# On local convergence of the method of alternating projections

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**Abstract**

The method of alternating projections is a classical tool to solve feasibility problems. Here we prove local convergence of alternating projections between subanalytic sets $A, B$ under a mild regularity hypothesis on one of the sets. We show that the speed of convergence is $\mathcal{O}(k^{-\rho})$ for some $\rho \in (0, \infty)$.
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## 1 Introduction

The method of alternating projections, introduced by von Neumann in 1950, is a classical tool to solve the following feasibility problem: Given closed sets $A, B$ in $\mathbb{R}^n$, find a point $x^* \in A \cap B$. The method generates sequences $a_k \in P_A(b_{k-1})$, $b_k \in P_B(a_k)$, where $P_A$, $P_B$ are the set-valued orthogonal projection operators on $A$ and $B$. If the alternating sequence $a_k, b_k$ is bounded and satisfies $a_k - b_k \to 0$, then each of its accumulation points is a solution of the feasibility problem. The fundamental question is when such a sequence converges to a single limit point $x^* \in A \cap B$.

For convex sets alternating projections are globally convergent as soon as $A \cap B \neq \emptyset$, and the survey [1] gives an excellent state-of-art of the convex theory. In one of the earliest contributions to the nonconvex case, Combettes and Trussell [7] proved in 1990 that the set of accumulation points of a bounded sequence of alternating projections with $a_k - b_k \to 0$ is either a singleton or a nontrivial compact continuum. In 2013 it was shown in [3] by way of an example that the continuum case may indeed occur. This shows that without convexity a sequence of alternating projections $a_k, b_k$ may fail to converge even when it is bounded and satisfies $a_k - b_k \to 0$.

In 2008 Lewis and Malick [9] proved that a sequence $a_k, b_k$ of alternating projections converges locally linearly if $A, B$ are $C^2$-manifolds intersecting at an angle. Expanding on this in 2009, Lewis *et al.* [10] proved local linear convergence for general $A, B$ intersecting non-tangentially in the sense of linear regularity, where one of the sets is superregular. In 2013 Bauschke *et al.* [2] investigate the case of non-tangential intersection further and prove linear convergence under weaker regularity hypotheses.

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Here we prove local convergence under less restrictive conditions, where $A, B$ may also intersect tangentially. We propose a new geometric concept, called *separable intersection*, which gives local convergence of alternating projections when combined with *Hölder regularity*, a mild hypothesis less restrictive than prox-regularity.

Separable intersection has wide scope for applications, as it not only includes non-tangential intersection, or intersection *at an angle* as we termed it, but goes beyond and allows also a large variety of cases where $A, B$ intersect tangentially. In particular, we prove that closed subanalytic sets $A, B$ *always* intersect separably. This leads to the central result that alternating projections between subanalytic sets converge locally with rate $\mathcal{O}(k^{-\rho})$ for some $\rho \in (0, \infty)$ if one of the sets is Hölder regular. As these hypotheses are satisfied in practical situations, we obtain a theoretical explanation for the fact, observed in practice, that even without convexity alternating projections converge well in the neighborhood of $A \cap B$.

The structure of the paper is as follows. Section 3 introduces the concept of separable intersection of two closed sets. In section 4 we discuss Hölder regularity and compare it to existing regularity concepts like prox-regularity, Clarke regularity, and superregularity. The central chapter 5 gives the convergence proof with rate for sets intersecting separably. In section 6 we show that subanalytic sets intersect separably and then deduce the convergence result for subanalytic sets. The final section gives limiting examples.

# 2 Preparation

We consider closed sets $A, B$ in $\mathbb{R}^n$ and the associated orthogonal projectors or nearest point mappings $P_A, P_B$. We use the notation $a \in P_A(b)$ if the operator is potentially set-valued, while $a = P_A(b)$ means the projection is unique.

A sequence of alternating projections satisfies $b_k \in P_B(a_k), a_{k+1} \in P_A(b_k)$. We occasionally switch to the following index-free notation, which is standard in optimization:

$$b \in P_B(a), a^+ \in P_A(b), b^+ \in P_B(a^+), \text{etc.}$$

The sequence of alternating projections is then $a, b, a^+, b^+, a^{++}, b^{++}, \ldots$. We refer to $a \to b \to a^+$, respectively $b \to a^+ \to b^+$, as the building blocks of the sequence, where it is always understood that $b \in P_B(a)$, $a^+ \in P_A(b)$, etc.

# 3 Tangential and non-tangential intersection

In this section we introduce the fundamental concept of separable intersection of sets $A, B$, which plays the central role in our convergence theory.

**Definition 1. (Separable intersection)**. *We say that $B$ intersects $A$ separably at $x^* \in A \cap B$ with exponent $\omega \in [0, 2)$ and constant $\gamma > 0$ if there exists a neighborhood $U$ of $x^*$ such that for every building block $b \to a^+ \to b^+$ in $U$, the condition*

$$\langle b - a^+, b^+ - a^+ \rangle \leq \left(1 - \gamma \|b^+ - a^+\|^\omega\right) \|b - a^+\| \|b^+ - a^+\| \tag{1}$$

*is satisfied.* □

We say that $B$ intersects $A$ separably at $x^*$ if (1) holds for *some* $\omega \in [0, 2)$, $\gamma > 0$. If it is also true that $A$ intersects $B$ separably, that is, if the analogue of (1) holds for building blocks $a \to b \to a^+$, then we obtain a symmetric condition, and in that case we say that $A, B$ intersect separably at $x^*$.

**Remark 1.** Condition (1) discloses itself if we introduce the angle $\alpha = \angle(b - a^+, b^+ - a^+)$ and rewrite (1) in the more suggestive form

$$\frac{1 - \cos\alpha}{\|a^+ - b^+\|^\omega} \geq \gamma, \tag{1'}$$

calling this the *angle condition* for the building block $b \to a^+ \to b^+$. For $\omega \in (0, 2)$ the interpretation of (1), or (1′), is that if the angle $\alpha$ between $b - a^+$ and $b^+ - a^+$ for two consecutive projection steps $b \to a^+ \to b^+$ shrinks down to 0 as the alternating sequence approaches $x^*$, then $\alpha$ should not shrink too fast. Namely, through (1′), the angle is linked to the shrinking distance between the sets. For $\omega = 0$ the meaning of (1′) is that the angle $\alpha$ stays away from 0.

**Remark 2.** Suppose $B$ intersects $A$ separably with exponent $\omega \in [0, 2)$ and constant $\gamma > 0$ at $x^*$. Let $\omega' \in (\omega, 2)$ and $\gamma' \in (\gamma, \infty)$. Then $B$ intersects $A$ also $\omega'$-separably with constant $\gamma'$. In consequence, 0-separability is the severest condition, while $\omega$-separability gets less restrictive as $\omega$ increases.

Informally, when the angle $\alpha = \angle(b - a^+, b^+ - a^+)$ between two consecutive projection steps shrinks to zero, $A, B$ must in some sense intersect *tangentially* at $x^*$. In contrast, when $\alpha$ stays away from 0, the case of 0-separability, one could say that $A, B$ intersect *at an angle*. In that case alternating projections are expected to behave well and converge linearly. Tangential intersection is the more embarrassing case, where convergence could be slowed down or even fail. Our concept of $\omega$-separability gives new insight into the case of tangential intersection.

There has been considerable effort in the literature to *avoid* tangential intersection by making regularity assumptions. We mention transversal intersection in [9], the generalized non-separation property in [11], linearly regular intersection in [10], or the notion of constraint qualification in [2]. In the following we relate these notions to 0-separability.

Bauschke *et al.* [2] introduce an extension of the Mordukhovich normal cone $N_A(a^*)$, called the $B$-restricted normal cone $N_A^B(a^*)$ to $A$ at $a^* \in A$. They define $u \in N_A^B(a^*)$ if there exist $a_n \in A$, $a_n \to a^*$, and $u_n \to u$ such that

$$u_n = \lambda_n (b_n - a_n)$$

for some $\lambda_n > 0$ and $b_n \in B$ with $a_n \in P_A(b_n)$. The authors say that $A, B$ satisfy the constraint qualification (CQ) at $a^* \in A \cap B$ if $N_A^B(a^*) \cap \left(-N_B^A(a^*)\right) \subset \{0\}$.

**Proposition 1. (CQ implies 0-separability).** *Suppose $A, B$ satisfy the constraint qualification at $x^* \in A \cap B$ in the sense that $N_A^B(x^*) \cap \left(-N_B^A(x^*)\right) \subset \{0\}$. Then $A, B$ intersect 0-separably with a constant $\gamma \in (0, 1)$.*

**Proof:** Following [2], the $B$-restricted proximal normal cone $\widehat{N}_A^B(a)$ of $A$ at $a \in A$ is the set of vectors $u$ of the form $u = \lambda(b - a)$ for some $\lambda > 0$ and some $b \in B$ satisfying $a \in P_A(b)$. Specialized to the case of two sets, the authors of [2] define

$$\theta_\delta(A, B, x^*) = \sup\left\{u^T v : u \in \widehat{N}_A^B(a), -v \in \widehat{N}_B^A(b), \|u\| \leq 1, \|v\| \leq 1, a, b \in B(x^*, \delta)\right\}$$

and they prove existence of the limit

$$\theta(A, B, x^*) = \lim_{\delta \to 0^+} \theta_\delta(A, B, x^*).$$

Then they show that $N_A^B(x^*) \cap \left(-N_B^A(x^*)\right) \subset \{0\}$ implies $\theta(A,B,x^*) < 1$.

