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Abstract

Adjustable robust optimization (ARO) is a technique to solve dynamic (multistage) optimization problems. In ARO, the decision in each stage is a function of the information accumulated from the previous periods on the values of the uncertain parameters. This information, however, is often inaccurate; there is much evidence in the information management literature that even in our Big Data era the data quality is often poor. Reliance on the data "as is" may then lead to poor performance of ARO, or in fact to any "data-driven" method. In this paper, we remedy this weakness of ARO by introducing a methodology that treats past data itself as an uncertain parameter. We show that algorithmic tractability of the robust counterparts associated with this extension of ARO is still maintained. The benefit of the new approach is demonstrated by a production-inventory application.

Key words: adjustable robust optimization, decision rules, inexact data, poor data quality.

JEL Classification: C00, C15, C44, C61, C63

1 Introduction

Multistage optimization problems, such as inventory/production control problems, are challenging computationally due to the so-called "curse of dimensionality" and the possible explosion of the event tree when the number of stages increases. Moreover, in practice, these problems are affected by uncertainty in their parameters (e.g., customers' demand). Often, the information on the values of these parameters is quite crude and it is in the form of "uncertainty sets". A methodology to address these difficulties was introduced in Ben-Tal et al. (2004). It employs a linear decision

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rule to express the functional dependence of the decision variables at stage t of the system on the revealed values of the parameters in the previous stages. The methodology, named *Affinely Adjustable Robust Counterpart* (AARC), has been successfully applied in a variety of applications (see the book *Robust Optimization* by Ben-Tal et al. (2009) and the surveys by Beyer and Sendhoff (2007), Bertsimas et al. (2011b)), and was extended to include some nonlinear decision rules (see Ben-Tal et al. (2004), Goh and Sim (2010), Bertsimas et al. (2011a)).

In all the above methods, the decision rules are based on the *exact* values of the revealed data from the past. However, even revealed information in multistage decision problems is often *inexact*. For instance, in inventory/production problems, the demand data of past periods may be known only partly; e.g., not all stores reported the demand values in time, or used reported sales as surrogates for demands, which then ignores lost sales. In more technical or medical applications the inexact nature of the revealed data is due to the fact that most of the information one can gather during the process are measurements of the real data, which are often prone to errors.

In general, even when it seems that the full data of the uncertain parameters is available at some stage, one cannot rely blindly on this information. Arguably, the developments in information technology have enabled firms to collect real-time data. However, despite these enormous developments in our Big Data era, poor data quality is still a big issue. In Redman (1998) it is estimated that 1–5% of data fields are erred, which led to a cost increase of 8–12% of revenue in some carefully studied cases, and to a consumption of 40–60% of the expenditure in service organizations. Haug et al. (2011) summarize the literature that deal with the big impact of poor data quality: "Less than 50% of companies claim to be very confident in the quality of their data", "75% of organizations have identified costs stemming from dirty data". See also Soffer (2010) for a general exploration of data inaccuracy in business processes. Other papers discuss the size and impact of poor data quality in specific environments. In DeHoratius and Raman (2008) results of an empirical study are reported; they found that 65% of the inventory records were inaccurate, and "the value of the inventory reflected by these inaccurate records amounted to 28% of the total value of the expected on-hand inventory". In Vawdrey et al. (2007) it is shown that the quality of data in many electronic medical records is often not very good. We summarize that there is abundant evidence that data may have poor quality. If one wants to apply the AARC methodology within this setting where the revealed information is inexact, then there are basically two options. First of all, one could discard all the obtained information that is inexact in the decision rules (similar to the RC approach). This may lead to worse solutions since we do not use the available information at all when making our decisions. Second, one could use the data in the decision rule assuming it is fully accurate and therefore neglecting the inexact nature of the information. This last option may lead to big infeasibilities as we will show via our examples.

In this paper, we therefore extend the adjustable robust counterpart (ARC) approach by Ben-Tal et al. (2004) to cope with situations where the revealed information is inexact. We call this the

Adjustable Robust Counterpart with decision rules based on Inexact Data (ARCID). In this ARCID approach, the decision rules for the wait-and-see variables are functions of the inexact information. We present the ARCID model and derive its tractable robust counterpart. Using a small, but illustrative example, we show that solutions of the ARO model may be infeasible if one neglects the inexact nature of the revealed information. Taking into account the inexact nature of the data, and using ARCID results in better solutions. It is noted that if the inaccuracy in the revealed information is too big, then there is no value in the information and the objective value of the ARCID coincides with the objective nature of the nonadjustable RO model. On the other hand, if the revealed information is fully accurate, then the ARCID approach turns out to be equivalent to the ARC approach by Ben-Tal et al. (2004).

