

B-475 Lagrangian-Conic Relaxations, Part I: A Unified Framework and Its Applications to Quadratic Optimization Problems

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Abstract. In Part I of a series of study on Lagrangian-conic relaxations, we introduce a unified framework for conic and Lagrangian-conic relaxations of quadratic optimization problems (QOPs) and polynomial optimization problems (POPs). The framework is constructed with a linear conic optimization problem (COP) in a finite dimensional vector space endowed with an inner product, where the cone used is not necessarily convex. By imposing a copositive condition on the COP, we establish fundamental theoretical results for the COP, its conic relaxations, its Lagrangian-conic relaxations, and their duals. A linearly constrained QOP with complementarity constraints and a general POP can be reduced to the COP satisfying the copositivity condition. Then, the conic and Lagrangian-conic relaxations of such a QOP and POP are discussed in a unified manner. The Lagrangian-conic relaxation takes one of the simplest forms, which is very useful to design efficient numerical methods. As for applications of the framework, we discuss the completely positive programming relaxation, and a sparse doubly nonnegative relaxation for a linearly constrained QOP with complementarity constraints. The unified framework is applied to general POPs in Part II.

Keywords. Lagrangian-conic relaxation, completely positive programming relaxation, doubly nonnegative relaxation, convexification, quadratic optimization problems, exploiting sparsity.

AMS Classification. 90C20, 90C25, 90C26.

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1 Introduction

We present a unified framework for conic and Lagrangian-conic relaxations of quadratic optimization problems (QOPs) and polynomial optimization problems (POPs). The unified framework is described as a primal-dual pair of conic (not necessarily convex) optimization problems (COPs) as follows:

$$\zeta^p(\mathbb{K}) := \inf \left\{ \langle \mathbf{Q}^0, \mathbf{X} \rangle \mid \begin{array}{l} \mathbf{X} \in \mathbb{K}, \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \\ \langle \mathbf{Q}^k, \mathbf{X} \rangle = 0 \ (k = 1, 2, \dots, m) \end{array} \right\} \quad (1)$$

$$\zeta^d(\mathbb{K}) := \sup \left\{ z_0 \mid \mathbf{Q}^0 + \sum_{k=1}^m \mathbf{Q}^k z_k - \mathbf{H}^0 z_0 \in \mathbb{K}^* \right\} \quad (2)$$

where \mathbb{K} is a (not necessarily convex nor closed) cone in a finite dimensional vector space \mathbb{V} endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. When applying this framework to QOPs and POPs, we will take \mathbb{V} to be the linear space of symmetric matrices with appropriate dimension. (This is why capital letters are used to denote vectors as \mathbf{Q} and \mathbf{X} in the space \mathbb{V} .) The primal COP minimizes a linear objective function $\langle \mathbf{Q}^0, \mathbf{X} \rangle$ subject to three types of constraints, a nonhomogeneous linear equality $\langle \mathbf{H}^0, \mathbf{X} \rangle = 1$, multiple homogeneous linear equalities $\langle \mathbf{Q}^k, \mathbf{X} \rangle = 0$ ($k = 1, 2, \dots, m$), and a cone constraint $\mathbf{X} \in \mathbb{K}$. We note that \mathbf{X} denotes a variable vector in \mathbb{V} , and \mathbf{H}^0 and \mathbf{Q}^k ($k = 1, 2, \dots, m$) are constant vectors in \mathbb{V} . A copositivity condition is imposed on \mathbf{H}^0 and \mathbf{Q}^k ($k = 1, 2, \dots, m$): $\mathbf{O} \neq \mathbf{H}^0 \in \mathbb{K}^*$ and $\mathbf{Q}^k \in \mathbb{K}^*$ ($k = 1, 2, \dots, m$) (Condition (I) in Section 2.3), where \mathbb{K}^* denotes the dual of \mathbb{K} , *i.e.*, $\mathbb{K}^* = \{ \mathbf{Y} \in \mathbb{V} : \langle \mathbf{X}, \mathbf{Y} \rangle \geq 0 \text{ for every } \mathbf{X} \in \mathbb{K} \}$. We assume that if the feasible region of (1) is empty, then $\zeta^p(\mathbb{K}) = \infty$, and that if the feasible region of (2) is empty, then $\zeta^d(\mathbb{K}) = -\infty$. By standard argument, we have the following weak duality result: $\zeta^d(\mathbb{K}) \leq \zeta^p(\mathbb{K})$.

Using this unified framework, we develop fundamental theoretical properties of the conic and Lagrangian-conic relaxations of QOPs and POPs in a unified manner. The Lagrangian-conic relaxation presented in this paper for QOPs and POPs is a generalization of the Lagrangian completely positive programming (Lagrangian-CPP) relaxation and the Lagrangian-doubly nonnegative (Lagrangian-DNN) relaxation for a class of linearly constrained QOPs in continuous and binary variables [3, 19]. The unified framework, its theoretical properties, and its applications to a class of linearly constrained QOPs with complementarity constraints are discussed in Part I, and applications of the framework to general POPs will be presented in Part II.

The copositive programming and the CPP relaxations (or formulations) have been important areas of active research for nonconvex QOPs in recent years; see the survey paper [9] and the references therein. In particular, Burer [8] formulated a class of linearly constrained QOPs in nonnegative continuous variables and binary variables as CPP problems, and showed that a QOP in the class has the same optimal objective value as its CPP formulation under certain assumptions. This result of Burer was extended to more general QOPs by [2, 10], and to POPs by [4, 25].

Indeed, many of those QOPs and POPs can be reformulated in the primal COP with a nonconvex cone \mathbb{K} satisfying the copositivity condition. This realization has led to the study on the unified framework in which the equivalence between the optimal values of the original QOP (or POP) and its CPP formulation (or its extended CPP formulation) can

be discussed as the equivalence $\zeta^p(\text{co } \mathbb{K}) = \zeta^p(\mathbb{K})$, where $\text{co } \mathbb{K}$ denotes the convex hull of \mathbb{K} . We provide a necessary and sufficient condition for this equivalence, which is one of the main theoretical contributions of this paper.

Although Burer's CPP formulation of a QOP and its extension are very strong in theory, they are numerically intractable. The computational difficulty arises from the numerical intractability of determining whether a given matrix lies in the CPP cone, which is known to be co-NP-complete [22]. If the CPP cone is replaced by the doubly nonnegative (DNN) cone, a numerically tractable DNN relaxation is obtained [14, 30]. Solving the resulting DNN relaxation by a primal-dual interior-point method [7, 12, 27, 28], however, is known to be numerically inefficient, especially for large-scale problems. More precisely, the DNN relaxation includes the nonnegativity constraints for the elements of the variable matrix in addition to the semidefinite constraint of the variable matrix. Thus, the number of the nonnegativity constraints grows quadratically as the size of the matrix variable increases. Many software packages currently available for nonconvex QOPs and POPs, for instance, GloptiPoly and SparsePOP [16, 29], are based on solving semidefinite relaxations of the problems using a primal-dual interior-point method. The size of the DNN relaxation problems that can be handled by a primal-dual interior point method is limited, particularly for POPs. Thus, it is essential to develop an efficient and effective numerical method for large-scale QOPs and POPs.

The Lagrangian-DNN relaxation of linearly constrained QOPs with complementarity constraints and a solution method using first-order algorithms were proposed in [19] to improve the numerical efficiency of solving the DNN relaxation of a class of linearly constrained QOPs with complementarity constraints. In fact, the Lagrangian-DNN relaxation was suggested originally in [3] as a numerically tractable method for the numerically intractable Lagrangian-CPP relaxation. The numerical results in [19] demonstrated that it was very efficient to solve the Lagrangian-DNN relaxation of the tested problems, including maximum stable set and quadratic assignment problems, by a bisection method combined with the proximal alternating direction multiplier method [11] and the accelerated proximal gradient method [6].

The primary goal of the current series of study is to develop an effective and efficient numerical method based on the Lagrangian-conic relaxation of large-scale QOPs and POPs, by generalizing the idea in the Lagrangian-DNN relaxation combined with first-order algorithms [19]. In the unified framework, we can reduce the primal COP satisfying the copositivity condition to an equivalent, but simpler COP; see Lemma 2.1. We note that the primal COP corresponds to the DNN relaxation of a nonconvex QOP when the framework is applied to QOPs. The precise forms of the simplified COP corresponding to (1) and (2) are given by

$$\eta^p(\mathbb{K}) := \inf \{ \langle \mathbf{Q}^0, \mathbf{X} \rangle \mid \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \langle \mathbf{H}^1, \mathbf{X} \rangle = 0, \mathbf{X} \in \mathbb{K} \} \quad (3)$$

$$\eta^d(\mathbb{K}) := \sup \{ y_0 \mid \mathbf{Q}^0 + \mathbf{H}^1 y_1 - \mathbf{H}^0 y_0 \in \mathbb{K}^* \}, \quad (4)$$

where the homogeneous linear equality constraints $\langle \mathbf{Q}^k, \mathbf{X} \rangle = 0$ ($k = 1, \dots, m$) in (1) are combined into a single a homogeneous linear equality $\langle \mathbf{H}^1, \mathbf{X} \rangle = 0$, with $\mathbf{H}^1 = \sum_{k=1}^m \mathbf{Q}^k$.

When the cone \mathbb{K} is closed and convex as in the cases of DNN and SDP cones, a primal-dual pair of Lagrangian-conic relaxation problems of the original COP can be obtained by applying the Lagrangian relaxation to the simplified COP. The primal Lagrangian-conic

relaxation problem minimizes the objective function $\langle \mathbf{Q}^0 + \lambda \mathbf{H}^1, \mathbf{X} \rangle$ subject to the single linear equality $\langle \mathbf{H}^0, \mathbf{X} \rangle = 1$ and the cone constraint $\mathbf{X} \in \mathbb{K}$, where $\lambda \in \mathbb{R}$ denotes a Lagrangian multiplier (or parameter). Let $\eta^p(\lambda, \mathbb{K})$ and $\eta^d(\lambda, \mathbb{K})$ denote the optimal values of the Lagrangian-conic relaxation problem with a parameter λ and its dual, respectively. The dual Lagrangian-conic relaxation problem can be further reduced to a problem of minimizing y_0 subject to a single equality constraint $g_\lambda(y_0) = 0$ in a single real variable y_0 , where $g_\lambda(y_0)$ is defined as the norm of the metric projection $\Pi_{\mathbb{K}}(-(\mathbf{Q}^0 + \lambda \mathbf{H}^1 - y_0 \mathbf{H}^0))$ of $-(\mathbf{Q}^0 + \lambda \mathbf{H}^1 - y_0 \mathbf{H}^0)$ onto \mathbb{K} . The primal-dual pair of Lagrangian-conic relaxation problems satisfy nice properties listed as follows:

- The optimal value $\eta^p(\lambda, \mathbb{K}) = \eta^d(\lambda, \mathbb{K})$ monotonically converges to the optimal value $\zeta^d(\mathbb{K})$ of the original dual COP (or the optimal value of the dual of the DNN relaxation of a nonconvex QOP in the QOP case) as λ tends to ∞ .
- For every $\lambda \in \mathbb{R}$, the primal problem is strictly feasible (*i.e.*, there exists a primal feasible solution that lies in the relative interior of the cone \mathbb{K}), and has an optimal solution with the optimal value $\eta^p(\lambda, \mathbb{K}) = \eta^d(\lambda, \mathbb{K})$ (the strong duality).
- $g_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ is a nonnegative continuous function such that $g_\lambda(y_0) = 0$ if and only if $y_0 \leq \eta^d(\lambda, \mathbb{K})$.
- g_λ is continuously differentiable, strictly increasing and convex in the interval $(\eta^d(\lambda, \mathbb{K}), \infty)$, and its derivative can be computed easily from $\Pi_{\mathbb{K}}(-(\mathbf{Q}^0 + \lambda \mathbf{H}^1 - y_0 \mathbf{H}^0))$.

The first property ensures that solving the Lagrangian-conic relaxation with a sufficiently large λ can generate a tight lower bound for the optimal value of the original COP. The rest of the properties provide the theoretical support to design efficient and stable numerical methods for solving the Lagrangian-conic relaxation problem and its dual. In fact, it was the first three properties that made it possible to apply a bisection method combined with first-order methods efficiently, stably, and effectively to the Lagrangian-DNN relaxation of QOPs in [19]. In addition to the bisection method, the last property makes it possible to apply an 1-dimensional Newton iteration from any initial point y_0 with $g_\lambda(y_0) > 0$ for computing $\eta^d(\lambda, \mathbb{K}) = \max \{y_0 : g_\lambda(y_0) = 0\}$. Furthermore the Newton iteration generates as a byproduct, a sequence of feasible solutions of the primal problem whose objective values tend to $\eta^p(\lambda, \mathbb{K}) = \eta^d(\lambda, \mathbb{K})$. The numerical efficiency can be further improved if the method proposed by [13, 23] for exploiting structured sparsity in SDPs is incorporated into the Lagrangian-DNN relaxations for QOPs.

The unified framework described in Section 2 consists of three primal-dual pairs of COPs over a (not necessarily convex nor closed) cone \mathbb{K} . The first pair is the primal-dual COPs (1) and (2) with the objective values $\zeta^p(\mathbb{K})$ and $\zeta^d(\mathbb{K})$, and it will be used for a unified model to represent nonconvex QOPs and general POPs as well as their convex relaxations in the subsequent discussions. The second pair is the simplified COPs (3) and (4) with the objective values $\eta^p(\mathbb{K})$ and $\eta^d(\mathbb{K})$. It is equivalent to the first one under the copositivity condition (Condition (I)). The third pair is the Lagrangian-conic relaxation of (3) and (4) with the objective values $\eta^p(\lambda, \mathbb{K})$ and $\eta^d(\lambda, \mathbb{K})$. We discuss the relations among their optimal values $\zeta^p(\mathbb{K})$, $\zeta^d(\mathbb{K})$, $\eta^p(\mathbb{K})$, $\eta^d(\mathbb{K})$, $\eta^p(\lambda, \mathbb{K})$ and $\eta^d(\lambda, \mathbb{K})$ in details.

In Section 3, the COP satisfying the copositive condition is considered for a nonconvex cone \mathbb{K} . We establish a necessary and sufficient condition for $\zeta^p(\mathbb{K}) = \zeta^d(\text{co } \mathbb{K})$. This

identity indicates that the optimal value of the COP over the nonconvex cone \mathbb{K} is attained by its convexification, *i.e.*, by the COP with replacing the nonconvex cone \mathbb{K} by its convex hull $\text{co } \mathbb{K}$. The result in Section 3 is applied to QOPs in Section 4.2 and to POPs in Section 3, Part II [5].

In Section 4, we convert the dual Lagrangian-conic relaxation problem into the above mentioned problem in a single real variable y_0 , and present some fundamental properties on the function g_λ for the bisection and the 1-dimensional Newton methods.

In Section 5, we deal with a class of linearly constrained QOPs with complementarity constraints, and derive some fundamental properties of its CPP and DNN relaxations of a QOP in the class. The results in this section are closely related to, but more general than, the ones obtained in [19] where the same class of QOPs was studied. In Section 6, we exploit structured sparsity in the DNN and the Lagrangian-DNN relaxations for the class of linearly constrained QOPs with complementarity constraints. Section 7 is devoted to concluding remarks.

2 A class of conic optimization problems and their relaxations

We note that conic optimization problems described in this section is not necessarily convex.