Using this, pick $\delta > 0$ such that $\theta_\delta(A,B,x^*) =: 1-\gamma < 1$. Consider a building block $b \to a^+ \to b^+$ with $b, a^+, b^+ \in B(x^*,\delta)$. Naturally, we have $b - a^+ \in \widehat{N}_A^B(a^+)$ and $a^+ - b^+ \in \widehat{N}_B^A(b^+)$. Hence also $u = (b-a^+)/\|b-a^+\| \in \widehat{N}_A^B(a^+)$ and $v = (b^+-a^+)/\|b^+-a^+\| \in -\widehat{N}_B^A(b^+)$. Therefore $\langle u,v\rangle = \cos\alpha \le 1-\gamma < 1$ by the definition of $\theta_\delta$, because $b, a^+, b^+ \in B(x^*,\delta)$ and $\|u\| = \|v\| = 1$. That shows $1-\cos\alpha \ge \gamma > 0$ and proves that $B$ intersects $A$ 0-separably at $x^*$. The estimate for building blocks $a \to b \to a^+$ is analogous. □

We will show in example 7.3 that the converse of proposition 1 is not true. In fact, 0-separability seems more versatile in applications, while still guaranteeing linear convergence. We conclude by noting that linear regular intersection [10] and transversality in the sense of [9] imply 0-separability. In view of [2] and Proposition 1 we have

**Corollary 1. (Linear regularity implies 0-separability)**. *Suppose $A, B$ intersect linearly regularly at $x^* \in A \cap B$. Then they intersect 0-separably at $x^*$.* □

**Corollary 2.** *Let $\mathcal{M}, \mathcal{N}$ be $C^2$-manifolds which intersect transversally at $x^*$ in the sense of* [9]*. Then $\mathcal{M}$ and $\mathcal{N}$ intersect 0-separably at $x^*$.* □

We will resume the discussion of separable intersection of sets in section 6.

# 4 Hölder regularity

In this section we introduce the concept of Hölder regularity. We then relate it to other regularity notions like Clarke regularity, prox-regularity, superregularity in the sense of [10], and its extension in [2].

**Definition 2. (Hölder regularity)**. *Let $\sigma \in [0,1)$. The set $B$ is $\sigma$-Hölder regular with respect to $A$ at $b^* \in A \cap B$ if there exists a neighborhood $U$ of $b^*$ and a constant $c > 0$ such that for every $a^+ \in A \cap U$ and every $b^+ \in P_B(a^+) \cap U$ one has*

$$B(a^+,(1+c)r) \cap \{b \in \mathbb{R}^n : \cos\angle(a^+-b^+, b-b^+) > \sqrt{c}r^\sigma\} \cap B = \emptyset, \tag{2}$$

*where $r = \|a^+-b^+\|$. We say that $B$ is Hölder regular with respect to $A$ if it is $\sigma$-Hölder regular with respect to $A$ for every $\sigma \in [0,1)$.* □

The remainder of this section is not needed for convergence, but relates Hölder regularity to older regularity concepts.

**Proposition 2. (0-Hölder regularity from superregularity)**. *Suppose $B$ is $(A,\epsilon,\delta)$-regular at $b^* \in A \cap B$ in the sense of* [2, Def. 2. 21]*. Then $B$ is 0-Hölder regular at $b^*$ with respect to $A$ with constant $c = \epsilon^2$. In particular, if $B$ is superregular at $b^*$ in the sense of* [10]*, then $B$ is 0-Hölder regular with respect to $A$ with constant $c$ that may be chosen arbitrarily small.*

**Proof:** Since superregularity of $B$ at $b^*$ implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that $B$ is $(A,\epsilon,\delta)$-regular at $b^*$, [2], it remains to prove the first part of the statement.

Let $U = B(b^*, \frac{\delta}{4(1+\epsilon^2)})$ and pick $a^+, b^+ \in U$ such that $b^+ \in P_B(a^+)$. Put $c = \epsilon^2$. That gives $r = \|b^+-a^+\| \le \frac{\delta}{2(1+c)}$. Since the ball $B(a^+,r)$ touches $B$ at $b^+$, we have $u :=$

$a^+ - b^+ \in \widehat{N}_B^A(b^+)$. Now let $b \in B$. We have to show that $b$ is not an element of the set in (2) for $\sigma = 0$. Suppose $b \in B(a^+, (1+c)r)$. Then we have to show $\cos\angle(a^+ - b^+, b - b^+) \leq \sqrt{c}$. But $\|b - b^*\| \leq \|b - a^+\| + \|a^+ - b^*\| \leq (1+c)r + \frac{\delta}{2(1+c)} \leq (1+c)\frac{\delta}{2(1+c)} + \frac{\delta}{4(1+c)} < \delta$. Hence $(A, \epsilon, \delta)$-regularity implies $\langle u, b - b^+\rangle \leq \epsilon\|u\|\|b - b^+\| = \sqrt{c}\|u\|\|b - b^+\|$, hence the claim. □

**Remark 3.** We shall see in example 7.1 that the converse of proposition 2 is not true. The difference between superregularity and its extension $(A, \epsilon, \delta)$-regularity on the one hand, and 0-Hölder regularity on the other, is the following: In (2) we exclude points in the intersection of a right circular cone with vertex $b^+$, axis $a^+ - b^+$, and aperture $\beta$ and the shrinking ball $B(a^+, (1+c)r)$. In contrast, $(A, \epsilon, \delta)$-regularity forbids many more points, namely all points in that same cone, but within the fixed ball $B(b^*, \delta)$. In consequence, this type of regularity is not suited to deal with singularities pointing inwards, like the prototype in example 7.1. □

**Remark 4.** If $B$ is $\sigma$-Hölder regular at $b^*$ with respect to $A$ with constant $c$ on the neighborhood $U$ of $b^*$, and if $\sigma' < \sigma$, then for every $c' \in (0, c)$ there exists a neighborhood $V \subset U$ of $b^*$ such that $B$ is $\sigma'$-Hölder regular at $b^*$ with constant $c'$. Indeed, if $b \in B(a^+, (1+c')r)$ in (2), then also $b \in B(a^+, (1+c)r)$, hence by assumption $\cos\beta \leq \sqrt{c}r^\sigma = \sqrt{c}r^{\sigma-\sigma'}r^{\sigma'} \leq \sqrt{c'}r^{\sigma'}$ if $V$ is chosen so that $\sqrt{c}r^{\sigma-\sigma'} < \sqrt{c'}$. □

Consider $b \in B$ and let $d \in N_B(b)$ be a unit limiting normal to $B$ at $b$. We define the reach of $B$ along $d$ as

$$R(b, d) = \sup\{R \geq 0 : b = P_B(b + td) \text{ for every } 0 \leq t \leq R\}.$$

Note that $R(b, d) \in [0, \infty]$; $R(b, d) = 0$ occurs when $b$ cannot be projected from outside. The case $R(b, d) = \infty$ occurs e.g. if $B$ is convex and $b \in \partial B$. We can say that $B(b + R(b, d)d, R(b, d))$ is the largest ball with its centre on $b + \mathbb{R}_+ d$ which touches $B$ in $b$ from outside. Here a closed ball $B(x, r)$ with $r \geq 0$ is said to touch the set $B$ at $b \in B$ from outside if $b \in B(x, r)$ and $B(x, r)$ contains no points from $B$ in its interior. Closed balls degenerated into a single point are admitted.

Following Federer [8] the set $B$ has positive reach $r > 0$ if $R(b, d) \geq r$ for every boundary point $b \in \partial B$ and every unit $d \in N_B(b)$. Sets of positive reach locally are also known as prox-regular sets, see [12]. We now relax this concept to sets where the reach may vanish at some boundary points, but slowly so.

**Definition 3.** *Let $\sigma \in (0, 1]$. The set $B$ has $\sigma$-slowly vanishing reach with respect to the set $A$ at $b^* \in A \cap B$ if there exists $0 \leq \tau < 1$ such that*

$$\limsup_{a \to b^*} \frac{\|a - b\|^\sigma}{R(b, d)} \leq \tau, \tag{3}$$

*where the limit superior is over $a \in A$, $b \in P_B(a)$, $(a - b)/\|a - b\| = d$. We say that the reach vanishes with exponent $\sigma$ and rate $\tau$.* □

**Proposition 3.** *If $B$ has positive reach at $b^* \in A \cap B$, then it has slowly vanishing reach with respect to $A$ with rate $\tau = 0$ and arbitrary $\sigma \in (0, 1]$.*

**Proof:** By hypothesis there exist $\epsilon > 0$ and $r > 0$ such that $R(b, d) \geq r$ for every $b \in B$ with $\|b - b^*\| \leq \epsilon$ and every $d \in \widehat{N}_B(b)$ with $\|d\| = 1$. Now let $a \in A \cap B(b^*, \epsilon)$, $b \in P_B(a)$, and $d = (a-b)/\|a-b\|$. Then by the above $R(b,d) \geq r$. As $b^* \in A \cap B$, we have $\|a-b\| = d_B(a) \leq \|a-b^*\| \to 0$ as $a \in A$ approaches $b^*$. Therefore the numerator in (3) tends to 0, while the denominator stays away from 0. Hence the limit superior in (3) is 0. $\square$

**Proposition 4. (Hölder regularity from slowly vanishing reach)**. *Let $\sigma \in [0,1)$. Suppose $B$ has $\sigma$-slowly vanishing reach with rate $\tau \in [0,1)$ with respect to $A$ at $b^* \in A \cap B$. Then $B$ is $1-\sigma$-Hölder regular with respect to $A$ with any constant $c > 0$ satisfying*

$$\frac{\tau}{2}\sqrt{2+c} < 1. \tag{4}$$

*In particular, $c$ may be chosen arbitrarily small.*

**Proof:** We have to show that there exists a neighborhood $U$ of $b^*$ on which condition (2) is satisfied with $c$ as above. Let $a^+ \in A \cap U$, $b^+ \in B \cap U$, $b^+ \in P_B(a^+)$. Let $b \in B$ and $\|a^+ - b\| \leq (1+c)r$, where $r = \|a^+ - b^+\|$. In order to verify (2), we have to show $\cos\beta \leq \sqrt{c} r^{1-\sigma}$, where $\beta = \angle(a^+ - b^+, b - b^+)$.

