integer programming and combinatorial optimization problems.

For the third property, assume that the polyhedron $\mathcal{P} := \{x \in \mathbb{R}^n : Ax \leq b^{(0)}\}$ is nonempty with empty interior. Then, the dimension of the affine hull of $\mathcal{P}$, say d, is less than n. Hence, we can represent the affine space of $\mathcal{P}$ as $\{x = h + By : y \in \mathbb{R}^d\}$, where B has full column rank d. We can define a polyhedron $\hat{\mathcal{P}} := \{x \in \mathbb{R}^d : \hat{A}x \leq \hat{b}\}$ with non-empty interior such that there is a one-to-one map between $\mathcal{P}$ and $\hat{\mathcal{P}}$, using the fact that B has full column rank. We can rewrite the problem in $\mathbb{R}^d$ with feasible region $\hat{\mathcal{P}}$.

In this paper, vectors and matrices are denoted, respectively, by lower and uppercase letters. The matrices Y and S represent diagonal matrices, having the components of vectors y and s on their main diagonals, respectively. The letter e and e_i denote a vector of ones and a vector that is everywhere zero except at the i-th entry with the appropriate dimension, respectively. The rows of a matrix are shown by superscripts of the row, i.e., $a^{(i)}$ is the i-th row of the matrix A. The inner product of two vectors $a, b \in \mathbb{R}^n$ is shown both by $\langle a, b\rangle$ and $a^\top b$. For a matrix A, we show the range of A with $\mathcal{R}(A)$ and the null space of A with $\mathcal{N}(A)$.

1.4. **Overview of the paper.** In this paper, we design new algorithms which alleviate some of the drawbacks of classical robust optimization approach mentioned above. We employ an interactive decision making approach to involve DM in the optimization process, and to increase the reliability of the information extracted from the DM. We also utilize the notion of weighted analytic centers, and implement our algorithms in the space of weight vectors which makes the interaction with the DM easier.

In Section 2, we explain our approach and the scheme of our algorithm. In Section 3, we consider the properties of the weight-space that help us to design the algorithm and perform the probabilistic analysis. In Section 4, we prove that our approach is as least as strong as the classical robust optimization approach. In Section 5, we design the cutting-plane algorithms, talk about the modifications of the algorithm, and explain some practical concerns of our approach. Section 6 is about the probabilistic analysis that is important for interaction with the decision maker. Some preliminary computational results are presented in Section 7. In Section 8, we briefly talk about the extension of the approach to semidefinite programming and quasi-concave utility functions, and then conclude the paper.

2. A utility function approach

In Section 1, we introduced different methods for dealing with LP problems under uncertainty. For each method, we explained the drawbacks and practical difficulties. In this section, we introduce our new approach that helps us overcome some of these difficulties. Let us focus on the robust optimization method that from many points of view is the strongest among the methods we introduced in Section 1. One of the main problems with robust optimization is that the uncertainty region must be specified before solving the problem. As we explained, in practice, even if the uncertainty is only in the RHS, expecting the DM to construct accurately an ellipsoid or a hypercube for uncertainty set may not be reasonable.

The proposed method removes DM's anxiety about determining the uncertainty set precisely, and a nominal value of the data is enough. We just need to have an estimate from the uncertainty set to evaluate the proposed solution and derive the probability bounds for our approach. We propose a tractable method to find a solution in the expected feasible region (nominal feasible region) which satisfies the expectations of the decision maker. Although in this approach the robustness of some constraints is ignored, the proposed solution is robust from DM's point of view. In this paper, we assume that A, $c^{(0)}$, and $b^{(0)}$ are the nominal values of the uncertain LP problem in (2). Hence, the nominal LP problem that we use to design our algorithm is (3).

The proposed solution is obtained efficiently by using the notion of weighted analytic centers. As we explain in Section 3, there is a correspondence between the feasible region and the weight-space. To any weight vector $w \in \mathbb{R}^m_{++}$, we can assign three vectors $(x(w), s(w), y(w))$, $s > 0$, $y > 0$, where $s = b - Ax$ is the slack vector. We will mention that for any feasible vector $\hat{x}$, there exists $w \in \mathbb{R}^m_{++}$ such that $x(w) = \hat{x}$. This property shows that we can sweep the whole feasible region by moving in the weight-space. Working in the weight-space is equivalent to working with the slack variables which gives a tangible understanding about how far we are from the boundary of the feasible region. As will be explained latter, we can also add a constraint to the problem for the objective function and translate the objective value to a slack variable. This helps us work just with the slack variables to solve the problem.

By using the notion of weighted center, we benefit from the differentiability. Weight-space and weighted-analytic-centers approach embeds a "highly differentiable" structure into the algorithms. Such tools are extremely useful in both the theory and applications of optimization. In contrast, classical robust optimization and other competing techniques usually end up delivering a final solution where differentiability cannot be expected; this happens because their potential solutions located on the boundary of some of the structures defining the problem. In this paper, we assume that the DM's preferences can be modeled by a utility function $U : \mathbb{R}^m \to \mathbb{R}$. By this assumption, we can write our problem as

$$\begin{array}{lll} & \max & U(s) \\ (4) & s.t. & s \in B_s, \end{array}$$

where B_s is the set of *centric* s-vectors that we define later in Section 3 (Definition 3.1). We do *not* have access to this utility function, however assume that, for a centric slack vector s, we can ask the DM for some information about the function. The questions we are going to ask are the supergradient of $U(s)$ at some points and some pairwise comparison questions if needed. The goal of our algorithm is to maximize this utility function. At each step, we use the information from the DM to produce a cut in the s-space or w-space to shrink the corresponding set such that an optimal solution is kept in the shrunken set. The design of the algorithms, some convergence theory, and some practical issues are covered in Section 5.

3. Weighted analytic centers

In this section, we first define the notion of weighted analytic center in Subsection 3.1. In Subsection 3.2, we prove many useful results about the properties of the weight-space, which are useful in the design of the algorithms in Section 5.

3.1. **Definition of weighted center.** For every $i \in \{1, 2, \cdots, m\}$, let $\mathcal{F}_i$ be a closed convex subset of $\mathbb{R}^n$ such that $\mathcal{F} := \bigcap_{i=1}^m \mathcal{F}_i$ is bounded and has nonempty interior.

Let $F_i : \text{int}(\mathcal{F}_i) \to \mathbb{R}$ be a self-concordant barrier for $\mathcal{F}_i$, $i \in \{1, 2, \cdots, m\}$ (For a definition of self-concordant barrier functions see [35]). For every $w \in \mathbb{R}^m_{++}$, we define the w-center of $\mathcal{F}$ as

$$\arg\min \left\{ \sum_{i=1}^{m} w_i F_i(x) : x \in \mathcal{F} \right\}.$$

Consider the special case when each $\mathcal{F}_i$ is a closed half-space in $\mathbb{R}^n$. Then the following result is well-known.

Theorem 3.1. *Suppose for every $i \in \{1, 2, \cdots, m\}$, $a^{(i)} \in \mathbb{R}^n \setminus \{0\}$ and $b_i \in \mathbb{R}$ are given such that:*

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n : \langle a^{(i)}, x \rangle \leq b_i, \forall i \in \{1, 2, .., m\} \right\},$$

is bounded and $\text{int}(\mathcal{F})$ *is nonempty. Also, for every $i \in \{1, 2, \cdots, m\}$ define $F_i(x) := -\ln(b_i - \langle a^{(i)}, x \rangle)$. Then for every $w \in \mathbb{R}^m_{++}$, there exists a unique w-center in the interior of $\mathcal{F}$, $x(w)$. Conversely, for every $x \in \text{int}(\mathcal{F})$, there exists some weight vector $w(x) \in \mathbb{R}^m_{++}$ such that x is the unique $w(x)$-center of $\mathcal{F}$.*

Define the following convex optimization problems:

$$\begin{aligned} \min \quad & \langle c, x \rangle - \sum_{i=1}^{m} w_i \ln(s_i) \\ \text{s.t.} \quad & Ax + s = b, \\ & s \in \mathbb{R}^m_{++}, \; x \in \mathbb{R}^n, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \min \quad & \langle b, y \rangle - \sum_{i=1}^{m} w_i \ln(y_i) \\ \text{s.t.} \quad & A^\top y = c, \\ & y \in \mathbb{R}^m_{++}, \end{aligned} \tag{6}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. For every weight vector $w > 0$, the objective functions of the above problems are strictly convex on their domains. Moreover, the objective function values tend to $+\infty$ along any sequence of their interior points (strictly feasible points), converging to a point on their respective boundary. So, the above problems have minimizers in the interior of their respective feasible regions. Since the objective functions are strictly convex, the minimizers are unique. Therefore, for every given $w > 0$, the above problems have unique solutions $(x(w), s(w))$ and $y(w)$. These solutions can be used to define many *primal-dual weighted-central-paths* as the solution set $\{(x(tw), y(tw), s(tw)) : t > 0\}$ of the following system of equations and strict inequalities:

$$\begin{aligned} & Ax + s = b, \; s > 0, \\ & A^\top y = c, \\ & Sy = w, \end{aligned} \tag{7}$$

where $S := \text{Diag}(s)$. When we set $w := e$ in $\{(x(tw), y(tw), s(tw)) : t > 0\}$, we obtain the usual primal-dual weighted-central-path. Figure 1 illustrates some weighted central paths.

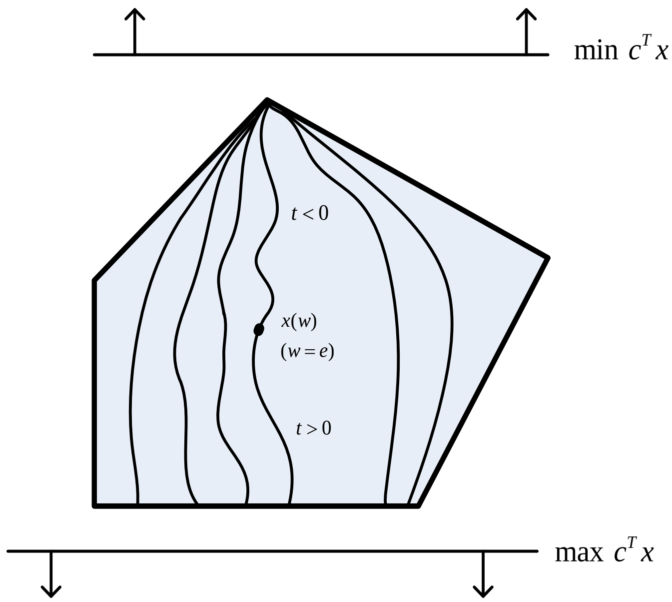

FIGURE 1. Primal-dual central paths.

In this paper, we prefer to start working only with the slack variables. Hence, we add a constraint that represents the objective function. This constraint is $\langle c, x\rangle \geq v$, where v is a lower bound specified by the information from the DM. For example, if the DM decides that the objective value must not be below a certain value, we can put v equal to that value. So, we change the definition of $\mathcal{F}$ as follows

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n : \langle c, x\rangle \geq v,\ \langle a^{(i)}, x\rangle \leq b_i, \forall i \in \{1, 2, .., m\} \right\}. \tag{8}$$

In the above formulation, the new matrix A is $(m+1)$-by-n. Now, we may redefine m and assume that $A \in \mathbb{R}^{m\times n}$ also contains the last added constraint. As we embedded the objective function in A, we can put $c := 0$, and solve the following set of equations to find the weighted analytic center:

$$\begin{aligned} &Ax + s = b,\ s > 0,\\ &A^\top y = 0,\\ &Sy = w, \end{aligned} \tag{9}$$

For every given weight vector w, $(x(w), y(w), s(w))$ is obtained uniquely and $x(w)$ is called the *weighted center* of w. We may also refer to $(x(w), y(w), s(w))$ as the weighted center of w. For every given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $y > 0$, that satisfy the above system, w and $s(w)$ are obtained uniquely. However, for a given $x \in \mathbb{R}^n$, there are many weight vectors w that give x as the w-center of the corresponding polytope.

Example 3.1. [30] *Let*

$$b := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad A := \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix},$$

then the point $x = (0.5\ 0.5)^\top$ *is both* $(0.25,\ 0.25,\ 0.25,\ 0.25)$*-center* $($*corresponding to* $y = 0.5e)$ *and* $(0.35,\ 0.35,\ 0.15,\ 0.15)$*-center* $($*corresponding to* $y = (0.7,\ 0.7,\ 0.3,\ 0.3)^\top)$ *of the polytope.*

The following well-known lemma is useful.

Lemma 3.1. *Let* (x, y, s) *and* $(\hat{x}, \hat{y}, \hat{s})$ *be the solutions of system* (9) *corresponding to the weight vectors* $w,\ \hat{w} \in \mathbb{R}^m_{++}$, *respectively. For every* $\bar{y}$ *in the null space of* $A^\top$ *we have:*

$$\langle \hat{s}, \bar{y} \rangle = \langle s, \bar{y} \rangle.$$

Proof. From (9), we have $s = b - Ax$ and $\hat{s} = b - A\hat{x}$, which results in $s - \hat{s} = A(x - \hat{x})$. Hence we have $s - \hat{s} \in \mathcal{R}(A)$. As the null space of $A^\top$ and the range of A are orthogonal, for every $\bar{y} \in \mathcal{N}(A^\top)$ we can write:

$$\langle s - \hat{s}, \bar{y} \rangle = 0 \quad \Rightarrow \quad \langle \hat{s}, \bar{y} \rangle = \langle s, \bar{y} \rangle.$$

□

Let $(\hat{x}, \hat{y}, \hat{s})$ be the solution of system (9) corresponding to the weight vector $\hat{w}$. Moreover, assume that $\bar{y} > 0$ is such that $A^\top \bar{y} = 0$. Then, by using Lemma 3.1, we can show that $(\hat{x}, \bar{y}, \hat{s})$ is the solution of system (9) corresponding to the weight vector $\bar{Y}(\hat{Y})^{-1}\hat{w}$. Hence, there may be many weight vectors that give the same w-center. A stronger result is the following lemma which shows that in some cases, we can find the weighted center for a combination of weight vectors by using the combination of their weighted centers.

Lemma 3.2. *Let* $(x^{(i)}, y^{(i)}, s^{(i)})$, $i \in \{1, \cdots, \ell\}$, *be solutions of system* (9), *corresponding to the weights* $w^{(i)}$. *Then, for every set of* $\beta_i \in [0, 1]$, $i \in \{1, \cdots, \ell\}$, *such that* $\sum_{i=1}^{\ell} \beta_i = 1$, *and for every* $j \in \{1, \cdots, \ell\}$, *we have* $(\sum_{i=1}^{\ell} \beta_i x^{(i)}, y^{(j)}, \sum_{i=1}^{\ell} \beta_i s^{(i)})$ *is the* w*-center of* $\mathcal{F}$, *where*

$$w := \sum_{i=1}^{\ell} \beta_i Y^{(j)} (Y^{(i)})^{-1} w^{(i)}.$$

Moreover,

$$\sum_{i=1}^{m} w_i = \sum_{i=1}^{m} w_i^{(j)}.$$

Proof. See Appendix A. □

Before starting the next subsection about the properties of w-space, without loss of generality, we restrict ourselves to the weights on the unit simplex, i.e., we consider weighted center (x, y, s) corresponding to weight vectors w such that $\sum_{i=1}^{m} w_i = 1$. A special case can be $w = \frac{1}{m}e$, where e

is the vector of all ones. We will show that this subset of weight vectors is enough to represent the feasible region. We call this simplex of weight vectors W:

$$W := \{w \in \mathbb{R}^m : w > 0,\ e^\top w = 1\}.$$

We can define the following notion for future reference:

Definition 3.1. *A vector $s \in \mathbb{R}^m$ or $y \in \mathbb{R}^m$ is called* centric *if there exists x such that (x, y, s) satisfies* (9) *for a weight vector $w > 0$ where $e^\top w = 1$.*

3.2. **Properties of w-space.** In this subsection, we study the structure of the w-space, which is important for the design of the algorithms in Section 5. Let s and y be centric. First, we note that the simplex of the weight vectors can be divided into regions of constant y-vector (W_y) and constant s-vector (W_s). By using Lemma 3.2, if $(\hat{x}, \hat{y}, \hat{s})$ is the solution of system (9) corresponding to the weight vector $\hat{w} \in W$, and $\bar{y} > 0$ is any centric y-vector, then $(\hat{x}, \bar{y}, \hat{s})$ is the solution of system (9) corresponding to the weight vector $\bar{Y}(\hat{Y})^{-1}\hat{w}$. This means that for every centric vector $\hat{s}$ and any centric vector y, $\hat{S}y$ is a weight vector in the simplex.