2.1 Notation and symbols

Let \mathbb{R} denote the set of real numbers, \mathbb{R}_+ the set of nonnegative real numbers, and \mathbb{Z}_+ the set of nonnegative integers. We use the following notation and symbols throughout the paper.

$$\begin{aligned}
\mathbb{V} &= \text{a finite dimensional vector space endowed with an inner product} \\
&\quad \langle \mathbf{Q}, \mathbf{X} \rangle \text{ and a norm } \|\mathbf{X}\| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} \text{ for every } \mathbf{Q}, \mathbf{X} \in \mathbb{V}, \\
\mathbb{K} &= \text{a nonempty (but not necessarily convex nor closed) cone in } \mathbb{V}, \\
\mathbb{L} &= \text{the subspace of } \mathbb{V} \text{ generated by } \mathbb{K}, \\
&\quad \text{(the minimal subspace of } \mathbb{V} \text{ that contains } \mathbb{K}\text{),} \\
\mathbb{K}^* &= \{\mathbf{Y} \in \mathbb{V} : \langle \mathbf{X}, \mathbf{Y} \rangle \geq 0 \text{ for every } \mathbf{X} \in \mathbb{K}\} \text{ (the dual of } \mathbb{K}\text{),} \\
\text{co } \mathbb{K} &= \text{the convex hull of } \mathbb{K}, \\
\mathbf{H}^0, \mathbf{Q}^k &\in \mathbb{V} \text{ (} k = 0, 1, 2, \dots, m\text{),} \\
F(\mathbb{K}) &= \left\{ \mathbf{X} \in \mathbb{V} \mid \begin{array}{l} \mathbf{X} \in \mathbb{K}, \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \\ \langle \mathbf{Q}^k, \mathbf{X} \rangle = 0 \text{ (} k = 1, 2, \dots, m\text{)} \end{array} \right\}.
\end{aligned}$$

2.2 Combining the homogeneous equalities in a single equality

We impose the following condition on the coefficient vectors \mathbf{H}^0 and \mathbf{Q}^k ($k = 1, 2, \dots, m$) of the equality constraints in (1).

Condition (I) $\mathbf{O} \neq \mathbf{H}^0 \in \mathbb{K}^*$, $\mathbf{Q}^k \in \mathbb{K}^*$ ($k = 1, 2, \dots, m$).

Lemma 2.1. *Suppose that Condition (I) is satisfied. Then, the following assertions hold.*

(i) $\mathbf{X} \in F(\mathbb{K})$ if and only if

$$\mathbf{X} \in \mathbb{K}, \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \langle \mathbf{H}^1, \mathbf{X} \rangle = 0, \quad (5)$$

Hence, $F(\mathbb{K}) = \{\mathbf{X} \in \mathbb{K} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \langle \mathbf{H}^1, \mathbf{X} \rangle = 0\}$.

(ii) $\zeta^p(\mathbb{K}) = \eta^p(\mathbb{K})$.

(iii) $\zeta^d(\mathbb{K}) = \eta^d(\mathbb{K})$.

Proof. We prove (i) and (iii) since (ii) follows directly from (i). (i) The only if part follows from the definitions of $F(\mathbb{K})$ and \mathbf{H}^1 . Assume that (5) holds. Thus,

$$0 = \langle \mathbf{H}^1, \mathbf{X} \rangle = \sum_{k=1}^m \langle \mathbf{Q}^k, \mathbf{X} \rangle.$$

By Condition (I) and $\mathbf{X} \in \mathbb{K}$, we know that $\langle \mathbf{Q}^k, \mathbf{X} \rangle \geq 0$ ($k = 1, 2, \dots, m$). Therefore, $\langle \mathbf{Q}^k, \mathbf{X} \rangle = 0$ ($k = 1, 2, \dots, m$), and $\mathbf{X} \in F(\mathbb{K})$.

(iii) If (y_0, y_1) is a feasible solution of (4) with the objective value y_0 , then $(z_0, z_1, \dots, z_m) = (y_0, y_1, \dots, y_1)$ is a feasible solution of (2) with the same objective value. Conversely if (z_0, z_1, \dots, z_m) is a feasible solution of (2) with the objective value z_0 , then

$$\begin{aligned} \mathbb{K}^* &\ni \mathbf{Q}^0 + \sum_{k=1}^m \mathbf{Q}^k z_k - \mathbf{H}^0 z_0 + \sum_{k=1}^m \mathbf{Q}^k \left(\max_j z_j - z_k \right) \quad (\text{by Condition (I)}) \\ &= \mathbf{Q}^0 + \left(\sum_{k=1}^m \mathbf{Q}^k \right) \max_j z_j - \mathbf{H}^0 z_0 = \mathbf{Q}^0 + \mathbf{H}^1 \max_j z_j - \mathbf{H}^0 z_0. \end{aligned}$$

Thus, $(y_0, y_1) = (z_0, \max_j z_j)$ is a feasible solution of (4) with the same objective value. Consequently, $\zeta^d(\mathbb{K}) = \eta^d(\mathbb{K})$ holds. \square

2.3 A Lagrangian-conic relaxation and its dual

Applying the Lagrangian relaxation to the simplified COP (3), we obtain the Lagrangian-conic relaxation of the COP (1) and its dual

$$\eta^p(\lambda, \mathbb{K}) := \inf \{ \langle \mathbf{Q}^0 + \lambda \mathbf{H}^1, \mathbf{X} \rangle \mid \mathbf{X} \in \mathbb{K}, \langle \mathbf{H}^0, \mathbf{X} \rangle = 1 \}, \quad (6)$$

$$\eta^d(\lambda, \mathbb{K}) := \sup \{ y_0 \mid \mathbf{Q}^0 + \lambda \mathbf{H}^1 - \mathbf{H}^0 y_0 \in \mathbb{K}^* \}, \quad (7)$$

where $\lambda \in \mathbb{R}$ denotes the Lagrangian multiplier for the homogeneous equality $\langle \mathbf{H}^1, \mathbf{X} \rangle = 0$ in (3).

Lemma 2.2. *Suppose that Condition (I) is satisfied. Then, the following assertions hold.*

(i) $\eta^d(\lambda, \mathbb{K}) \leq \eta^p(\lambda, \mathbb{K})$ for every $\lambda \in \mathbb{R}$.

(ii) $\eta^p(\lambda_1, \mathbb{K}) \leq \eta^p(\lambda_2, \mathbb{K}) \leq \eta^p(\mathbb{K})$ if $\lambda_1 < \lambda_2$.

(iii) $\eta^d(\lambda_1, \mathbb{K}) \leq \eta^d(\lambda_2, \mathbb{K}) \leq \eta^d(\mathbb{K})$ if $\lambda_1 < \lambda_2$, and $\lim_{\lambda \rightarrow \infty} \eta^d(\lambda, \mathbb{K}) = \eta^d(\mathbb{K})$.

Proof. Since the weak duality relation (i) is straightforward, we only prove assertions (ii) and (iii).

(ii) The first inequality follows from the inequality $\langle \mathbf{H}_1, \mathbf{X} \rangle \geq 0$ for every $\mathbf{X} \in \mathbb{K}$. To show the second inequality, suppose that $\mathbf{X} \in \mathbb{K}$ is a feasible solution of (3) with objective value $\langle \mathbf{Q}^0, \mathbf{X} \rangle$. Then, it is a feasible solution of (6) with the same objective value for any $\lambda \in \mathbb{R}$. Hence $\eta^p(\lambda_2, \mathbb{K}) \leq \eta^p(\mathbb{K})$.

(iii) Suppose that $\lambda_1 < \lambda_2$. If y_0 is a feasible solution of (7) with $\lambda = \lambda_1$, then it is a feasible solution of (7) with $\lambda = \lambda_2$ because $\mathbf{H}^1 \in \mathbb{K}^*$. This implies the first inequality. To show the second inequality, suppose that y_0 is a feasible solution of (7) with $\lambda = \lambda_2$. Then (y_0, y_1) with $y_1 = \lambda_2$ is a feasible solution of (4), and the second inequality follows. If (y_0, y_1) is a feasible solution of (4), then y_0 is a feasible solution of (7) with $\lambda = y_1$. Therefore, we obtain $\lim_{\lambda \rightarrow \infty} \eta^d(\lambda, \mathbb{K}) \geq \eta^d(\mathbb{K})$. \square

2.4 Strong duality relations

We assume the following condition to discuss the strong duality between (6) and (7), and between (1) and (2) in this subsection.

Condition (II) \mathbb{K} is closed and convex.

Lemma 2.3. *Suppose that Conditions (I) and (II) are satisfied. Then, the following assertions hold.*

(i) $\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K})$ for every $\lambda \in \mathbb{R}$. Moreover, if $\eta^p(\lambda, \mathbb{K})$ is finite, then (7) has an optimal solution with the objective value $\eta^p(\lambda, \mathbb{K})$.

(ii) $(\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K})) \uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K})$. Here \uparrow means “increases monotonically as $\lambda \rightarrow \infty$ ”.

Proof. Assertion (ii) follows from assertion (i) and Lemma 2.2. Thus, we only have to show (i). Let $\lambda \in \mathbb{R}$ be fixed. We know by the weak duality that $\eta^d(\lambda, \mathbb{K}) \leq \eta^p(\lambda, \mathbb{K})$. By $\mathbf{O} \neq \mathbf{H}^0 \in \mathbb{K}^*$ from Condition (I), $\eta^p(\lambda, \mathbb{K}) < \infty$. If $\eta^p(\lambda, \mathbb{K}) = -\infty$, then it is obvious that the equality holds. Thus, we assume that $\eta^p(\lambda, \mathbb{K})$ takes a finite value, and prove the assertion by the duality theorem. We notice, however, that \mathbb{K} may not have an interior point with respect to \mathbb{V} . In this case, the standard duality theorem can not be directly used (see, for example, Theorem 4.2.1 in [24]). Let \mathbb{L} denote the subspace of \mathbb{V} generated by \mathbb{K} , i.e., the minimal subspace of \mathbb{V} that contains \mathbb{K} . Then \mathbb{K} has an interior-point with respect to \mathbb{L} . Now, (6) and its dual (7) can be converted into conic optimization problems within the space \mathbb{L} :

$$\hat{\eta}^p(\lambda, \mathbb{K}) := \inf \left\{ \langle \widehat{\mathbf{Q}}^0 + \lambda \widehat{\mathbf{H}}^1, \mathbf{X} \rangle \mid \mathbf{X} \in \mathbb{K}, \langle \widehat{\mathbf{H}}^0, \mathbf{X} \rangle = 1 \right\}, \quad (8)$$

$$\hat{\eta}^d(\lambda, \mathbb{K}) := \sup \left\{ y_0 \mid \widehat{\mathbf{Q}}^0 + \lambda \widehat{\mathbf{H}}^1 - \widehat{\mathbf{H}}^0 y_0 \in \mathbb{K}^* \cap \mathbb{L} \right\}, \quad (9)$$

where $\widehat{\mathbf{Q}}^0$, $\widehat{\mathbf{H}}^0$ and $\widehat{\mathbf{H}}^1$ are the metric projections of \mathbf{Q}^0 , \mathbf{H}^0 and \mathbf{H}^1 onto \mathbb{L} , respectively. We can easily see that (6) is equivalent to (8). We also see that

$$\begin{aligned} \mathbf{Q}^0 + \lambda \mathbf{H}^1 - \mathbf{H}^0 y_0 &\in \mathbb{K}^* \\ \text{i.e., } \langle \mathbf{Q}^0 + \lambda \mathbf{H}^1 - \mathbf{H}^0 y_0, \mathbf{X} \rangle &\geq 0 \text{ for every } \mathbf{X} \in \mathbb{K} \end{aligned}$$

if and only if

$$\begin{aligned} \widehat{\mathbf{Q}}^0 + \lambda \widehat{\mathbf{H}}^1 - \widehat{\mathbf{H}}^0 y_0 &\in \mathbb{K}^* \cap \mathbb{L} \\ \text{i.e., } \langle \widehat{\mathbf{Q}}^0 + \lambda \widehat{\mathbf{H}}^1 - \widehat{\mathbf{H}}^0 y_0, \mathbf{X} \rangle &\geq 0 \text{ for every } \mathbf{X} \in \mathbb{K} \cap \mathbb{L}. \end{aligned}$$

Thus, (7) is equivalent to (9). It suffices to show by the duality theorem that $\hat{\eta}^p(\lambda, \mathbb{K}) = \hat{\eta}^d(\lambda, \mathbb{K})$. By $\mathbf{O} \neq \mathbf{H}^0 \in \mathbb{K}^*$ from Condition (I), there exists an $\widehat{\mathbf{X}} \in \mathbb{K}$ such that $\langle \widehat{\mathbf{H}}^0, \widehat{\mathbf{X}} \rangle > 0$. We can take such an $\widehat{\mathbf{X}}$ from the interior of \mathbb{K} with respect to \mathbb{L} . Then, $\widehat{\mathbf{X}} / \langle \widehat{\mathbf{H}}^0, \widehat{\mathbf{X}} \rangle$ is an interior feasible solution of (8). Recall that $\hat{\eta}^p(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K})$ is assumed to be finite. By the duality theorem, the dual problem (9) (hence (7)) has an optimal solution with the objective value $\hat{\eta}^p(\lambda, \mathbb{K})$. \square

The following lemma shows the difficulty of proving the strong duality for the pair of the problems (1) and (2) and the pair of the problems (3) and (4) in the same way as in the proof above for the pair of (6) and (7) by the duality theorem.

Lemma 2.4. *Suppose that Conditions (I) and (II) are satisfied and that $F(\mathbb{K})$ is a proper subset of $\{\mathbf{X} \in \mathbb{K} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 1\}$. Then, the feasible region $F(\mathbb{K})$ of (1) (and (3)) contains no interior point of \mathbb{K} with respect to \mathbb{L} (= the subspace of \mathbb{V} generated by \mathbb{K}).*

Proof. We assume that $F(\mathbb{K}) \neq \emptyset$ since otherwise the assertion is trivial. By Condition (I) and the assumption that $F(\mathbb{K})$ is a proper subset of $\{\mathbf{X} \in \mathbb{K} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 1\}$, there exists $k \in \{1, 2, \dots, m\}$ and $\mathbf{X} \in \mathbb{K}$ such that $\langle \mathbf{Q}^k, \mathbf{X} \rangle > 0$. Let $\widehat{\mathbf{Q}}^k$ be the metric projection of \mathbf{Q}^k onto \mathbb{L} . Then, $\widehat{\mathbf{Q}}^k \in \mathbb{K}^* \cap \mathbb{L}$ and $\langle \widehat{\mathbf{Q}}^k, \mathbf{X} \rangle = \langle \mathbf{Q}^k, \mathbf{X} \rangle > 0$. Let $\widetilde{\mathbf{X}}$ be an arbitrary interior point of \mathbb{K} with respect to \mathbb{L} . Then, there exists a positive number ϵ such that $\widetilde{\mathbf{X}} - \epsilon \widehat{\mathbf{Q}}^k$ remains in \mathbb{K} . Thus, $\langle \widehat{\mathbf{Q}}^k, \widetilde{\mathbf{X}} - \epsilon \widehat{\mathbf{Q}}^k \rangle \geq 0$. It follows that

$$\langle \mathbf{Q}^k, \widetilde{\mathbf{X}} \rangle = \langle \widehat{\mathbf{Q}}^k, \widetilde{\mathbf{X}} \rangle > \langle \widehat{\mathbf{Q}}^k, \widetilde{\mathbf{X}} \rangle - \epsilon \langle \widehat{\mathbf{Q}}^k, \widehat{\mathbf{Q}}^k \rangle = \langle \widehat{\mathbf{Q}}^k, \widetilde{\mathbf{X}} - \epsilon \widehat{\mathbf{Q}}^k \rangle \geq 0.$$

Therefore, any interior point of \mathbb{K} with respect to \mathbb{L} cannot be contained in $F(\mathbb{K})$. \square

We need an additional condition to ensure the strong duality between (1) and (2).