We work in the plane spanned by $b^+, a^+, b$. Choose local euclidian coordinates $b^+ = (0,0)$, $a^+ = (0,r)$, and $b = (x,y)$ with $y \geq 0$. Then $x = y \tan\beta$. Moreover,

$$\|b - a^+\|^2 = \|(x,y) - (0,r)\|^2 = x^2 + (y-r)^2 \leq (1+c)^2 r^2.$$

That gives $x^2 + y^2 - 2yr \leq (2c + c^2) r^2$. Consider the circle with centre $(0,R)$ and radius $R > 0$ passing through $b = (x,y)$. That means

$$x^2 + (y-R)^2 = R^2$$

or $x^2 + y^2 = 2yR$. That implies $y = \frac{2R}{1+\tan^2\beta} = 2R\cos^2\beta$. Substituting we find

$$2y(R-r) \leq (2c + c^2) r^2,$$

hence with $y = 2R\cos^2\beta$,

$$R(R-r) \leq r^2 \, \frac{2c + c^2}{4\cos^2\beta}.$$

Completing squares, this gives

$$R \leq r\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2c+c^2}{\cos^2\beta}}\right).$$

Let $d = (a^+ - b^+)/\|a^+ - b^+\|$, then the ball in $\mathbb{R}^n$ with centre $b^+ + Rd$ and radius $R$ contains $b \in B$, hence $R \geq R(b^+, d)$.

Now observe that by (4) we can choose $\tau' \in (\tau, 1)$ and a small $\epsilon > 0$ such that $\frac{\tau'}{2}\left(\epsilon + \sqrt{\epsilon^2 + 2 + c}\right) < 1$. Then by (3) there exists a neighborhood $U$ of $b^*$ such that if $a^+, b^+ \in U$, then $r^\sigma / R(b^+, d) < \tau'$, and hence also $r^\sigma / R < \tau'$. Moreover, on shrinking $U$ if necessary, we may also arrange that $a^+, b^+ \in U$ implies $r^{1-\sigma} = \|a^+ - b^+\|^{1-\sigma} < \epsilon$. Combining this with the above estimate for $R$ and dividing by $R$ gives

$$1 \leq r^{1-\sigma}\tau'\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2c+c^2}{\cos^2\beta}}\right).$$

Suppose now that contrary to what is claimed we have $\cos\beta > \sqrt{c}r^{1-\sigma}$. Then

$$
\begin{aligned}
1 &< r^{1-\sigma}\tau'\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2c+c^2}{cr^{2(1-\sigma)}}}\right)\\
&= \frac{\tau'}{2}\left(r^{1-\sigma}+\sqrt{r^{2(1-\sigma)}+2+c}\right)\\
&< \frac{\tau'}{2}\left(\epsilon+\sqrt{\epsilon^2+2+c}\right) < 1,
\end{aligned}
$$

a contradiction. That proves the result. □

**Corollary 3. (Hölder regularity from prox-regularity)**. *Let $B$ be prox-regular. Then $B$ is $\sigma$-Hölder regular for every $\sigma \in [0,1)$ with a constant $c > 0$ that may be chosen arbitrarily small.* □

Consider the case of a Lipschitz domain $B$. Here Hölder regularity may be related to a property of the boundary $\partial B$.

**Proposition 5. (Hölder regularity from Hölder smooth boundary)**. *Let $\sigma \in (0,1)$. Suppose $B = \{(x,\xi) \in \mathbb{R}^n : \xi \leq f(x)\}$ is the hypograph of a locally Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$. Let $x^* \in \mathbb{R}^{n-1}$ and suppose there exists a neighborhood $U$ of $x^*$ and $\mu > 0$ such that for every $x_0 \in U$ and every supergradient $g \in \partial^+ f(x_0)$ the Hölder estimate $f(x) \leq f(x_0) + \langle g, x - x_0\rangle + \mu\|x - x_0\|^{1+\sigma}$ is satisfied for every $x \in U$. Then $B$ is $\sigma$-Hölder regular at $(x^*, f(x^*)) \in B$ with respect to any closed set $A$ containing $(x^*, f(x^*))$ for any constant $c > 0$ satisfying $\mu < \sqrt{c}$.*

**Proof:** Suppose $b^+ = (x_0, f(x_0)) \in B$ with $x_0 \in U$ is projected from $a^+ \notin B$. Then $a^+ = (x_1, \xi_1)$ with $\xi_1 > f(x_1)$. This means there exists a supergradient $g \in \partial^+ f(x_0)$ such that $b^+ + t(-g, 1) = (x_1, \xi_1) = a^+$ for some $t > 0$; cf. [11]. Since $\|a^+ - b^+\| = r$, we have

$$
a^+ - b^+ = \left(-\frac{rg}{\sqrt{1+\|g\|^2}}, \frac{r}{\sqrt{1+\|g\|^2}}\right).
$$

Now let $B(x^*, \epsilon) \subset U$ and put $V = B((x^*, f(x^*)), \epsilon)$. Then for $r$ sufficiently small the ball $B(a^+, (1+c)r)$ is contained in $V$. In order to verify (2), consider $b = (y, f(y)) \in B \cap B(a^+, (1+c)r)$. We have to show $\cos\angle(a^+ - b^+, b - b^+) \leq \sqrt{c}\, r^\sigma$. Now

$$
\begin{aligned}
\cos\angle(a^+ - b^+, b - b^+) &= \frac{-\langle g, y - x_0\rangle + f(y) - f(x_0)}{\sqrt{\|y - x_0\|^2 + (f(y) - f(x_0))^2}}\\
&\leq \frac{-\langle g, y - x_0\rangle + f(y) - f(x_0)}{\|y - x_0\|}\\
&\leq \mu\|y - x_0\|^\sigma
\end{aligned}
$$

since $f(y) \leq f(x_0) + \langle g, y - x_0\rangle + \mu\|y - x_0\|^{1+\sigma}$ for every $y \in U$. By the choice of $c$ this shows $\cos\angle(a^+ - b^+, b - b^+)) \leq \sqrt{c}r^\sigma$. □

The nomenclature in Proposition 5 can be explained as follows. A function $f$ is called Lipschitz-smooth at $x_0$ if the second difference quotient is bounded above

$$
\Delta_2(x) = \frac{f(x) - f(x_0) - \langle g, x - x_0\rangle}{\|x - x_0\|^2} \leq \mu < \infty
$$

in a neighborhood of $x_0$. In analogy, we call $f$ $\sigma$-Hölder smooth at $x_0$ for some $\sigma \in (0,1)$ if the weaker $\Delta_{1+\sigma}(x) \leq \mu$ holds in a neighborhood of $x_0$, and this is the condition above.

We consider the following natural modification of amenability from [12]. The set $B \subset \mathbb{R}^n$ is called $\sigma$-Hölder amenable at $x^* \in B$ if there exists a neighborhood $U$ of $x^*$, a class $C^{1,\sigma}$-mapping $G : \mathbb{R}^n \to \mathbb{R}^m$, and a closed convex set $C \subset \mathbb{R}^m$, such that $B \cap U = \{x \in U : G(x) \in C\}$ and $N_C(G(x^*)) \cap \ker\left(DG(x^*)^T\right) = \{0\}$. A typical example in optimization is when $B$ is defined by $C^{1,\sigma}$ equality and inequality constraints, where the Mangasarian-Fromowitz constraint qualification holds at $x^*$.

**Proposition 6. (Hölder regularity from Hölder amenability)**. *Suppose the closed set $B$ is $\sigma'$-Hölder amenable at $x^*$. Then $B$ is $\sigma$-Hölder regular at $x^*$ with respect to any closed set $A$ containing $x^*$ for every $\sigma < \sigma'$, and with arbitrary constant $c$.* □

The proof may be adopted from on [10, Prop. 4.8] with minor changes, and we skip the details. This result suggests that Hölder regularity is settled between the weaker superregularity and the stronger prox-regularity. This is true as long as we consider this type of regularity as a property of $B$ alone. We stress, however, that it is the combination with $A$ and the shrinking distance between the sets in (2) which makes our definition 2 truly versatile in applications.

# 5 Convergence

In this section we prove the main convergence result. Alternating projections converge locally for sets which intersect separably, if one of the sets is Hölder regular. The proof requires two preparatory lemmas.