For every pair of centric vectors s and y, W_s and W_y are convex. To see this, let $(x, \bar{y}, s)$ and (x, y, s) be the weighted centers of $\hat{w}$ and w. Then, it is easy to see that for every $\beta \in [0, 1]$, $(x, \beta\bar{y} + (1-\beta)y, s)$ is the weighted center of $\beta\hat{w} + (1-\beta)w$. With a similar reasoning, W_y is convex for every centric y.

Using (9), we can express W_s and W_y as follows:

$$\begin{aligned} W_y &= \left\{Y(b - Ax) \ :\ Ax < b,\ y^\top(b - Ax) = 1\right\} \\ &= \left\{w > 0 \ :\ YAx + w = Yb,\ e^\top w = 1\right\} \\ &= Y[(\mathcal{R}(A) + b) \cap \mathbb{R}^m_{++}] \cap B_1(0, 1), \end{aligned} \tag{10}$$

$$\begin{aligned} W_s &= \left\{Sy \ :\ A^\top y = 0,\ y > 0\ ,\ s^\top y = 1\right\} \\ &= \left\{w > 0 \ :\ A^\top S^{-1} w = 0,\ e^\top w = 1\right\} \\ &= S[\mathcal{N}(A^\top) \cap \mathbb{R}^m_{++}] \cap B_1(0, 1), \end{aligned} \tag{11}$$

where $B_1(0, 1)$ is the unit ball in 1-norm centered at zero vector. Here, we want to find another formulation for W_y that might work better in some cases. We use the following lemma.

Lemma 3.3. *Assume that the rows of $B_y \in \mathbb{R}^{(m-n)\times m}$ make a basis for the null space of $A^\top Y$. Then there exists $x \in \mathbb{R}^n$ such that $YAx + w = Yb$ if and only if $B_y w = B_y Yb$. I.e., $(Yb - w) \in \mathcal{R}(YA)$ iff $(Yb - w) \in \mathcal{N}(B_y)$.*

Proof. Assume that there exists x such that $YAx + w = Yb$. By multiplying both sides with B_y from the left and using the fact that $B_y YA = 0$ we have the result. For the other direction, assume that $B_y w = B_y Yb$. Then $B_y(w - Yb) = 0$ which means $w - Yb$ is in the null space of B_y. Then, using the orthogonal decomposition theorem, we have $\mathcal{N}(B_y) = \mathcal{R}(B_y^\top)^\perp = \mathcal{N}(A^\top Y)^\perp = \mathcal{R}(YA)$. Thus, there exists x such that $YAx + w = Yb$. □

Assume that $B \in \mathbb{R}^{(m-n)\times m}$ is such that its rows make a basis for the null space of $A^\top$. For every vector y, we have $A^\top y = A^\top Y(Y^{-1}y)$, so if y is in the null space of $A^\top$, $Y^{-1}y$ is in the null space of $A^\top Y$. Hence, if the rows of B make a basis for the null space of $A^\top$, the rows of BY^{-1} make a basis for the null space of $A^\top Y$ and we can write $B_y = BY^{-1}$. Using Lemma 3.3, there exists x such that $YAx + w = Yb$ if and only if $BY^{-1}w = BY^{-1}Yb = Bb$, and we can write (10) as:

$$\begin{aligned} W_y &= \left\{w > 0 \;:\; YAx + w = Yb,\; e^\top w = 1\right\} \\ &= \left\{w > 0 \;:\; BY^{-1}w = Bb,\; e^\top w = 1\right\}. \end{aligned} \tag{12}$$

Let us denote the affine hull with aff(.). We can prove the following lemma about W_s and W_y.

Lemma 3.4. *Assume that s and y are centric, we have*

$$W_s = \operatorname{aff}(W_s) \cap W \quad \text{and} \quad W_y = \operatorname{aff}(W_y) \cap W.$$

Proof. See Appendix A. □

We conclude that W is sliced in two ways by W_y's and W_s's for centric s and y vectors. For each centric s and each centric y, W_y and W_s intersect at a single point Sy on the simplex. We want to prove that the smallest affine subspace containing W_s and W_y is $\operatorname{aff}(W) = \{w : e^\top w = 1\}$. To that end, we prove some results on the intersection of affine subspaces. We start with the following definition:

Definition 3.2. *The* recession cone *of a convex set $C \in \mathbb{R}^n$ is denoted by* $\operatorname{rec}(C)$ *and defined as:*

$$\operatorname{rec}(C) := \{y \in \mathbb{R}^n \;:\; (x+y) \in C, \quad \forall x \in C\}.$$

The lineality space *of a convex set C is denoted by* $\operatorname{lin}(C)$ *and defined as:*

$$\operatorname{lin}(C) := (\operatorname{rec}(C)) \cap (-\operatorname{rec}(C)).$$

Let U be an affine subspace of $\mathbb{R}^m$. If $y \in \operatorname{rec}(U)$, then $-y \in \operatorname{rec}(U)$, which means $(\operatorname{rec}(U)) = (-\operatorname{rec}(U))$. Therefore, by Definition 3.2, we have $\operatorname{lin}(U) = \operatorname{rec}(U)$. Then, by using the definition of the affine space we have:

$$\operatorname{lin}(U) := \{u_1 - u_2 : \forall u_1, u_2 \in U\}. \tag{13}$$

In other words, $\operatorname{lin}(U)$ is a linear subspace such that $U = u + \operatorname{lin}(U)$ for all $u \in U$ where $'+'$ is the Minkowski sum. The following two lemmas are standard, see, for instance, [20].

Lemma 3.5. *Given a pair of nonempty affine subspaces U and V in $\mathbb{R}^n$, the following facts hold:*
(1) $U \cap V \neq \emptyset$ iff for every $u \in U$ and $v \in V$, we have $(v-u) \in \operatorname{lin}(U) + \operatorname{lin}(V)$.
(2) $U \cap V$ consists of a single point iff for every $u \in U$ and $v \in V$, we have

$$(v-u) \in \operatorname{lin}(U) + \operatorname{lin}(V) \quad \text{and} \quad \operatorname{lin}(U) \cap \operatorname{lin}(V) = \{0\}.$$

(3) For every $u \in U$ and $v \in V$, we have

$$\operatorname{lin}(\operatorname{aff}(U \cup V)) = \operatorname{lin}(U) + \operatorname{lin}(V) + \{\alpha(v-u) :\; \alpha \in \mathbb{R}\}.$$

Lemma 3.6. *Let U and V be nonempty affine subspaces in $\mathbb{R}^n$. Then we have the following properties:*

(1) if $U \cap V = \emptyset$, then

$$\dim(\text{aff}(U \cup V)) = \dim(U) + \dim(V) + 1 - \dim(\text{lin}(U) \cap \text{lin}(V)),$$

(2) if $U \cap V \neq \emptyset$, then

$$\dim(\text{aff}(U \cup V)) = \dim(U) + \dim(V) - \dim(U \cap V).$$

Using the above lemmas, we deduce the following proposition.

Proposition 3.1. *Assume that s and y are centric s-vector and y-vector, respectively. Then the smallest affine subspace containing W_s and W_y is* $\text{aff}(W) = \{w : e^\top w = 1\}$.

Proof. See Appendix A. □

The following simple examples make the geometry of W_s and W_y clearer.

Example 3.2. *Here, we bring two examples for $m = 3,\ n = 1$. For the first example, let $b := [1\ 0\ 0]^\top$ and $A := [1\ \ -1\ \ -1]^\top$. By using (9), the set of centric s-vectors is $B_s = \{[(1-x),\ x,\ x]^\top : x \in (0, 1)\}$. The set of centric y-vectors is specified by solving $A^\top y = 0$ and $b^\top y = 1$, while $y > 0$. We can see that in this example, as shown in Figure 2, W_ss are parallel line segments while W_ys are line segments which all intersect at $[1\ 0\ 0]^\top$. For the second example, let $A := [1\ \ -1\ 0]^\top$ and $b := [1\ 0\ 1]^\top$. The*

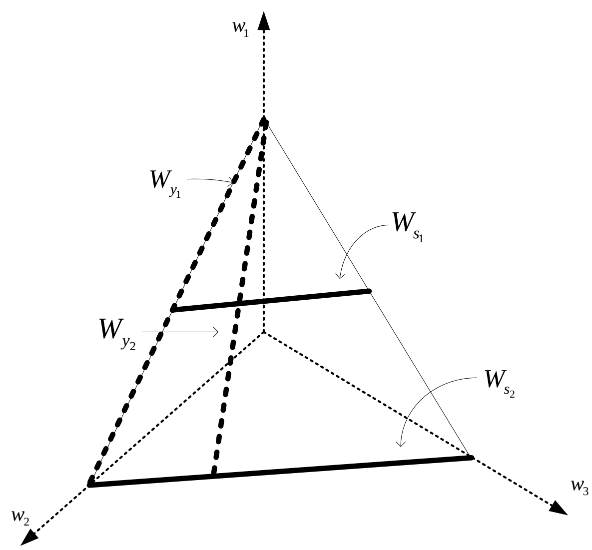

FIGURE 2. W_ss and W_ys for the first example in Example 3.2.

set of W_ss and W_ys are shown in Figure 3 derived by solving (9). As can be seen, this time W_ys are parallel line segments and W_ss are line segments which intersect at the point $[0\ 0\ 1]^\top$.

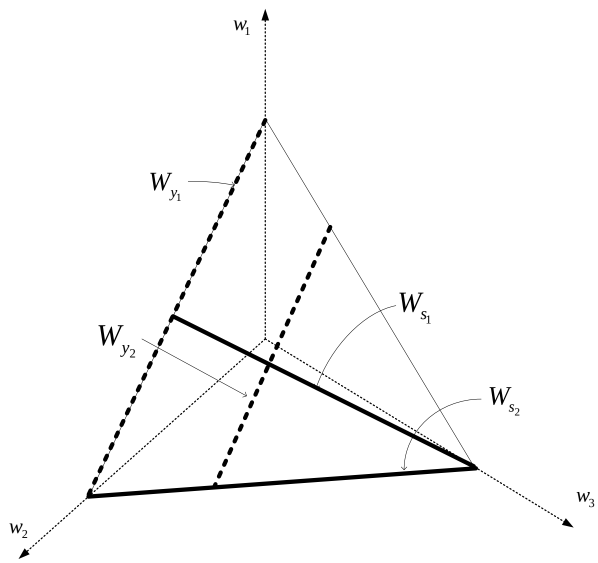

FIGURE 3. W_ss and W_ys for the second example in Example 3.2.

These examples show that the affine hulls of W_{y^1} and W_{y^2} might not intersect for two centric y-vectors y^1 and y^2. This is also true for the affine hulls of W_{s^1} and W_{s^2} for two centric s-vectors s^1 and s^2.

Example 3.3. *For the second example, let* $A := [3\ -3\ -2]^\top$ *and* $b := [1\ 1\ 0]^\top$. *The set of* $W_s s$ *and* $W_y s$ *are shown in Figure 4, derived by solving* (9). *In this example, none of* $W_y s$, $W_s s$, *or their affine hulls intersect in a single point.*

4. Improvised Robust Optimization via utility functions

In previous sections, we introduced our new methodology to deal with LP problems with uncertainty. We explained in Section 2 that our approach has many good features in terms of interaction with the decision maker and usability, and its practical advantages over the classical robust optimization approach are clear. In this section, we prove that the robust optimal solutions generated by our algorithms are at least as desirable to the decision maker as any solution generated by many other robust optimization algorithms.

In most of the papers in the robust optimization literature, the uncertainty is considered in the coefficient matrix A while we consider it in the RHS. We want to show that by choosing a suitable utility function $U(s)$ we can model many of the classical robust formulations. In other words, we can find a solution of a classical robust optimization problem by solving

$$\begin{array}{rl} \max & g(x) := U(b - Ax) \\ \text{s.t.} & a_i^\top x \le b_i, \ \ i \in \{1, \cdots, m\}. \end{array} \tag{14}$$

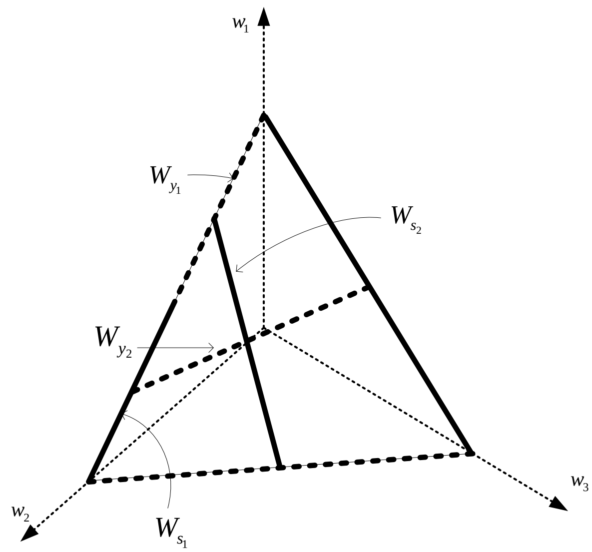

FIGURE 4. W_ss and W_ys for Example 3.3.

Many classical robust optimization models and their approximations can be written as follows

$$\begin{array}{lll} (15) \qquad & \max & c^\top x \\ & \text{s.t.} & a_i^\top x + f_i(x) \leq b_i, \;\; i \in \{1, \cdots, m\}, \end{array}$$

where $f_i(x)$, $i \in \{1, \cdots, m\}$, is a convex function such that $f_i(x) \geq 0$ for all feasible x. By changing $f_i(x)$, different formulations can be derived. In the following we bring some examples. Assume that for each entry A_{ij} of matrix A we have $A_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$. It can easily be seen [12] that the classical robust optimization problem is equivalent to (15) for $f_i(x) = \hat{a}_i^\top |x|$.

For the second example, assume that $A \in \{A \,:\, \left\|M(\text{vec}(A) - \text{vec}(\bar{A}))\right\| \leq \Delta\}$ for a given $\bar{A}$ where $\|.\|$ is a general norm and M is an invertible matrix. vec(A) is a vector in $\mathbb{R}^{mn\times 1}$ created by stacking the columns of A on top of one another. It is proved in [9] that many approximate robust optimization models can be formulated by changing the norm. It is also proved in [9] that this robust optimization model can be formulated as (15) by $f_i(x) = \Delta \left\|M^{-T} x_i\right\|_*$, where $\|.\|_*$ is the dual norm and $x_i \in \mathbb{R}^{mn\times 1}$ is a vector that contains x in entries $(i-1)n+1$ through in, and 0 everywhere else. Now, utilizing Karush-Kuhn-Tucker (KKT) theorem, we prove that for every robust optimization problem that can be put into form (15), there exists a concave utility function U for which (14) has the same optimal solution as (15).