Condition (III) $\{\mathbf{X} \in F(\mathbb{K}) : \langle \mathbf{Q}^0, \mathbf{X} \rangle \leq \tilde{\zeta}\}$ is nonempty and bounded for some $\tilde{\zeta} \in \mathbb{R}$.

Lemma 2.5. *Suppose that Conditions (I), (II) and (III) are satisfied. Then, the following assertions hold.*

$$(i) \quad \lim_{\lambda \rightarrow \infty} \eta^p(\lambda, \mathbb{K}) = \eta^p(\mathbb{K}).$$

$$(ii) \quad (\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K})) \uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) = \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K}).$$

Proof. Assertion (ii) follows from assertion (i) and Lemma 2.3, thus we only prove (i). We first show that the set $L(\lambda) = \{\mathbf{X} \in \mathbb{K} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \langle \mathbf{Q}^0 + \lambda \mathbf{H}^1, \mathbf{X} \rangle \leq \zeta\}$ is nonempty, closed, and bounded (hence $-\infty < \eta^p(\lambda, \mathbb{K})$) for every sufficiently large λ . The closedness of $L(\lambda)$ follows from Condition (II). By Conditions (I) and (III), we see that

$$\emptyset \neq \left\{ \mathbf{X} \in \mathbf{F}(\mathbb{K}) : \langle \mathbf{Q}^0, \mathbf{X} \rangle \leq \tilde{\zeta} \right\} \subset L(\lambda_2) \subset L(\lambda_1) \text{ if } 0 < \lambda_1 < \lambda_2.$$

Next, we show that $L(\lambda)$ is bounded for every sufficiently large $\lambda > 0$. Assume on the contrary that there exists a sequence $\{(\lambda^k, \mathbf{X}^k) \in \mathbb{R}_+ \times \mathbb{K}\}$ such that $\mathbf{X}^k \in L(\lambda^k)$, $0 < \lambda^k \rightarrow \infty$ and $0 < \|\mathbf{X}^k\| \rightarrow \infty$ as $k \rightarrow \infty$. Then, we have

$$\begin{aligned} \frac{\mathbf{X}^k}{\|\mathbf{X}^k\|} \in \mathbb{K}, \quad \langle \mathbf{H}^1, \frac{\mathbf{X}^k}{\|\mathbf{X}^k\|} \rangle \geq 0, \quad \langle \mathbf{Q}^0, \frac{\mathbf{X}^k}{\|\mathbf{X}^k\|} \rangle \leq \frac{\tilde{\zeta}}{\|\mathbf{X}^k\|}, \\ \langle \mathbf{H}^0, \frac{\mathbf{X}^k}{\|\mathbf{X}^k\|} \rangle = \frac{1}{\|\mathbf{X}^k\|} \quad \text{and} \quad \langle \mathbf{Q}^0, \frac{\mathbf{X}^k}{\lambda^k \|\mathbf{X}^k\|} \rangle + \langle \mathbf{H}^1, \frac{\mathbf{X}^k}{\|\mathbf{X}^k\|} \rangle \leq \frac{\tilde{\zeta}}{\lambda^k \|\mathbf{X}^k\|}. \end{aligned}$$

We may assume without loss of generality that $\mathbf{X}/\|\mathbf{X}^k\|$ converges to a nonzero $\mathbf{D} \in \mathbb{K}$. By taking the limit as $k \rightarrow \infty$, we obtain that

$$\mathbf{O} \neq \mathbf{D} \in \mathbb{K}, \quad \langle \mathbf{H}^0, \mathbf{D} \rangle = 0, \quad \langle \mathbf{H}^1, \mathbf{D} \rangle = 0, \quad \langle \mathbf{Q}^0, \mathbf{D} \rangle \leq 0.$$

Thus, if we choose an \mathbf{X} from the set $\left\{ \mathbf{X} \in \mathbf{F}(\mathbb{K}) : \langle \mathbf{Q}^0, \mathbf{X} \rangle \leq \tilde{\zeta} \right\}$, then $\{\mathbf{X} + \mu \mathbf{D} : \mu \geq 0\}$ forms an unbounded ray contained in the set by Condition (II). This contradicts Condition (III). Therefore, we have shown that $L(\tilde{\lambda})$ is bounded for some sufficiently large $\tilde{\lambda} > 0$ and $\emptyset \neq L(\lambda) \subset L(\tilde{\lambda})$ for every $\lambda \geq \tilde{\lambda}$.

Let $\{\lambda^k \geq \tilde{\lambda}\}$ be a divergent sequence to ∞ . Since the nonempty and closed level set $L(\lambda^k)$ is contained in a bounded set $L(\tilde{\lambda})$, the problem (6) with each $\lambda = \lambda^k$ has an optimal solution \mathbf{X}^k with the objective value $\eta^p(\lambda^k) = \langle \mathbf{Q}^0 + \lambda^k \mathbf{H}^1, \mathbf{X}^k \rangle$ in the level set $L(\tilde{\lambda})$. We may assume without loss of generality that \mathbf{X}^k converges to some $\tilde{\mathbf{X}} \in L(\tilde{\lambda})$. Since $\eta^p(\lambda^k, \mathbb{K}) \leq \eta^p(\mathbb{K})$ by Lemma 2.2, it follows that

$$\langle \mathbf{H}^0, \mathbf{X}^k \rangle = 1, \quad \left\langle \frac{\mathbf{Q}^0}{\lambda^k} + \mathbf{H}^1, \mathbf{X}^k \right\rangle \leq \frac{\eta^p(\mathbb{K})}{\lambda^k}, \quad \langle \mathbf{H}^1, \mathbf{X}^k \rangle \geq 0, \quad \langle \mathbf{Q}^0, \mathbf{X}^k \rangle \leq \eta^p(\mathbb{K}).$$

By taking the limit as $k \rightarrow \infty$, we obtain that

$$\tilde{\mathbf{X}} \in \mathbb{K}, \quad \langle \mathbf{H}^0, \tilde{\mathbf{X}} \rangle = 1, \quad \langle \mathbf{H}^1, \tilde{\mathbf{X}} \rangle = 0, \quad \langle \mathbf{Q}^0, \tilde{\mathbf{X}} \rangle \leq \eta^p(\mathbb{K}).$$

This implies that $\tilde{\mathbf{X}}$ is an optimal solution of the problem (3), hence, $\langle \mathbf{Q}^0, \mathbf{X}^k \rangle$ converges to $\eta^p(\mathbb{K})$ as $k \rightarrow \infty$. We also see from

$$\langle \mathbf{Q}^0, \mathbf{X}^k \rangle \leq \eta^p(\lambda^k, \mathbb{K}) = \langle \mathbf{Q}^0 + \lambda^k \mathbf{H}^1, \mathbf{X}^k \rangle \leq \eta^p(\mathbb{K})$$

that $\eta^p(\lambda^k, \mathbb{K})$ converges to $\eta^p(\mathbb{K})$ as $k \rightarrow \infty$. Thus, we have shown assertion (i). \square

Alternatively, the strong duality can be established by incorporating the linear constraint $\langle \mathbf{H}^1, \mathbf{X} \rangle = 0$ into the cone \mathbb{K} for the problem (3). Define $\mathbb{M} = \{\mathbf{X} \in \mathbb{V} : \langle \mathbf{H}^1, \mathbf{X} \rangle = 0\}$ and $\tilde{\mathbb{K}} = \mathbb{K} \cap \mathbb{M}$, and consider the following primal-dual pair:

$$\begin{aligned}\tilde{\eta}^p &= \inf \left\{ \langle \mathbf{Q}_0, \mathbf{X} \rangle \mid \mathbf{X} \in \tilde{\mathbb{K}}, \langle \mathbf{H}^0, \mathbf{X} \rangle = 1 \right\}. \\ \tilde{\eta}^d &= \sup \left\{ y_0 \mid \mathbf{Q}_0 - \mathbf{H}_0 y_0 \in \tilde{\mathbb{K}}^* \right\}.\end{aligned}$$

By the same argument in the proof of Lemma 2.3, we can prove that there is no duality gap between these problems, *i.e.*, $\tilde{\eta}^p = \tilde{\eta}^d$, and that if their common optimal value is finite, then the dual problem has an optimal solution. Some readers may find this proof more clear than the one presented in Lemma 2.5. However, it should be mentioned that the dual problem is not equivalent to (4) although the primal problem is equivalent to (3). In fact, we know that $\tilde{\mathbb{K}}^* = \text{cl}(\mathbb{K}^* + \mathbb{M}^\perp)$ while (4) is equivalent to the dual problem above with replacing $\tilde{\mathbb{K}}^*$ by $\mathbb{K}^* + \mathbb{M}^\perp$. In general, $\mathbb{K}^* + \mathbb{M}^\perp$ may not be closed and may be a proper subset of $\tilde{\mathbb{K}}^*$. Such an example was given in Section 3.3 of [3].

The following theorem summarizes the results in this section.

Theorem 2.1.

- (i) $\eta^d(\lambda, \mathbb{K})\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) \leq \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$ and $(\eta^d(\lambda, \mathbb{K}) \leq \eta^p(\lambda, \mathbb{K}))\uparrow \leq \eta^p(\mathbb{K})$ under Condition (I).
- (ii) $(\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K}))\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) \leq \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$ under Conditions (I) and (II).
- (iii) $(\eta^d(\lambda, \mathbb{K}) = \eta^p(\lambda, \mathbb{K}))\uparrow = \eta^d(\mathbb{K}) = \zeta^d(\mathbb{K}) = \eta^p(\mathbb{K}) = \zeta^p(\mathbb{K})$ under Conditions (I), (II) and (III).

3 Convexification

We focus on the COP (1) with nonconvex cone $\mathbf{\Gamma}$ in \mathbb{V} in this section. Assuming that Condition (I) holds for $\mathbb{K} = \mathbf{\Gamma}$, we drive a necessary and sufficient condition for the equivalence between COP (1) with $\mathbb{K} = \mathbf{\Gamma}$ and COP (1) with $\mathbb{K} = \text{co } \mathbf{\Gamma}$.

Since $\mathbf{\Gamma} \subset \text{co } \mathbf{\Gamma}$, we immediately see that $\zeta^p(\mathbf{\Gamma}) \geq \zeta^p(\text{co } \mathbf{\Gamma})$. Suppose that Condition (I) is satisfied for $\mathbb{K} = \mathbf{\Gamma}$. Then it also holds for $\mathbb{K} = \text{co } \mathbf{\Gamma}$. As a result, we can consistently define the simplified COP (3), the Lagrangian-conic relaxation (8) and their duals for $\mathbb{K} = \text{co } \mathbf{\Gamma}$, and all results established in Lemmas 2.1, 2.2 and 2.3 remain valid for $\mathbb{K} = \text{co } \mathbf{\Gamma}$.

Let

$$\mathbf{\Delta}_1 = \{\mathbf{X} \in \mathbf{\Gamma} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 1\} \quad \text{and} \quad \mathbf{\Delta} = \{\mu \mathbf{X} : \mu \geq 0, \mathbf{X} \in \mathbf{\Delta}_1\}. \quad (10)$$

We assume throughout this section that $\mathbf{\Delta}_1 \neq \emptyset$, because otherwise COP (1) with $\mathbb{K} = \mathbf{\Gamma}$ is obviously infeasible. Then, $\mathbf{\Delta}$ forms a cone in \mathbb{V} . Suppose that Condition (I) holds for $\mathbb{K} = \mathbf{\Gamma}$. Then, $\langle \mathbf{H}^0, \mathbf{X} \rangle \geq 0$ for every $\mathbf{X} \in \mathbf{\Gamma}$. This geometrically implies that $\{\mathbf{X} \in \mathbb{V} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 0\}$ forms a supporting hyperplane for the cone $\mathbf{\Gamma}$. We can easily verify that

$$\mathbf{\Delta} = \{\mathbf{X} \in \mathbf{\Gamma} : \langle \mathbf{H}^0, \mathbf{X} \rangle > 0\} \cup \{\mathbf{O}\}, \quad \mathbf{\Delta} \cap \mathbf{\Delta}_0 = \{\mathbf{O}\} \quad \text{and} \quad \mathbf{\Gamma} = \mathbf{\Delta} \cup \mathbf{\Delta}_0,$$

where $\Delta_0 = \Gamma \cap \{\mathbf{X} \in \mathbb{V} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 0\} = \{\mathbf{X} \in \Gamma : \langle \mathbf{H}^0, \mathbf{X} \rangle = 0\}$. Since $\Gamma \supset \Delta$, even if the cone $\mathbb{K} = \Gamma$ is replaced by the cone $\mathbb{K} = \Delta$, Condition (I) is satisfied. Thus, all assertions of Lemmas 2.1, 2.2 and 2.3 remain valid for $\mathbb{K} = \Delta$.

To characterize the convexification of COP (1) with $\mathbb{K} = \Gamma$, we introduce the following COP:

$$\zeta_0^p(\mathbb{K}) = \inf \{ \langle \mathbf{Q}^0, \mathbf{X} \rangle \mid \mathbf{X} \in F_0(\mathbb{K}) \}, \quad (11)$$

where $F_0(\mathbb{K}) = \{\mathbf{X} \in \mathbb{K} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 0, \langle \mathbf{Q}^k, \mathbf{X} \rangle = 0 (k = 1, 2, \dots, m)\}$ and \mathbb{K} denotes a cone in \mathbb{V} . We will assume Condition (I) and the following condition for $\mathbb{K} = \Gamma$ to ensure $\zeta^p(\text{co } \Gamma) = \zeta^p(\Gamma)$ in Lemma 3.1.

Condition (IV) $\langle \mathbf{Q}^0, \mathbf{X} \rangle \geq 0$ for every $\mathbf{X} \in F_0(\mathbb{K})$.

Lemma 3.1. *Assume that Condition (I) holds. Then,*

$$\zeta_0^p(\mathbb{K}) = \zeta_0^p(\text{co } \mathbb{K}) = \begin{cases} 0 & \text{if Condition (IV) holds,} \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. We first prove that Condition (IV) is equivalent to the condition

$$\langle \mathbf{Q}^0, \mathbf{X} \rangle \geq 0 \text{ for every } \mathbf{X} \in F_0(\text{co } \mathbb{K}). \quad (12)$$

Since the condition above implies Condition (IV), we only need to show that Condition (IV) implies the condition above. Assume that $\mathbf{X} \in F_0(\text{co } \mathbb{K})$. Then there are $\mathbf{X}^i \in \mathbb{K}$ ($i = 1, 2, \dots, r$) such that

$$\begin{aligned} \mathbf{X} &= \sum_{i=1}^r \mathbf{X}^i, \quad 0 = \langle \mathbf{H}^0, \mathbf{X} \rangle = \sum_{i=1}^r \langle \mathbf{H}^0, \mathbf{X}^i \rangle, \\ 0 &= \langle \mathbf{Q}^k, \mathbf{X} \rangle = \sum_{i=1}^r \langle \mathbf{Q}^k, \mathbf{X}^i \rangle \quad (k = 1, 2, \dots, m). \end{aligned}$$

By Condition (I), we know that $\langle \mathbf{H}^0, \mathbf{X}^i \rangle \geq 0$ and $\langle \mathbf{Q}^k, \mathbf{X}^i \rangle \geq 0$ ($i = 1, 2, \dots, r$, $k = 1, 2, \dots, m$). Thus, each \mathbf{X}^i ($i = 1, 2, \dots, r$) satisfies

$$\mathbf{X}^i \in \mathbb{K}, \quad \langle \mathbf{H}^0, \mathbf{X}^i \rangle = 0, \quad \langle \mathbf{Q}^k, \mathbf{X}^i \rangle = 0 \quad (k = 1, 2, \dots, m),$$

or $\mathbf{X}^i \in F_0(\mathbb{K})$ ($i = 1, 2, \dots, r$). By Condition (IV), $\langle \mathbf{Q}^0, \mathbf{X} \rangle = \sum_{i=1}^r \langle \mathbf{Q}^0, \mathbf{X}^i \rangle \geq 0$ holds.