**Lemma 1. (Three-point estimate)**. *Suppose $B$ intersects $A$ separably at $x^* \in A \cap B$ with exponent $\omega \in [0,2)$ and constant $\gamma > 0$ on the neighborhood $U$ of $x^*$. Suppose $B$ is also $\omega/2$-Hölder regular at $x^* \in A \cap B$ with respect to $A$ on $U$ with constant $c$ satisfying $c < \frac{\gamma}{2}$. Then there exists $0 < \ell < 1$, depending only on $\gamma, c$ and $U$, such that*

$$\|a^+ - b^+\|^2 + \ell\|b - b^+\|^2 \leq \|a^+ - b\|^2 \tag{5}$$

*for every building block $b \to a^+ \to b^+$ in $U$.*

**Proof:** 1) By the cosine theorem we have

$$\|a^+ - b\|^2 = \|a^+ - b^+\|^2 + \|b - b^+\|^2 - 2\|a^+ - b^+\|\|b - b^+\|\cos\beta,$$

where $\beta = \angle(b - b^+, a^+ - b^+)$. Hence in order to assure (5) we have to find $\ell \in (0,1)$ such that

$$\tfrac{1-\ell}{2}\|b - b^+\| \geq \|a^+ - b^+\|\cos\beta \tag{6}$$

for all building blocks $b \to a^+ \to b^+$ in $U$. We consider two cases. Case I is when $\beta \in (\frac{\pi}{2}, \pi]$. Case II is $\beta \in [0, \frac{\pi}{2}]$.

2) We start by discussing case I. For angles $\beta \in (\frac{\pi}{2}, \pi]$ we have $\cos\beta < 0$, hence (6) is trivially true if we choose any $0 < \ell < 1$. For instance $\ell = \frac{1}{2}$ will do. To establish (6) we may now concentrate on case II, where $\beta \in [0, \frac{\pi}{2}]$.

3) In case II we want to use $\omega/2$-Hölder regularity of $B$ with respect to $A$. We subdivide case II in two subcases. Case IIa is when $\cos\beta \leq \sqrt{c}\|a^+ - b^+\|^{\frac{\omega}{2}}$. Case IIb is when $\cos\beta > \sqrt{c}\|a^+ - b^+\|^{\frac{\omega}{2}}$.

Let us start with case IIa, where $\cos\beta \le \sqrt{c}\|a^+ - b^+\|^{\frac{\omega}{2}}$. Observe that:

$$\begin{aligned}\|b-b^+\|^2 &= \|b-a^+\|^2 + \|a^+-b^+\|^2 - 2\langle b-a^+, b^+-a^+\rangle \\ &= (\|b-a^+\| - \|a^+-b^+\|)^2 + 2\|b-a^+\|\|a^+-b^+\|\,(1-\cos\alpha) \\ &\ge 2\|b-a^+\|\|a^+-b^+\|\,(1-\cos\alpha)\,,\end{aligned}$$

where $\alpha = \angle(b-a^+, b^+-a^+)$. By the angle condition (1) we have $1-\cos\alpha \ge \gamma\|a^+-b^+\|^\omega$ for every building block $b \to a^+ \to b^+$ in $U$. Hence

$$\|b-b^+\|^2 \;\ge\; 2\gamma\|b-a^+\|\|a^+-b^+\|^{1+\omega} \ge 2\gamma\|a^+-b^+\|^{2+\omega},$$

where the last estimate uses $b^+ \in P_B(a^+)$. Altogether we obtain

$$\|b-b^+\| \;\ge\; \sqrt{2\gamma}\|a^+-b^+\|^{1+\frac{\omega}{2}} \ge \sqrt{\frac{2\gamma}{c}}\|a^+-b^+\|\cos\beta,$$

bearing in mind that we are in case IIa. To assure (6) we put $\ell = 1-\sqrt{\frac{2c}{\gamma}}$. Then $\ell \in (0,1)$ because of the hypothesis $c < \frac{\gamma}{2}$.

4) Let us now deal with case IIb, where $\cos\beta > \sqrt{c}\|a^+ - b^+\|^{\frac{\omega}{2}}$. By Hölder regularity (2) of $B$ with respect to $A$, we have $b \notin B(a^+, (1+c)r)$. In other words, $\|b-a^+\| > (1+c)\|a^+-b^+\|$. Using this and the cosine theorem again, we have

$$\begin{aligned}\|b-b^+\|^2 &= \|b-a^+\|^2 - \|a^+-b^+\|^2 + 2\|b-b^+\|\|a^+-b^+\|\cos\beta \\ &\ge c(c+2)\|a^+-b^+\|^2 + 2\|b-b^+\|\|a^+-b^+\|\cos\beta.\end{aligned}$$

Since $a^+ \ne b^+$, this may be rearranged as

$$\frac{\|b-b^+\|^2}{\|a^+-b^+\|^2} - 2\frac{\|b-b^+\|}{\|a^+-b^+\|}\cos\beta - c(c+2) \ge 0. \tag{7}$$

Hence (7) implies that the polynomial $P(X) = X^2 - 2X\cos\beta - c(c+2)$ is nonnegative at $X = \frac{\|b-b^+\|}{\|a^+-b^+\|}$. But for nonnegative $X$, nonnegativity $P(X) \ge 0$ is equivalent to

$$X \;\ge\; \cos\beta + \sqrt{\cos^2\beta + c(c+2)} = \cos\beta\left(1+\sqrt{1+\frac{c(c+2)}{\cos^2\beta}}\right).$$

Hence for $X = \frac{\|b-b^+\|}{\|a^+-b^+\|}$ we know that

$$\frac{\|b-b^+\|}{\|a^+-b^+\|} \ge \cos\beta\left(1+\sqrt{1+\frac{c(c+2)}{\cos^2\beta}}\right). \tag{8}$$

Putting $\Theta_{r,\beta} = 1+\sqrt{1+\frac{c(c+2)}{\cos^2\beta}}$, we have $\Theta_{r,\beta} \ge c+2$. Then $\ell = \frac{c}{2+c}$ will do in case IIb.

5) In conclusion, it suffices to put $\ell = \min\left\{\frac{1}{2}, 1-\sqrt{\frac{2c}{\gamma}}, \frac{c}{2+c}\right\}$, with $c < \frac{\gamma}{2}$, to cover all cases. □

**Theorem 1. (Local convergence)**. *Suppose $B$ intersects $A$ separably at $x^* \in A\cap B$ with exponent $\omega \in [0,2)$ and constant $\gamma$ and is $\omega/2$-Hölder regular at $x^*$ with respect to $A$ and constant $c < \frac{\gamma}{2}$. Then there exists a neighborhood $V$ of $x^*$ such that every sequence of alternating projections between $A$ and $B$ which reaches $V$ converges to a point $a^* \in A\cap B$.*

**Proof:** 1) By hypothesis there exists a neighborhood $U = B(x^*, 4\epsilon)$ of $x^* \in A \cap B$ such that every building block $b \to a^+ \to b^+$ with $b, a^+, b^+ \in U$ satisfies the angle condition $1 - \cos\alpha \geq \gamma \|b^+ - a^+\|^\omega$, where $\alpha = \angle(b - a^+, b^+ - a^+)$. In addition, by shrinking $U$ if necessary, we may assume that $B$ is $\omega/2$-Hölder regular at $x^*$ on $U$ with constant $c < \frac{\gamma}{2}$. Then by the three-point lemma (Lemma 1) there exists $\ell \in (0,1)$ depending only on $c, \gamma$ and $U$, such that $\|a^+ - b^+\|^2 + \ell\|b - b^+\|^2 \leq \|a^+ - b\|^2$ for every such building block. Since $\|a^+ - b\| \leq \|a - b\|$, we deduce the following four-point estimate

$$d_B(a^+)^2 + \ell\|b - b^+\|^2 \leq d_B(a)^2$$

for building blocks $b \to a^+ \to b^+$ in $U$.

2) Define the constants $\theta = (\omega + 2)/4$ and $C = 1/((1-\theta)\ell\sqrt{2\gamma}))$. Choose $\delta > 0$ such that

$$9\delta + C2^{2(1-\theta)}\delta^{2(1-\theta)} < \tfrac{\epsilon}{4},$$

which implies $16\delta < \epsilon$. Then define the neighborhood $V$ as $V = B(x^*, \delta)$. We have to show that if the alternating sequence reaches $V$, then it converges to a unique limit $b^* \in A \cap B$. Assume without loss that there exists $k_1 \in \mathbb{N}$ such that $b_{k_1 - 1} \in V = B(x^*, \delta)$. The case where the $a_k$'s reach $V$ is treated analogously.

We shall prove by induction that for every $k \geq k_1$, we have

$$b_k, a_{k+1}, b_{k+1} \in B(x^*, \epsilon) \tag{9}$$

and

$$\sum_{j=k_1}^{k} \|b_j - b_{j+1}\| \leq \frac{1}{2}\sum_{j=k_1}^{k} \|b_{j-1} - b_j\| + \frac{C}{2}\left(d_B(a_{k_1})^{2(1-\theta)} - d_B(a_{k+1})^{2(1-\theta)}\right). \tag{10}$$

Let us first do the induction step and suppose that hypotheses (9), (10) are true at $k-1$. We have to show that they also hold at $k$.