Theorem 4.1. *Assume that* (15) *has Slater points. Then, there exists a concave function $g(x)$ (or equivalently $U(s)$) such that optimization problems* (14) *and* (15) *have the same optimal solutions.*

Proof. For the optimality condition of (15) we have: There exists $\lambda \in \mathbb{R}^m_+$ such that

$$\begin{aligned} & c - \sum_{i=1}^{m} \lambda_i (a_i + \nabla f_i(x)) = 0 \\ & \lambda_i (a_i^\top x + f_i(x) - b_i) = 0, \quad i \in \{1, \cdots, m\}. \end{aligned} \tag{16}$$

Since the Slater condition holds for (15), optimality conditions (16) are necessary and sufficient. Let x^* be an optimal solution of (15), and let $J \subseteq \{1, \cdots, m\}$ denote the set of indices for which $\lambda_i \neq 0, i \in J$. Let $h(x)$ be an increasing concave function such that its domain contains the positive orthant. We define $g(x)$ as follows:

$$g(x) := c^\top x + \sum_{i \in J} \mu_i h(b_i + t_i - a_i^\top x - f_i(x)), \tag{17}$$

where $t_i > 0, i \in J$, are arbitrary numbers. We claim that $g(x)$ is concave. $b_i + t_i - a_i^\top x - f_i(x)$ is a concave function and $h(x)$ is increasing concave, hence $h(b_i + t_i - a_i^\top x - f_i(x))$ is a concave function for $i \in \{1, \cdots, m\}$. $g(x)$ is the summation of an affine function and some concave functions and so is concave. The gradient of $g(x)$ is

$$\nabla g(x) = c - \sum_{i \in J} \mu_i h'(b_i + t_i - a_i^\top x - f_i(x))(a_i + \nabla f_i(x)). \tag{18}$$

Now choose $\mu_i, i \in J$, such that

$$\mu_i h'(b_i + t_i - a_i^\top x^* - f_i(x^*)) = \lambda_i. \tag{19}$$

By using (19) and comparison of (18) and (16), we conclude that x^* is a solution of (14), as we wanted. The other direction can be proved similarly. □

For example, let $h(x) := \ln(x)$, then we have

$$\begin{aligned} & g(x) := c^\top x + \sum_{i \in J} \mu_i \ln(b_i + t_i - a_i^\top x - f_i(x)) \\ \Rightarrow \quad & \nabla g(x) = c - \sum_{i \in J} \frac{\mu_i}{b_i + t_i - a_i^\top x - f_i(x)} (a_i + \nabla f_i(x)). \end{aligned} \tag{20}$$

Therefore, choosing

$$\mu_i := \lambda_i \left[b_i + t_i - a_i^\top x^* - f_i(x^*) \right], \quad \forall i \in \{1, \ldots, m\}$$

works. The above argument proves the existence of a suitable utility function. A remaining question is that can we construct such a utility function without having a solution of (16)? In the following, we construct a function with objective value arbitrarily close to the objective value of (15). Assume that strong duality holds for (15). Let us define $g(x) := c^\top x + \mu \sum_{i=1}^m \ln(b_i - a_i^\top x - f_i(x))$ and assume that $\hat{x}$ is the maximizer of $g(x)$. We have

$$\nabla g(\hat{x}) = c - \sum_{i=1}^{m} \frac{\mu}{b_i - a_i^\top \hat{x} - f_i(\hat{x})} (a_i + \nabla f_i(\hat{x})) = 0. \tag{21}$$

This means that $\hat{x}$ is the maximizer of the Lagrangian of the problem in (15), $L(\lambda, x)$, for $\hat{\lambda}_i := \mu/(b_i - a_i^\top \hat{x} - f_i(\hat{x}))$, $i \in \{1, \cdots, m\}$. So by strong duality, we have

$$\begin{aligned} c^\top x^* \leq L(\hat{\lambda}, \hat{x}) &= c^\top \hat{x} + \sum_{i=1}^{m} \frac{\mu}{b_i - a_i^\top \hat{x} - f_i(\hat{x})}(b_i - a_i^\top \hat{x} - f_i(\hat{x})) \\ &= c^\top \hat{x} + m\mu. \end{aligned} \tag{22}$$

(22) shows that by choosing μ small enough, we can construct $g(x)$ such that the optimal objective value of (14) is arbitrarily close to the optimal objective value of (15).

5. Algorithms

In this section, we develop the cutting-plane algorithms that find an optimal solution for the DM, using the facts we proved in the previous sections. As we mentioned in Section 2, we assume that the DM's preferences, knowledge, wisdom, expertise, etc. can be modeled by a utility function (as a function of the slack variables s), i.e., $U(s)$, and our problem is to maximize this utility function over the set of centric (Definition 3.1) s-vectors B_s. (Of course, we do not assume to have access to this function U, except through our limited interactions with the DM.) Therefore, our problem becomes

$$\begin{aligned} \max \quad & U(s) \\ s.t. \quad & s \in B_s. \end{aligned} \tag{23}$$

In the following, we denote an optimal solution of (23) with s^{opt}. In many applications, it is possible to capture choices with concave, quasi-concave, or nondecreasing utility functions. We are going to start with the assumption of concave $U(s)$. We see in Subsection 8.2 that the algorithm can easily be refined to be used for quasi-concave functions. Here, we want to use the concept of supergradient we introduce shortly. Supergradients (subgradients for convex functions) have been widely used before to design cutting-plane and ellipsoid algorithms. Our goal is to use the concept to design cutting-plane algorithms.

Assume that we start the algorithm from a point $w^0 \in \mathbb{R}^m$ with the corresponding s-vector $s^0 \in \mathbb{R}^m$. By using the idea of supergradient, we can introduce cuts in the s-space or w-space to shrink the set of s-vectors or w-vectors, such that the shrunken space contains an optimal point. In the following subsections, we discuss these algorithms in s-space and w-space. Our main algorithm is the one in the w-space, however, the s-space algorithm helps us understand the second one better.

As mentioned above, our algorithms are based on the notion of supergradient of a concave function. Therefore, before starting to talk about the algorithms, we express a summary of the results we want to use. These properties are typically proven for convex functions in the literature [38, 16], however we can translate all of them to concave functions.

Theorem 5.1. *For a concave function $f : \mathbb{R}^n \to \mathbb{R}$, any local maximizer is also a global maximizer. If a strictly concave function attains its global maximizer, it is unique.*

The following theorem is fundamental for developing our cutting-plane algorithms.

Theorem 5.2. *Assume that* $f : \mathbb{R}^n \to \mathbb{R}$ *is a concave function and let* $x^0 \in \operatorname{relint}(\operatorname{dom} f)$. *Then there exists* $g \in \mathbb{R}^n$ *such that*

$$f(x) \leq f(x^0) + g^\top (x - x^0), \quad \forall x \in \mathbb{R}^n. \tag{24}$$

If f *is differentiable at* x^0, *then* g *is unique, and* $g = \nabla f(x^0)$.

The vector g that satisfies (24) is called the *supergradient* of f at x^0. The set of all supergradients of f at x_0 is called the *superdifferential* of f at x^0, and is denoted $\partial f(x^0)$. By Theorem 5.2, if f is differentiable at x^0, then $\partial f(x^0) = \{\nabla f(x^0)\}$.

The following lemma about supergradient, which is a simple application of the chain rule, is also useful.

Lemma 5.1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a concave function, and* $D \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$ *be arbitrary matrices. Then,* $g(x) := f(Dx + b)$ *is a concave function and we have:*

$$\partial g(x) = D^\top \partial f(Dx + b)$$

5.1. **Cutting-plane algorithm in the s-space.** Assume that we have a starting point s^0 and we can obtain a supergradient of U at s^0 from the DM, e.g. g^0, ($g^0 = \nabla U(s^0)$ if U is differentiable at s^0). By using (24), for every s,

$$U(s) - U(s^0) \geq 0 \quad \Rightarrow \quad (g^0)^\top (s - s^0) \geq 0. \tag{25}$$

This means that all optimal points are in the half-space $(g^0)^\top (s - s^0) \geq 0$. So, by adding this cut, we can shrink the s-space and guarantee that there exists an optimal solution in the shrunken part. We can translate this cut to a cut in the x-space by using (9):

$$(g^0)^\top (s - s^0) = (g^0)^\top (b - Ax - b + Ax^0) = (g^0)^\top A(x^0 - x).$$

Using this equation, we can consider the cut as a new constraint of the original problem; $(g^0)^\top Ax \leq (g^0)^\top Ax^0$. Let us define $a^{(m+1)} = (g^0)^\top A$ and $b_{m+1} = (g^0)^\top Ax^0$. We can redefine $\mathcal{F}$ in (8) by adding this new constraint and find the weighted center for a chosen weight vector w^1. The step-by-step algorithm is as follows:

S-space Algorithm:

- Step 1: Set $w^0 = \frac{1}{m}e$ and find the w^0-centers (x^0, y^0, s^0) with respect to $\mathcal{F}$.
- Step 2: Set $k = 0$, $A_0 = A$, $b^0 = b$, and $\mathcal{F}_0 = \mathcal{F}$.
- Step 3: If s^k satisfies the DM, return (x^k, y^k, s^k) and **stop**.
- Step 4: Set $k = k + 1$. Find g_{k-1}, the supergradient of $U(s)$ at s^{k-1}. Set

$$A_k = \begin{bmatrix} A_{k-1} \\ g_{k-1}^\top A_{k-1} \end{bmatrix}, \qquad b^k = \begin{bmatrix} b^{k-1} \\ g_{k-1}^\top A_{k-1} x^{k-1} \end{bmatrix},$$

$$\mathcal{F}_k := \left\{ x \in \mathbb{R}^n : \langle a_k^{(i)}, x \rangle \leq b_i^k, \forall i \in \{1, 2, \cdots, m + k\} \right\}. \tag{26}$$

- Step 5: Set $w_i^k = \frac{1}{m^2}$ for $i \in \{m + 1, \cdots, m + k\}$ and $w_i^k = \frac{1}{m} - \frac{k}{m^2}$ for $i \in \{1, \cdots, m\}$. Find the w^k-center (x^k, y^k, s^k) with respect to $\mathcal{F}_k$. Return to Step 3.

The logic behind Step 5 is that we want to give smaller weights to the new constraints than the original ones (however, our choices above are just examples; implementers should make suitable, practical choices that are tailored to their specific application). A main problem with the algorithm is that the dimension of the weight-space is increased by one every time we add a constraint. We show that this problem is solved by our w-space algorithm in the following subsections.

5.2. **Cutting-plane algorithm in the w-space.** In this subsection, we consider the cuts in the w-space. To do that, we first try a natural way of extending the algorithm in the s-space to the one in the w-space. We will show that this extension does not work for all utility functions. Then, we develop an algorithm applicable to all concave utility functions.

Like the s-space, we try to use the supergradients of $U(s)$. Let U_w denote the utility function as a function of w. From (9) we have $Ys = w$; so, $U_w(w) = U(s) = U(Y^{-1}w)$. If Y were constant for all weight vectors, $U_w(w)$ would be a concave function, and we could use Lemma 5.1 to find the supergradient at each point. The problem here is that Y is not necessarily the same for different weight vectors. Assume that we confine ourselves to weight vectors in the simplex W with the same y-vector (W_y). $U_w(w)$ is a concave function on W_y, so, we can define its supergradient. By Lemma 5.1, we conclude that $\partial U_w(w) = Y^{-1}\partial U(s)$ for all $w \in W_y$.

Suppose we start at w^0 with the weighted center (x^0, y^0, s^0). Let us define $g^{0w} := (Y^0)^{-1}g^0$, where g_0 is a supergradient of $U(s)$ at s^0. Then from (24) we have,

$$U_w(w) \leq U_w(w^0) + (g^{0w})^\top (w - w^0), \quad \forall w \in W_{y_0}. \tag{27}$$

If we confine the weight-space to W_y, by the same procedure used for s-space, we can introduce cuts in the w-space by using (27). The problem is that we do not have a proper characterization of W_y. On the other hand, U_w may not be a concave function on the whole simplex.

Assume that s^{opt} is an optimal solution of (23), and $W_{s^{opt}}$ is the set of weight vectors in the simplex with s-vector s^{opt}. It is easy to see that $W_{s^{opt}}$ is convex. We also have the following lemma:

Lemma 5.2. *Let (x', y', s') be the weighted center corresponding to w', s^{opt} be an optimal solution of (23) , and g' be the supergradient of $U(s)$ at s'. Then $S^{opt}y'$ is in the half-space ${g'}_w^\top (w - w') \geq 0$, where $g'_w = Y'^{-1}g'$.*

Proof. We have ${g'}_w^\top (S^{opt}y' - w') = g'^\top Y'^{-1}(S^{opt}y' - S'y') = g'^\top (s^{opt} - s') \geq 0$. The last inequality follows from the fact that s^{opt} is a maximizer and g' is a supergradient of $U(s)$ at s'. □

The above lemma shows that using hyperplanes of the form $g'^\top Y'^{-1}(w - w')$, we can always keep a point from $W_{s^{opt}}$. Now, by using the fact that $W_{s^{opt}}$ is convex and the above lemma, the question is: if we use a sequence of these hyperplanes, can we always keep a point from $W_{s^{opt}}$? A simpler question is: We start with w^0 and shrink the simplex W into the intersection of the half-space $(g^{0w})^\top (w - w^0) \geq 0$ and the simplex, say W_0. Then we choose an arbitrary weight vector w^1 with weighted center (x^1, y^1, s^1) from the shrunken space W_0. If g^1 is a supergradient of $U(s)$ at s^1, then we shrink W_0 into the intersection of W_0 and the half-space $(g^{1w})^\top (w - w^1) \geq 0$, where $g^{1w} = (Y^1)^{-1}g^1$, and call the last shrunken space W_1. Is it always true that a weight vector with s-vector s^{opt} exists in W_1?

In the following, we show that this is true for some utility functions, but not true in general. We define a special set of functions that have good properties for cuts in the w-space, and the above algorithm works for them.

Definition 5.1. *A function $f : \mathbb{R}^m_{++} \to \mathbb{R}$ is called* Non-Decreasing under Affine Scaling (NDAS) *if for every $d \in \mathbb{R}^m_{++}$ we have:*

(1) $f(s) \ \leq \max\{f(Ds), f(D^{-1}s)\}, \quad \forall s \in \mathbb{R}^m_{++}$.
(2) If for a single $s^0 \in \mathbb{R}^m_{++}$ we have $f(s^0) \leq f(Ds^0)$, then $f(s) \leq f(Ds)$ for all $s \in \mathbb{R}^m_{++}$.

For every $t \in \mathbb{R}^m$ the function $f_1(s) := \sum_{i=1}^m t_i \log s_i$ is NDAS. Indeed, for every $s, d \in \mathbb{R}^m_{++}$ we have:

$$\begin{aligned} f_1(s) - f_1(Ds) &= -\sum_{i=1}^m t_i \log d_i, \\ f_1(s) - f_1(D^{-1}s) &= -\sum_{i=1}^m t_i \log \frac{1}{d_i} = \sum_{i=1}^m t_i \log d_i, \end{aligned}$$

and so we have $2f_1(s) = f_1(Ds) + f_1(D^{-1}s)$. The second property is also easy to verify and the function is NDAS. $f_1(s)$ is also important due to its relation to a family of classical utility functions in mathematical economics; Cobb-Douglas production function which is defined as $U_{cd}(s) = \prod_{i=1}^m s_i^{t_i}$, where $t \in \mathbb{R}^m_{++}$. Using this function to simulate problems in economics goes back to 1920's. Maximization of $U_{cd}(s)$ is equivalent to the maximization of its logarithm which is equal to $f_1(s) = \ln(U_{cd}(s)) = \sum_{i=1}^m t_i \log s_i$.

Authors in [27] considered the Cobb-Douglas utility function to present an algorithm for evaluating and ranking items with multiple attributes. [27] is related to our work as the proposed algorithm is a cutting-plane one. [27] also used the idea of weight-space as the utility function is the weighted sum of the attributes. However, our algorithm uses the concept of weighted analytic center which is different. Now, we have the following proposition.

Proposition 5.1. *Assume that $U(s)$ is a NDAS concave function. Let (x^0, y^0, s^0) and (x^1, y^1, s^1) be the weighted centers of w^0 and w^1, and g^0 and g^1 be the supergradients of $U(s)$ at s^0 and s^1, respectively. Then we have*

$$\Big\{w : \ (g^{0w})^\top (w - w^0) \geq 0, \ (g^{1w})^\top (w - w^1) \geq 0\Big\} \cap W_{s^{opt}} \neq \ \phi,$$

where $g^{0w} = (Y^0)^{-1} g^0$ and $g^{1w} = (Y^1)^{-1} g^1$.

Proof. See Appendix A. □

By Proposition 5.1, using the first two hyperplanes, the intersection of the shrunken space and $W_{s^{opt}}$ is not empty. Now, we want to show that we can continue shrinking the space and have nonempty intersection with $W_{s^{opt}}$.

Proposition 5.2. *Assume that $U(s)$ is a NDAS concave function. Let (x^i, y^i, s^i) be the weighted centers of w^i, $i \in \{0, \cdots, k\}$, and g^i be the supergradients of $U(s)$ at s^i. Let us define*

$$W^i := \left\{ w : \ (g^{iw})^\top (w - w^i) \geq 0 \right\} \cap W,$$

where $g^{iw} = (Y^i)^{-1} g^i$. Assume we picked the points such that

$$w^i \in \operatorname{relint} \left(\bigcap_{j=0}^{i-1} W^j \right), \qquad i \in \{1, \cdots, k\}. \tag{28}$$

Then we have

$$\left(\bigcap_{j=0}^{k} W^j \right) \ \cap \ W_{s^{opt}} \neq \ \phi, \tag{29}$$

where s^{opt} is an optimal solution of (23).

Proof. See Appendix A. □

Proposition 5.2 shows that the above-mentioned cutting-plane algorithm works for the NDAS functions. It would be very helpful in designing a cutting-plane algorithm in the w-space if this Proposition were true in general. However, this is not true for a general concave function. For a counter example, see Example A.1 in Appendix A. To be able to perform a cutting-plane algorithm in the w-space, we have to modify the definition of cutting hyperplanes. In the next two propositions, we introduce a new set of cutting-planes.