Since the objective function of the problem (11) is linear and its feasible region forms a cone, we know that $\zeta_0^p(\mathbb{K}) = 0$ or $-\infty$ and that $\zeta_0^p(\mathbb{K}) = 0$ if and only if the objective value is nonnegative for all feasible solutions, *i.e.*, Condition (IV) holds. Similarly, $\zeta_0^p(\text{co } \mathbb{K}) = 0$ or $-\infty$, and $\zeta_0^p(\text{co } \mathbb{K}) = 0$ if and only if the condition (12), which has been shown to be equivalent to Condition (IV), holds. \square

Before presenting the main result of this section, we show a simple illustrative example.

Example 3.1. Let $\mathbb{V} = \mathbb{R}^2$, $\mathbf{Q}^0 = (0, \alpha)$, $\mathbf{H}^0 = (1, 0)$, $m = 0$, and

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_2 - x_1 = 0 \text{ or } x_1 = 0\},$$

where α denotes a parameter to be specified. Then, COP (1) with $\mathbb{K} = \mathbf{\Gamma}$ is of the form

$$\begin{aligned}\zeta^p(\mathbf{\Gamma}) &= \inf \{ \alpha x_1 \mid (x_1, x_2) \in \mathbf{\Gamma}, \langle \mathbf{H}^0, (x_1, x_2) \rangle = x_1 = 1 \} \\ &= \inf \{ \alpha x_1 \mid (x_1, x_2) \in \mathbb{R}_+^2, x_2 - x_1 = 0, x_1 = 1 \}.\end{aligned}$$

Thus, the feasible region of COP (1) with $\mathbb{K} = \mathbf{\Gamma}$ consists of a single point $\mathbf{x} = (1, 1)$. We further see that

$$\begin{aligned}\Delta_1 &= \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_1 = x_2 = 1 \}, \\ \Delta_0 &= F_0(\mathbf{\Gamma}) = \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_1 = 0 \}, \\ \text{co } \Delta &= \Delta = \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_2 - x_1 = 0 \}, \\ \text{co } \mathbf{\Gamma} &= \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_2 - x_1 \geq 0 \}, \\ \zeta_0^p(\mathbf{\Gamma}) &= \inf \{ \alpha x_2 \mid (x_1, x_2) \in \mathbb{R}_+^2, x_1 = 0 \}, \\ \zeta^p(\text{co } \mathbf{\Gamma}) &= \inf \{ \alpha x_2 \mid (x_1, x_2) \in \mathbb{R}_+^2, x_2 - x_1 \geq 0, x_1 = 1 \}.\end{aligned}$$

If $\alpha < 0$, then Condition (IV) is not satisfied for $\mathbb{K} = \mathbf{\Gamma}$, and $-\infty = \zeta^p(\text{co } \mathbf{\Gamma}) < \zeta^p(\text{co } \Delta) = \zeta^p(\mathbf{\Gamma}) = \zeta^p(\Delta) = \alpha$. Otherwise, Condition (IV) is satisfied for $\mathbb{K} = \mathbf{\Gamma}$, and $\zeta^p(\text{co } \mathbf{\Gamma}) = \zeta^p(\text{co } \Delta) = \zeta^p(\mathbf{\Gamma}) = \zeta^p(\Delta) = \alpha$.

Theorem 3.1. *Suppose that Condition (I) holds for $\mathbb{K} = \mathbf{\Gamma}$ and that $\Delta_1 \neq \emptyset$. Then,*

- (i) $F(\mathbf{\Gamma}) = F(\Delta)$. Thus, $\zeta^p(\mathbf{\Gamma}) = \zeta^p(\Delta)$.
- (ii) $\text{co } F(\mathbf{\Gamma}) = F(\text{co } \Delta)$ and $\zeta^p(\mathbf{\Gamma}) = \zeta^p(\text{co } \Delta)$.
- (iii) $\zeta^p(\text{co } \mathbf{\Gamma}) = \zeta^p(\mathbf{\Gamma}) + \zeta_0^p(\mathbf{\Gamma})$.
- (iv) Assume that $\zeta^p(\mathbf{\Gamma})$ finite. Then,

$$\zeta^p(\text{co } \mathbf{\Gamma}) = \begin{cases} \zeta^p(\mathbf{\Gamma}) & \text{if Condition (IV) holds for } \mathbb{K} = \mathbf{\Gamma}, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. (i) Since $\mathbf{\Gamma} = \Delta \cup \Delta_0$ and $F(\Delta_0) = \emptyset$, we know that $F(\mathbf{\Gamma}) = F(\Delta \cup \Delta_0) = F(\Delta)$.

(ii) Since $F(\text{co } \Delta)$ is a convex set containing $F(\Delta)$ and $F(\mathbf{\Gamma}) = F(\Delta)$ by (i), we obtain that $\text{co } F(\mathbf{\Gamma}) = \text{co } F(\Delta) \subset F(\text{co } \Delta)$. To show the converse inclusion relation, suppose that $\mathbf{X} \in F(\text{co } \Delta)$. Then,

$$\begin{aligned}\mathbf{O} \neq \mathbf{X} &= \sum_{i=1}^r \mathbf{X}^i, \quad \mathbf{O} \neq \mathbf{X}^i \in \Delta \quad (i = 1, 2, \dots, r), \\ 1 = \langle \mathbf{H}^0, \mathbf{X} \rangle &= \sum_{i=1}^r \langle \mathbf{H}^0, \mathbf{X}^i \rangle, \quad 0 = \langle \mathbf{Q}^k, \mathbf{X} \rangle = \sum_{i=1}^r \langle \mathbf{Q}^k, \mathbf{X}^i \rangle \quad (k = 1, 2, \dots, m)\end{aligned}$$

for some $\mathbf{X}^i \in \Delta$ ($i = 1, 2, \dots, m$). By the definition of Δ , $\mathbf{O} \neq \mathbf{X}^i = \mu^i \mathbf{Y}^i$ for some $\mu^i > 0$ and $\mathbf{Y}^i \in \Delta_1$ ($i = 1, 2, \dots, r$). Thus,

$$\begin{aligned}\mathbf{X} &= \sum_{i=1}^r \mu^i \mathbf{Y}^i, \quad \mathbf{Y}^i \in \Delta_1 \subset \mathbf{\Gamma}, \quad \langle \mathbf{H}^0, \mathbf{Y}^i \rangle = 1 \quad (i = 1, 2, \dots, r), \\ 1 = \sum_{i=1}^r \langle \mathbf{H}^0, \mathbf{X}^i \rangle &= \sum_{i=1}^r \mu^i, \quad 0 = \sum_{i=1}^r \mu^i \langle \mathbf{Q}^k, \mathbf{Y}^i \rangle \quad (k = 1, 2, \dots, m).\end{aligned}$$

Since $\mathbf{Q}^k \in \Gamma^*$ ($k = 1, 2, \dots, m$) by Condition (I) and $\mathbf{Y}^i \in \Gamma$ ($i = 1, 2, \dots, r$), we see that $\langle \mathbf{Q}^k, \mathbf{Y}^i \rangle \geq 0$ ($k = 1, 2, \dots, m, i = 1, 2, \dots, r$). Hence the last equality above is equivalent to $\langle \mathbf{Q}^k, \mathbf{Y}^i \rangle = 0$ ($k = 1, 2, \dots, m, i = 1, 2, \dots, r$). It follows that $\mathbf{Y}^i \in F(\Gamma)$ ($i = 1, 2, \dots, r$) and $\mathbf{X} \in \text{co } F(\Gamma) = \text{co } F(\Delta)$. Thus, we have shown that $\text{co } F(\Delta) \supset F(\text{co } \Delta)$. Since the objective function is linear, we obtain that

$$\begin{aligned} \zeta^p(\text{co } \Delta) &= \inf \{ \langle \mathbf{Q}_0, \mathbf{X} \rangle \mid \mathbf{X} \in F(\text{co } \Delta) \} \\ &= \inf \{ \langle \mathbf{Q}_0, \mathbf{X} \rangle \mid \mathbf{X} \in \text{co } F(\Gamma) \} = \zeta^p(\Gamma). \end{aligned}$$

(iii) We first prove that

$$\text{co } \Gamma = \text{co } \Delta + \text{co } \Delta_0. \quad (13)$$

Recall that Γ , Δ and Δ_0 are all nonempty cones. Let $\mathbf{X} \in \text{co } \Delta + \text{co } \Delta_0$. Then there exist $\mathbf{Y}^i \in \Delta$ ($i = 1, 2, \dots, p$) and $\mathbf{Z}^j \in \Delta_0$ ($j = 1, 2, \dots, q$) such that $\mathbf{X} = \sum_{i=1}^p \mathbf{Y}^i + \sum_{j=1}^q \mathbf{Z}^j$. Since $\Gamma = \Delta \cup \Delta_0$, we obtain that $\mathbf{X} \in \text{co } \Gamma$. Now suppose that $\mathbf{X} \in \text{co } \Gamma$. Then there exist $\mathbf{X}^i \in \Gamma$ ($i = 1, 2, \dots, p$) such that $\mathbf{X} = \sum_{i=1}^p \mathbf{X}^i$. Let $I_+ = \{i : \langle \mathbf{H}^0, \mathbf{X}^i \rangle > 0\}$ and $I_0 = \{i : \langle \mathbf{H}^0, \mathbf{X}^i \rangle = 0\}$. Since $\langle \mathbf{H}^0, \mathbf{X}^i \rangle \geq 0$ ($i = 1, 2, \dots, p$) by Condition (I), we know that $I_+ \cup I_0 = \{1, 2, \dots, p\}$ and $I_+ \cap I_0 = \emptyset$. Hence $\mathbf{X} = \sum_{i \in I_+} \mathbf{X}^i + \sum_{j \in I_0} \mathbf{X}^j$. We also see that $\mathbf{X}^i = \langle \mathbf{H}^0, \mathbf{X}^i \rangle (\mathbf{X}^i / \langle \mathbf{H}^0, \mathbf{X}^i \rangle) \in \Delta$ ($i \in I_+$) and $\mathbf{X}^j \in \Delta_0$ ($j \in I_0$). Therefore, $\mathbf{X} \in \text{co } \Delta + \text{co } \Delta_0$, and we have shown (13). We now observe that

$$\begin{aligned} \zeta^p(\text{co } \Gamma) &= \inf \left\{ \langle \mathbf{Q}^0, \mathbf{X} \rangle \mid \begin{array}{l} \mathbf{X} \in \text{co } \Gamma, \langle \mathbf{H}^0, \mathbf{X} \rangle = 1, \\ \langle \mathbf{Q}^k, \mathbf{X} \rangle = 0 \ (k = 1, 2, \dots, m) \end{array} \right\} \\ &= \inf \left\{ \langle \mathbf{Q}^0, \mathbf{Y} + \mathbf{Z} \rangle \mid \begin{array}{l} \mathbf{Y} \in \text{co } \Delta, \mathbf{Z} \in \text{co } \Delta_0, \\ \langle \mathbf{H}^0, \mathbf{Y} + \mathbf{Z} \rangle = 1, \\ \langle \mathbf{Q}^k, \mathbf{Y} + \mathbf{Z} \rangle = 0 \ (k = 1, 2, \dots, m) \end{array} \right\} \quad (\text{by (13)}) \\ &= \inf \left\{ \langle \mathbf{Q}^0, \mathbf{Y} + \mathbf{Z} \rangle \mid \begin{array}{l} \mathbf{Y} \in \text{co } \Delta, \mathbf{Z} \in \text{co } \Delta_0, \\ \langle \mathbf{H}^0, \mathbf{Y} \rangle = 1, \\ \langle \mathbf{Q}^k, \mathbf{Y} \rangle = 0, \langle \mathbf{Q}^k, \mathbf{Z} \rangle = 0 \ (k = 1, 2, \dots, m) \end{array} \right\} \\ &\quad (\text{since } \mathbf{Y} \in \text{co } \Delta \subset \text{co } \Gamma, \mathbf{Z} \in \text{co } \Delta_0 \subset \text{co } \Gamma \\ &\quad \text{and } \mathbf{Q}^k \in \Gamma^* \ (k = 1, 2, \dots, m)) \\ &= \zeta^p(\text{co } \Delta) + \inf \left\{ \langle \mathbf{Q}^0, \mathbf{Z} \rangle \mid \mathbf{Z} \in \text{co } \Delta_0, \langle \mathbf{Q}^k, \mathbf{Z} \rangle = 0 \ (k = 1, 2, \dots, m) \right\} \\ &= \zeta^p(\text{co } \Delta) + \zeta_0^p(\text{co } \Delta_0) \\ &\quad (\text{since } \langle \mathbf{H}^0, \mathbf{Z} \rangle = 0 \text{ for every } \mathbf{Z} \in \text{co } \Delta_0) \\ &= \zeta^p(\Gamma) + \zeta_0^p(\Delta_0) \quad (\text{by assertion (ii) and Lemma 3.1}) \\ &= \zeta^p(\Gamma) + \zeta_0^p(\Gamma) \quad (\text{by the definitions of } \Delta_0 \text{ and } \zeta_0^p(\mathbb{K})). \end{aligned}$$

(iv) follows from assertion (iii) and Lemma 3.1. \square

In [2, 4], the following condition was assumed for nonconvex COP induced from various classes of QOPs and POPs, in addition to Condition (I).

Condition (IV)'

$$F_0(\mathbb{K}) \subset F(\mathbb{K})^\infty = \left\{ \mathbf{D} \in \mathbb{V} : \begin{array}{l} \exists (\mu_r, \mathbf{X}_r) \in \mathbb{R}_+ \times F(\mathbb{K}) \ (r = 1, 2, \dots,); \\ (\mu_r, \mu_r \mathbf{X}_r) \rightarrow (0, \mathbf{D}) \text{ as } r \rightarrow \infty \end{array} \right\}$$

(the horizontal cone generated by $F(\mathbb{K})$)

Assume that Condition (I) holds for $\mathbb{K} = \mathbf{\Gamma}$ and that $\mathbf{\Delta}_1 \neq \emptyset$. Then Condition (IV) is a necessary and sufficient for the identity $\zeta^p(\mathbb{K}) = \zeta^p(\text{co } \mathbb{K})$, while Condition (IV)' is merely sufficient for the identity. Thus, Condition (IV)' implies Condition (IV). In particular, if Condition (IV)' holds, then Condition (IV) is satisfied for any $\mathbf{Q}^0 \in \mathbb{V}$. In fact, we can prove the following lemma.