2.1) Firstly in order to check (9) at $k$, we have to prove $a_{k+1}, b_{k+1} \in B(x^*, \epsilon)$. We claim that $b_k \in B(x^*, \frac{\epsilon}{4})$. Indeed, using the induction hypothesis (10) at $k-1$ gives

$$\sum_{j=k_1}^{k-1} \|b_j - b_{j+1}\| \leq \frac{1}{2}\sum_{j=k_1}^{k-1} \|b_j - b_{j-1}\| + \frac{C}{2}\left(d_B(a_{k_1})^{2(1-\theta)} - d_B(a_k)^{2(1-\theta)}\right).$$

Hence

$$\begin{aligned}
\sum_{j=k_1}^{k-1} \|b_j - b_{j+1}\| &\leq \|b_{k_1-1} - b_{k_1}\| + C\, d_B(a_{k_1})^{2(1-\theta)} - C\, d_B(a_k)^{2(1-\theta)} - \|b_{k-1} - b_k\| \\
&\leq \|b_{k_1-1} - b_{k_1}\| + C\, d_B(a_{k_1})^{2(1-\theta)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|b_k - x^*\| &\leq \|b_k - b_{k_1}\| + \|b_{k_1} - x^*\| \leq \sum_{j=k_1}^{k-1} \|b_{j+1} - b_j\| + \|b_{k_1} - x^*\| \\
&\leq \|b_{k_1-1} - b_{k_1}\| + C\, d_B(a_{k_1})^{2(1-\theta)} + \|b_{k_1} - x^*\|. \qquad (11)
\end{aligned}$$

Since $b_{k_1-1} \in B(x^*, \delta)$, we have $\|b_{k_1-1} - a_{k_1}\| \leq \delta$, hence $a_{k_1} \in B(x^*, 2\delta)$. Then $\|a_{k_1} - b_{k_1}\| \leq \|a_{k_1} - x^*\| \leq 2\delta$, which gives $\|b_{k_1} - x^*\| \leq 4\delta$. It follows that $\|b_{k_1-1} - b_{k_1}\| \leq 5\delta$. Now since $d_B(a_{k_1}) = \|a_{k_1} - b_{k_1}\| \leq 2\delta$, going back to (11), we obtain

$$\|b_k - x^*\| \leq 9\delta + C2^{2(1-\theta)}\delta^{2(1-\theta)} < \frac{\epsilon}{4},$$

which implies

$$\|a_{k+1} - x^*\| \leq \|a_{k+1} - b_k\| + \|b_k - x^*\| \leq 2\|b_k - x^*\| < \frac{\epsilon}{2} < \epsilon,$$

and

$$\|b_{k+1} - x^*\| \leq \|a_{k+1} - b_{k+1}\| + \|a_{k+1} - x^*\| \leq 2\|a_{k+1} - x^*\| < \epsilon.$$

This proves the claim $a_{k+1} \in B(x^*, \epsilon)$ and $b_{k+1} \in B(x^*, \epsilon)$.

2.2) Let us now prove that (10) is true at $k$. Using the induction hypothesis $b_{k-1}$, $a_k$, $b_k \in B(x^*, \epsilon)$, we apply part 1) to the building block $b_{k-1} \to a_k \to b_k$, which gives

$$\frac{1 - \cos\alpha_k}{\|a_k - b_k\|^\omega} \geq \gamma, \tag{12}$$

where $\alpha_k = \angle(b_{k-1} - a_k, b_k - a_k)$. By part 2.1), which is already proved, we have $b_k$, $a_{k+1}$, $b_{k+1} \in B(x^*, \epsilon)$ and $B(x^*, \epsilon) \subset U$, so that we can apply the four-point estimate of part 1) to the building block $b_k \to a_{k+1} \to b_{k+1}$. This gives

$$d_B(a_{k+1})^2 + \ell\|b_k - b_{k+1}\|^2 \leq d_B(a_k)^2. \tag{13}$$

Now using the cosine theorem and (12) we obtain

$$\begin{aligned}
\|b_{k-1} - b_k\|^2 &= \|b_{k-1} - a_k\|^2 + \|a_k - b_k\|^2 - 2\|b_{k-1} - a_k\|\|a_k - b_k\|\cos\alpha_k \\
&= (\|b_{k-1} - a_k\| - \|a_k - b_k\|)^2 + 2\|b_{k-1} - a_k\|\|a_k - b_k\|(1 - \cos\alpha_k) \\
&\geq (\|b_{k-1} - a_k\| - \|a_k - b_k\|)^2 + 2\gamma\|b_{k-1} - a_k\|\|a_k - b_k\|^{\omega+1} \\
&\geq 2\gamma\|b_{k-1} - a_k\|\|a_k - b_k\|^{\omega+1}.
\end{aligned}$$

Since $b_k \in P_B(a_k)$ and $b_{k-1} \in B$, we have $\|b_{k-1} - a_k\| \geq \|a_k - b_k\| = d_B(a_k)$. Hence $\|b_{k-1} - b_k\|^2 \geq 2\gamma d_B(a_k)^{\omega+2}$, or what is the same

$$\|a_k - b_k\|^{-(\omega+2)/2}\|b_{k-1} - b_k\| \geq \sqrt{2\gamma}. \tag{14}$$

Recalling that $\theta = (\omega + 2)/4$ we have $\theta \in (\frac{1}{2}, 1)$, because of $\omega \in (0, 2)$. By concavity of the function $s \mapsto s^{1-\theta}/(1-\theta)$ we have $s_1^{1-\theta} - s_2^{1-\theta} \geq (1-\theta)s_1^{-\theta}(s_1 - s_2)$. Applying this to $s_1 = d_B(a_k)^2$ and $s_2 = d_B(a_{k+1})^2$, we obtain

$$\begin{aligned}
d_B(a_k)^{2(1-\theta)} - d_B(a_{k+1})^{2(1-\theta)} &\geq (1-\theta)d_B(a_k)^{-2\theta}\left(d_B(a_k)^2 - d_B(a_{k+1})^2\right) \\
&= (1-\theta)\|a_k - b_k\|^{-2\theta}\left(\|a_k - b_k\|^2 - \|a_{k+1} - b_{k+1}\|^2\right) \\
&\geq (1-\theta)\ell\sqrt{2\gamma}\,\frac{\|b_k - b_{k+1}\|^2}{\|b_{k-1} - b_k\|},
\end{aligned}$$

where the last estimate uses (13) and (14). Recalling that $C = 1/((1-\theta)\ell\sqrt{2\gamma})$ this gives

$$C\,\left(d_B(a_k)^{2(1-\theta)} - d_B(a_{k+1})^{2(1-\theta)}\right) \geq \frac{\|b_k - b_{k+1}\|^2}{\|b_k - b_{k-1}\|}.$$

By comparison of the arithmetic and geometric mean, that implies

$$\|b_k - b_{k+1}\| \le \frac{1}{2}\|b_k - b_{k-1}\| + \frac{C}{2}\left(d_B(a_k)^{2(1-\theta)} - d_B(a_{k+1})^{2(1-\theta)}\right). \tag{15}$$

By the induction hypothesis we have (10) at $k-1$, that is,

$$\sum_{j=k_1}^{k-1}\|b_j - b_{j+1}\| \le \frac{1}{2}\sum_{j=k_1}^{k-1}\|b_{j-1} - b_j\| + \frac{C}{2}\left(d_B(a_{k_1})^{2(1-\theta)} - d_B(a_k)^{2(1-\theta)}\right).$$

Adding this and (15) gives (10) at index $k$.

2.3) Let us now prove that the hypotheses (9) and (10) hold at $k_1$. Concerning (9), since $b_{k_1} \in B(x^*, 4\delta)$ and $\|b_{k_1} - a_{k_1+1}\| \le 4\delta \le \frac{\epsilon}{4}$, we have $a_{k_1+1} \in B(x^*, \frac{\epsilon}{2})$. Then using $\|a_{k_1+1} - b_{k_1+1}\| \le \frac{\epsilon}{2}$ gives $b_{k_1+1} \in B(x^*, \epsilon)$, so (9) is true at $k_1$.

Concerning the validity of (10) at $k_1$, observe that using $b_{k_1-1}, b_{k_1}, a_{k_1} \in B(x^*, \frac{\epsilon}{4})$ we may repeat the argument in the induction step starting before formula (13) with $k_1$ in the place of $k$. The conclusion is formula (15) at $k_1$, that is,

$$\|b_{k_1} - b_{k_1+1}\| \le \frac{1}{2}\|b_{k_1} - b_{k_1-1}\| + \frac{C}{2}\left(d_B(a_{k_1})^{2(1-\theta))} - d_B(a_{k_1+1})^{2(1-\theta)}\right),$$

and this is precisely (10) at $k = k_1$. This concludes the induction argument.

3) Having proved (9), (10) for all indices $k \ge k_1$, we see from (11) that the series $\sum_{j=k_1}^{\infty}\|b_j - b_{j+1}\|$ converges, which means $b_k$ is a Cauchy sequence, which converges to a limit $b^* \in B \cap B(x^*, \epsilon)$. Using relation (14) we conclude that $a_k$ converges to the same limit $b^* \in A \cap B$. □

**Corollary 4. (Rate of convergence)**. *Under the hypotheses of Theorem 1, the convergence rates are* $\|b_k - b^*\| = \mathcal{O}\left(k^{-\frac{1-\theta}{2\theta-1}}\right) = \mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right)$ *and* $\|a_k - b^*\| = \mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right)$.