Proposition 5.3. *For every point $Y^0 s^0 \in W$, there exists a hyperplane P passing through it such that:*
1- P contains all the points in W_{s^0}, and
2- P cuts W_{y^0} the same way as $(g^0)^\top (Y^0)^{-1}(w - Y^0 s^0) = 0$ cuts it; the intersections of P and $(g^0)^\top (Y^0)^{-1}(w - Y^0 s^0) = 0$ with W_{y^0} is the same, and the projections of their normals onto W_{y^0} have the same direction.

Proof. Assume that $w^0 = Y^0 s^0$ is the point that is chosen and let u^0 be the normal vector to the desired hyperplane P. First, we want the hyperplane to contain W_{s^0}. This means that for all centric $\hat{y}$, the vector $S^0 y^0 - S^0 \hat{y}$ is on P, i.e., we have $(u^0)^\top S^0 (y^0 - \hat{y}) = 0$. Since $A^\top (y^0 - \hat{y}) = 0$, we can put $u^0 = (S^0)^{-1} A h^0$ with an arbitrary h^0 and we have:

$$(u^0)^\top S^0 (y^0 - \hat{y}) = (h^0)^\top A^\top (S^0)^{-1} S^0 (y^0 - \hat{y}) = 0.$$

Now, we want to find h^0 such that $(u^0)^\top (w - Y^0 s^0)$ cuts W_{y^0} the same way as $(g^0)^\top (Y^0)^{-1}(w - Y^0 s^0)$ cuts it. We actually want to find h^0 which satisfies the stronger property that $(u^0)^\top (w - Y^0 s^0) = (g^0)^\top (Y_0)^{-1}(w - Y^0 s^0)$ for all $w \in W_{y^0}$. All the points in W_{y^0} are of the form $Y^0 \hat{s}$, so we must have $(u^0)^\top Y^0 (\hat{s} - s^0) = (g^0)^\top (\hat{s} - s^0)$. Since $(\hat{s} - s^0)$ is in the range of A, this equation is true if and only if:

$$(u^0)^\top Y^0 A x = (g^0)^\top A x \ \Rightarrow \ ((u^0)^\top Y^0 - (g^0)^\top) A x = 0, \quad \forall x \in \mathbb{R}^n.$$

This means that $Y^0u^0 - g^0$ must be in the $\mathcal{R}(A)^\perp = \mathcal{N}(A^\top)$, which means $A^\top(Y^0u^0 - g^0) = 0$. However, we had from above that $u^0 = (S^0)^{-1}Ah^0$ and hence:

$$A^\top Y^0u^0 = A^\top g^0 \;\Rightarrow\; A^\top Y^0(S^0)^{-1}Ah^0 = A^\top g^0 \;\Rightarrow\; h^0 = (A^\top Y^0(S^0)^{-1}A)^{-1}A^\top g^0. \tag{30}$$

So, the hyperplane with normal vector $u^0 = (S^0)^{-1}Ah^0$, where $h^0 = (A^\top Y^0(S^0)^{-1}A)^{-1}A^\top g^0$ has the required properties. Since this hyperplane cuts W_{y^0} the same way as $(g^0)^\top(Y^0)^{-1}(w - Y^0s^0)$ does, we conclude that $(u^0)^\top(Y^0s^{opt} - Y^0s^0) \geq 0$. Therefore, Y^0s^{opt} is in the half-space $(u^0)^\top(w - Y^0s^0) \geq 0$. □

The normal of the hyperplane derived in Proposition 5.3 has a nice interpretation with respect to orthogonal projection and the primal-dual scaling $Y^{-1}S$. We have:

$$\begin{aligned} u^0 &= (S^0)^{-1}A(A^\top Y^0(S^0)^{-1}A)^{-1}A^\top g^0 \\ &= (Y^0)^{-1/2}(S^0)^{-1/2} \\ &\qquad \underbrace{[((Y^0)^{1/2}(S^0)^{-1/2}A)(A^\top Y^0(S^0)^{-1}A)^{-1}(A^\top(S^0)^{-1/2}(Y^0)^{1/2})]}_{\Pi}(Y^0)^{-1/2}(S^0)^{1/2}g_0 \\ &= (Y^0)^{-1/2}(S^0)^{-1/2}P(Y^0)^{-1/2}(S^0)^{1/2}g_0, \end{aligned} \tag{31}$$

where Π is the orthogonal projection onto the range of $(Y^0)^{1/2}(S^0)^{-1/2}A$. Note that a main benefit of the hyperplane in Proposition 5.3 is that when we choose a point, we can cut away all the points with the same s-vector. Now, we prove the following proposition which shows we can cut the simplex with a sequence of hyperplanes such that the intersection of their corresponding half-spaces contain a point from $W_{s^{opt}}$.

Proposition 5.4. *Assume that we choose the points $Y^0s^0, Y^1s^1 \in W$. The hyperplane P passing through Y^1s^1, with the normal vector $u^1 := (S^1)^{-1}Ah^1$, $h^1 = (A^\top Y^0(S^1)^{-1}A)^{-1}A^\top g^1$ satisfies the following properties:*
1- P contains all the points in W_{s^1}, and
2- $(u^1)^\top(Y^0s^{opt} - Y^1s^1) \geq 0$ for every feasible maximizer of $U(s)$.

Proof. See Appendix A. □

By Proposition 5.4, we can create a sequence of points and hyperplanes such that the corresponding half-spaces contain Y^0s^{opt}. The algorithm is as follows:

W-space Algorithm:

- Step 1: Set $w^0 = \frac{1}{m}e$ and find the w^0-centers (x^0, y^0, s^0) with respect to $\mathcal{F}$.
- Step 2: Set $k = 0$, and $W_0 = W$.
- Step 3: If s^k satisfies the optimality condition, return (x^k, y^k, s^k) and **stop**.
- Step 4: Find g^k, the supergradient of $U(s)$ at s^k. Find h^k by solving the following equation

$$A^\top Y^0(S^k)^{-1}Ah^k = A^\top g^k. \tag{32}$$

- Step 5: Set $u^k = (S^k)^{-1}Ah^k$ and $W_{k+1} = W_k \cap \{w :\ (u^k)^\top(w - w^k) \geq 0\}$. Pick an arbitrary point w^{k+1} from W_{k+1} and find the w^{k+1}-center $(x^{k+1}, y^{k+1}, s^{k+1})$ with respect to $\mathcal{F}$. Set $k = k + 1$ and return to Step 3.

A clear advantage of this algorithm over the one in the s-space is that we do not have to increase the dimension of the w-space at each step and subsequently we do not have to assign weights to the new added constraints. So, the above algorithm is straightforward to implement.

5.3. **Choosing the next weight vector.** In the above algorithm, we did not explain about choosing the next weight vector in the shrunken space. In the case of little information about the function, different centers can be chosen to achieve better convergence, as we explain in Subsection 5.5. If we have enough information about the utility function, we might be able to choose a more appropriate weight vector. Assume that $U(s) := \sum_{i=1}^{m} t_i \log s_i$ where $t \in \mathbb{R}^m_{++}$. By comparison of (23) with the optimization problem for the weighted analytic center, we see that our problem is actually finding the weighted analytic center for the weight vector t. Hence, if we had t, our problem would be finding the weighted center of t. However, t can be computed by using the gradient of the function.

Assume that we start with w^0 with the weighted center (x^0, s^0, y^0). Defining $g^0 := \nabla U(s^0)$, it is easy to see that $t = S^0 g^0$. Now we can choose $w^1 = \beta S^0 g^0$ where β is the scaling factor such that $\beta e^\top S^0 g^0 = 1$, and the s-vector of w^1 is the solution of the problem. The same idea can be used if we know that the utility function is close to the sum of the logarithms.

Assume that $U(s)$ is non-decreasing on each entry, i.e., $\nabla U(s) \geq 0$ for all $s \in R^m_+$. Consider W-space algorithm introduced above and the point $w^k = S^k y^k$ from the simplex. The corresponding half-space is $(u^k)^\top (w - w^k) \geq 0$ where $u^k = (S^k)^{-1} A h^k$, and h^k is the solution of $A^\top Y^0 (S^k)^{-1} A h^k = A^\top g^k$. It is easy to show that $S^k g^k$ lies in that half space. We have:

$$\begin{aligned} (u^k)^\top (S^k g^k - w^k) &= (u^k)^\top (S^k g^k) = (h^k)^\top A^\top (S^k)^{-1} (S^k g^k) \\ &= (g^k)^\top A (A^\top Y^0 (S^k)^{-1} A)^{-1} A^\top g^k \geq 0, \end{aligned} \tag{33}$$

where the last inequality is from the fact that $A(A^\top Y^0 (S^k)^{-1} A)^{-1} A^\top$ is positive semidefinite. The problem here is that $\beta^k S^k g^k$ may not be in the shrunken space, where β^k is again the scaling factor. So, we can perform a line search to find a point on the line segment $[S^k y^k, \beta S^k g^k]$ in the interior of the shrunken space.

5.4. **Modified algorithm in the w-space.** We designed a cutting-plane algorithm in the w-space for maximizing the utility function. In this subsection, we are going to use the properties of the weighted center we derived in Section 2 to improve the performance of the algorithm. We introduce two modified versions of the w-space algorithms in this subsection.

5.4.1. *First modified algorithm.* As we proved in Section 2, for every centric y-vector $\hat{y}$ and any centric s-vector $\hat{s}$, $\hat{w} = \hat{Y} \hat{s}$ is a weight vector in the simplex W. As we are maximizing $U(s)$ over s, roughly speaking, only the s-vector of the weighted center is important for us for each $w \in W$. This is somehow explicit in our algorithm as, for example, the normal to the cutting-plane at each step, given in Proposition 5.4, depends on s and y^0 which is the y-vector of the starting point w^0. The algorithm also guarantees to keep $Y^0 s^{opt}$ in the shrunken region at each step. Hence, we lose nothing if we try to work with weight vectors with $y = y^0$.

Consider Lemma 3.2 which is about the convex combination of weight vectors. Assume that we have weight vectors w^i, $i \in \{1, \cdots, l\}$, with weighted centers (x^i, y^0, s^i), which means they have the same y-vector. By Lemma 3.2, for every set of $\beta_i \in [0,1]$, $i \in \{1,\cdots,l\}$, such that $\sum_{i=1}^{l} \beta_i = 1$, we have $(\sum_{i=1}^{l} \beta_i x^i, y^0, \sum_{i=1}^{l} \beta_i s^i)$ is the w-center where $w := \sum_{i=1}^{l} \beta_i w^i$. In other words, when the y-vectors are the same, s-vector (equivalently x-vector) of the convex combination of w^i is equal to the convex combination of s^i, $i \in \{1,\cdots,l\}$. This is interesting because if we can update the weight vectors by using the convex combination, we do not need to compute the weighted center. We are going to use this to modify our algorithm.

Assume that the starting point is w^0 with weighted center (x^0, y^0, s^0). The modified algorithm is similar to the algorithm in Subsection 5.2 and the normal to the cutting-plane is derived by using (63). However, in the modified one, all w^i have y-vector equal to y^0. The modified algorithm has two modules:

Module 1: Assume that at Step i, we have $w^i = Y^0 s^i$ and $w^{i-1} = Y^0 s^{i-1}$ with the corresponding normals of the cutting-planes u^i and u^{i-1}. By the choice of w^i, we must have $(u^{i-1})^\top (w^i - w^{i-1}) \geq 0$. In the modified algorithm, if we have $(u^i)^\top (w^{i-1} - w^i) \geq 0$, then we put $w^{i+1} = (w^i + w^{i-1})/2$ (it is easy to see this weight vector is in the required cut simplex). In this case, we have $y^{i+1} = y^0$ and $s^{i+1} = (s^i + s^{i-1})/2$.

If we have $(u^i)^\top (w^{i-1} - w^i) \geq 0$, then the line segment $[w^{i-1}, w^i]$ is no longer in the required cut simplex. However, there exists $t > 0$ such that $\hat{w} := w^i + t(w^i - w^{i-1})$ is in the required cut simplex. We can do a line search to find t and then we set $w^{i+1} = \hat{w}$. In this case, we have $y^{i+1} = y^0$ and $s^{i+1} = s^i + t(s^i - s^{i-1})$.

Module 2: In Module 1, the algorithm always moves along a single line. When the weight vectors in Module 1 get close to each other, we perform Module 2 to get out of that line. To do that, we choose a constant $\epsilon > 0$ and whenever in Module 1 we have $\|w^i - w^{i-1}\|_2 \leq \epsilon$, we perform Module 2. In Module 2, like the algorithm in Subsection 5.2, we pick an arbitrary weight vector $\hat{w}$ in the remaining cut simplex and compute the weighted center $(\hat{x}, \hat{y}, \hat{s})$. The problem now is that y-vector is not necessarily equal to y^0. However, we said that $\hat{s}$ is important for our algorithm; hence, we consider the weight vector $Y^0 \hat{s}$. This new weight vector is not necessarily in the required cut simplex. To solve this problem, we use the same technique as in Module 1. We consider the line containing the line segment $[w^i, Y^0 \hat{s}]$ and do a line search to find an appropriate weight vector on this line. To simplify the line search, we consider $(u^i)^\top (Y^0 \hat{s} - w^i) \geq 0$ and $(u^i)^\top (Y^0 \hat{s} - w^i) < 0$ separately.

At the end of Module 2, we again come back to Module 1 to continue the algorithm. As can be seen, we only have to find a weighted center in Module 1 which makes the modified algorithm computationally more efficient than the original algorithm, in practice.

5.4.2. *Second modified algorithm.* Consider the main algorithm and the proof of Proposition 5.4. We constructed normal vectors that satisfy (62). By using the supergradient inequality, $(g^1)^\top (\hat{s} - s^1) \leq 0$ results in $U(\hat{s}) \leq U(s^1)$. Assume that a sequence of s-vectors $\{s^0, s^1, \cdots, s^j\}$ has been created by the algorithm up to iteration j. We may not have access to the value of $U(s_i)$, $i \in \{1,\cdots,j\}$, however, we know that there exists $p \in \{1,\cdots,j\}$ such that $U(s^p) \geq U(s^i)$ for all $i \in \{1,\cdots,j\}$. By the supergradient inequality we must have $(g^i)^\top (s^p - s^i) \geq 0$ for all $i \in \{1,\cdots,j\}$ and from (62)

$$(u^i)^\top (Y^0 s^p - Y^i s^i) = (g^i)^\top (s^p - s^i) \geq 0, \quad \forall i \in \{1,\cdots,j\}.$$

This means that $Y^0 s^p$ is a weight vector in the desired cut simplex. Let $\{p_1, \cdots, p_k\}$ be the indices that $(g^i)^\top(s^{p_l} - s^i) \geq 0$ for all $i \in \{1, \cdots, j\}, l \in \{1, \cdots, k\}$. By the above explanation, we know that $k \geq 1$ (the s-vector with the largest value so far is in this set.). The idea of the modified algorithm is that when $k > 1$, we put a convex combination of these s-vectors as the new s-vector. We can divide the new algorithm into three modules.

Module 1: $k > 1$: Define $s^{j+1} := \frac{1}{k}(s^{p_1} + \cdots + s^{p_k})$ and $w^{j+1} := Y^0 s^{j+1}$.

Module 2: $k = 1$. We only have one point s^p that $(g^i)^\top(s^p - s^i) \geq 0$ for all $i \in \{1, \cdots, j\}$ and by the above explanation we have $U(s^p) \geq U(s^i)$ for all $i \in \{1, \cdots, j\}$. Hence, s^p is our best point so far and we use it to find the next one. To do that, we choose a direction ds such that $s^{j+1} = s^p + \alpha ds$. $s^{j+1} - s^p = \alpha ds$ must be in $\mathcal{R}(A)$ and therefore a good choice is the projection of g^p on $\mathcal{R}(A)$. Let us define P_A as the projection matrix to $\mathcal{R}(A)$, then we define $ds = P_A g^p$ and do a line search to find the appropriate α such that $s^{j+1} := s^p + \alpha ds$ is in the desired cut simplex. We also have $w^{j+1} := Y^0 s^{j+1}$.

Module 3: In the first two modules, we do not have to calculate the weighted center. In this module, like the first modified algorithm, when $\|w^{j+1} - w^j\|$ in Module 1 or 2 is smaller than a specified value, we perform an iteration like the original algorithm; pick an arbitrary point inside the cut simplex and compute the weighted center for that.