Lemma 3.2. *Suppose that $\zeta^p(\mathbb{K})$ is finite. Then, Condition (IV)' implies Condition (IV) for any $\mathbf{Q}^0 \in \mathbb{V}$.*

Proof. Assume on the contrary that $\langle \mathbf{Q}^0, \mathbf{D} \rangle < 0$ for some $\mathbf{O} \neq \mathbf{D} \in F_0(\mathbb{K})$. By Condition (IV)', there exists a sequence $\{(\mu_r, \mathbf{X}_r) \in \mathbb{R}_+ \times F(\mathbb{K}) : r = 1, 2, \dots\}$ such that $(\mu_r, \mu_r \mathbf{X}_r)$ converges to $(0, \mathbf{D})$ as $r \rightarrow \infty$. Thus, there is a positive number δ such that $\langle \mathbf{Q}^0, \mu_r \mathbf{X}_r \rangle < -\delta$ for every sufficiently large r . Therefore, $\langle \mathbf{Q}^0, \mathbf{X}_r \rangle < -\delta/\mu_r \rightarrow -\infty$ along a sequence $\{\mathbf{X}_r : r = 1, 2, \dots\}$ of feasible solutions of COP (1). But this contradicts the assumption that $\zeta^p(\mathbb{K})$ is finite. \square

By (ii) of Theorem 3.1, we may replace the cone $\mathbb{K} = \mathbf{\Gamma}$ by $\mathbb{K} = \mathbf{\Delta}$ in COP (1) to have an equivalent COP whose convexification attains the same optimal objective value $\zeta^p(\mathbf{\Gamma})$ without assuming Condition (IV) for the cone $\mathbb{K} = \mathbf{\Gamma}$. We note, however, that even when $\mathbf{\Gamma}$ is closed, neither $\mathbf{\Delta}$ nor $\text{co } \mathbf{\Delta}$ is closed in general.

The results in this section are summarized as the following theorem.

Theorem 3.2. $\zeta^p(\mathbb{K}) = \zeta^p(\text{co } \mathbb{K})$ under Conditions (I) and (IV).

4 Numerical methods for solving the primal dual-pair of COPs (6) and (7)

In this section, we take \mathbb{K} to be a closed convex cone. For every $\mathbf{G} \in \mathbb{V}$, let $\Pi(\mathbf{G})$ and $\Pi^*(\mathbf{G})$ denote the metric projection of \mathbf{G} onto the cone \mathbb{K} and \mathbb{K}^* , respectively:

$$\begin{aligned} \Pi(\mathbf{G}) &= \operatorname{argmin} \{ \|\mathbf{G} - \mathbf{X}\| \mid \mathbf{X} \in \mathbb{K} \}, \\ \Pi^*(\mathbf{G}) &= \operatorname{argmin} \{ \|\mathbf{G} - \mathbf{Z}\| \mid \mathbf{Z} \in \mathbb{K}^* \}. \end{aligned}$$

In addition to Conditions (I), (II) and (III), we assume the following condition throughout this section.

Condition (V) For every $\mathbf{G} \in \mathbb{V}$, $\Pi(\mathbf{G})$ can be computed.

Under these four conditions, we briefly present two types of numerical methods for solving the primal dual-pair of COPs (6) and (7) with a fixed λ . The first is based on a bisection method, which was proposed in [19], and the second is an 1-dimensional Newton method, which is newly proposed in this paper.

Remark 4.1. When the unified framework is applied to QOPs in Section 5, and to POPs in Section 4 of Part II [5], the cone \mathbb{K} is given as the intersection of two closed convex cones \mathbb{K}_1 and \mathbb{K}_2 in the space of symmetric matrices. In such cases, we can utilize the accelerated proximal gradient method [6] to compute the metric projection onto $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$ based on those metric projections onto \mathbb{K}_1 and \mathbb{K}_2 as in Algorithm C of [19].

Let $\lambda \in \mathbb{R}$ be fixed arbitrarily. For every $y_0 \in \mathbb{R}$, define

$$\begin{aligned} \mathbf{G}_\lambda(y_0) &= \mathbf{Q}^0 + \lambda \mathbf{H}^1 - y_0 \mathbf{H}^0, \\ g_\lambda(y_0) &= \min \{ \|\mathbf{G}_\lambda(y_0) - \mathbf{Z}\| \mid \mathbf{Z} \in \mathbb{K}^* \}, \\ \widehat{\mathbf{Z}}_\lambda(y_0) &= \Pi^*(\mathbf{G}_\lambda(y_0)), \quad \widehat{\mathbf{X}}_\lambda(y_0) = \Pi(-\mathbf{G}_\lambda(y_0)) \end{aligned}$$

By the decomposition theorem of Moreau [21], we know that

$$\widehat{\mathbf{Z}}_\lambda(y_0) - \widehat{\mathbf{X}}_\lambda(y_0) = \mathbf{G}_\lambda(y_0) \quad \text{and} \quad \langle \widehat{\mathbf{X}}_\lambda(y_0), \widehat{\mathbf{Z}}_\lambda(y_0) \rangle = 0; \quad (14)$$

hence

$$g_\lambda(y_0) = \|\mathbf{G}_\lambda(y_0) - \widehat{\mathbf{Z}}_\lambda(y_0)\| = \|\widehat{\mathbf{X}}_\lambda(y_0)\|, \quad (15)$$

$$(g_\lambda(y_0))^2 = \|\widehat{\mathbf{X}}_\lambda(y_0)\|^2 = \langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(y_0) \rangle y_0 - \langle \mathbf{Q}^0 + \lambda \mathbf{H}^1, \widehat{\mathbf{X}}_\lambda(y_0) \rangle \quad (16)$$

hold for every y_0 . By definition, $g_\lambda(y_0) \geq 0$ for every $y_0 \in \mathbb{R}$, and y_0 is a feasible solution of COP (7) if and only if $g_\lambda(y_0) = 0$. Therefore we can rewrite COP (7) as

$$\eta^d(\lambda, \mathbb{K}) := \sup \{ y_0 \mid g_\lambda(y_0) = 0 \}.$$

Thus we can easily design a standard bracketing and bisection method for computing $\eta^d(\lambda, \mathbb{K})$. The details are omitted here. See [19].

To describe the 1-dimensional Newton method for computing $\eta^d(\lambda, \mathbb{K})$, we need the following lemma, which exhibits some fundamental properties of the function g_λ .

Lemma 4.1. *Let $\lambda \in \mathbb{R}$ be fixed.*

(i) $g_\lambda : \mathbb{R} \rightarrow \mathbb{R}_+$ is continuous and convex.

(ii) Assume that $-\infty < \eta^d(\lambda, \mathbb{K}) < y_0$. Then $\langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(y_0) \rangle > 0$.

(iii) Assume that $-\infty < \eta^d(\lambda, \mathbb{K}) < y_0$. Then $dg_\lambda(y_0)/dy_0 = \langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(y_0) \rangle / g_\lambda(y_0) > 0$; hence $g_\lambda : (\eta^d(\lambda, \mathbb{K}), \infty) \rightarrow \mathbb{R}$ is continuously differentiable and strictly increasing.

Proof. Consider the distance function $\theta(\mathbf{x}) = \min \{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in C \}$ from $\mathbf{x} \in \mathbb{V}$ to a closed convex subset C of \mathbb{V} and the metric projection $\mathbf{P}(\mathbf{x}) = \operatorname{argmin} \{ \|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in C \}$ of $\mathbf{x} \in \mathbb{V}$ onto C in general. It is well-known and also easily proved that θ is convex and continuous (see for example [17, 31]). It is also known that $\theta^2(\cdot)$ is continuously differentiable with $\nabla \theta^2(\mathbf{x}) = 2(\mathbf{x} - \mathbf{P}(\mathbf{x}))$ (see for example [26, Proposition 2.2]).

Since $g_\lambda(y_0) = \theta(\mathbf{G}_\lambda(y_0))$ and $\mathbf{G}_\lambda(y_0)$ is linear with respect to $y_0 \in \mathbb{R}$, the assertion (i) follows. In addition, we have that

$$\frac{dg_\lambda^2(y_0)}{dy_0} = 2\langle \mathbf{G}_\lambda(y_0) - \Pi^*(\mathbf{G}_\lambda(y_0)), -\mathbf{H}^0 \rangle = 2\langle \widehat{\mathbf{X}}_\lambda(y_0), \mathbf{H}^0 \rangle. \quad (17)$$

Next we prove assertion (ii) for $y_0 > \eta^d(\lambda, \mathbb{K})$. Note that by the definition of $\eta^d(\lambda, \mathbb{K})$, $g_\lambda(y_0) > 0$. Assume on the contrary that $\langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(y_0) \rangle = 0$. Then we see by (16) that

$$\langle \mathbf{Q}^0 + \lambda \mathbf{H}^1, \widehat{\mathbf{X}}_\lambda(y_0) \rangle = -\|\widehat{\mathbf{X}}_\lambda(y_0)\|^2 = -g_\lambda(y_0)^2 < 0.$$

Hence $\widehat{\mathbf{X}}_\lambda(y_0) \neq \mathbf{O}$ is a direction along which the objective function of (7) tends to $-\infty$. This contradicts to the assumption $-\infty < \eta^d(\lambda, \mathbb{K})$. Therefore $\langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(y_0) \rangle > 0$.

Finally we prove assertion (iii). Again, note that $g_\lambda(y_0) > 0$ for $y_0 > \eta^d(\lambda, \mathbb{K})$. By (17), we get

$$\frac{dg_\lambda(y_0)}{dy_0} = \frac{d\sqrt{g_\lambda^2(y_0)}}{dy_0} = \frac{1}{2g_\lambda(y_0)} \frac{dg_\lambda^2(y_0)}{dy_0} = \frac{\langle \widehat{\mathbf{X}}_\lambda(y_0), \mathbf{H}^0 \rangle}{g_\lambda(y_0)} > 0.$$

From here, the remaining assertions follow. \square

Suppose that $g_\lambda(\bar{y}_0) > 0$ for some $\bar{y}_0 \in \mathbb{R}$. Then the Newton iteration for computing $\eta^d(\lambda, \mathbb{K})$ is given by

$$\begin{aligned} \bar{y}_0^+ &= \bar{y}_0 - \frac{g_\lambda(\bar{y}_0)}{dg_\lambda(\bar{y}_0)/dy_0} \\ &= \bar{y}_0 - \frac{\langle \widehat{\mathbf{X}}_\lambda(\bar{y}_0), \widehat{\mathbf{X}}_\lambda(\bar{y}_0) \rangle}{\langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(\bar{y}_0) \rangle} \\ &= \bar{y}_0 - \frac{\langle \widehat{\mathbf{Z}}(\bar{y}_0) + \bar{y}_0 \mathbf{H}^0 - \mathbf{Q}^0 - \lambda \mathbf{H}^1, \widehat{\mathbf{X}}_\lambda(\bar{y}_0) \rangle}{\langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(\bar{y}_0) \rangle} \quad (\text{by (14)}) \\ &= \left\langle \mathbf{Q}^0 + \lambda \mathbf{H}^1, \widetilde{\mathbf{X}}_\lambda(\bar{y}_0) \right\rangle \geq \eta^p(\lambda, \mathbb{K}). \end{aligned}$$

where $\widetilde{\mathbf{X}}_\lambda(\bar{y}_0) = \frac{\widehat{\mathbf{X}}_\lambda(\bar{y}_0)}{\langle \mathbf{H}^0, \widehat{\mathbf{X}}_\lambda(\bar{y}_0) \rangle}$ denotes a feasible solution of the primal COP (6). Thus, using this Newton iteration formula, we can design a numerical method for computing optimal solutions of the primal-dual pair of COPs (6) and (7). The details are omitted here.

5 Applications to a class of quadratic optimization problems

Here we take \mathbb{V} to be the linear space of $(1+n) \times (1+n)$ symmetric matrices \mathbb{S}^{1+n} with the inner product $\langle \mathbf{Q}, \mathbf{X} \rangle = \text{Trace } \mathbf{Q}^T \mathbf{X} = \sum_{i=0}^n \sum_{j=0}^n Q_{ij} X_{ij}$. We assume that the row and column indices of each matrix in \mathbb{S}^{1+n} range from 0 to n . We are particularly interested in

the following cones in the space of \mathbb{S}^{1+n} :

$$\begin{aligned}
\mathbb{S}_+^{1+n} &= \text{the cone of positive semidefinite matrices in } \mathbb{S}^{1+n}, \\
\mathbb{N}^{1+n} &= \{ \mathbf{X} \in \mathbb{S}^{1+n} : X_{ij} \geq 0 \ (1 \leq i \leq j \leq 1+n) \}, \\
\mathbb{C}^{1+n} &= \{ \mathbf{A} \in \mathbb{S}^{1+n} : \langle \mathbf{A}, \mathbf{x}\mathbf{x}^T \rangle \geq 0 \text{ for every } \mathbf{x} \in \mathbb{R}^{1+n} \} \\
&= \text{(the copositive cone),} \\
(\mathbb{C}^{1+n})^* &= \left\{ \sum_{i=1}^r \mathbf{x}_i \mathbf{x}_i^T : \mathbf{x}_i \in \mathbb{R}_+^{1+n} \ (i = 1, 2, \dots, r), \ r \in \mathbb{Z}_+ \right\} \\
&= \text{(the completely positive (CPP) cone),} \\
\mathbb{D}^{1+n} &= \mathbb{S}_+^{1+n} \cap \mathbb{N}^{1+n} \text{ (the doubly nonnegative cone).}
\end{aligned}$$

5.1 A class of linearly constrained quadratic optimization problems with complementarity constraints

We first introduce the following linearly constrained QOP with complementarity constraints [19] (see also [2, 3, 8]):

$$\zeta^* = \inf \left\{ \mathbf{u}^T \mathbf{Q} \mathbf{u} + 2\mathbf{c}^T \mathbf{u} \mid \begin{array}{l} \mathbf{u} \in \mathbb{R}_+^n, \ \mathbf{A}\mathbf{u} + \mathbf{b} = \mathbf{0}, \\ u_i u_j = 0 \ ((i, j) \in \mathcal{E}) \end{array} \right\}, \quad (18)$$

where $\mathbf{A} \in \mathbb{R}^{q \times n}$, $\mathbf{b} \in \mathbb{R}^q$, $\mathbf{c} \in \mathbb{R}^n$ and $\mathcal{E} \subset \{(i, j) : 1 \leq i < j \leq n\}$ are given data. Since the binary constraint $u_i(1 - u_i) = 0$ can be converted to a complementarity constraint $u_i v_i = 0$ with a slack variable $v_i = 1 - u_i \geq 0$, QOP (18) can represent nonconvex QOPs with linear, binary, and complementarity constraints [3, 8].