**Proof:** Summing (15) from $k = N$ to $k = M$ gives

$$-\frac{1}{2}\|b_N - b_{N-1}\| + \frac{1}{2}\sum_{k=N}^{M-1}\|b_k - b_{k+1}\| + \|b_M - b_{M+1}\| \le \frac{C}{2}\left(d_B(a_N)^{2(1-\theta)} - d_B(a_{M+1})^{2(1-\theta)}\right).$$

Passing to the limit $M \to \infty$ gives

$$-\frac{1}{2}\|b_N - b_{N-1}\| + \frac{1}{2}\sum_{k=N}^{\infty}\|b_k - b_{k+1}\| \le \frac{C}{2}d_B(a_N)^{2(1-\theta)}.$$

Introducing $S_N = \sum_{k=N}^{\infty}\|b_k - b_{k+1}\|$, this becomes

$$-\frac{1}{2}\left(S_{N-1} - S_N\right) + \frac{1}{2}S_N \le \frac{C}{2}d_B(a_N)^{2(1-\theta)}.$$

Equivalently,

$$\frac{1}{2}S_N \le \frac{1}{2}(S_{N-1} - S_N) + C\, d_B(a_N)^{2(1-\theta)}.$$

Now using estimate (14), we have $d_B(a_N)^{2(1-\theta)} \le (2\gamma)^{-\frac{1-\theta}{2\theta}}\|b_{N-1} - b_N\|^{\frac{1-\theta}{\theta}}$. Putting $C' := C(2\gamma)^{-\frac{1-\theta}{2\theta}}$ and substituting this gives

$$\frac{1}{2}S_N \le \frac{1}{2}(S_{N-1} - S_N) + C'\left(S_{N-1} - S_N\right)^{\frac{1-\theta}{\theta}}. \tag{16}$$

Since $\theta \in (\frac{1}{2}, 1)$, we have $(1-\theta)/\theta \in (0,1)$. Moreover, $S_N \to 0$, so the second term $(S_{N-1}-S_N)^{\frac{1-\theta}{\theta}}$ dominates the first term $S_{N-1}-S_N$. That means, there exists another constant $C''$ such that

$$S_N^{\frac{\theta}{1-\theta}} \leq C''(S_{N-1}-S_N).$$

We claim that there exists yet another constant $C'''$ such that

$$1 \leq C''(S_{N-1}-S_N)S_N^{-\frac{\theta}{1-\theta}} \leq C''' \left( S_N^{-\frac{2\theta-1}{1-\theta}} - S_{N-1}^{-\frac{2\theta-1}{1-\theta}} \right). \tag{17}$$

Assuming this proved, summation of (17) from $N=1$ to $N=M$ gives

$$M \leq C''' \left( S_M^{-\frac{2\theta-1}{1-\theta}} - S_1^{-\frac{2\theta-1}{1-\theta}} \right).$$

Hence for yet two other constants $C'''', C'''''$,

$$S_M \leq C'''' \left[ S_1^{-\frac{2\theta-1}{1-\theta}} + M \right]^{-\frac{1-\theta}{2\theta-1}} \leq C''''' M^{-\frac{1-\theta}{2\theta-1}}.$$

Since $\|b_M - b^*\| \leq S_M$ by the triangle inequality, that proves the claimed speed of convergence.

In order to prove (17) we divide the set of indices into $\mathcal{I} = \{N : 2S_N \geq S_{N-1}\}$ and $\mathcal{J} = \{N : 2S_N < S_{N-1}\}$. For $N \in \mathcal{I}$ we have

$$\begin{aligned} (S_{N-1}-S_N)S_N^{-\frac{\theta}{1-\theta}} &\leq 2^{\frac{\theta}{1-\theta}}(S_{N-1}-S_N)S_{N-1}^{-\frac{\theta}{1-\theta}} \\ &\leq 2^{\frac{\theta}{1-\theta}} \int_{S_N}^{S_{N-1}} S^{-\frac{\theta}{1-\theta}}\, dS \\ &= 2^{\frac{\theta}{1-\theta}} \frac{1-\theta}{2\theta-1} \left( S_N^{-\frac{2\theta-1}{1-\theta}} - S_{N-1}^{-\frac{2\theta-1}{1-\theta}} \right), \end{aligned}$$

proving (17). In contrast, for $N \in \mathcal{J}$ we have

$$S_N^{-\frac{2\theta-1}{1-\theta}} - S_{N-1}^{-\frac{2\theta-1}{1-\theta}} \geq 2^{\frac{2\theta-1}{1-\theta}} S_{N-1}^{-\frac{2\theta-1}{1-\theta}} - S_{N-1}^{-\frac{2\theta-1}{1-\theta}} = \left(2^{\frac{2\theta-1}{1-\theta}} - 1\right) S_{N-1}^{-\frac{2\theta-1}{1-\theta}} \to \infty$$

in view of $S_{N-1} \to 0$, $\frac{2\theta-1}{1-\theta} > 0$ and $2^{\frac{2\theta-1}{1-\theta}} - 1 > 0$. So on the set $\mathcal{J}$ estimate (17) is trivially satisfied. Finally, the same estimate for $a_k$ follows from $\|a_{k+1} - b^*\| \leq \|a_{k+1} - b_k\| + \|b_k - b^*\| \leq 2\|b_k - b^*\|$. $\square$

**Theorem 2. (Local convergence with linear rate)**. *Let $A, B$ intersect 0-separably at $x^*$ with constant $\gamma \in (0,2)$. Suppose in addition that $B$ is 0-Hölder regular at $x^*$ with respect to $A$ with constant $c < \frac{\gamma}{2}$. Then there exists a neighborhood $V$ of $x^*$ such that every sequence of alternating projections that reaches $V$ converges R-linearly.*

**Proof:** Applying Lemma 1 and Theorem 1 in the case $\omega = 0$, we obtain convergence of the sequence $a_k, b_k$ from summability of $\sum_k \|b_{k-1} - b_k\|$. Now from the proof of Corollary 4, we see that in the case $\theta = \frac{1}{2}$ equation (16) simplifies to

$$\frac{1}{2}S_N \leq \frac{1}{2}(S_{N-1}-S_N) + C'(S_{N-1}-S_N),$$

or what is the same

$$S_N \leq \frac{1+2C'}{2+2C'} S_{N-1}.$$

This proves Q-linear convergence of $S_N$ to 0, hence R-linear convergence of $b_N \to b^*$. □

**Remark 5.** Theorem 2 extends the results in [10, Thm. 5.2] and [2, Thm. 3.14] in two ways. Firstly, as seen in example 7.1, 0-Hölder regularity includes sets $B$ which have singularities pointing inwards, where superregularity [10] and its extension in [2] fail. Secondly, 0-separability is weaker than linear regularity or the CQ in [2], see example 7.4.

When specialized to the case of two sets $A, B$, we obtain results proved in [10, 2].

**Corollary 5. (Bauschke et al. [2])**. *Suppose $A, B$ satisfy the constraint qualification at $x^* \in A \cap B$ in the sense that $N_A^B(x^*) \cap \left(-N_B^A(x^*)\right) \subset \{0\}$. Moreover, suppose for every $\epsilon > 0$ there exists $\delta > 0$ such that $B$ is $(A, \epsilon, \delta)$ regular at $x^*$. Then there exists a neighborhood $V$ of $x^*$ such that every alternating sequence which enters $V$ converges R-linearly.* □

**Corollary 6. (Lewis, Luke, Malick [10])**. *Suppose $A, B$ intersect linearly regularly and $B$ is superregular. Then alternating projections converge locally R-linearly.* □

# 6 Subanalytic sets

Following [4], a subset $A$ of $\mathbb{R}^n$ is called *semianalytic* if for every $x \in \mathbb{R}^n$ there exists an open neighborhood $V$ of $x$ such that

$$A \cap V = \bigcup_{i \in I} \bigcap_{j \in J} \{x \in V : \phi_{ij}(x) = 0, \psi_{ij}(x) > 0\}$$

for finite sets $I, J$ and real-analytic functions $\phi_{ij}, \psi_{ij} : V \to \mathbb{R}$. The set $B$ in $\mathbb{R}^n$ is called *subanalytic* if for every $x \in \mathbb{R}^n$ there exist a neighborhood $V$ of $x$ and a bounded semianalytic subset $A$ of some $\mathbb{R}^n \times \mathbb{R}^m$, $m \geq 1$, such that $B \cap V = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m (x, y) \in A\}$. Finally, an extended-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called subanalytic if its graph is a subanalytic subset of $\mathbb{R}^n \times \mathbb{R}$.

We consider the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined as $f(x) = i_A(x) + \frac{1}{2} d_B(x)^2$, where $i_A$ is the indicator function of $A$.