5.5. **Convergence of the algorithm.** Introduction of cutting-plane algorithms goes back to 1960's and one of the first appealing ones is the center of gravity version [36]. The center of gravity algorithm has not been used in practice because computing the center of gravity, in general, is difficult. However, it is noteworthy due to its theoretical properties. For example, Grünbaum [24] proved that by using any cutting-plane through the center, more than 0.37 of the feasible set is cut out which guarantees a geometric convergence rate with a sizeable constant. Many different types of centers have been proposed in the literature. A group of algorithms use the center of a specific localization set, which is updated at each step. One of them is the ellipsoid method [44] where the localization set is represented by an ellipsoid containing an optimal solution. Ellipsoid method can be related to our algorithm as we can use it to find the new weight vectors at each iteration. The cutting-plane method which is most relevant to our algorithm is the analytic center one, see [23] for a survey. In this method, the new point at each iteration is an approximate analytic center of the remaining polytope. The complexity of such algorithms has been widely studied in the literature. Nesterov [34] proved the ϵ-accuracy bound of $O(\frac{L^2R^2}{\epsilon^2})$ when the objective function is Lipschitz continuous with constant L, and the optimal set lies in a ball of diameter R. Goffin, Luo, and Ye [22] considered the feasibility version of the problem and derived an upper bound of $O(\frac{n^2}{\epsilon^2})$ calls to the cutting-plane oracle.

Another family of cutting-plane algorithms are based on *volumetric barriers* or *volumetric centers* [42, 43, 1]. Vaidya used the volumetric center to design a new algorithm for minimizing a convex function over a convex set [42]. More sophisticated algorithms have been developed based on Vaidya's volumetric cutting plane method [43, 1].

5.6. **Solutions for Practical Concerns.** In the previous subsections, we introduced an algorithm that is highly cooperative with the DM and proved many interesting features about it. In this subsection, we set forth some practical concerns about our algorithm and introduce solutions for them.

5.6.1. *Driving factors.* As we mentioned, one of the main criticisms of classical robust optimization is that it is not practical to ask the DM to specify an m-dimensional ellipsoid for the uncertainty set. Our approach improves this situation by asking much simpler questions. However, as the DM might not be a technical expert, asking the supergradient or gradient of a function in $\mathbb{R}^m$ still has many practical difficulties. Here, we are going to show how to solve this problem.

The idea is similar to those used in the area of multi-criteria optimization. Consider the system of inequalities $Ax \leq b$ and the corresponding slack vector $s = b - Ax$ representing the problem. What happens in practice is that the DM might prefer to directly consider only a few factors that really matter, we call them *Driving Factors.* For example, the driving factors for a DM might be budget amount, profit, human resources, etc. We can represent k driving factors by $(c^i)^\top x$, $i \in \{1, \ldots, k\}$, and the problem for the DM is to maximize the utility function $U((c^1)^\top x, \ldots, (c^k)^\top x)$. Similar to the way we added the objective of the linear program to the constraints, we can add k constraints to problem and write (23) as:

$$\begin{aligned} \max \quad & U(\xi_1, \ldots, \xi_k) + \tilde{U}(s_1, \ldots, s_m) \\ s.t. \quad & \xi_i = \hat{b}_i - (c^i)^\top x \geq 0, \quad i \in \{1, \cdots, k\} \\ & s = b - Ax. \end{aligned} \tag{34}$$

As can be seen, the supergradient vector has only k nonzero elements which makes it much easier for the DM to specify it for $k << m$. In problem (34), the utility function $\tilde{U}$ can be managed by the technical people (perhaps picking relatively uniform weights and only adjusting it globally with strategic input from the DM).

5.6.2. *Approximate gradients.* In the previous subsection, we derived a cutting-plane algorithm in the w-space. As can be seen from Propositions 5.3 and 5.4, for the algorithm we need the supergradients of the utility function $U(s)$. However, we usually do not have an explicit formula for $U(s)$ and our knowledge about it comes from the interaction with the DM. Supplying supergradient information on preferences (i.e., the utility function) might still be a difficult task for the DM. So, we have to simplify our questions for the DM and try to adapt our algorithm accordingly.

We try to derive approximate supergradients based on simple questions from the DM. The idea is similar to the one used by Arbel and Oren in [2]. Assume that $U(s)$ is differentiable which means the supergradient at each point is unique and equal to the gradient of the function at that point. Assume that the algorithm is at the point s. By Taylor's Theorem (first order expansion) for arbitrarily small scalars $\epsilon_i > 0$ we have:

$$\begin{aligned} & u_i := U(s + \epsilon_i e_i) \approx U(s) + \frac{\partial U(s)}{\partial s_i} \epsilon_i \\ & \Rightarrow \frac{\partial U(s)}{\partial s_i} \approx \frac{u_i - u_0}{\epsilon_i}, \qquad u_0 := U(s). \end{aligned} \tag{35}$$

Assume that we have $m + 1$ points s and $s + \epsilon_i e_i$, $i \in \{1, \cdots, m\}$. By the above equations, if we have the value of $U(s)$ at these points, we can find the approximate gradient. But in the absence of true utility function, we have to find these values through proper questions from the DM. Here, we assume that we can ask the DM about the relative preference for the value of the function at these $m + 1$ points. For example, DM can use a method called Analytic Hierarchy Process (AHP) to assess relative preference. We use these relative preferences to find the approximate gradient.

Assume that the DM provides us with the priority vector p, then we have the following relationship between p and u_i's

$$\frac{u_i}{u_j} = \frac{p_i}{p_j}, \qquad i,j \in \{0,\cdots,m\},$$
$$\Rightarrow \qquad \frac{u_i - u_0}{u_0} = \frac{p_i - p_0}{p_0},$$
$$\Rightarrow \qquad u_i - u_0 = \beta_0(p_i - p_0), \qquad \beta_0 := \frac{u_0}{p_0}. \tag{36}$$

Now, we can substitute the values of $u_i - u_0$ from (36) into (35) and we have

$$\nabla U(s) = \beta_0 \begin{bmatrix} \frac{p_1 - p_0}{\epsilon_1} & \cdots & \frac{p_m - p_0}{\epsilon_m} \end{bmatrix}^\top. \tag{37}$$

The problem here is that we do not have the parameter β_0. However, this parameter is not important in our algorithm because we are looking for normals to our proper hyperplanes and, as it can be seen in Propositions 5.3 and 5.4, a scaled gradient vector can also be used to calculate h^0 and h^1. Therefore, we can simply ignore β_0 in our algorithm.

6. Probabilistic Analysis

Probabilistic analysis is tied to robust optimization. One of the recent trends in robust optimization research is the attempt to try reducing conservatism to get better results, and at the same time keeping a good level of robustness. In other words, we have to show that our proposed answer has a low probability of infeasibility. In this section, we derive some probability bounds for our algorithms based on weight and slack vectors. These bounds can be given to the DM with each answer and the DM can use them to improve the next feedback.

6.1. **Representing the robust feasible region with weight vectors.** Before starting the probabilistic analysis, want to relate the notion of weights to the parameters of the uncertainty set. As we explained in Subsection 1.3, we consider our uncertainty sets as follows:

$$B_i := \left\{ \tilde{b}_i \ : \ \exists \tilde{z} = (\tilde{z}_i^1, \cdots, \tilde{z}_i^{N_i}) \in [-1,1]^{N_i} \ s.t. \ \tilde{b}_i = b_i^{(0)} + \sum_{l=1}^{N_i} \Delta b_i^l \tilde{z}_i^l \right\}, \tag{38}$$

where $\{\tilde{z}_i^l\}_{l=1}^{N_i}$, $i \in \{1,\cdots,m\}$ are independent random variables, and Δb_i^l is the scaling factor of $\tilde{z}_i^l$. We assume that the support of $\tilde{z}_i^l$ contains $\tilde{z}_i^l = -1$, i.e., $Pr\{\tilde{z}_i^l = -1\} \neq 0$. Let us define another set which is related to the weight vectors:

$$\mathcal{W} := \left\{ (w_1,\cdots,w_m) : w_i \in [y_i(w)\|\Delta b_i\|_1, 1), \ \sum_{i=1}^m w_i = 1 \right\}, \tag{39}$$

where $y(w)$ is the y-vector of w. Our goal is to explicitly specify a set of weights whose corresponding w-center makes the feasible solution of the robust counterpart.

Proposition 6.1. *Let x satisfy $Ax \leq \tilde{b}$ for every $\tilde{b} \in B_1 \times B_2 \times \cdots \times B_m$. Then there exists some $w \in \mathcal{W}$, so that x is the weighted analytic center with respect to the weight vector w, i.e., $x = x(w)$. In other words,*

$$\left\{x \ : \ Ax \leq \tilde{b}, \ \forall \tilde{b} \in B_1 \times B_2 \times \cdots \times B_m\right\} \subseteq \{x(w) \ : \ w \in \mathcal{W}\}.$$

Proof. See Appendix A. □

The above proposition shows that when the robust counterpart problem with respect to the uncertainty set $B_1 \times B_2 \times \cdots \times B_m$ is feasible, the set $\mathcal{W}$ is nonempty. In the next proposition we prove that the equality holds in the above inclusion.

Proposition 6.2. *(a)We have*

$$\{x \ : \ Ax \leq \tilde{b}, \ \forall \tilde{b} \in B_1 \times B_2 \times \cdots \times B_m\} = \{x(w) \ : \ w \in \mathcal{W}\}.$$

(b) Assume that $w > 0$ satisfies $\sum_{i=1}^m w_i = 1$, and y is its corresponding y-vector. For every $i \in \{1, \cdots, m\}$, we have

$$w_i \geq y_i \|\Delta b_i\|_1 \Rightarrow \langle a_i, x(w) \rangle \leq \tilde{b}_i, \quad \forall \tilde{b}_i \in B_i.$$

Proof. See Appendix A. □

6.2. **Probability bounds.** Suppose we wish to find a robust feasible solution with respect to the uncertainty set $B_1 \times B_2 \times \cdots \times B_m$, where B_i was defined in (38). By Proposition 6.2, it is equivalent to finding the weighted center for a $w \in \mathcal{W}$, where $\mathcal{W}$ is defined in (39). However, finding such a weight vector is not straight forward as we do not have an explicit formula for $\mathcal{W}$. Assume that we pick an arbitrary weight vector $w > 0$ such that $\sum_{i=1}^m w_i = 1$, with the weighted center (x, y, s). Let us define the vector δ for w as

$$\delta_i = \frac{w_i}{y_i \|\Delta b_i\|_1}, \quad i \in \{1, 2, \cdots, m\},$$

where Δb_i was defined in (38). For each $i \in \{1, \cdots, m\}$, if $1 \leq \delta_i$, by Proposition 6.2-(b) we have $\langle a_i, x(w) \rangle \leq \tilde{b}_i$ for all $\tilde{b}_i \in B_i$. So, the problem is with the constraints that $1 > \delta_i$. For every such constraint, we can find a bound on the probability that $\langle a_j, x(w) \rangle > \tilde{b}_j$. As in the proof of Proposition 6.2-(b), in general we can write:

$$\begin{aligned}
\Pr\{\langle a_j, x \rangle > \tilde{b}_j\} &= \Pr\left\{-y_i \sum_{l=1}^{N_i} \Delta b_i^l \ \tilde{z}_i^l > w_i = y_i \delta_i \|\Delta b_i\|_1\right\} \\
&= \Pr\left\{-\sum_{l=1}^{N_i} \Delta b_i^l \ \tilde{z}_i^l > \delta_i \|\Delta b_i\|_1\right\} \\
&\leq \exp\left(-\frac{\delta_i^2 (\|\Delta b_i\|_1)^2}{2 \sum_{l=1}^{N_i} (\Delta b_i^l)^2}\right), \qquad (40)
\end{aligned}$$

where the last inequality is derived by using Hoeffding's inequality:

Lemma 6.1. *(*Hoeffding's inequality[25]*) Let $v_1, v_2, \cdots, v_n$ be independent random variables with finite first and second moments, and for every $i \in \{1, 2, \cdots, n\}$, $\tau_i \le v_i \le \rho_i$. Then for every $\varphi > 0$*

$$\Pr\left\{\sum_{i=1}^{n} v_i - E\left(\sum_{i=1}^{n} v_i\right) \ge n\varphi\right\} \le \exp\left[\frac{-2n^2\varphi^2}{\sum_{i=1}^{n}(\rho_i - \tau_i)^2}\right].$$

Bertsimas and Sim [12] derived the best possible bound, i.e., a bound that is achievable. The corresponding lemma proved in [12] is as follows:

Lemma 6.2. *(a) If $\tilde{z}_i^l$, $l \in \{1, \cdots, N_i\}$, are independent and symmetrically distributed random variables in $[-1, 1]$, p is a positive constant, and $\gamma_{il} \le 1$, $l \in \{1, \cdots, N_i\}$, then*

$$\Pr\left\{\sum_{l=1}^{N_i} \gamma_{il}\ \tilde{z}_i^l \ge p\right\} \le B(N_i, p), \tag{41}$$

where

$$B(N_i, p) = \frac{1}{2^{N_i}}\left[(1-\mu)\binom{N_i}{\lfloor \nu \rfloor} + \sum_{i=\lfloor \nu \rfloor + 1}^{N_i} \binom{N_i}{i}\right],$$

where $\nu := (N_i + p)/2$, and $\mu := \nu - \lfloor \nu \rfloor$.
(b) The bound in (41) *is tight for $\tilde{z}_i^l$ having a discrete probability distribution:*
$\Pr\{\tilde{z}_i^l = 1\} = \Pr\{\tilde{z}_i^l = -1\} = 1/2$*, $\gamma_{il} = 1$, $l \in \{1, \cdots, N_i\}$, an integral value of $p \ge 1$, and $p + N_i$ being even.*

We can use the bound for our relation (40) as follows. Assume that $\tilde{z}_i^l$, $l \in \{1, \cdots, N_i\}$, are independent and symmetrically distributed random variables in $[-1, 1]$. Also denote by $\max(\Delta b_i)$, the maximum entry of Δb_i. Using (40), We can write

$$\begin{aligned}
\Pr\{\langle a_j, x\rangle > \tilde{b}_j\} &= \Pr\left\{\sum_{l=1}^{N_i} \Delta b_i^l\ \tilde{z}_i^l > \delta_i \|\Delta b_i\|_1\right\} \\
&\le \Pr\left\{\sum_{l=1}^{N_i} \frac{\Delta b_i^l}{\max(\Delta b_i)}\ \tilde{z}_i^l \ge \delta_i \frac{\|\Delta b_i\|_1}{\max(\Delta b_i)}\right\} \\
&\le B\left(N_i, \delta_i \frac{\|\Delta b_i\|_1}{\max(\Delta b_i)}\right). \qquad (42)
\end{aligned}$$

To compare these two bounds, assume that all the entries of Δb_i are equal. Bound (40) reduces to $\exp(-\delta_i^2 N_i/2)$, and bound (42) reduces to $B(N_i, \delta_i N_i)$. Figure 5 is the comparison of these two bounds for $\delta_i = 0.8$. Bound (42) dominates bound (40). Moreover, bound (42) is somehow the best possible bound as it can be achieved by a special probability distribution as in Lemma 6.2.

The above probability bounds do not take part in our algorithm explicitly. However, for each solution, we can present these bounds to the DM and s/he can use them to improve the feedback to the algorithm. As an example of how these bounds may be used for the DM, we show how to construct a concave utility function $U(s)$ based on these probability bounds.

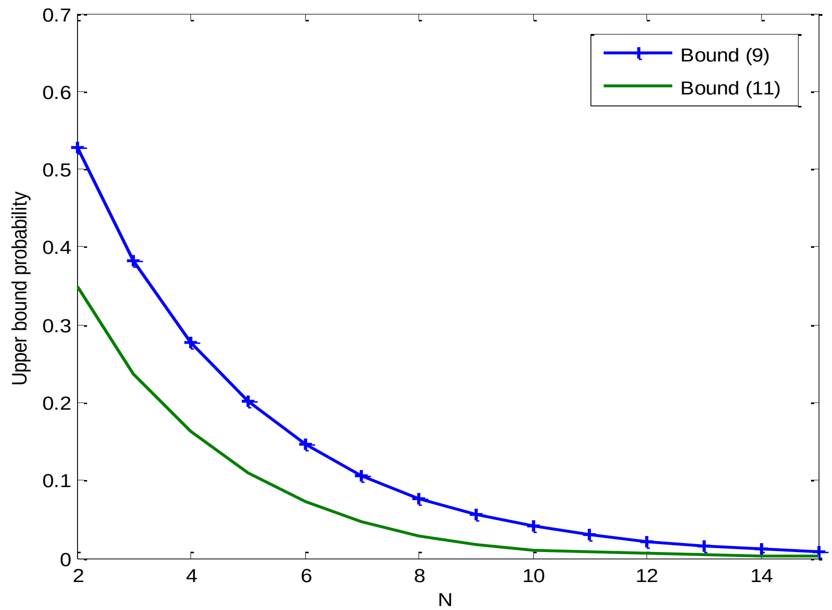

FIGURE 5. Comparison of bounds (40) and (42).