5.2 Representing QOP (18) as a conic LOP over a nonconvex cone and its convexification

Let

$$\mathbf{Q}^0 = \begin{pmatrix} 0 & \mathbf{c} \\ \mathbf{c} & \mathbf{Q} \end{pmatrix} \in \mathbb{S}^{1+n}, \quad \mathbf{Q}^1 = \begin{pmatrix} \mathbf{b}^T \mathbf{b} & \mathbf{b}^T \mathbf{A} \\ \mathbf{A}^T \mathbf{b} & \mathbf{A}^T \mathbf{A} \end{pmatrix} \in \mathbb{S}^{1+n}, \quad (19)$$

\mathbf{C}^{ij} = the $n \times n$ matrix with 1 at the (i, j) th element and 0 elsewhere $((i, j) \in \mathcal{E})$,

$$\mathbf{Q}^{ij} = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{C}^{ij} + (\mathbf{C}^{ij})^T \end{pmatrix} \in \mathbb{S}^{1+n} \ ((i, j) \in \mathcal{E}),$$

\mathbf{H}^0 = the $(1+n) \times (1+n)$ matrix whose (i, j) th element H_{ij}^0 is given by

$$H_{ij}^0 = \begin{cases} 1 & \text{if } i = j = 0 \\ 0 & \text{otherwise} \end{cases} \quad (0 \leq i, j \leq n),$$

$$\Delta_1 = \left\{ \mathbf{U} = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}^T = \begin{pmatrix} 1 & \mathbf{u}^T \\ \mathbf{u} & \mathbf{u}\mathbf{u}^T \end{pmatrix} \in \mathbb{S}^{1+n} : \mathbf{u} \in \mathbb{R}_+^n \right\}.$$

We renumber the superscript ij of \mathbf{Q}^{ij} ($(i, j) \in \mathcal{E}$) to $2, \dots, m$ for some m . Then, we can rewrite QOP (18) as follows:

$$\zeta^* = \inf \left\{ \langle \mathbf{Q}^0, \mathbf{U} \rangle \mid \mathbf{U} \in \Delta_1, \langle \mathbf{Q}^k, \mathbf{U} \rangle = 0 \ (k = 1, 2, \dots, m) \right\}. \quad (20)$$

By definition, we know that

$$\mathbf{O} \neq \mathbf{H}^0 \in \mathbb{S}_+^{1+n} + \mathbb{N}^{1+n} = (\mathbb{D}^{1+n})^*, \quad \mathbf{Q}^k \in (\mathbb{D}^{1+n})^* \ (k = 1, 2, \dots, m). \quad (21)$$

The set Δ_1 can be embedded in a nonconvex cone in two different ways. First, we can simply take the conic hull of Δ_1 , *i.e.*, $\Delta = \{\lambda \mathbf{U} : \mathbf{U} \in \Delta_1, \lambda \geq 0\}$, and alternatively, we can homogenize Δ_1 as

$$\Gamma = \left\{ \mathbf{X} = \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix}^T = \begin{pmatrix} x_0^2 & x_0 \mathbf{x}^T \\ x_0 \mathbf{x} & \mathbf{x} \mathbf{x}^T \end{pmatrix} \in \mathbb{S}^{1+n} : \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \in \mathbb{R}_+^{1+n} \right\}.$$

Obviously, both Δ and Γ are cones in \mathbb{S}^{1+n} .

Now, we consider the COP (1) for $\mathbb{K} = \Delta, \Gamma, \text{co } \Delta,$ and $\text{co } \Gamma$. The feasible regions $F(\Gamma)$ and $F(\Delta)$ of the first two COPs coincide with the feasible region of QOP (20), resulting in $\zeta^p(\Gamma) = \zeta^p(\Delta) = \zeta^*$. The third and fourth COPs with convex feasible regions $F(\text{co } \Delta)$ and $F(\text{co } \Gamma)$ correspond to their convexifications. In particular, $\text{co } \Gamma$ coincides with *the completely positive programming* (CPP) cone $(\mathbb{C}^{1+n})^*$ and the COP with $\mathbb{K} = \text{co } \Gamma = (\mathbb{C}^{1+n})^*$ is called *a completely positive programming* (CPP) relaxation of QOP (18) (or QOP (20)) [2, 3, 8, 19]. Since $\text{co } \Gamma \supset \Gamma$, $\zeta^p(\text{co } \Gamma)$ always provides a lower bound for the optimal value ζ^* of QOP (20).

Next we describe a condition that characterizes the equivalence between $\zeta^p(\text{co } \Gamma)$ and ζ^* . By construction, $\Delta, \Gamma, \text{co } \Delta, \text{co } \Gamma \subset \mathbb{D}^{1+n}$. From (21), we see that Condition (I) is satisfied for $\mathbb{K} = \Delta, \Gamma, \text{co } \Delta, \text{co } \Gamma$. Thus, we can consistently define the simplified COP (3), the Lagrangian-conic relaxation (6) and their duals for $\mathbb{K} = \Delta, \Gamma, \text{co } \Delta, \text{co } \Gamma$, and apply 2.1 and 2.2.

Remark 5.1. In the previous discussion, we introduced the set Δ_1 to describe the problem (20) that is equivalent to QOP (18), and then defined the cones Δ and Γ . We can define, however, the cone Γ first, and then define Δ_1 and Δ by (10). In the process, we can apply the discussions in Section 2.4.

To present main results of this section, we consider the following problem:

$$\zeta_0^* = \inf \left\{ \mathbf{u}^T \mathbf{Q} \mathbf{u} \mid \begin{array}{l} \mathbf{u} \in \mathbb{R}_+^n, \mathbf{A} \mathbf{u} = \mathbf{0}, \\ u_i u_j = 0 \ ((i, j) \in \mathcal{E}) \end{array} \right\}. \quad (22)$$

The set of feasible solutions of this problem forms a cone in \mathbb{R}_+^n . Hence, we see that $\zeta_0^* = 0$ or $\zeta_0^* = -\infty$, and that $\zeta_0^* = 0$ if and only if

$$\mathbf{u}^T \mathbf{Q} \mathbf{u} \geq 0 \text{ for every feasible solution } \mathbf{u} \text{ of (22)}. \quad (23)$$

Assume that the set $\{\mathbf{u} \in \mathbb{R}_+^n : \mathbf{A} \mathbf{u} + \mathbf{b} = \mathbf{0}\}$ of \mathbf{u} satisfying the linear constraints in QOP (18) is bounded. Then $\{\mathbf{u} \in \mathbb{R}_+^n : \mathbf{A} \mathbf{u} = \mathbf{0}\} = \{\mathbf{0}\}$, which implies the condition (23). Furthermore, for every cone $\mathbb{K} \subset \mathbb{S}^{1+n}$ satisfying $\mathbb{K} \subset \mathbb{D}^{1+n}$, we see that

$$\begin{aligned} \mathbf{O} &\in \left\{ \mathbf{X} \in \mathbb{K} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 0, \langle \mathbf{Q}^k, \mathbf{X} \rangle = 0 \ (k = 1, 2, \dots, m) \right\} \\ &\subset \left\{ \mathbf{X} \in \mathbb{D}^{1+n} : \langle \mathbf{H}^0, \mathbf{X} \rangle = 0, \langle \mathbf{Q}^k, \mathbf{X} \rangle = 0 \ (k = 1, 2, \dots, m) \right\} = \{\mathbf{O}\}, \end{aligned} \quad (24)$$

which implies that the feasible region $F(\mathbb{K})$ of COP (1) is bounded. As a result, Condition (III) holds. (see Lemma 2.1 of [19] and its proof for the last identity of (24)).

Lemma 5.1.

(i) $\zeta^p(\text{co } \mathbf{\Delta}) = \zeta^*$.

(ii) Assume that ζ^* is finite. Then,

$$\zeta^p(\text{co } \mathbf{\Gamma}) = \zeta^* + \zeta_0^* = \begin{cases} \zeta^* & \text{if the condition (23) holds,} \\ -\infty & \text{otherwise.} \end{cases}$$

(iii) Assume that the set $\{\mathbf{u} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{u} + \mathbf{b} = \mathbf{0}\}$ is bounded and the feasible region of QOP (18) is nonempty. Then,

$$(\eta^d(\lambda, \text{co } \mathbf{\Gamma}) = \eta^p(\lambda, \text{co } \mathbf{\Gamma}))\uparrow = \zeta^d(\text{co } \mathbf{\Gamma}) = \zeta^p(\text{co } \mathbf{\Gamma}) = \zeta^*.$$

Proof. (i) We have already seen $\zeta^* = \zeta^p(\mathbf{\Gamma})$. Hence, $\zeta^p(\text{co } \mathbf{\Delta}) = \zeta^*$ follows from (ii) of Theorem 3.1.

(ii) We apply (iv) of Lemma 3.1. Observe that $\mathbf{X} \in \mathbf{\Gamma}$ and $\langle \mathbf{H}^0, \mathbf{X} \rangle = 0$ if and only if $\mathbf{X} = \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{u} \end{pmatrix}^T$ for some $\mathbf{u} \in \mathbb{R}_+^n$. With this correspondence, we see that $\mathbf{u}^T \mathbf{Q}\mathbf{u} = \langle \mathbf{Q}^0, \mathbf{X} \rangle$, and that \mathbf{u} is a feasible solution of (22) if and only if \mathbf{X} is a feasible solution of (11) with $\mathbb{K} = \mathbf{\Gamma}$. Thus, $\zeta_0^* = \zeta_0^p(\mathbf{\Gamma})$ and the condition (23) corresponds to Condition (IV) with $\mathbb{K} = \mathbf{\Gamma}$. The desired result follows from (iv) of Theorem 3.1.

(iii) The CPP cone $\text{co } \mathbf{\Gamma} = (\mathbb{C}^{1+n})^*$ is known to be a closed convex cone, thus, Condition (II) holds for $\mathbb{K} = \text{co } \mathbf{\Gamma}$. We have observed that the assumption implies that Condition (III) holds for $\mathbb{K} = \mathbf{\Gamma}$ and that $F(\text{co } \mathbb{K})$ is bounded. The desired result follows from (iii) of Theorem 2.1 and assertion (ii). \square

5.3 DNN and Lagrangian-DNN relaxations

In this subsection, we present DNN and Lagrangian-DNN relaxations of QOP (20). First, we let \mathbb{K} be the doubly nonnegative cone $\mathbb{D}^{1+n} = \mathbb{S}_+^{1+n} \cap \mathbb{N}^{1+n}$. Note the relation

$$\text{co } \mathbf{\Gamma} = (\mathbb{C}^{1+n})^* \subset \mathbb{D}^{1+n} \subset \mathbb{S}_+^{1+n} + \mathbb{N}^{1+n} = (\mathbb{D}^{1+n})^* \subset \mathbb{C}^{1+n}.$$

Hence, $\zeta^p(\mathbb{D}^{1+n}) \leq \zeta^p((\mathbb{C}^{1+n})^*) \leq \zeta^*$. By (21), Condition (I) is satisfied for $\mathbb{K} = \mathbb{D}^{1+n}$. As a result, we can introduce the simplified COP (3), the Lagrangian-conic relaxation (6) and their duals for $\mathbb{K} = \mathbb{D}^{1+n}$.

Lemma 5.2.

(i) Assume that the set $\{\mathbf{u} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{u} + \mathbf{b} = \mathbf{0}\}$ is bounded. Then,

$$\zeta^* \geq \zeta^d(\mathbb{D}^{1+n}) = \zeta^p(\mathbb{D}^{1+n}) = (\eta^d(\lambda, \mathbb{D}^{1+n}) = \eta^p(\lambda, \mathbb{D}^{1+n}))\uparrow$$

(ii) Assume that the condition (23) does not hold. Then, $\zeta^p(\mathbb{D}^{1+n}) = -\infty$.

Proof. (i) Since Conditions (I), (II) and (III) hold for $\mathbb{K} = \mathbb{D}^{1+n}$, the desired result follows from Theorem 2.1. Note that Condition (III) was verified in the paragraph above Lemma 5.1.

(ii) Since $\text{co } \mathbf{\Gamma} \subset \mathbb{D}^{1+n}$, we know that $\zeta^* \geq \zeta^p(\text{co } \mathbf{\Gamma}) \geq \zeta^p(\mathbb{D}^{1+n})$. Hence $\zeta^p(\mathbb{D}^{1+n}) = -\infty$ follows from (ii) of Lemma 5.1. \square

In the recent paper [19], Kim, Kojima and Toh proposed a numerical method for QOPs of the form (18) based on its Lagrangian-DNN relaxation (6) with $\mathbb{K} = \mathbb{D}^{1+n}$ and a bisection method, combined with the proximal alternating direction multiplier method [11] and the accelerated proximal gradient method [6], for solving the relaxation problem. It was assumed in [19] that the linear constraint set $\{\mathbf{u} \in \mathbb{R}^n : \mathbf{A}\mathbf{u} + \mathbf{b} = \mathbf{0}\}$ is bounded. Thus, by Lemma 5.2, a common optimal value $\eta^p(\lambda, \mathbb{D}^{1+n}) = \eta^d(\lambda, \mathbb{D}^{1+n})$ of the primal-dual pair (6) and (7) of Lagrangian-DNN relaxation converges to the optimal value $\zeta^p(\mathbb{D}^{1+n})$ of the DNN relaxation of (3) (which is derived from the QOP (20)) as $\lambda \rightarrow \infty$. In addition, for each fixed Lagrange parameter $\lambda \in \mathbb{R}_+$, the primal problem (6) is an unconstrained problem with a DNN matrix variable whose upper left corner element is fixed to 1, and its dual, (7) with $\mathbb{K} = \mathbb{D}^{1+n}$, becomes a simple problem with just a single real variable. Furthermore, the primal problem, COP (6) with $\mathbb{K} = \mathbb{D}^{1+n}$ is strictly feasible (*i.e.*, its feasible region intersect with the interior of the DNN cone). These properties contributed to the effectiveness, efficiency, and stability of the numerical method proposed in [19]. Notice that Assertion (ii) of Lemma 5.2 shows that their method will not work if condition (23) does not hold.

6 Exploiting sparsity in the DNN and Lagrangian-DNN relaxations for QOP (18)

Let $N_0 = \{0, 1, \dots, n\}$. We say that a subset $\mathcal{G} \subset N_0 \times N_0$ is symmetric if $(i, j) \in \mathcal{G}$ implies $(j, i) \in \mathcal{G}$. For every symmetric subset \mathcal{G} of $N_0 \times N_0$ and every cone $\mathbb{J} \subset \mathbb{S}^{1+n}$, let

$$\begin{aligned} \mathcal{G}^c &= \{(i, j) \in N_0 \times N_0 : (i, j) \notin \mathcal{G}\}, \\ \mathbb{S}^{1+n}(\mathcal{G}, 0) &= \{\mathbf{X} \in \mathbb{S}^{1+n} : X_{ij} = 0 \text{ if } (i, j) \notin \mathcal{G}\}, \\ \mathbb{J}(\mathcal{G}, 0) &= \mathbb{J} \cap \mathbb{S}^{1+n}(\mathcal{G}, 0), \\ \mathbb{J}(\mathcal{G}, ?) &= \mathbb{J} + \mathbb{S}^{1+n}(\mathcal{G}^c, 0) \\ &= \{\mathbf{X} \in \mathbb{S}^{1+n} : X_{ij} = \bar{X}_{ij} \text{ } ((i, j) \in \mathcal{G}) \text{ for some } \bar{\mathbf{X}} \in \mathbb{J}\}. \end{aligned}$$

Obviously, $\mathbb{S}^{1+n}(\mathcal{G}, 0)$ forms a linear subspace of \mathbb{S}^{1+n} . $\mathbb{J}(\mathcal{G}, 0)$ and $\mathbb{J}(\mathcal{G}, ?)$ are cones in \mathbb{S}^{1+n} , and

$$\mathbb{S}^{1+n}(\mathcal{G}, 0)^\perp = \mathbb{S}^{1+n}(\mathcal{G}, 0)^* = \mathbb{S}^{1+n}(\mathcal{G}^c, 0) \quad \text{and} \quad \mathbb{J}(\mathcal{G}, 0) \subset \mathbb{J} \subset \mathbb{J}(\mathcal{G}, ?). \quad (25)$$

We use $\mathbb{J}^*(\mathcal{G}, 0)$ for $\mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)$, and $\mathbb{J}(\mathcal{G}, 0)^*$ for the dual of $\mathbb{J}(\mathcal{G}, 0)$.