Inspired by the results in DeHoratius and Raman (2008) on inaccuracy in inventory records, we illustrate the benefits of our new approach by revisiting the inventory problem that was used in the first paper on ARO (Ben-Tal et al. (2004)).

The rest of this paper is organized as follows. Section 2 introduces the ARO model and explains how the ARCID model is related to the ARO model and to the nonadjustable RO model. In Section 3 we derive robust counterparts for the nonadjustable RO, the ARO and the ARCID. Section 4 presents our test case and the corresponding ARCID approach. The numerical results for our test case are given and analyzed in Section 5. Section 6 concludes and gives directions for further research.

Notation and definitions. The convex conjugate of a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$f^*(y) = \sup_x \{y^\top x - f(x)\}$$

and the *concave conjugate* of a function $g : \mathbb{R}^m \to \mathbb{R}$ is defined similarly

$$g_*(y) = \inf_x \{y^\top x - g(x)\}.$$

Note that f^* is a convex function, since it is the pointwise maximum of linear functions and that f^* is even convex if the original function f was not convex. Similarly, g_* is concave for any function g. A special function is the *indicator function* of the set S which is defined as:

$$\delta(x|S) = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{otherwise.} \end{cases}$$

The convex conjugate of the indicator function, the *support function*, is:

$$\delta^*(y|S) = \sup_{x \in S} y^\top x.$$

2 Towards the adjustable robust counterpart based on inexact data

In Section 2.1 we give the robust optimization model (RO). The adjustable robust optimization model with decision rules based on exact data (ARO) is presented in Section 2.2. Finally, in Section 2.3 we introduce the adjustable robust counterpart with decision rules based on inexact data (ARCID), which is the main subject of this paper. A simple example is used throughout the entire section to illustrate the differences between these formulations.

2.1 The robust optimization model

A constraint can be affected by uncertainty in the parameters. In robust optimization the distribution of the uncertain parameters need not to be known, only that they can take any value in a given uncertainty set.

Example 1. *Consider the following toy instance*

$$
\begin{aligned}
\max_{x,y} \quad & x \\
subject\ to \quad (1+a)x + y \ &\leq 1, \qquad \forall a \in [0,\theta] \\
-ax \ &\leq y, \qquad \forall a \in [0,\theta] \\
x \geq 0, &
\end{aligned}
\tag{RO-toy}
$$

where $a \in [0,\theta]$ and $\theta \geq 0$ being a constant that indicates the level of uncertainty. A solution of (RO-toy) *must be feasible for all values of a in the uncertainty set $[0,\theta]$. From the second constraint we obtain $y \geq 0$ for the worst case realization ($a = 0$). In the first constraint we clearly need y to be as small as possible, so $y = 0$ must be optimal. Therefore, we get $x \leq \frac{1}{1+a}$, where the right hand side is equal to $\frac{1}{1+\theta}$ in the worst case ($\theta = a$). So $(x,y) = (\frac{1}{1+\theta}, 0)$ gives the optimal solution for* (RO-toy) *with objective value OPT(RO-toy) $= \frac{1}{1+\theta}$.* $\square$

In general, we consider the following constraints in a RO model:

$$
f(a,x) \leq 0 \qquad \forall a \in \mathcal{U}_0
$$

or equivalently

$$
\sup_{a \in \mathcal{U}_0} f(a,x) \leq 0.
\tag{2.1}
$$

4

To derive tractable robust counterparts for general constraints, we normally assume that $f : \mathcal{U}_0 \times \mathbb{R}^n \to \mathbb{R}$ is a closed and concave function in its uncertain parameter a, which resides in the nonempty, closed and convex set $\mathcal{U}_0$. We can write "max" instead of "sup" in (2.1) because $\mathcal{U}_0$ is closed and bounded, so the supremum is attained (if it exists). Another technical condition is needed on the set $\mathcal{U}_0$ and the domain of f to derive tractable counterparts by Fenchel duality as is done in this paper:

$$\text{ri}(\mathcal{U}_0) \cap \text{ri}(\text{dom } f(., x)) \neq \emptyset, \ \forall x, \qquad (2.2)$$

where ri denotes the relative interior. We refer to (2.2) as the *regularity condition*.