Bounds (40) and (42) are functions of $\delta_i = \frac{w_i}{y_i \|\Delta b_i\|_1} = \frac{s_i}{\|\Delta b_i\|_1}$ and as a result, functions of s. Now, assume that based on the probability bounds, the DM defines a function $u_i(s_i)$ for each slack variable s_i as shown in Figure 6. $u_i(s_i)$ increases as s_i increases, and then at the point ϵ_i^1 becomes flat. At $s_i = \epsilon_i^2$ it starts to decrease to reach zero. Parameters ϵ_i^1 and ϵ_i^2 are specified by the DM's desired bounds. Now, we can define the utility function as $U(s) := \prod_{j=1}^{m} u_i(s_i)$. This function is not concave, but maximization of it is equivalent to the maximization of $\ln(U(s))$ which is concave.

7. Illustrative preliminary computational experiments

In this section, we present some numerical results to show the performance of the algorithms in the w-space designed in Section 5. LP problems we use are chosen from the NETLIB library of LPs. Most of these LP problems are not in the format we have used throughout the paper which is the standard inequality form. Hence, we convert each problem to the standard equality form and then use the dual problem. In this section, the problem $\max\{(c^{(0)})^\top x : \ Ax \leq b^{(0)}\}$ is the converted one. In the following, we consider several numerical examples.

Example 1: In this example, we consider a simple problem of maximizing a quadratic function. Consider the ADLITTLE problem (in the converted form) with 139 constraints and 56 variables. We apply the algorithm to function $U_{ij}(s) = -(s_i - s_j)^2$ which makes two slack variables as close as possible. This function may not have any practical application, however, shows a simple example difficult to solve by classical robust optimization.

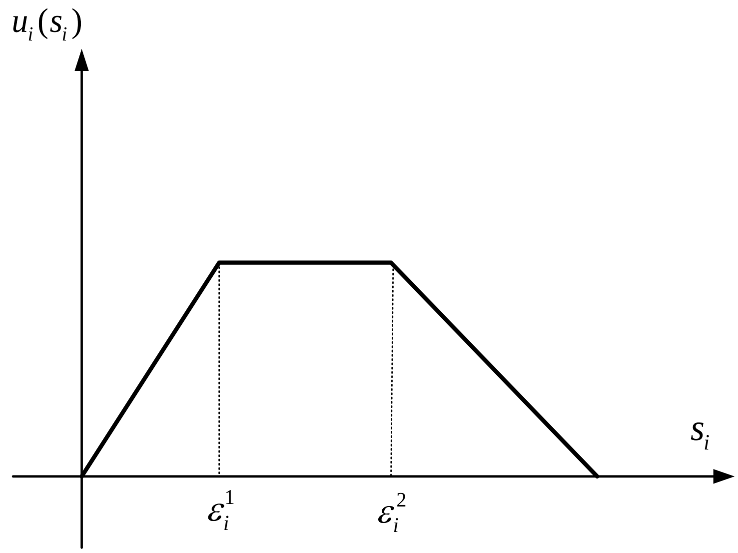

FIGURE 6. The function $u_i(s_i)$ defined for the slack variable s_i

The stopping criteria is $\|g\| \leq 10^{-6}$. For U_{23} the algorithm takes 36 iterations and returns $U_{23} = -5 \times 10^{-11}$. For U_{34} the algorithm takes 35 iterations and returns $U_{34} = -2.4 \times 10^{-12}$.

Example 2: Consider the ADLITTLE problem and assume that three constraints $\{68, 71, 74\}$ are important for the DM. Assume that the DM estimates that there is 20 percent uncertainty in the RHS of these inequalities. We have $(b_{68}, b_{71}, b_{74}) = (500, 493, 506)$ and so the desired slack variables are around $(s_{68}, s_{71}, s_{74}) = (100, 98, 101)$. By using the classical robust optimization method that satisfies the worst case scenario, the optimal objective value is $obj_c = 1.6894 \times 10^5$.

Now assume that the following utility function represents DM's preferences:

$$U_1(s) = t_{68} \ln(s_{68}) + t_{71} \ln(s_{71}) + t_{74} \ln(s_{74}) + t_m \ln(s_m).$$

This function is a NDAS function that we defined in Definition 5.1. Assume that the DM set $t_m = 10$ and $t_{68} = t_{71} = t_{74} = 1$. By using our algorithm, we get the objective value of $obj_1 = 1.7137 \times 10^5$ with the slack variables $(s_{68}, s_{71}, s_{74}) = (82, 83, 132)$. As we observe, the objective value is higher than the classical robust optimization method while two of the slack conditions are not satisfied. However, the slack variables are close to the desired ones. If the DM sets $t_m = 20$, we get the objective value of $obj_2 = 1.9694 \times 10^5$ with the slack variables $(s_{68}, s_{71}, s_{74}) = (40, 41, 79)$. However, all the iterates might be interesting for the DM. The following results are also returned by the algorithm before the optimal one:

$$obj_3 = 1.8847 \times 10^5, \quad (s_{68}, s_{71}, s_{74}) = (56, 58, 83),$$
$$obj_4 = 1.7 \times 10^5, \quad (s_{68}, s_{71}, s_{74}) = (82, 84, 125).$$

Now assume that the DM wants to put more weight on constraints 68 and 71 and so set $t_{68} = t_{71} = 2$, $t_{74} = 1$ and $t_m = 20$. In this case, the algorithm returns $obj_5 = 1.8026 \times 10^5$ with the slack variables $(s_{68}, s_{71}, s_{74}) = (82, 84, 64)$.

Example 3: In this example, we consider the DEGEN2 problem (in the converted form) with 757 constraints and 442 variables. The optimal solution of this LP is $obj_1 = -1.4352 \times 10^3$. Assume that constraints 245, 246, and 247 are important for the DM who wants them as large as possible, however, at the optimal solution we have $s(245) = s(246) = s(247) = 0$. The DM also wants the optimal objective value to be at least -1.5×10^3. As we stated before, we add the objective function as a constraint to the system. To have the objective value at least -1.5×10^3, we can add this constraint as $c^\top x = -1500 + s_{m+1}$. For the utility function, the DM can use the NDAS function

$$U(s) = \ln(s_{245}) + \ln(s_{246}) + \ln(s_{247}).$$

By running the algorithm for the above utility function, we get
$(s_{245}, s_{246}, s_{247}) = (7.75, 17.31, 17.8)$ with objective value $obj_2 \approx -1500$ after 50 iterations and
$(s_{245}, s_{246}, s_{247}) = (15.6, 27.58, 27.58)$ with $obj_3 \approx -1500$ after 100 iterations.

Example 4: We include a stopping criterion in the algorithm based on the norm of the super-gradient. The DM should also have some control over the stopping criteria (perhaps because of being satisfied or getting tired of the process). In this example, we consider the SCORPION problem (in the converted form) with 466 constraints and 358 variables. The optimal objective value of this LP is $obj_1 = 1.8781 \times 10^3$. We consider the following two NDAS utility functions:

$$U_1(s) = \ln(s_{m+1}) + \sum_{i=265}^{274} \ln(s_i),$$

$$U_2(s) = 5\ln(s_{m+1}) + \sum_{i=265}^{274} \ln(s_i). \tag{43}$$

In this example, we apply the original and the 2nd modified algorithms to both $U_1(s)$ and $U_2(s)$. The improvement in the utility function value after 200 iterations is shown in Figure 7. As can be

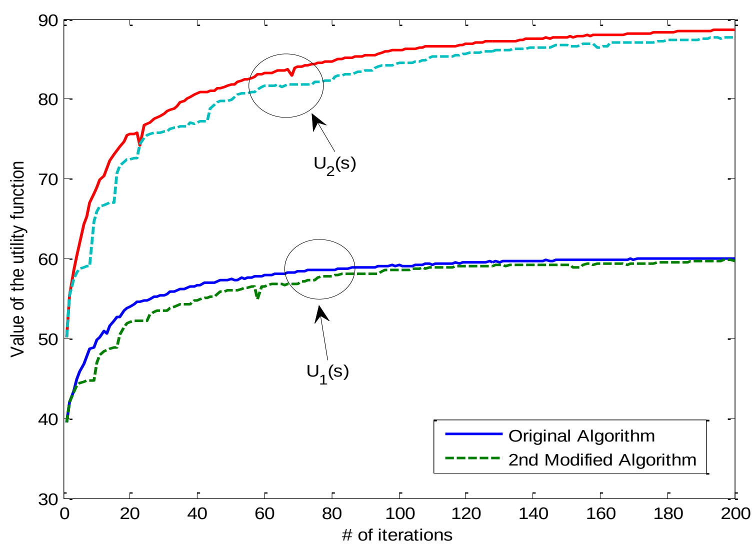

FIGURE 7. Value of the utility function versus the number of iterations.

seen, the rate of increase in the utility function is decreasing after each iteration. For this problem, the algorithm does not stop by itself and continues until the satisfaction of the DM. The DM can stop the algorithm, for example, when the rate of increase is less than a specified threshold. For this example, the rate of improvement for the 2nd modified algorithm is almost as good as the original one. However, in the modified algorithm, the weighted center is computed around 40 times during the 200 iterations which is much less computational work.

Example 5: In this example, we consider utility functions introduced at the end of Section 6. Consider problem SCORPION with optimal objective value of $obj_1 = 1.8781 \times 10^3$. Assume that the uncertainty in constraints 211 to 215 are important for the DM and we have $\|\Delta b_i\|_1 = 0.7b_i^{(0)}$, $i \in \{211, \cdots, 215\}$, where Δb_i was defined in (38). Let $\hat{x}$ be the solution of MATLAB's LP solver, then we have $s_{211} = \cdots = s_{215} = 0$ which is not satisfactory for the DM. Besides, assume that the DM wants the objective value to be at least 1800. To satisfy that, we add the $(m+1)$th constraint as $s_{m+1} = -1800 + (c^{(0)})^\top x$ which guarantees $(c^{(0)})^\top x \geq 1800$. For the utility function, first we define $u_i(s_i)$, $i \in \{211, \cdots, 215\}$ similar to Figure 6 with $\epsilon_i^1 = \|\Delta b_i\|_1 = 0.7b_i^{(0)}$ and $\epsilon_i^2 = \infty$. So we have for $i \in \{211, \cdots, 215\}$:

$$u_i(s_i) = \begin{cases} s_i & s_i < \|\Delta b_i\|_1 \\ \|\Delta b_i\|_1 & s_i \geq \|\Delta b_i\|_1. \end{cases} \tag{44}$$

Now, we can define $U(s) := \sum_{i=211}^{215} \ln u_i(s_i)$. By running the algorithm, the supergradient goes to zero after 65 iterations and the algorithm stops. Denote the solution by x^*, then the results are as follows:

$$\begin{aligned} &(c^{(0)})^\top x^* = 1800.3, \\ &b_{211}^{(0)} = 3.86, \quad b_{212}^{(0)} = 48.26, \quad b_{211}^{(0)} = 21.81, \quad b_{211}^{(0)} = 48.26, \quad b_{211}^{(0)} = 3.86, \\ &s_{211}^* = 3.29, \quad s_{212}^* = 19.47, \quad s_{211}^* = 7.39, \quad s_{211}^* = 16.97, \quad s_{211}^* = 3.24. \end{aligned} \tag{45}$$

Now, assume that the DM wants the objective value to be at least 1850 and the $(m+1)$th constraint becomes $s_{m+1} = -1850 + (c^{(0)})^\top x$. In this case, the norm of the supergradient reaches zero, after 104 iterations. The norm of supergradients versus the number of iterations are shown in Figure 8 for these two cases. Denote the solution after 100 iterations by $\bar{x}^*$, then we have:

$$\begin{aligned} &(c^{(0)})^\top \bar{x}^* = 1850, \\ &\bar{s}_{211}^* = 1.22, \quad \bar{s}_{212}^* = 16.74, \quad \bar{s}_{211}^* = 6.80, \quad \bar{s}_{211}^* = 14.54, \quad \bar{s}_{211}^* = 1.25. \end{aligned} \tag{46}$$

Let $\bar{x}$ be the returned value in the second case after 65 iterations. It is clearly not robust feasible; however, we can use bound (42) to find an upper bound on the probability of infeasibility. Assume that $N = 10$ and all the entries of Δb_i are equal. Then, bound (42) reduces to $B(N, \delta_i N)$, where $\delta_i = \frac{s_i}{\|\Delta b_i\|_1}$. The probabilities of infeasibility of $\bar{x}$ for constraints 211 to 215 are given in Table 7 (using bound (42)).

8. Extensions and conclusion

8.1. **Extension to Semidefinite Optimization (SDP).** Semidefinite Programming is a special case of Conic Programming where the cone is a direct product of semidefinite cones. Many convex optimization problems can be modeled by SDP. Since our method is based on a barrier function for

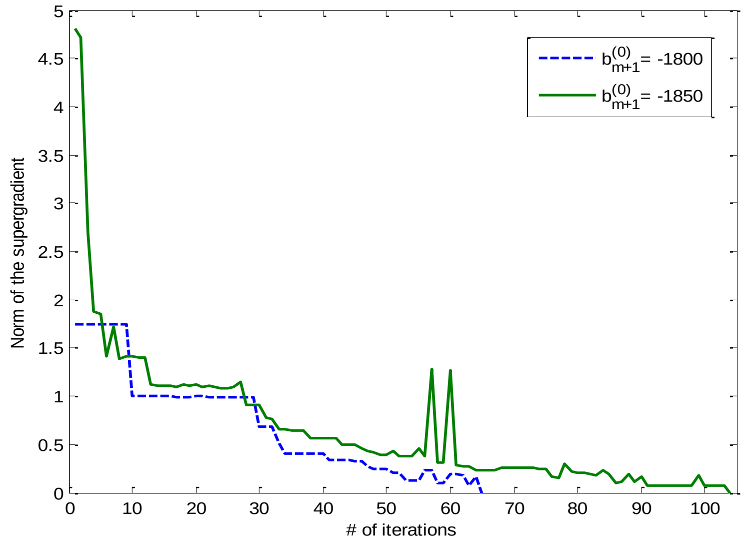

FIGURE 8. Norm of the supergradient versus the number of iterations for Example 5.

i	$\Pr(\langle a_j, \bar{x}\rangle > \tilde{b}_j)$
211	0
212	0.0827
213	0.0018
214	0.0866
215	0

TABLE 1. The probability of infeasibility of $\bar{x}$ for constraints 211 to 215.

a polytope in $\mathbb{R}^n$, it can be generalized and used as an approximation method for robust semidefinite programming that is NP-hard for ellipsoidal uncertainty sets. An SDP problem can be formulated as follows

$$\begin{aligned} \sup \quad & \langle \tilde{c}, x \rangle, \\ \text{s.t.} \quad & \sum_{j=1}^{t_i} A_i^{(j)} x_j + S_i = \tilde{B}_i, \quad \forall i \in \{1, 2, ..., m\}, \\ & S_i \succeq 0, \quad \forall i \in \{1, 2, ..., m\}, \end{aligned}$$

where $A_i^{(j)}$ and $\tilde{B}_i$ are symmetric matrices of appropriate size, and $\succeq$ is the Löwner order; for two square, symmetric matrices C_1 and C_2 with the same size, we have $C_1 \succeq C_2$ iff $C_1 - C_2$ is a semidefinite matrix. For every $i \in \{1, \cdots, m\}$, define

$$\mathcal{F}_i := \{x \in \mathbb{R}^n : \sum_{j=1}^{t_i} A_i^{(j)} x_j \preceq \tilde{B}_i\}.$$

Assume that $\text{int}(\mathcal{F}_i) \neq \emptyset$ and let $F_i : \text{int}(\mathcal{F}_i) \to \mathcal{R}$ be a self-concordant barrier for $\mathcal{F}_i$. The typical self-concordant barrier for SDP is $F_i(x) = -\log\det(\tilde{B}_i - \sum_{j=1}^{t_i} A_i^{(j)} x_j)$. Assume

$$\mathcal{F} := \bigcap_{i=1}^{m} \mathcal{F}_i$$

is bounded and its interior is nonempty. Now, as in the definition of the weighted center for LP, we can define a weighted center for SDP. For every $w \in \mathbb{R}^m_{++}$, we can define the weighted center as follows:

$$\arg\min \left\{ \sum_{i=1}^{m} w_i F_i(x) : x \in \mathcal{F} \right\} \tag{47}$$

The problem with this definition is that we do not have many of the interesting properties we proved for LP. The main one is that the weighted centers do not cover the relative interior of the whole feasible region and we cannot sweep the whole feasible region by moving in the w-space. There are other notions of weighted centers that address this problem; however, they are more difficult to work with algorithmically. Extending the results we derived for LP to SDP can be a good future research direction to follow.