Lemma 6.1. *Let \mathcal{G} be a symmetric subset $N_0 \times N_0$ and \mathbb{J} a cone in \mathbb{S}^{1+n} . Then, the following assertions hold.*

(i) $\mathbb{J}(\mathcal{G}, ?)^* = \mathbb{J}^*(\mathcal{G}, 0)$.

(ii) Moreover, if \mathbb{J} is a closed convex cone, then $(\mathbb{J}^*(\mathcal{G}, 0))^* = \text{cl}(\mathbb{J}(\mathcal{G}, ?))$.

Proof. (i) Suppose that $\mathbf{X} \in \mathbb{J}^*(\mathcal{G}, 0) = \mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)$. Then, for every $\mathbf{Y} + \mathbf{Z} \in \mathbb{J}(\mathcal{G}, ?)$ with $\mathbf{Y} \in \mathbb{J}$ and $\mathbf{Z} \in \mathbb{S}^{1+n}(\mathcal{G}^c, 0)$, we see that $\langle \mathbf{X}, \mathbf{Y} + \mathbf{Z} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle + \langle \mathbf{X}, \mathbf{Z} \rangle \geq 0$. Hence, $\mathbf{X} \in \mathbb{J}(\mathcal{G}, ?)^*$. Now suppose that $\mathbf{X} \in \mathbb{S}^{1+n}$ and $\mathbf{X} \notin \mathbb{J}^*(\mathcal{G}, 0) = \mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0)$. Then we have either

$$\mathbf{X} \notin \mathbb{J}^* \quad \text{and} \quad \mathbf{X} \in \mathbb{S}^{1+n}(\mathcal{G}, 0) \quad (26)$$

or

$$\mathbf{X} \notin \mathbb{S}^{1+n}(\mathcal{G}, 0), \quad \text{i.e.,} \quad X_{ij} \neq 0 \quad \text{for some} \quad (i, j) \in \mathcal{G}^c. \quad (27)$$

In the case of (26), there exists a $\mathbf{Y} \in \mathbb{J} \subset \mathbb{J}(\mathcal{G}, ?)$ such that $\langle \mathbf{X}, \mathbf{Y} \rangle < 0$. Thus, $\mathbf{X} \notin \mathbb{J}(\mathcal{G}, ?)^*$. For the case of (27), let $\mathbf{Y} \in \mathbb{S}^{1+n}$ be such that

$$Y_{ij} = \begin{cases} 0 & \text{if } (i, j) \in \mathcal{G}, \\ -X_{ij} & \text{if } (i, j) \in \mathcal{G}^c. \end{cases}$$

Then, $\mathbf{Y} \in \mathbb{J} + \mathbb{S}^{1+n}(\mathcal{G}^c, 0) = \mathbb{J}(\mathcal{G}, ?)$ and

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{(i,j) \in \mathcal{G}^c} X_{ij} Y_{ij} = \sum_{(i,j) \in \mathcal{G}^c} X_{ij} (-X_{ij}) = - \sum_{(i,j) \in \mathcal{G}^c} X_{ij}^2 < 0.$$

Consequently, $\mathbf{X} \notin \mathbb{J}(\mathcal{G}, ?)^*$.

(ii) It is known in general that $(A^*)^* = A$ and $(A \cap B)^* = \text{cl}(A^* + B^*)$ if A and B are closed convex cone in \mathbb{S}^{1+n} . Thus, we obtain that

$$\begin{aligned} (\mathbb{J}^*(\mathcal{G}, 0))^* &= (\mathbb{J}^* \cap \mathbb{S}^{1+n}(\mathcal{G}, 0))^* = \text{cl}((\mathbb{J}^*)^* + \mathbb{S}^{1+n}(\mathcal{G}, 0)^\perp) \\ &= \text{cl}(\mathbb{J} + \mathbb{S}^{1+n}(\mathcal{G}^c, 0)) = \text{cl}(\mathbb{J}(\mathcal{G}, ?)). \end{aligned}$$

□

6.1 Sparse DNN and Lagrangian relaxation

We can utilize the last inclusion relation $\mathbb{J} \subset \mathbb{J}(\mathcal{G}, ?)$ of (25) with $\mathbb{J} = \mathbb{D}^{1+n}$ to construct sparse DNN and Lagrangian-DNN relaxations of QOP (20). As seen in the previous section, COP (1) with $\mathbb{K} = \mathbb{D}^{1+n}$ serves as the DNN relaxation of QOP (20), hence, the DNN relaxation of QOP (18). If \mathcal{G} is a symmetric subset of $N_0 \times N_0$ and \mathbb{K}_1 is a convex cone in \mathbb{S}^{1+n} satisfying $\mathbb{D}^{1+n}(\mathcal{G}, ?) \subset \mathbb{K}_1$, then COP (1) with $\mathbb{K} = \mathbb{K}_1$ serves as a sparse DNN relaxation. To derive effective and efficient DNN and Lagrangian-DNN relaxations of QOP (20), some additional restrictions on \mathcal{G} and \mathbb{K}_1 are necessary. In particular, Condition (I) with $\mathbb{K} = \mathbb{K}_1$ is necessary for the sparse Lagrangian-DNN relaxation (6) with $\mathbb{K} = \mathbb{K}_1$. We also want to choose a symmetric subset \mathcal{G} of $N_0 \times N_0$ so that it properly reflects the sparsity of the matrices \mathbf{Q}^k ($k = 0, 1, \dots, m$) for the resulting DNN and Lagrangian-DNN relaxations to be solved efficiently.

For such a symmetric subset \mathcal{G} of $N_0 \times N_0$, we introduce the sparsity pattern (undirected) graph $G(N_0, \mathcal{E}_0)$ such that

$$\mathcal{E}_0 = \{(i, j) \in N_0 \times N_0 : i \neq j, Q_{ij}^k \neq 0 \text{ for some } k \in \{0, 1, \dots, m\}\}.$$

We identify $(i, j) \in \mathcal{E}_0$ and $(j, i) \in \mathcal{E}_0$ so that $G(N_0, \mathcal{E}_0)$ forms an undirected graph. Let $\bar{G}(N_0, \bar{\mathcal{E}}_0)$ be a chordal extension of $G(N_0, \mathcal{E}_0)$. Consider the set of maximal cliques C_1, \dots, C_r of $\bar{G}(N_0, \bar{\mathcal{E}}_0)$, where each maximal clique is denoted by a subset of N_0 . It is known that the number r of the maximal cliques is not greater than the size $1 + n$ of the node set N_0 , and that the maximal cliques can be renumbered so that they can satisfy the running intersection property

$$\forall p \in \{1, \dots, r-1\}, \exists q > p; C_p \cap (C_{p+1} \cup \dots \cup C_r) \subset C_q. \quad (28)$$

Let $\square C_p = C_p \times C_p$ ($p = 1, \dots, r$) and $\mathcal{E} = \bigcup_{p=1}^r \square C_p$. In this case, we can apply the following lemma to determine whether a matrix $\mathbf{X} \in \mathbb{S}^{1+n}$ belongs to $\mathbb{S}_+^{1+n}(\mathcal{E}, ?)$ and $\mathbb{S}_+^{1+n}(\mathcal{E}, 0)$. If $\mathbf{X} \in \mathbb{S}_+^{1+n}(\mathcal{E}, ?)$, then X_{ij} ($(i, j) \in \mathcal{E}^c$) may be regarded as elements with undetermined values, but their values can be assigned so that the completed matrix belongs to \mathbb{S}_+^{1+n} . Assigning appropriate values is known as the positive semidefinite matrix completion in the literature [15]. Techniques for exploiting sparsity in SDPs based on the positive semidefinite matrix completion were proposed in [13, 23]; see also [18, 20]. We can utilize those techniques for an efficient implementation of the method proposed in this section.

Lemma 6.2. *Let $\mathbf{X} \in \mathbb{S}^{1+n}$.*

(i) $\mathbf{X} \in \mathbb{S}_+^{1+n}(\mathcal{E}, ?)$ if and only if

$$\mathbf{X} \in \bigcap_{p=1}^r \mathbb{S}_+^{1+n}(\square C_p, ?) = \bigcap_{p=1}^r \{ \mathbf{X} \in \mathbb{S}^{1+n} : (X_{ij} : \square C_p) \text{ is positive semidefinite} \}.$$

(ii) $\mathbf{X} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ if and only if $\mathbf{X} \in \sum_{p=1}^r \mathbb{S}_+^{1+n}(\square C_p, 0)$.

(iii) $\mathbf{X} \in \mathbb{N}_+^{1+n}(\mathcal{E}, ?)$ if and only if

$$\mathbf{X} \in \bigcap_{p=1}^r \mathbb{N}_+^{1+n}(\square C_p, ?) = \bigcap_{p=1}^r \{ \mathbf{X} \in \mathbb{N}^{1+n} : (X_{ij} : \square C_p) \text{ is a nonnegative matrix} \}.$$

(iv) $\mathbf{X} \in \mathbb{N}_+^{1+n}(\mathcal{E}, 0)$ if and only if $\mathbf{X} \in \sum_{p=1}^r \mathbb{N}_+^{1+n}(\square C_p, 0)$.

Proof. See [15] and [1] for assertions (i) and (ii), respectively. Assertions (iii) and (iv) are straightforward. \square

Assertion (ii) may be regarded as a dual of (i) since

$$\begin{aligned} \mathbb{S}_+^{1+N}(\mathcal{E}, 0) &= (\mathbb{S}_+^{1+n}(\mathcal{E}, ?))^* \quad (\text{by (i) of Lemma 6.1}) \\ &= \left(\bigcap_{p=1}^r \mathbb{S}_+^{1+n}(\square C_p, ?) \right)^* \quad (\text{by (i)}) \\ &= \text{cl} \left(\sum_{p=1}^r \mathbb{S}_+^{1+n}(\square C_p, 0) \right) = \sum_{p=1}^r \mathbb{S}_+^{1+n}(\square C_p, 0). \end{aligned}$$

Note that the closeness of the cone $\sum_{p=1}^r \mathbb{S}_+^{1+n}(\square C_p, 0)$ can be proved easily.

We employ the convex cone $\mathbb{D}(\mathcal{E}) = \mathbb{S}_+^{1+n}(\mathcal{E}, ?) \cap \mathbb{N}_+^{1+n}(\mathcal{E}, ?)$ for COP (1) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ as a sparse DNN relaxation of QOP (20).

Lemma 6.3.

(i) $\mathbb{D}(\mathcal{E}) = \bigcap_{p=1}^r (\{\mathbf{X} \in \mathbb{S}^{1+n} : (X_{ij} : \square C_p) \text{ is doubly nonnegative}\})$, and $\mathbb{D}(\mathcal{E})$ is closed.

(ii) $\mathbb{D}(\mathcal{E})^* = \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) = \sum_{p=1}^r (\mathbb{S}_+^{1+n}(\square C_p, 0) + \mathbb{N}^{1+n}(\square C_p, 0))$, and $\mathbb{D}(\mathcal{E})^*$ is closed.

Proof. (i) By Lemma 6.2,

$$\begin{aligned} \mathbb{D}(\mathcal{E}) &= \mathbb{S}_+^{1+n}(\mathcal{E}, ?) \cap \mathbb{N}(\mathcal{E}, ?) \\ &= \bigcap_{p=1}^r (\mathbb{S}_+^{1+n}(\square C_p, ?) \cap \mathbb{N}^{1+n}(\square C_p, ?)) \\ &= \bigcap_{p=1}^r (\{\mathbf{X} \in \mathbb{S}^{1+n} : (X_{ij} : \square C_p) \text{ is doubly nonnegative}\}). \end{aligned}$$

The closeness of $\mathbb{D}(\mathcal{E})$ follows from the above identity.

(ii) By definition, Lemmas 6.1, 6.2 and assertion (i),

$$\begin{aligned} \mathbb{D}(\mathcal{E})^* &= (\mathbb{S}_+^{1+n}(\mathcal{E}, ?) \cap \mathbb{N}^{1+n}(\mathcal{E}, ?))^* \\ &= \text{cl} (\mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0)) \\ &= \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) \\ &= \sum_{p=1}^r (\mathbb{S}_+^{1+n}(\square C_p, 0) + \mathbb{N}^{1+n}(\square C_p, 0)). \end{aligned}$$

We need to prove the closeness of $\mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0)$ for the third identity above. Suppose that $\mathbf{X}^s = \mathbf{Y}^s + \mathbf{Z}^s$, $\mathbf{Y}^s \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$, $\mathbf{Z}^s \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$ ($s = 1, 2, \dots$) and $\mathbf{X}^s \rightarrow \overline{\mathbf{X}}$ as $s \rightarrow \infty$ for some $\overline{\mathbf{X}} \in \mathbb{S}^{1+n}$. We will prove that $\overline{\mathbf{X}} = \overline{\mathbf{Y}} + \overline{\mathbf{Z}}$ for some $\overline{\mathbf{Y}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ and $\overline{\mathbf{Z}} \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$, so that $\overline{\mathbf{X}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0)$. First we show that the sequence $\{\mathbf{Y}^s\} \subset \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ is bounded. Assume on the contrary that there is a subsequence of $\{\mathbf{Y}^s\}$ along which $\|\mathbf{Y}^s\|$ diverges. We may assume without loss of generality that $\mathbf{Y}^s / \|\mathbf{Y}^s\| \rightarrow \overline{\mathbf{Y}}$ for some nonzero $\overline{\mathbf{Y}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ as $s \rightarrow \infty$. Then $\mathbb{N}^{1+n}(\mathcal{E}, 0) \ni \mathbf{Z}^s / \|\mathbf{Y}^s\| = \mathbf{X}^s / \|\mathbf{Y}^s\| - \mathbf{Y}^s / \|\mathbf{Y}^s\| \rightarrow -\overline{\mathbf{Y}}$ as $s \rightarrow \infty$. Since $\mathbb{N}^{1+n}(\mathcal{E}, 0)$ is closed, we obtain $-\overline{\mathbf{Y}} \in \mathbb{N}^{1+n}(\mathcal{E}, 0)$, which implies all the diagonal elements of $\overline{\mathbf{Y}} \in \mathbb{S}_+^{1+n}(\mathcal{E}, 0)$ vanish. Therefore $\overline{\mathbf{Y}} = \mathbf{O}$, which is a contradiction. Thus $\{\mathbf{Y}^s\}$ is bounded. As a result $\{\mathbf{Z}^s\}$ is also bounded. From here, the required result follows. \square

By the construction of \mathcal{E} , it follows that

$$\begin{aligned} \mathbf{O} \neq \mathbf{H}^0 &\in \mathbb{S}_+^{1+n}(\mathcal{E}, 0) \cap \mathbb{N}^{1+n}(\mathcal{E}, 0) \subset \mathbb{D}(\mathcal{E})^*, \\ \mathbf{Q}^k &\in \mathbb{S}_+^{1+n}(\mathcal{E}, 0) + \mathbb{N}^{1+n}(\mathcal{E}, 0) = \mathbb{D}(\mathcal{E})^* \quad (k = 1, 2, \dots, m). \end{aligned}$$

Thus, the closed convex cone $\mathbb{K} = \mathbb{D}(\mathcal{E})$ satisfies Conditions (I) and (II). COP (1) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ can be introduced consistently as a sparse DNN relaxation of QOP (20) and