2.2 The adjustable robust optimization model based on exact data

The decision variables in the robust optimization model are chosen prior to knowing the realization of the uncertain parameter. This can be very conservative if there are some *adjustable* or *wait-and-see* variables. One can decide upon these adjustable variables after the true value of $a \in \mathcal{U}_0$ reveals itself as is shown in the next example.

Example 2. *Reconsider the linear program* (RO-toy) *from Example 1. Assume that y is adjustable, i.e., we can decide upon y after we know the realization of the data $a \in [0, \theta]$. We use a general decision rule $y = y(a)$, where a is the actual realization. The model now reads*

$$
\begin{aligned}
\max_{x, y(a)} \quad & x \\
\text{subject to} \quad & (1 + a)x + y(a) \leq 1, \quad && \forall a \in [0, \theta] \\
& -ax \leq y(a), \quad && \forall a \in [0, \theta] \\
& x \geq 0.
\end{aligned}
\qquad \text{(ARO-toy)}
$$

Since y has to be as small as possible, $y(a)$ has to be as small as possible as well. From the second constraint we obtain $y(a) \geq -ax$. Hence, it is optimal to set $y(a) = -ax$. For the first constraint this results in $x \leq 1$ and therefore $OPT(\text{ARO-toy}) = 1$. Note that

$$OPT(\text{ARO-toy})/OPT(\text{RO-toy}) \to \infty \ as \ \theta \to \infty.$$

$\square$

The robust optimization model where one uses decision functions to decide upon adjustable variables is called the *adjustable robust optimization model* or ARO model. A typical constraint with adjustable variables in this type of models is:

$$f(a, x) + d^\top y(a) \leq 0 \qquad \forall a \in \mathcal{U}_0, \qquad (2.3)$$

where $y(a) \in \mathbb{R}^p$ is the adjustable variable, $d \in \mathbb{R}^p$ and $f : \mathcal{U}_0 \times \mathbb{R}^n \to \mathbb{R}$ is a function of the uncertain parameter and all the so-called *here-and-now* variables $x \in \mathbb{R}^n$. These variables have to be set prior to knowing the actual realization of the uncertain parameter a. The function $f(a, x)$ is assumed to be concave in $a \in \mathbb{R}^m$ for all $x \in \mathbb{R}^n$. Note that (2.3) is the case of *fixed recourse*, meaning that $d \in \mathbb{R}^p$ is certain. In general it is difficult from a computational point of view to let $y(a)$ be any function of a, since it is hard to optimize over all real functions. Therefore, we restrict ourselves to affine decision rules as in Ben-Tal et al. (2009): $y(a) = q + Qa$, where $q \in \mathbb{R}^p$ and $Q = [q^1 \ldots q^m]$ with $q, q^1, \ldots, q^m \in \mathbb{R}^p$ as the new nonadjustable variables. Using this decision rule, constraint (2.3) can be rewritten to obtain the *affinily adjustable robust counterpart* (AARC):

$$f(a, x) + d^\top (q + Qa) \leq 0 \qquad \forall a \in \mathcal{U}_0, \tag{2.4}$$

or equivalently

$$\max_{a \in \mathcal{U}_0} \{ f(a, x) + d^\top (q + Qa) \} \leq 0. \tag{2.5}$$

Note that for $\mathcal{U}_0$ we can also use uncertainty sets as in Ben-Tal et al. (2012):

$$\mathcal{U}_0 = \{ a = a^0 + A\zeta \mid \zeta \in \mathcal{V}_0 \}, \tag{2.6}$$

where $A \in \mathbb{R}^{m \times L}$, $\mathcal{V}_0 \subset \mathbb{R}^L$ is a nonempty, convex and closed set, $a^0 \in \mathbb{R}^m$ the *nominal value* and ζ the vector of *primitive uncertainties*. Now one can also use a decision rule that is based on ζ, the vector of primitive uncertainties, instead of a. All results presented in this paper can be readily extended to this type of uncertainty sets and decision rules.

2.3   The adjustable robust model based on inexact data

In this section we present the adjustable robust optimization model with decision rules based on inexact data. The ARO model based on exact data assumes that there is one time moment where the data $a \in \mathcal{U}_0$, used to decide upon the variable y, is known exactly. However, in many practical applications only an estimate $\hat{a}$ of the true value a is available. In that case we have inexact data and $\hat{a}$ is not exactly equal to a, but we assume that $\hat{a} - a \in \widehat{\mathcal{Z}}$, where $\widehat{\mathcal{Z}}$ is the (closed and convex) set indicating the estimation error. The next example illustrates that if we neglect this inexact nature of the observation and use the ARC, we can get infeasible solutions. In this example we also present a decision rule based on inexact data that leads to a better solution than the decision rule that does not use this inexact data.