**Lemma 2.** *Let $f = i_A + \frac{1}{2} d_B^2$. Let $a^+ \in A$ be projected from $b \in B$ and $v = \lambda(b - a^+) \in N_A(a^+)$, where $\lambda \geq 0$. Then $v + a^+ - P_B(a^+) \subset \widehat{\partial} f(a^+)$.*

**Proof:** Since $a^+ \in P_A(b)$, the ball $B(b, r)$ with $r = \|a^+ - b\|$ touches $A$ at $a^+$ from outside. Now let $b^+ \in P_B(a^+)$. We note that $w := a^+ - b^+ \in \widehat{\partial} \left(\frac{1}{2} d_B^2\right)(a^+)$, see e.g. [11, 12]. We wish to show $v + w \in \widehat{\partial} f(a^+)$, which means we have to check

$$\liminf_{a \to a^+} \frac{f(a) - f(a^+) - \langle v + w, a - a^+ \rangle}{\|a - a^+\|} \geq 0.$$

Since $f(a) = \infty$ for points $a \notin A$, it suffices to consider points $a \in A$. Moreover, since $a \to a^+$, we may assume $\|a - a^+\| \leq r$. Now since $B(b, r)$ touches $A$ at $a^+$, we have $\|a - b\| \geq r$. We claim that

$$-\frac{\langle v, a - a^+ \rangle}{\|a - a^+\|} \geq -\frac{\|v\| \|a - a^+\|}{2r}. \tag{18}$$

Indeed, using the cosine theorem for the angle $\beta = \angle(a - a^+, b - a^+)$ and $r = \|b - a^+\|$, we have

$$\begin{aligned}
\frac{\langle v, a - a^+\rangle}{\|a - a^+\|} &= \|v\| \cos\beta = \|v\| \frac{\|b - a^+\|^2 + \|a - a^+\|^2 - \|a - b\|^2}{2\|b - a^+\|\|a - a^+\|} \\
&= \frac{\|v\|\|a - a^+\|}{2r} \left(1 + \frac{r^2 - \|a - b\|^2}{\|a - a^+\|^2}\right) \\
&\leq \frac{\|v\|\|a - a^+\|}{2r},
\end{aligned}$$

the latter since $\|a - b\| \geq r$. Now we obtain

$$\begin{aligned}
\frac{f(a) - f(a^+) - \langle v + w, a - a^+\rangle}{\|a - a^+\|} &= \frac{f(a) - f(a^+) - \langle w, a - a^+\rangle}{\|a - a^+\|} - \frac{\langle v, a - a^+\rangle}{\|a - a^+\|} \\
&\geq \frac{f(a) - f(a^+) - \langle w, a - a^+\rangle}{\|a - a^+\|} - \frac{\|v\|\|a - a^+\|}{2r}.
\end{aligned}$$

Here the rightmost term converges to 0 as $a \to a^+$ due to (18), which shows

$$\liminf_{a \to a^+} \frac{f(a) - f(a^+) - \langle v + w, a - a^+\rangle}{\|a - a^+\|} \geq \liminf_{a \to a^+} \frac{f(a) - f(a^+) - \langle w, a - a^+\rangle}{\|a - a^+\|} \geq 0,$$

the latter because $f = \frac{1}{2}d_B^2$ on $A$ and $w \in \widehat{\partial}\left(\frac{1}{2}d_B^2\right)(a^+)$. This proves the claim. □

**Definition 4.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be lower semi-continuous with closed domain such that $f|_{\mathrm{dom} f}$ is continuous. We say that $f$ satisfies the Łojasiewicz inequality with exponent $\theta \in [0, 1)$ at the critical point $x^*$ of $f$ if there exists $\gamma > 0$ and a neighborhood $U$ of $x^*$ such that $|f(x) - f(x^*)|^{-\theta}\|g\| \geq \gamma$ for every $x \in U$ and every $g \in \widehat{\partial} f(x)$.* □

Here $x^*$ is critical in the sense of the limiting subdifferential, see [11, 12].

**Lemma 3.** *Suppose $f = i_A + \frac{1}{2}d_B^2$ satisfies the Łojasiewicz inequality with exponent $\theta \in [0, 1)$ at $a^* = b^* \in A \cap B$ and constant $\gamma > 0$. Then in fact $\theta > \frac{1}{2}$. Moreover, $B$ intersects $A$ separably with exponent $\omega = 4\theta - 2 \in (0, 2)$ and constant $\gamma' = 2^{-2\theta-1}\gamma^2$.*

**Proof:** Note that $f(a^*) = 0$. Therefore there exists a neighborhood $U$ of $a^* \in A \cap B$ such that

$$f(a^+)^{-\theta}\|g\| \geq \gamma > 0 \tag{19}$$

for every $a^+ \in A \cap U$ and every $g \in \widehat{\partial} f(a^+)$. Now let $a \to b \to a^+ \to b^+$ be a building block with $a, b, a^+, b^+ \in U$. From Lemma 2, $g = v + a^+ - b^+ \in \widehat{\partial} f(a^+)$ for every $v \in N_A(a^+)$ of the form $v = \lambda(b - a^+)$. This uses the fact that $a^+ \in P_A(b)$. Hence by (19) we have

$$2^\theta d_B(a^+)^{-2\theta}\|\lambda(b - a^+) + a^+ - b^+\| \geq \gamma > 0$$

for every $\lambda \geq 0$. We can replace this by

$$d_B(a^+)^{-2\theta} \min_{\lambda \geq 0} \|\lambda(b - a^+) + a^+ - b^+\| \geq 2^{-\theta}\gamma. \tag{20}$$

Let us for the time being consider angles $\alpha = \angle(b - a^+, b^+ - a^+)$ smaller than $90°$. Then the minimum value in (20) is $d_B(a^+)^{-2\theta}\|a^+ - b^+\| \sin\alpha$. Therefore

$$\frac{\sin\alpha}{d_B(a^+)^{2\theta-1}} \geq 2^{-\theta}\gamma. \tag{21}$$

Since $1 - \cos\alpha \geq \frac{1}{2}\sin^2\alpha$, estimate (21) implies

$$\frac{1-\cos\alpha}{d_B(a^+)^{4\theta-2}} \geq 2^{-2\theta-1}\gamma^2. \tag{22}$$

This shows that we must have $\theta > \frac{1}{2}$, because the numerator tends to 0, so the denominator has to go to zero, too, which it does for $4\theta - 2 > 0$.

Let us now discuss the case where $\alpha \geq 90^0$. We claim that the same estimate (22) is still satisfied. Since $\cos\alpha < 0$, the numerator $1 - \cos\alpha$ in (22) is $\geq 1$. Moreover, the infimum in (20) is now attained at $\lambda = 0$ with the value $\|a^+ - b^+\| = d_B(a^+)$. Hence estimate (20) implies $d_B(a^+)^{1-2\theta} \geq 2^{-\theta}\gamma$, hence $d_B(a^+)^{2-4\theta} \geq 2^{-2\theta}\gamma^2 > 2^{-2\theta-1}\gamma^2$, so that (22) is satisfied. This completes the proof.

□

**Theorem 3.** *Let $A, B$ be closed subanalytic sets. Then $A, B$ intersect separably.*

**Proof:** We assume $A \cap B \neq \emptyset$, otherwise there is nothing to prove. Consider the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defined as $f(x) = i_A(x) + \frac{1}{2}d_B(x)^2$. Then $f$ has closed domain $A$ and is continuous on $A$, which makes it amenable to definition 4. Every $x^* \in A \cap B$ is a critical point of $f$. Since $A, B$ are subanalytic sets, $f$ is subanalytic. That can be seen as follows. First observe that $d_B$ is subanalytic by [13, I.2.1.11]. Then $d_B^2$ is subanalytic as the product of two subanalytic functions [13, I.2.1.9]. Finally, $\text{graph}(f) = (A \times \mathbb{R}) \cap \text{graph}(\frac{1}{2}d_B^2)$ shows that $f$ is subanalytic.

Now we invoke Theorem 3.1 of [5], which asserts that $f$ satisfies the Łojasiewicz inequality at $x^*$ for some $\theta \in (0, 1)$. But we have already seen in the proof of Lemma 3 that this can only be true for $\theta > \frac{1}{2}$. Applying Lemma 3, we deduce that $B$ intersects $A$ separably with $\omega = 4\theta - 2$. Interchanging the roles of $A$ and $B$, it follows also that $A$ intersects $B$ separably. □

**Corollary 7. (Local convergence for subanalytic sets)**. *Let $A, B$ be subanalytic. Suppose $B$ is Hölder regular at $x^* \in A \cap B$ with respect to $A$. Then there exists a neighborhood $V$ of $x^*$ such that every sequence of alternating projections $a_k, b_k$ meeting $V$ converges to some $b^* \in A \cap B$ with rate $\|b_k - b^*\| = \mathcal{O}(k^{-\rho})$ for some $\rho \in (0, \infty)$.* □

**Corollary 8.** *Let $A, B$ be closed subanalytic sets and suppose $B$ has slowly vanishing reach with respect to $A$. Then every bounded sequence of alternating projections $a_k, b_k$ satisfying $a_k - b_k \to 0$ converges with rate $\|b_k - b^*\| = \mathcal{O}(k^{-\rho})$ for some $\rho \in (0, \infty)$.* □

**Corollary 9. (Borwein,Li,Yao [6])**. *Let $A, B$ be closed convex semialgebraic sets with nonempty intersection. Then there exists $\rho \in (0, \infty)$ such that every sequence of alternating projections converges with rate $\|b_k - b^*\| = \mathcal{O}(k^{-\rho})$.* □

# 7 Examples

**Example 7.1. (Packman gulping an ice-cream cone the wrong way).** Consider packman $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \leq |y|\}$ the instant before it scarfs down the ice-cream cone section $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x^2 + y^2 \leq 1, 2|y| \leq x\}$ fitting symmetrically into its notch. We have $A \cap B = \{(0,0)\}$, leaving an angular gap of 15° on both sides.

Let $a^+ = (x, \frac{1}{2}x) \in \partial A$, then $b^+ = P_B(a^+) = (\frac{3}{4}x, \frac{3}{4}x) \in \partial B$, which means $r = \|b^+ - a^+\| = \frac{\sqrt{2}}{4}x$. It is easy to see that condition (2) is satisfied for every $c < 1$ and arbitrary $\sigma \in [0,1)$, i.e., $B$ is Hölder regular with respect to $A$. This example shows that Hölder regularity applies to sets which have inward corners and fail Clarke regularity.