COP (6) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ as a Lagrangian-DNN relaxation of QOP (20). Since Conditions (I) and (II) are satisfied for $\mathbb{K} = \mathbb{D}(\mathcal{E})$, Lemmas 2.1, 2.2 and 2.3 can be applied for $\mathbb{K} = \mathbb{D}(\mathcal{E})$. In particular, we obtain the relation

$$(\eta^d(\lambda, \mathbb{D}(\mathcal{E})) = \eta^p(\lambda, \mathbb{D}(\mathcal{E}))) \uparrow = \zeta^d(\mathbb{D}(\mathcal{E})) \leq \zeta^p(\mathbb{D}(\mathcal{E})). \quad (29)$$

A slight modification is necessary to ensure that Condition (III) holds. Assume that the set $\{\mathbf{u} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{u} + \mathbf{b} = \mathbf{0}\}$ is bounded as in (iii) of Lemma 5.1, and that $0 \in C_p$ ($p = 1, 2, \dots, r$). Then, for each $p = 1, 2, \dots, r$, a positive number γ_p can be chosen such that $\sum_{i \in C_p \setminus \{0\}} u_i + v_p = \gamma_p$ for every $\mathbf{u} \in \{\mathbf{u} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{u} + \mathbf{b} = \mathbf{0}\}$, where $v_p \geq 0$ is the slack variable. For simplicity of notation, we assume that the nonnegative slack variable v_p has already been included in the variables u_i ($i \in C_p \setminus \{0\}$) ($p = 1, 2, \dots, r$). Then, the linear constraints $\sum_{i \in C_p \setminus \{0\}} u_i - \gamma_p = 0$ ($p = 1, 2, \dots, r$) can be added to QOP (18) without changing its feasible region. For each $p = 1, 2, \dots, r$, let

$$\begin{aligned} \mathbf{g}^p &= (g_0^p, g_1^p, \dots, g_n^p) \in \mathbb{R}^{1+n}, \text{ where } g_i^p = \begin{cases} -\gamma_p & \text{if } i = 0, \\ 1 & \text{if } i \in C_p \setminus \{0\}, \\ 0 & \text{otherwise,} \end{cases} \\ \mathbf{G}^p &= \mathbf{g}^p (\mathbf{g}^p)^T \in \mathbb{S}^{1+n}. \end{aligned}$$

Then, the linear constraints added to QOP (18) can be rewritten as quadratic constraints

$$\mathbf{U} = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}^T \in \mathbf{\Delta}_1, \langle \mathbf{G}^p, \mathbf{U} \rangle = 0 \quad (p = 1, 2, \dots, r).$$

We assume that the quadratic constraints above have already been included in the QOP (20). Now we prove that the feasible region $F(\mathbb{D}(\mathcal{E}))$ of COP (1) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ is bounded. Notice that adding those quadratic constraints would not affect the results obtained up to this point, including Lemmas 6.1, 6.2, 6.3, and the identity (29), because $\mathbf{G}_p \in \mathbf{S}_+^{1+n}(\square C_p, 0)$ ($p = 1, 2, \dots, r$) by construction. Assume on the contrary that $\|\mathbf{X}^s\| \rightarrow \infty$ as $s \rightarrow \infty$ for some sequence $\{\mathbf{X}^s\} \subset F(\mathbb{D}(\mathcal{E}))$. Then $\|(X_{ij}^s : \square C_p)\| \rightarrow \infty$ as $s \rightarrow \infty$ for some $p \in \{1, 2, \dots, r\}$. For $\mathbf{X}^s \in F(\mathbb{D}(\mathcal{E}))$, we have that

$$\mathbf{X}^s \in \mathbb{D}(\mathcal{E}), \langle \mathbf{H}^0, \mathbf{X}^k \rangle = 1, \langle \mathbf{G}^p, \mathbf{X}^s \rangle = 0 \quad (s = 1, 2, \dots).$$

Hence,

$$\begin{aligned} (X_{ij}^s : \square C_p) &\text{ is doubly nonnegative,} \\ X_{00}^s &= 1, \langle (G_{ij}^p : \square C_p), (X_{ij}^s : \square C_p) \rangle = 0 \quad (s = 1, 2, \dots). \end{aligned}$$

We may assume without loss of generality that $(X_{ij}^s : \square C_p) / \|(X_{ij}^s : \square C_p)\|$ converges to some doubly nonnegative $(D_{ij} : \square C_p) \neq \mathbf{O}$ as $s \rightarrow \infty$. Thus, dividing the identities above by $\|(X_{ij}^s : \square C_p)\|$ and taking the limit as $s \rightarrow \infty$, we obtain that

$$\begin{aligned} \mathbf{O} &\neq (D_{ij} : \square C_p) \text{ is doubly nonnegative,} \\ D_{00} &= 0, \langle (G_{ij}^p : \square C_p), (D_{ij} : \square C_p) \rangle = 0 \quad (s = 1, 2, \dots). \end{aligned}$$

Since $(D_{ij} : \square C_p)$ is positive semidefinite and $D_{00} = 0$, we see that $D_{0i} = D_{i0} = 0$, $i \in C_p$. Thus, we obtain that

$$\begin{aligned} \mathbf{O} \neq (D_{ij} : \square(C_p \setminus \{0\})) \text{ is doubly nonnegative,} \\ 0 = \left\langle (G_{ij}^p : \square(C_p \setminus \{0\})), (D_{ij} : \square(C_p \setminus \{0\})) \right\rangle = \sum_{(i,j) \in \square C_p \setminus \{0\}} D_{ij}. \end{aligned}$$

This is a contradiction to $\mathbf{O} \neq \mathbf{D}$. Thus we have shown that the feasible region $F(\mathbb{D}(\mathcal{E}))$ of COP (1) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ is bounded, and that Condition (III) holds. By applying Lemma 2.5 with $\mathbb{K} = \mathbb{D}(\mathcal{E})$, we obtain the identity $\zeta^p(\mathbb{D}(\mathcal{E})) = \zeta^d(\mathbb{D}(\mathcal{E}))$ in addition to (29).

6.2 A brief discussion on the applications of the bisection and the 1-dimensional Newton methods

We recall that the computation of the metric projection of each $\mathbf{G} \in \mathbb{V}$ onto \mathbb{K} , which is assumed in Condition (V), is the key to apply the bisection and the 1-dimensional Newton methods presented in Section 4 to the primal-dual pair of Lagrangian-conic relaxation problems (6) and (7). See also Remark 4.1. We will transform the problems with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ to ones with a closed convex cone onto which the metric projection can easily be constructed below.

Since the variables X_{ij} ($(i, j) \notin \mathcal{E}$) are redundant in both of the equality constraints and the cone constraint $\mathbf{X} \in \mathbb{D}(\mathcal{E})$ of the primal COPs (1), (3) and (6) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$, those variables can be eliminated from the primal COPs. On the other hand, all matrices \mathbf{Q}^k ($k = 0, 1, \dots, m$), \mathbf{H}^0 , \mathbf{H}^1 and the cone $\mathbb{D}(\mathcal{E})^*$ are contained in $\mathbb{S}^{1+n}(\mathcal{E}, 0)$ in the dual COPs (2), (4) and (7) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$. This implies that the coordinates H_{ij}^0 , H_{ij}^1 , Q_{ij}^k ($k = 0, 1, 2, \dots, m$) ($(i, j) \notin \mathcal{E}$) are redundant in the inclusion constraints of the dual COPs. Furthermore, whether $\mathbf{X} \in \mathbb{S}^{1+n}$ belongs to $\mathbb{S}_+^{1+n}(\mathcal{E}, ?)$ can be determined by checking whether its sub matrices $(X_{ij} : (i, j) \in \square C_p)$ ($p = 1, 2, \dots, r$) are all positive semidefinite (Lemma 6.3). We note that some elements may appear in a pair of these submatrices, *i.e.*, $\square C_p \cap \square C_q \neq \emptyset$ for some p, q .

Let

$$\begin{aligned} \mathbb{S}^{C_p} &= \{ \mathbf{Y}^p = (Y_{ij}^p : (i, j) \in \square C_p) : Y_{ij}^p = Y_{ji}^p \in \mathbb{R} \} \quad (p = 1, 2, \dots, r), \\ \mathbb{S}_+^{C_p} &= \{ \mathbf{Y}^p \in \mathbb{S}^{C_p} : \text{positive semidefinite} \} \quad (p = 1, 2, \dots, r), \\ \mathbb{S}^{\mathcal{E}} &= \prod_{p=1}^r \mathbb{S}^{C_p} = \{ \mathbf{Y} = (\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^r) : \mathbf{Y}^p \in \mathbb{S}^{C_p} \quad (p = 1, 2, \dots, r) \}, \\ \mathbb{S}_+^{\mathcal{E}} &= \prod_{p=1}^r \mathbb{S}_+^{C_p} = \{ \mathbf{Y} = (\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^r) : \mathbf{Y}^p \in \mathbb{S}_+^{C_p} \quad (p = 1, 2, \dots, r) \}, \\ \mathbb{L}^{\mathcal{E}} &= \{ \mathbf{Y} = (\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^r) \in \mathbb{S}^{\mathcal{E}} : Y_{ij}^p = Y_{ij}^q \text{ if } (i, j) \in \square C_p \cap \square C_q \}, \\ \mathbb{K}_1 &= \mathbb{S}_+^{\mathcal{E}}, \\ \mathbb{K}_2 &= \left\{ \mathbf{Y} = (\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^r) \in \mathbb{S}^{\mathcal{E}} : \begin{array}{l} Y_{ij}^p \geq 0 \quad (i, j) \in \square C_p \\ (p = 1, 2, \dots, r) \end{array} \right\} \cap \mathbb{L}^{\mathcal{E}}. \end{aligned}$$

Each $\mathbf{Y} \in \mathbb{S}^\mathcal{E}$ may be regarded as a block diagonal matrix with diagonal blocks $\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^r$. We use $\langle \widetilde{\mathbf{U}}, \mathbf{Y} \rangle = \sum_{p=1}^r \langle \widetilde{\mathbf{U}}^p, \mathbf{Y}^p \rangle$ for the inner product of $\widetilde{\mathbf{U}} = (\widetilde{\mathbf{U}}^1, \widetilde{\mathbf{U}}^2, \dots, \widetilde{\mathbf{U}}^r)$, $\mathbf{Y} = (\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^r) \in \mathbb{S}^\mathcal{E}$.

We now associate each $\mathbf{X} \in \mathbb{S}^{1+n}(\mathcal{E}, ?)$ with $\widetilde{\mathbf{X}} = (\widetilde{\mathbf{X}}^1, \widetilde{\mathbf{X}}^2, \dots, \widetilde{\mathbf{X}}^r) \in \mathbb{S}^\mathcal{E}$ by

$$\widetilde{\mathbf{X}}^p = (X_{ij} : (i, j) \in \square C_p) \quad (p = 1, 2, \dots, r).$$

This correspondence yields that $\mathbf{X} \in \mathbb{D}(\mathcal{E})$ if and only if $\widetilde{\mathbf{X}} \in \mathbb{K}_1 \cap \mathbb{K}_2$. It is also possible to choose $\widetilde{\mathbf{Q}}^0, \widetilde{\mathbf{H}}^0, \widetilde{\mathbf{H}}^1 \in \mathbb{S}^\mathcal{E}$ such that

$$\langle \widetilde{\mathbf{Q}}^0, \widetilde{\mathbf{X}} \rangle = \langle \mathbf{Q}^0, \mathbf{X} \rangle, \quad \langle \widetilde{\mathbf{H}}^0, \widetilde{\mathbf{X}} \rangle = \langle \mathbf{H}^0, \mathbf{X} \rangle \quad \text{and} \quad \langle \widetilde{\mathbf{H}}^1, \widetilde{\mathbf{X}} \rangle = \langle \mathbf{H}^1, \mathbf{X} \rangle.$$

Consequently, we obtain the following primal-dual pair of COPs, which are equivalent to the primal-dual of COPs with $\mathbb{K} = \mathbb{D}(\mathcal{E})$.

$$\tilde{\zeta}^p(\lambda) = \inf \left\{ \langle \widetilde{\mathbf{Q}}^0 + \lambda \widetilde{\mathbf{H}}^1, \widetilde{\mathbf{X}} \rangle \mid \begin{array}{l} \widetilde{\mathbf{X}} = (\widetilde{\mathbf{X}}^1, \widetilde{\mathbf{X}}^2, \dots, \widetilde{\mathbf{X}}^r) \in \mathbb{K}_1 \cap \mathbb{K}_2, \\ \langle \widetilde{\mathbf{H}}^0, \widetilde{\mathbf{X}} \rangle = 1 \end{array} \right\} \quad (30)$$

$$\tilde{\zeta}^d(\lambda) = \sup \left\{ y_0 \mid \widetilde{\mathbf{Q}}^0 + \lambda \widetilde{\mathbf{H}}^1 - \widetilde{\mathbf{H}}^0 y_0 \in \mathbb{K}_1^* + \mathbb{K}_2^* \right\}. \quad (31)$$

Then the metric projections Π_i from $\mathbb{S}^\mathcal{E}$ onto \mathbb{K}_i ($i = 1, 2$) are expressed as

$$\begin{aligned} \Pi_i(\widetilde{\mathbf{X}}) &= (\Pi_{i1}(\widetilde{\mathbf{X}}), \Pi_{i2}(\widetilde{\mathbf{X}}), \dots, \Pi_{ir}(\widetilde{\mathbf{X}})) \quad (i = 1, 2), \\ \Pi_{1p}(\widetilde{\mathbf{X}}) &= \text{the metric projection of } \widetilde{\mathbf{X}}^p \in \mathbb{S}^{C_p} \text{ onto } \mathbb{S}_+^{C_p} \quad (p = 1, 2, \dots, r), \\ \left(\Pi_{2p}(\widetilde{\mathbf{X}}) \right)_{ij} &= \max \left\{ \frac{\sum_{p \in P(i,j)} \widetilde{\mathbf{X}}_{ij}^p}{\#P(i,j)}, 0 \right\} \quad ((i, j) \in \square C_p, p = 1, 2, \dots, r), \end{aligned}$$

where $P(i, j) = \{p : (i, j) \in \square C_p\}$ ($((i, j) \in \mathcal{E})$). We refer to [13, 23] for details on the conversion from the primal-dual pair (6) and (7) with $\mathbb{K} = \mathbb{D}(\mathcal{E})$ to the primal-dual pair (30) and (31), and [6, 19] for numerical methods for computing the metric projection onto $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$ from those onto Π_1 and Π_2 .

7 Concluding remarks

We have provided a unified framework expressed in a primal-dual pair of COPs. It provides a convenient and effective tool to develop the theory and methods originated from the completely positive programming relaxation of QOPs. By imposing the copositivity condition on the primal-dual pair of COPs, equivalent but simpler primal-dual pair of COPs and their Lagrangian-conic relaxations have been derived. We have investigated theoretical properties of the three primal-dual pairs of COPs and the conditions which yield the equivalence for the optimal values of the COPs. When the cone \mathbb{K} involved in the first primal-dual pair of COPs is nonconvex, we have provided a necessary and sufficient condition for the equivalence between the primal COP and its convexification, *i.e.*, the COP obtained by replacing

the cone \mathbb{K} by its convex hull. This result has been applied to a class of linearly constrained QOPs with complementarity constraints.

In our recent paper [19], some promising numerical results were reported on the Lagrangian-DNN relaxation for QOPs using a bisection method. But any sparsity was not utilized there. In the current paper, we have proposed the sparse Lagrangian-DNN relaxation for the same class of QOPs, and the 1-dimensional Newton method for solving the primal-dual pair of Lagrangian-conic relaxation problems (6) and (7). If exploiting sparsity and the 1-dimensional Newton method are implemented, we could expect to solve large-scale QOPs more efficiently. We hope to present numerical results for this subject in the future.

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