8.2. **Quasi-concave utility functions.** The definition of the quasi-concave function is as follows:

Definition 8.1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is* quasi-concave *if its domain is convex, and for every $\alpha \in \mathbb{R}$, the set*

$$\{x \in \text{dom} f \ : \ f(x) \geq \alpha\}$$

is also convex.

All concave functions are quasi-concave, however, the converse is not true. Quasi-concave functions are important in many fields such as game theory and economics. In microeconomics, many utility functions are modeled as quasi-concave functions. For differentiable functions, we have the following useful proposition:

Proposition 8.1. *A differentiable function f is quasi-concave if and only if the domain of f is convex and for every x and y in* dom$f$ *we have:*

$$f(y) \geq f(x) \quad \Rightarrow \quad (\nabla f(x))^\top (y - x) \geq 0 \tag{48}$$

(48) is similar to (25), which is the property of the supergradient we used to design our algorithms. The whole point is that for a differentiable quasi-concave function $U(s)$ and any arbitrary point s^0, the maximizers of $U(s)$ are in the half-space
$(\nabla U(s^0))^\top (s - s^0) \geq 0$. This means that we can extend our algorithms to differentiable quasi-concave utility functions simply by replacing supergradient with gradient, and all the results for s-space and w-space stay valid.

8.3. **Conclusion.** In this paper, we presented new algorithms in a framework for robust optimization designed to mitigate some of the major drawbacks of robust optimization in practice. Our algorithms have the potential of increasing the applicability of robust optimization. Some of the advantages of our new algorithms are:

(1) Instead of a single, isolated, and very demanding interaction with the DM, our algorithms interact continuously with the DM throughout the optimization process with more reasonable demands from the DM in each iteration. One of the benefits of our approach is that the DM "learns" what is feasible to achieve throughout the process. Another benefit is that the DM is more likely to be satisfied (or at least be content) with the final solution. Moreover, being personally involved in the production of the final solution, the DM bears some responsibility for it and is more likely to adapt it in practice.
(2) Our algorithms operate in the weight-space using only driving factors with the DM. This helps reduce the dimension of the problem, simplify the demands on the DM while computing the most important aspect of the problem at hand.
(3) Weight-space and weighted-analytic-centers approach embeds a "highly differentiable" structure into the algorithms. Such tools are extremely useful in both the theory and applications of optimization. In contrast, classical robust optimization and other competing techniques usually end up delivering a final solution where differentiability cannot be expected.

Developing similar algorithms for semidefinite programming is left as a future research topic. As we explained in Subsection 8.1, we can define a similar notion of weighted center for SDP. However, these weighted centers do not have many properties we used for LP, and we may have to switch to other notions of weighted centers that are more difficult to work with algorithmically, and have fewer desired properties compared to the LP setting.

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Appendix A

Lemma 3.2 *Let $(x^{(i)}, y^{(i)}, s^{(i)})$, $i \in \{1, \cdots, \ell\}$, be solutions of system (9), corresponding to the weights $w^{(i)}$. Then for every set of $\beta_i \in [0,1]$, $i \in \{1, \cdots, \ell\}$, such that $\sum_{i=1}^{\ell} \beta_i = 1$, and for every $j \in \{1, \cdots, \ell\}$, we have $(\sum_{i=1}^{\ell} \beta_i x^{(i)}, y^{(j)}, \sum_{i=1}^{\ell} \beta_i s^{(i)})$ is the w-center of $\mathcal{F}$, where*

$$w := \sum_{i=1}^{\ell} \beta_i Y^{(j)} (Y^{(i)})^{-1} w^{(i)}.$$

Moreover

$$\sum_{i=1}^{m} w_i = \sum_{i=1}^{m} w_i^{(j)}.$$

Proof. According to the assumptions, for every $i \in \{1, \cdots, \ell\}$, we have

$$\begin{aligned} &Ax^{(i)} + s^{(i)} = b^{(0)}, \quad s > 0, \\ &A^\top y^{(i)} = 0, \\ &S^{(i)} y^{(i)} = w^{(i)}. \end{aligned}$$

Now, it can be seen that $(\sum_{i=1}^{\ell} \beta_i x^{(i)}, y^{(j)}, \sum_{i=1}^{\ell} \beta_i s^{(i)})$ satisfies the system:

$$\begin{aligned} &A(\sum_{i=1}^{\ell} \beta_i x^{(i)}) + (\sum_{i=1}^{\ell} \beta_i s^{(i)}) = b^{(0)}, \quad (\sum_{i=1}^{\ell} \beta_i s^{(i)}) > 0, \\ &A^\top y^{(j)} = 0, \\ &(\sum_{i=1}^{\ell} \beta_i S^{(i)}) y^{(j)} = \sum_{i=1}^{\ell} \beta_i Y^{(j)} (Y^{(i)})^{-1} w^{(i)}. \end{aligned} \tag{49}$$

Since the w-center of $\mathcal{F}$ is unique, the proof for the first part is done.
For the second part, from (49) we can write

$$\sum_{i=1}^{m} w_i = \sum_{i=1}^{m} (\sum_{p=1}^{\ell} \beta_p s_i^{(p)}) y_i^{(j)} = \sum_{p=1}^{\ell} \beta_p (\sum_{i=1}^{m} s_i^{(p)} y_i^{(j)}) = \sum_{p=1}^{\ell} \beta_p \langle s^{(p)}, y^{(j)} \rangle.$$

By Lemma 3.1, we have $\langle s^{(p)}, y^{(j)} \rangle = \langle s^{(i)}, y^{(j)} \rangle$. Therefore, we can continue the above series of equations as follows:

$$\sum_{i=1}^{m} w_i = \sum_{p=1}^{\ell} \beta_p \langle s^{(j)}, y^{(j)} \rangle = \sum_{p=1}^{\ell} \beta_p (\sum_{i=1}^{m} s_i^{(j)} y_i^{(j)}) = (\sum_{i=1}^{m} w_i^{(j)}) \sum_{p=1}^{\ell} \beta_p = \sum_{i=1}^{m} w_i^{(j)}.$$

□

Lemma 3.4 *Assume that s and y are centric, we have*

$$W_s = \mathrm{aff}(W_s) \cap W \quad \text{and} \quad W_y = \mathrm{aff}(W_y) \cap W.$$

Proof. We prove the first one and our proof for the second one is the same. Clearly we have $W_s \subseteq \text{aff}(W_s) \cap W$. To prove the other side, assume by contradiction that there exist $w \in \text{aff}(W_s) \cap W$ such that $w \notin W_s$. Pick an arbitrary $\hat{w} \in \text{relint}(W_s)$ and consider all the points $w(\beta) = \beta w + (1-\beta)\hat{w}$ for $\beta \in [0,1]$. Both w and $\hat{w}$ are in $\text{aff}(W_s)$, so all the points $w(\beta)$ are also in $\text{aff}(W_s)$. $w(0) \in W_s$ and $w(1) \notin W_s$, so let $\hat{\beta}$ be $\sup\{\beta : w(\beta) \in W_s\}$.

Note that all the points in W_s has the same s-vector, so we have $w(\beta) = Sy(\beta)$ for $\beta \in [0, \hat{\beta})$. By using (9) we must also have $w(\hat{\beta}) \in W_s$. We want to prove that $\hat{\beta} = 1$. Assume that $\hat{\beta} < 1$. All the points on the line segment between $w(0)$ and $w(\hat{\beta})$ have the same s-vector and we can write them as $S(\gamma y(0) + (1-\gamma)y(\hat{\beta}))$ for $\gamma \in [0,1]$. But note that $y(\hat{\beta}) > 0$, so there is a small enough $\epsilon > 0$ such that $y_\epsilon = (-\epsilon y(0) + (1+\epsilon)y(\hat{\beta})) > 0$ and hence Sy_ϵ is a weight vector in W_s. However, it is also a vector on the line segment between $w(\hat{\beta})$ and w which is a contradiction to $\hat{\beta} = \sup\{\beta : w(\beta) \in W_s\}$. So $\hat{\beta} = 1$ and $w = w(1) \in W_s$ which is a contradiction. Hence $W_s \supseteq \text{aff}(W_s) \cap W$ and we are done. □

Proposition 3.1 *Assume that s and y are centric s-vector and y-vector, respectively. Then the smallest affine subspace containing W_s and W_y is* $\text{aff}(W) = \{w : e^\top w = 1\}$.

Proof. We assumed that $A \in \mathbb{R}^{m \times n}$ has full column rank, i.e., $\text{rank}(A) = n \le m$ and the interior of $\{x : Ax \le b\}$ is not empty. Let B_s denote the set of all centric s-vectors, i.e., the set of s-vectors for which there exist (x, y, s) satisfies all the equations in (9). We claim that $B_s = \{s > 0 : s = b - Ax\}$. For every $s \in \{s > 0 : s = b - Ax\}$, pick an arbitrary $y > 0$ such that $A^\top y = 0$. For every scalar α we have $A^\top(\alpha y) = 0$, so we can choose α such that $\alpha y^\top s = 1$. Hence $(x, \alpha y, s)$ satisfies (9) and we conclude that $B_s = \{s > 0 : s = b - Ax\}$. The range of A has dimension n and since B_s is not empty; it is easy to see that the dimension of B_s is also n. Moreover, we have $W_y = YB_s$ and since Y is non-singular, we have $\dim(W_y) = n$.

Now denote by B_y the set of centric y-vectors. By (9), we have $A^\top y = 0$. The dimension of the null space of $A^\top$ is $(n - m)$. In addition, we have to consider the restriction $e^\top w = 1$; we have

$$1 = e^\top w = e^\top(Ys) = s^\top y = (b - Ax)^\top y = b^\top y - x^\top A^\top y = b^\top y.$$

So, we have $b^\top y = 1$ for centric y-vectors which reduces the dimension by one (since $b \notin \mathcal{R}(A)$), and $\dim(B_y) = m - n - 1$. We have $W_s = SB_y$ and so by the same explanation $\dim(W_s) = m - n - 1$.

We proved that W_s and W_y intersect at only a single point $w = Sy$, so $\dim(W_s \cap W_y) = 0$. By using Lemma 3.6-(2) the dimension of the smallest affine subspace containing W_s and W_y is

$$\dim(W_s) + \dim(W_y) - \dim(W_s \cap W_y) = n + m - n - 1 = m - 1.$$

The dimension of $\text{aff}(W)$ is also $m-1$, so by Lemma 3.4 $\text{aff}(W)$ is the least affine subspace containing W_s and W_y. □

Proposition 5.1 *Assume that $U(s)$ is a NDAS concave function. Let (x^0, y^0, s^0) and (x^1, y^1, s^1) be the weighted centers of w^0 and w^1, and g^0 and g^1 be the supergradients of $U(s)$ at s^0 and s^1, respectively. Then we have*

$$\left\{w : \ (g^{0w})^\top(w - w^0) \ge 0, \ (g^{1w})^\top(w - w^1) \ge 0\right\} \cap W_{s^{opt}} \neq \ \phi,$$

where $g^{0w} = (Y^0)^{-1}g^0$ *and* $g^{1w} = (Y^1)^{-1}g^1$.

Proof. Consider the weight vectors $Y^0 s^{opt}$ and $Y^1 s^{opt}$. Our two hyperplanes are

$$\begin{aligned} P_0 &:= \{w : \quad (g^0)^\top (Y^0)^{-1}(w - Y^0 s^0) = 0\}, \\ P_1 &:= \{w : \quad (g^1)^\top (Y^1)^{-1}(w - Y^1 s^1) = 0\}. \end{aligned}$$

By Lemma 5.2, $Y^0 s^{opt}$ is in the half-space $(g^0)^\top (Y^0)^{-1}(w - Y^0 s^0) \geq 0$ and $Y^1 s^{opt}$ is in the half-space $(g^1)^\top (Y^1)^{-1}(w - Y^1 s_1) \geq 0$. If one of these two points is also in the other half-space, then we are done. So, assume that

$$(g^0)^\top (Y^0)^{-1}(Y^1 s^{opt} - Y^0 s^0) < 0 \quad \text{and} \quad (g^1)^\top (Y^1)^{-1}(Y^0 s^{opt} - Y^1 s^1) < 0$$

(we are seeking contradiction), which is equivalent to

$$(g^0)^\top ((Y^0)^{-1} Y^1 s^{opt} - s^0) < 0 \quad \text{and} \quad (g^1)^\top ((Y^1)^{-1} Y^0 s^{opt} - s^1) < 0. \tag{50}$$

Using (27) and (50) we conclude that

$$\begin{aligned} &U((Y^0)^{-1} Y^1 s^{opt}) < U(s^0) \leq U(s^{opt}) \quad \text{and} \\ &U((Y^1)^{-1} (Y^0) s^{opt}) < U(s^1) \leq U(s^{opt}). \end{aligned}$$

However, note that $(Y^0)^{-1} Y^1 = ((Y^1)^{-1} Y^0)^{-1}$ and this is a contradiction to Definition 5.1. So (50) is not true and at least one of $Y^0 s^{opt}$ and $Y^1 s^{opt}$ is in

$$\{w : \ (g^{0w})^\top (w - w^0) \geq 0, \ (g^{1w})^\top (w - w^1) \geq 0\}.$$

□

Proposition 5.2 *Assume that $U(s)$ is a NDAS concave function. Let (x^i, y^i, s^i) be the weighted centers of w^i, $i \in \{0, \cdots, k\}$, and g^i be the supergradients of $U(s)$ at s^i. Let us define*

$$W^i := \left\{ w : \ (g^{iw})^\top (w - w^i) \geq 0 \right\} \cap W,$$

where $g^{iw} = (Y^i)^{-1} g^i$. *Assume we picked the points such that*

$$w^i \in \operatorname{relint} \left(\bigcap_{j=0}^{i-1} W^j \right), \qquad i \in \{1, \cdots, k\}. \tag{51}$$

Then we have

$$\left(\bigcap_{j=0}^{k} W^j \right) \ \cap \ W_{s^{opt}} \neq \ \phi, \tag{52}$$

where s^{opt} is an optimal solution of (23).