Example 3. *Let us reconsider* (ARO-toy) *from Example 2, but now the decision rule is based on an inexact measurement $\hat{a}$ instead of the exact value of a. We only know that the true value a lies in the interval generated by the estimation error of $\hat{a}$ (and in $[0, \theta]$). To state it more formally, the*

true value a lies in the intersection of the interval $[0, \theta]$ and $[\hat{a} - \rho, \hat{a} + \rho]$, where ρ is a positive real number indicating the size of the estimation error. The model now becomes

$$
\begin{aligned}
\max_{x, y(\hat{a})} \quad & x \\
\text{subject to} \quad & (1 + a)x + y(\hat{a}) && \leq 1, && \forall a \in [\hat{a} - \rho, \hat{a} + \rho] \cap [0, \theta], && \hat{a} \in [0, \theta] \\
& -ax && \leq y(\hat{a}), && \forall a \in [\hat{a} - \rho, \hat{a} + \rho] \cap [0, \theta], && \hat{a} \in [0, \theta] \\
& x \geq 0.
\end{aligned}
$$

(ARCID -toy)

To find suitable decision rules based on the estimate, let us first try to neglect the inexact nature in the decision rule. If we use the rule that assumes the information is exact from Example 2, $y(\hat{a}) = -\hat{a}x$, then the second constraint reads

$$
-ax \leq -\hat{a}x, \qquad \forall a \in [\hat{a} - \rho, \hat{a} + \rho] \cap [0, \theta] \qquad \forall \hat{a} \in [0, \theta].
$$

Clearly, this particular decision rule from the ARO model is infeasible for all predetermined values of $\rho > 0$. Hence, the decision rule has to be changed. The optimal decision rule for the new case with the estimation error can be derived in the same fashion as in done in Example 2 from the second constraint. Doing so we obtain

$$
y(\hat{a}) \geq \min\{-(\hat{a} - \rho)x, 0\}. \tag{2.7}
$$

One can readily check that for this piecewise linear decision rule the second constraint holds for all $x \geq 0$ and $\hat{a} \in [0, \theta]$ and that this decision rule is optimal. Note that the decision rule is not based on the exact data a, but on its estimate $\hat{a}$. From the first constraint we still want y to be as small as possible, so it is optimal to take the decision rule (2.7) (with equality). The first constraint of (ARCID -toy) can then be written as

$$
(1 + a)x + \min\{-(\hat{a} - \rho)x, 0\} \leq 1, \qquad \forall a \in [\hat{a} - \rho, \hat{a} + \rho] \cap [0, \theta], \qquad \hat{a} \in [0, \theta],
$$

or equivalently

$$
(1 + \min\{\hat{a} + \rho, \theta\})x + \min\{-(\hat{a} - \rho)x, 0\} \leq 1, \qquad \forall \hat{a} \in [0, \theta].
$$

One can check the cases $\hat{a} + \rho \leq \theta$, $\hat{a} + \rho \geq \theta$ combined with the cases $\hat{a} - \rho \leq 0$, $\hat{a} - \rho \geq 0$ to obtain

$$
x \leq \max\{\tfrac{1}{1+2\rho}, \tfrac{1}{1+\theta}\}. \tag{2.8}
$$

Hence, $OPT(\text{ARCID -toy}) = \max\{\tfrac{1}{1+2\rho}, \tfrac{1}{1+\theta}\}$. For fixed ρ, $OPT(\text{ARCID -toy})/OPT(\text{RO-toy}) \rightarrow$

∞ *as $\theta \to \infty$, similar to the result obtained in the ARO model.*[1]

If we have exact information: $\rho = 0$, then (2.8) boils down to $x \leq 1$, so the optimal value of the adjustable robust model based on inexact data and the adjustable robust model (based on exact data) coincide. For large estimation errors ($2\rho > \theta$) we have $x \leq \frac{1}{1+\theta}$, so we could just as well take the RC model in this case (as this is obviously also feasible in the ARO models).