Note that since $B$ is not Clarke regular at $x^* = (0,0)$, it is not superregular in the sense of [10]. What is more, $B$ is not $(A, \epsilon, \delta)$-regular in the sense of [2] at $x^* = (0,0)$, regardless how $\epsilon, \delta > 0$ are chosen, because the cone $b^+ + \{v : \langle a^+ - b^+, v\rangle \leq \epsilon \|a^+ - b^+\| \|v\|\}$ with vertex at the projected point $b^+ = (\frac{3}{4}x, \frac{3}{4}x) \in B$ hits $B$ at points $b' \in B$ other than $b$ on the opposite side of $A$, regardless how small $\epsilon$ is chosen. And this cannot be prevented by shrinking the neighborhood $B(x^*, \delta)$.

Note that $A, B$ intersect 0-separably at $(0,0)$, hence alternating projections converge linearly by Theorem 2. This cannot be obtained from the results in [10, 2]. □

**Example 7.2. (Regularity cannot be dispensed with)**. Following [3], consider the spiral $z(\phi) = (1 + e^{-\phi})e^{i\phi}$, $\phi \in [0, \infty)$ in the complex plane which approaches the unit circle $S = \{|z| = 1\}$ form outside. Define a sequence $z_n = z(\phi_n)$ with $\phi_1 < \phi_2 < \cdots \to \infty$ such that $\|z_{n+1} - z_n\| < \|z_n - z_{n-1}\| \to 0$, $P_{\{z_k : k \neq n\} \cup S}(z_n) = z_{n+1}$, and such that every $z \in S$ is an accumulation point of the $z_n$. In [3] an explicit construction with these properties is obtained recursively as

$$z_n = z(\phi_n), \quad \|z(\phi_{n+1}) - z(\phi_n)\| = r_n, \quad r_{n+1} = e^{-\phi_{n+1}} \frac{1 - e^{-2\pi}}{2}. \tag{23}$$

Let $A = \{z_{2n} : n \in \mathbb{N}\} \cup S$, $B = \{z_{2n-1} : n \in \mathbb{N}\} \cup S$, then $A \cap B = S$. Note that for starting points $|z_0| > 1$, the sequence of alternating projections between $A$ and $B$ is a tail of the sequence $z_n$, so none of the alternating sequences converges. Note that $\angle(z_{n+1} - z_n, z_{n-1} - z_n) \to \pi$, hence $A, B$ intersect 0-separably at every $x^* \in S = A \cap B$. The CQ in the sense of [2] is satisfied at every $x^* \in A \cap B$. Namely, for $z \in S$, $N_A^B(z) = N_B^A(z) = \mathbb{R}_+(-iz)$. Indeed, as $a_n = P_A(b_n)$ approaches $z$, the direction $u_n = (b_n - a_n)/\|b_n - a_n\|$ approaches a direction perpendicular to $z$, and since the spiral turns counterclockwise, this direction is $-iz$. Therefore $N_A^B(z) \cap (-N_B^A(z)) = \{0\}$ for every $z \in S$.

Since the sequence $z_n$ fails to converge, we conclude that this must be due to the lack of regularity at points in $S$. Indeed, Hölder regularity fails for every $0 \leq \sigma < 1$. This can be seen as follows. Since the angle $\angle(b - a^+, b^+ - a^+)$ for the building block $b \to a^+ \to b^+$ approaches $\pi$, the corresponding angle $\beta = \angle(b - b^+, a^+ - b^+)$ goes to 0, so $\cos\beta \to 1$, and for $\sigma \in (0,1)$ we cannot find $c > 0$ such that $\cos\beta \leq \sqrt{c r^\sigma}$. Therefore, in order to assure (2), we would need $b \notin B(a^+, (1+c)r)$, where $r = \|a^+ - b^+\|$. This however, would imply linear convergence of the alternating sequence, which fails. As a consequence of Proposition 2, the other regularity concepts fail, too. □

**Example 7.3. (Discrete spiral I).** We consider a discrete approximation of the logarithmic spiral, generated by 8 equally spaced rays emanating from the origin. Starting on one of the rays, we project perpendicularly on the neighboring ray, going counterclockwise. We label the projected points as $a_1, b_1, a_2, \ldots$. This defines two sets $A = \{a_i : i \in \mathbb{N}\} \cup \{(0,0)\}$

and $B = \{b_i : i \in \mathbb{N}\} \cup \{(0,0)\}$ with $A \cap B = \{(0,0)\}$ such that $P_B(a_i) = b_i$ and $P_A(b_i) = a_{i+1}$. Every sequence of alternating projections between $A$ and $B$ not starting at the origin is a tail of the sequence $a_n, b_n$ and converges to $(0,0)$.

As is easy to see, $\alpha = \angle(b - a^+, b^+ - a^+) = 135°$, so $A, B$ intersect 0-separably, at $x^* = (0,0)$. We check whether the intersection satisfies the CQ in the sense on [2]. Consider one of the rays on which a point $a^+$ is situated. Then $u = b - a^+ \in \widehat{N}_A^B(a^+)$ is perpendicular to $a^+ - x^*$, i.e., perpendicular to the ray in question. As $u$ is the same for all $a^+$ on that ray, we have $u \in N_A^B(0,0)$. Altogether, $N_A^B(0,0) = \text{lin}\{u_1, u_3, -u_1, -u_3\}$ for four directions spaced 90°. Similarly, $N_B^A(0,0) = \{u_2, u_4, -u_2, -u_4\}$ spaced 90°, and intertwined with the directions of $N_A^B(0,0)$. We have $N_A^B(0,0) = -N_A^B(0,0)$, and similarly for $N_B^A(0,0)$, and since $N_B^A(0,0) \cap N_A^B(0,0) = \{(0,0)\}$, the intersection does indeed satisfy the CQ in the sense of [2].

How about regularity at $(0,0)$? Naturally, $A, B$ are not superregular at $(0,0)$, because they are not Clarke regular. Concerning $(A, \epsilon, \delta)$-regularity of $B$ in the sense of [2], suppose in a building block $b \to a^+ \to b^+$ we wish to set up a cone with apex $b^+$ and axis $b^+ + \mathbb{R}_+(a^+ - b^+)$ by choosing its aperture small enough through the choice of $\epsilon$ such that all previous points of $A$ are avoided, then we have to choose smaller and smaller angles $\beta$ to do this, so this type of regularity fails.

On the other hand, we have $\sigma$-Hölder regularity for every $\sigma \in [0,1)$. Suppose we start at $a_1 = (1,0)$, then $b_1 = (\frac{1}{2}, \frac{1}{2})$ and $a_2 = (0, \frac{1}{2})$. After a tour of 360°, the spiral comes back to the horizontal ray $\mathbb{R}^+(1,0)$ at $a_5 = (\frac{1}{16}, 0)$. So while at the beginning the spiral turns within the square $[-1,1]^2$, from the second tour onward it will stay in the square $[-\frac{1}{16}, \frac{1}{16}]^2$. As the circle $B(b_1, \frac{7}{16}\sqrt{2})$ contains no points of the small square in its interior, the distance of $b_1$ to the small square being $R = \frac{7}{16}\sqrt{2}$, writing $R = (1+c)r$ we conclude that we can take $c = \frac{7}{8}\sqrt{2} - 1 > 0$ in (2). Now up to a scaling and a rotation the situation is precisely the same for *every* building block $a \to b \to a^+$ starting in a square of length $2\|a\|$. After one 360°-tour we end up at $a^{++++}$ on the same ray as $a$, and from there on the spiral will stay in that smaller square of length $2\frac{1}{16}\|a\| = 2\|a^{++++}\|$. As a consequence of theorem 2, the sequence converges to $(0,0)$ with linear rate. None of the approaches of [10, 9, 2] allows to derive this. □

**Example 7.4. (Discrete spiral II)**. We can modify the above construction by fixing $\phi \in (0, \frac{\pi}{4})$ and generating rays $k\phi$, $k \in \mathbb{N}$. Turning counterclockwise, and keeping only the projected points, we generate iterates $a_k, b_k$ with the property that $a_k$ has angle $2k\phi$ mod $2\pi$ with the horizontal, $b_k$ has angle $(2k+1)\phi$ mod $2\pi$. We put $A = \{a_k : k \in \mathbb{N}\} \cup \{(0,0)\}$, $B = \{b_k : k \in \mathbb{N}\} \cup \{(0,0)\}$, then $A \cap B = \{(0,0)\}$ and $P_B(a_k) = b_k$, $P_A(b_k) = a_{k+1}$ by adapting the argument in example 7.3. The sequence represents again a discrete version of the logarithmic spiral, turning inwards counterclockwise. However, if we now choose $\phi$ such that $\phi/(2\pi)$ is irrational, there will be no periodicity, and the set of directions $a_k/\|a_k\|$ will be dense in $\mathbb{S}^1$, and so for $b_k/\|b_k\|$. We have $\angle(b^+ - a^+, b - a^+) = \pi - \phi$, which means $A, B$ intersect 0-separably at $(0,0)$. They intersect *at an angle*, this angle being $\pi - \phi$. However, $A, B$ do not intersect linearly regularly in the sense of [10, 2]. Indeed, let us fix $\psi \in [0, 2\pi)$ and $u = (\cos\psi, \sin\psi)$. Then there exist rays $2k\phi$ arbitrarily close to $\psi$ and $a_k$ on these rays, projected from $b_{k-1}$ on ray $(2k-1)\phi$. That means, $u_k = (b_k - a_k)/\|b_k - a_k\|$ gets arbitrarily close to the direction $u^\perp = (-\sin\psi, \cos\psi)$, so $u^\perp \in N_A^B(0,0)$. This shows $N_A^B(0,0) = \mathbb{R}^2$. The same holds for $N_B^A(0,0)$, and so linear regularity and extensions fail. □

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