Proof. Among the three representations of W_s were given in (11), we use the second one in the following. If (52) is not true, then the following system is infeasible:

$$\begin{aligned} &A^\top (S^{opt})^{-1} w = 0, \quad e^\top w = 1, \quad w \geq 0, \\ &(g^{iw})^\top (w - w^i) \geq 0, \quad i \in \{0, \cdots, k\}. \end{aligned} \tag{53}$$

By Farkas' Lemma, there exist $v \in \mathbb{R}^n$, $p \in \mathbb{R}$, and $q \in \mathbb{R}^k_+$ such that:

$$(S^{opt})^{-1}Av + pe - \sum_{i=0}^{k} q_i g^{iw} \geq 0 \quad \equiv \quad Av + ps^{opt} - \sum_{i=0}^{k} q_i S^{opt}(Y^i)^{-1}g^i \geq 0,$$

$$p - \sum_{i=0}^{k} q_i (g^{iw})^\top w^i < 0 \quad \equiv \quad p - \sum_{i=0}^{k} q_i (g^i)^\top s^i < 0. \tag{54}$$

Now for each $j \in \{0, \cdots, k\}$, we multiply both sides of the first inequality in (54) with $e^\top Y^j$, then we have:

$$p - \sum_{i=0}^{k} q_i (s^{opt})^\top Y^j (Y^i)^{-1} g^i \geq 0, \quad \forall j \in \{0, \cdots, k\},$$

$$p - \sum_{i=0}^{k} q_i (g^i)^\top s^i < 0, \tag{55}$$

where we used the facts that $e^\top Y^j Av = (A^\top y^j)^\top v = 0$ and $e^\top Y^j s^{opt} = 1$. If we multiply the first set of inequalities in (55) with -1 and add it to the second one we have

$$q_j (g^j)^\top (s^{opt} - s^j) + \sum_{i \neq j} q_i (g^i)^\top (Y^j (Y^i)^{-1} s^{opt} - s^i) < 0, \tag{56}$$

for all $j \in \{0, \cdots, k\}$. $q \in \mathbb{R}^k_+$ and $(g^j)^\top (s^{opt} - s^j) \geq 0$ by supergradient inequality. Hence, from (56), for each $j \in \{0, \cdots, k\}$, there exists $\phi_j \in \{0, \cdots, k\} \backslash \{j\}$ such that $(g^{\phi_j})^\top (Y^j (Y^{\phi_j})^{-1} s^{opt} - s^{\phi_j}) < 0$ which, using (25), means $U(Y^j (Y^{\phi_j})^{-1} s^{opt}) < U(s^{\phi_j}) \leq U(s^{opt})$. Therefore, by the first property of NDAS functions, we must have

$$U(Y^{\phi_j} (Y^j)^{-1} s^{opt}) \geq U(s^{opt}). \tag{57}$$

Now, it is easy to see that there exists a sequence $j_1, \cdots, j_t \in \{0, \cdots, k\}$ such that $\phi_{j_i} = j_{i+1}$ and $\phi_{j_t} = j_1$. By using (57) and the second property of NDAS functions $t-1$ times we can write:

$$\begin{aligned} U(s^{opt}) &\leq U(Y^{j_2}(Y^{j_1})^{-1}s^{opt}) \\ &\leq U(Y^{j_3}(Y^{j_2})^{-1}Y^{j_2}(Y^{j_1})^{-1}s^{opt}) \\ &\leq \cdots \leq U(Y^{j_t}(Y^{j_{t-1}})^{-1} \cdots Y^{j_2}(Y^{j_1})^{-1}s^{opt}) \\ &= U(Y^{j_t}(Y^{j_1})^{-1}s^{opt}). \end{aligned} \tag{58}$$

However, we had $U(Y^{j_t}(Y^{j_1})^{-1}s^{opt}) = U(Y^{j_t}(Y^{\phi_{j_t}})^{-1}s^{opt}) < U(s^{opt})$ which is a contradiction to (58). This means the system (53) is feasible and we are done. □

Example A.1. *The statement of Proposition 5.1 is not true for a general concave function.*

Proof. Consider the first example of Example 3.2. We have $m = 3$, $n = 1$, $A = [1,\ -1,\ -1]^\top$, and $b = [1,\ 0,\ 0]^\top$. Using (9), the set of centric s-vectors is

$$B_s = \{[1-z, \quad z, \quad z]^\top : z \in (0,1)\}.$$

The set of centric y-vectors, B_y, is specified by solving $A^\top y = 0$ and $y^\top b = 1$ while $y > 0$ and we can see that $B_y = \{[1, \quad z, \quad 1-z]^\top : z \in (0,1)\}$. As shown in Figure 2, W_ss are parallel line segments while W_ys are line segments that all intersect at $[1,\ 0,\ 0]^\top$.

Now, assume that the function $U(s)$ is as follows (does not depend on s_3)

$$U(s) = \begin{cases} 3s_1 - s_2, & \text{if} \quad s_1 \leq s_2; \\ -s_1 + 3s_2, & \text{if} \quad s_1 > s_2. \end{cases} \tag{59}$$

This function is piecewise linear and it is easy to see that it is concave. $U(s)$ is also differentiable at all the points except the points $s_1 = s_2$. At any point that the function is differentiable, the supergradient is equal to the gradient of the function at that point. Hence, we have $\partial U(s) = \{[3,\ -1,\ 0]^\top\}$ for $s_1 < s_2$ and $\partial U(s) = \{[-1,\ 3,\ \ 0]^\top\}$ for $s_1 > s_2$.

If we consider $U(s)$ on B_s, we can see that the maximum of the function is attained at the point that $s_1 = s_2$, so $s_{opt} = [1/2,\ 1/2,\ 1/2]^\top$. Now assume that we start at $w^0 = S^0 y^0 = [0.4,\ 0.1,\ 0.5]^\top$. Because we have $y_1 = 1$ for all centric y-vectors, $w_1^0 = s_1^0$, and we can easily find s^0 and y^0 as $s^0 = [0.4,\ 0.6,\ 0.6]^\top$ and $y^0 = [1,\ 1/6,\ 5/6]^\top$. The hyperplane passing through w^0 is $(g^0)^\top (Y^0)^{-1}(w - w^0) = 0$ and since $s_1^0 < s_2^0$ we have

$$(g^0)^\top (Y^0)^{-1} = [3,\ -1,\ \ 0](Y^0)^{-1} = [3,\ -6,\ \ 0], \tag{60}$$

and we can write the hyperplane as $3(w_1 - 0.4) - 6(w_2 - 0.1) = 0$. In the next step, we have to choose a point w^1 such that $(g^0)^\top (Y^0)^{-1}(w^1 - w^0) \geq 0$. Let us pick $w^1 = [0.6,\ 0.19,\ 0.21]^\top$ for which we can easily find $s^1 = [0.6,\ 0.4,\ 0.4]^\top$ and $y^1 = [1,\ 0.475,\ 0.525]^\top$. For this point we have $s_1^1 > s_2^2$, so $(g^1)^\top (Y^1)^{-1} = [-1,\ 6.32,\ 0]^\top$ and the hyperplane passing through w^1 is $-(w_1 - 0.6) + 6.32(w_2 - 0.19) = 0$. The intersection of two hyperplanes on the simplex can be found by solving the following system of equations:

$$\begin{cases} 3w_1 - 6w_2 = 0.6 \\ -w_1 - 6w_2 = 0.6 \\ w_1 + w_2 + w_3 = 1 \end{cases} \Rightarrow\ w^* = \begin{bmatrix} 0.57 \\ 0.185 \\ 0.245 \end{bmatrix}. \tag{61}$$

The intersection of simplex and the hyperplanes $(g^0)^\top (Y^0)^{-1}(w-w^0) = 0$ and $(g^1)^\top (Y^1)^{-1}(w-w^1) = 0$ are shown in Figure 9. The intersection of simplex with $\{w :\ (g^0)^\top (Y^0)^{-1}(w - w^0) \geq 0,\ (g^1)^\top (Y^1)^{-1}(w - w^1) \geq 0\}$ is shown by hatching lines. As can be seen, we have:

$$\left\{w :\ (g^{0w})^\top (w - w^0) \geq 0,\ (g^{1w})^\top (w - w^1) \geq 0\right\} \cap W_{s_{op}} = \ \phi.$$

□

Proposition 5.4 *Assume that we choose the points $Y^0 s^0, Y^1 s^1 \in W$. The hyperplane P passing through $Y^1 s^1$, with the normal vector $u^1 := (S^1)^{-1} A h^1$, $h^1 = (A^\top Y^0 (S^1)^{-1} A)^{-1} A^\top g^1$ satisfies the following properties:*
1- P contains all the points in W_{s^1}, and
2- $(u^1)^\top (Y^0 s^{opt} - Y^1 s^1) \geq 0$ for every feasible maximizer of $U(s)$.

Proof. As in the proof of Proposition 5.3, if we set $u^1 = (S^1)^{-1} A h^1$, then the hyperplane contains all the points in W_{s^1}. To satisfy the second property, we want to find h^1 with the stronger property that

$$(u^1)^\top (Y^0 \hat{s} - Y^1 s^1) = (g^1)^\top (\hat{s} - s^1), \tag{62}$$

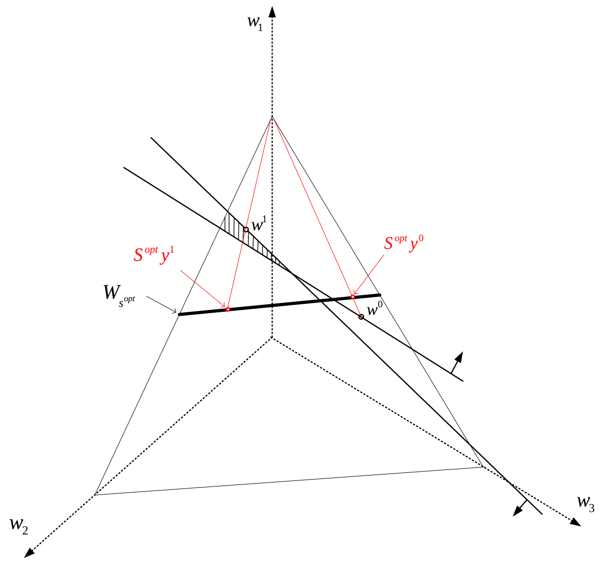

FIGURE 9. Intersection of simplex and the hyperplanes $(g^0)^\top (Y^0)^{-1}(w - w^0) = 0$ and $(g^1)^\top (Y^1)^{-1}(w - w^1) = 0$ in Example A.1.

for all the centric $\hat{s}$. The reason is that we already have $(g^1)^\top (s_{opt} - s^1) \geq 0$. By the choice of $u^1 = (S^1)^{-1}Ah^1$, for every centric y we have

$$(u^1)^\top S^1 y = (h^1)^\top A^\top (S^1)^{-1} S^1 y = (h^1)^\top A^\top y = 0.$$

So, we have $(u^1)^\top Y^1 s^1 = (u^1)^\top Y^0 s^1 = 0$ and we can continue the above equation as follows:

$$\begin{aligned}(g^1)^\top (\hat{s} - s^1) &= (u^1)^\top (Y^0 \hat{s} - Y^1 s^1) = (u^1)^\top (Y^0 \hat{s}) \\ &= (u^1)^\top (Y^0 \hat{s} - Y^0 s^1) \\ &= (u^1)^\top Y^0 (\hat{s} - s^1).\end{aligned}$$

Now we can continue in a similar way as in the proof of Proposition 5.3. Since $(\hat{s} - s^0)$ is in the range of A, we must have:

$$((u^1)^\top Y^0 - (g^1)^\top)Ax = 0, \quad \forall x \in \mathbb{R}^n.$$

By the same reasoning, we have:

$$A^\top Y^0 u^1 = A^\top g^1 \;\Rightarrow\; A^\top Y^0 (S^1)^{-1} A h^1 = A^\top g^1 \;\Rightarrow\; h^1 = (A^\top Y^0 (S^1)^{-1} A)^{-1} A^\top g^1. \tag{63}$$

So, the hyperplane with normal vector $u^1 = (S^1)^{-1}Ah^1$, where $h^1 = (A^\top Y^0 (S^1)^{-1} A)^{-1} A^\top g^1$ has the required properties. □

Propositions 6.1 *Let x satisfy $Ax \leq \tilde{b}$ for every $\tilde{b} \in B_1 \times B_2 \times \cdots \times B_m$. Then there exists some $w \in \mathcal{W}$, so that x is the weighted analytic center with respect to the weight vector w, i.e., $x = x(w)$. In other words,*

$$\left\{x \;:\; Ax \leq \tilde{b},\ \forall \tilde{b} \in B_1 \times B_2 \times \cdots \times B_m\right\} \subseteq \{x(w) \;:\; w \in \mathcal{W}\}.$$

Proof. Let $\hat{w} > 0$ be an arbitrary vector such that $\sum_{i=1}^{m} \hat{w}_i = 1$, and let $(\hat{x}, \hat{y}, \hat{s})$ be the weighted center corresponding to it. Assume that x is in the robust feasible region; we must have $\langle a_i, x\rangle \leq b_i^{(0)} + \langle \Delta b_i, \tilde{z}_i\rangle$ for every $\tilde{z}_i$ with nonzero probability, particularly for $\tilde{z}_i = -e$ where e is all ones vector. So

$$\langle a_i, x\rangle - b_i^{(0)} \leq \langle \Delta b_i, \tilde{z}_i\rangle = \langle \Delta b_i, -e\rangle = -\|\Delta b_i\|.$$

Define $s_i := b_i^{(0)} - \langle a_i, x\rangle$. Thus, from the above equation, for every $i \in \{1, \cdots, m\}$ we have

$$0 < \|\Delta b_i\|_1 \leq s_i,$$

and consequently $\hat{y}_i \|\Delta b_i\|_1 \leq \hat{y}_i s_i$ using the fact that $\hat{y}_i > 0$. For every $i \in \{1, \cdots, m\}$, we set

$$w_i := \hat{y}_i s_i.$$

Since $(x, \hat{y}, s)$ satisfies the optimality conditions, we have $x = x(w)$. It remains to show that $w \in \mathcal{W}$. First note that:

$$\sum_{i=1}^{m} w_i = \sum_{i=1}^{m} s_i \hat{y}_i = \sum_{i=1}^{m} \hat{s}_i \hat{y}_i = \sum_{i=1}^{m} \hat{w}_i = 1,$$

where for the second equality we used Lemma 3.1. Now, using the fact that $w_i \geq 0$ for every $i \in \{1, \cdots, m\}$, we have $w_i < \sum_{j=1}^{m} w_j = 1$. We already proved that $\hat{y}_i \|\Delta b_i\|_1 \leq \hat{y}_i s_i = w_i$. These two inequalities prove that $w_i \in [\hat{y}_i \|\Delta b_i\|_1, 1)$. ☐

Proposition 6.2 *(a) We have* $\{x \in \mathbb{R}^n \ : \ Ax \leq \tilde{b}, \ \forall \tilde{b} \in B_1 \times B_2 \times \cdots \times B_m\} = \{x(w) \ : \ w \in \mathcal{W}\}$. *(b) Assume that* $w > 0$ *satisfies* $\sum_{i=1}^{m} w_i = 1$, *and* y *is its corresponding* y*-vector. For every* $i \in \{1, \cdots, m\}$, *we have* $w_i \geq y_i \|\Delta b_i\|_1 \Rightarrow \langle a_i, x(w)\rangle \leq \tilde{b}_i, \quad \forall \tilde{b}_i \in B_i$.

Proof. (a) $\subseteq$ part was proved in Proposition 6.1. For $\supseteq$, let $w \in \mathcal{W}$ and (x, y, s) be its corresponding weighted center. By $w \in \mathcal{W}$ we have

$$y_i \|\Delta b_i\|_1 \leq w_i = s_i y_i = (b_i^{(0)} - \langle a_i, x\rangle) y_i \Longrightarrow \|\Delta b_i\|_1 \leq (b_i^{(0)} - \langle a_i, x\rangle).$$

Therefore, for all $\tilde{z}_i \in \times_{i=1}^{m} [-1, 1]$,

$$\langle a_i, x\rangle \leq b_i^{(0)} - \|\Delta b_i\|_1 \leq b_i^{(0)} - \sum_{l=1}^{N} \Delta b_i^l \ \tilde{z}_i^l = b_i^{(0)} + \langle \tilde{z}_i, \Delta b_i\rangle,$$

which proves x is a robust feasible solution with respect to the uncertainty set $B_1 \times B_2 \times \cdots \times B_m$.
(b) Assume that $w > 0$ satisfies $\sum_{i=1}^{m} w_i = 1$, y is its corresponding y-vector, and there exists $i \in \{1, \cdots, m\}$ such that $w_i \geq y_i \|\Delta b_i\|_1$. If there exists $\tilde{b}_i \in B_i$ such that $\langle a_i, x(w)\rangle > \tilde{b}_i$ where

$\tilde{b}_i = b_i^{(0)} + \sum_{l=1}^{N_i} \Delta b_i^l \tilde{z}_i^l$, by using $\tilde{z}_i^l \geq -1$ we have

$$\begin{aligned}
\langle a_i, x(w)\rangle > \tilde{b}_i \quad &\Rightarrow \quad \langle a_i, x(w)\rangle > b_i^{(0)} + \sum_{l=1}^{N_i} \Delta b_i^l \ \tilde{z}_i^l \ \geq \ b_i^{(0)} - \sum_{l=1}^{N_i} \Delta b_i^l \\
&\Rightarrow \quad \sum_{l=1}^{N_i} \Delta b_i^l > b_i^{(0)} - \langle a_i, x(w)\rangle = s_i(w) \\
&\Rightarrow \quad y_i \sum_{l=1}^{N_i} \Delta b_i^l > y_i s_i(w) = w_i \geq y_i \sum_{l=1}^{N_i} \Delta b_i^l \\
&\Rightarrow \quad \sum_{l=1}^{N_i} \Delta b_i^l \ > \ \sum_{l=1}^{N_i} \Delta b_i^l,
\end{aligned}$$

which is a contradiction. We conclude that $\langle a_i, x(w)\rangle \leq \tilde{b}_i$ for all $\tilde{b}_i \in B_i$. □