Note that we can also take a linear decision rule. The parameter ρ is known beforehand, so we can set $y(a) = -(\hat{a} - \rho)x$ if $\rho \leq \frac{1}{2}\theta$ and $y(a) = 0$ otherwise. One can derive that this decision rule is feasible for all $\hat{a} \in [0, \theta]$ and leads to the same constraint as (2.8), and is therefore an optimal decision rule as well. $\qquad\qquad\square$

In this paper we investigate a generalization of the ARO models where the decisions are based on an estimate $\hat{a}$ of the actual parameter a, as illustrated in Example 3. We refer to this robust counterpart as the adjustable robust counterpart with decision rules *based on inexact data*, or ARCID. A typical constraint with decision rules based on inexact data is given below:

$$f(a, x) + d^\top (q + Q\hat{a}) \leq 0, \qquad \forall (a, \hat{a}) \in \mathcal{U}, \tag{2.9}$$

or equivalently

$$\max_{(a,\hat{a}) \in \mathcal{U}} \{f(a, x) + d^\top (q + Q\hat{a})\} \leq 0. \tag{2.10}$$

The uncertain parameter and its estimate $(a, \hat{a})$ lie in the uncertainty set $\mathcal{U}$, which is given by

$$\mathcal{U} = \left\{ (a, \hat{a}) : a \in \mathcal{U}_0, \ \hat{a} \in \mathcal{U}_1, \ (a - \hat{a}) \in \widehat{\mathcal{Z}} \right\}, \tag{2.11}$$

where $\mathcal{U}_0$ is the closed and convex uncertainty set for the parameter a, $\mathcal{U}_1$ the closed and convex set that specifies the *estimation range* of the estimate $\hat{a}$, and the closed and convex set $\widehat{\mathcal{Z}}$ specifies the error of the estimate $\hat{a}$. Usually we assume that $\mathcal{U}_0 = \mathcal{U}_1$ and $\{0\} \in \text{ri}(\widehat{\mathcal{Z}})$. In that case, the regularity condition for the ARCID is the same as (2.2). For the general case (so $\mathcal{U}_0 \neq \mathcal{U}_1$ or $\{0\} \notin \text{ri}(\widehat{\mathcal{Z}})$) we have to write the regularity condition for the ARCID as:

$$\text{ri}(\mathcal{U}) \cap \{\text{ri}(\text{dom } f(., x)) \times \mathbb{R}^m\} \neq \emptyset, \ \forall x \in \mathbb{R}^n. \tag{2.12}$$

The relation between the RC, ARCID and ARC concepts in terms of the inexactness in the revealed information, is depicted in Figure 1. In the RC none of the revealed information is used, so it assumes the parameter can still take any value in the uncertainty set. The ARCID uses the revealed information and takes into account that the data used in the decision rule is inexact and

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[1]This results holds also for non fixed ρ, as long as the uncertainty region expands faster than the estimation error, i.e., if for the sequence $\{\theta_k, \rho_k\}_{k=1}^\infty$ we have $\frac{\theta_k}{\rho_k} \to \infty$ as $k \to \infty$.

therefore is still uncertain to some extent. The ARC model also uses the revealed information, but assumes that this information is exact.
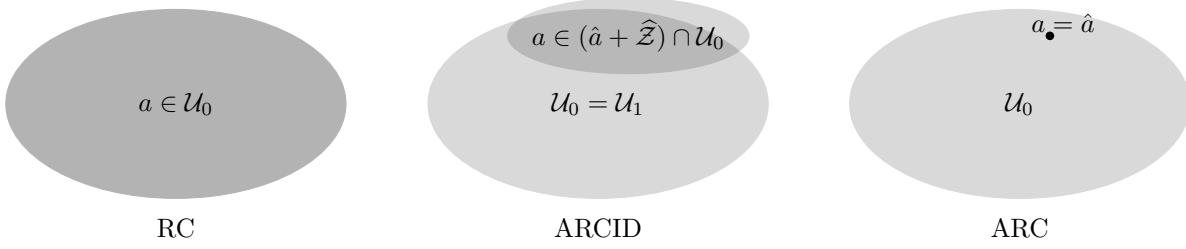

Figure 1: Comparison between uncertainty of the revealed information in the RC, ARCID and ARC concepts.

In Example 3 we observed equivalence between the ARCID and the ARC if there is no estimation error. This also holds in general: If the measurements are exact, so $\widehat{\mathcal{Z}} = \{0\}$ or $a = \hat{a}$ for all possible realizations of $a \in \mathcal{U}_0$, then the ARC and the ARCID coincide. From this fact one can also easily see that if a solution is feasible for the ARCID (2.9), then it is also feasible for the ARC (2.4)[2]. Similarly, we can see that if (x, y) is feasible for the nonadjustable RC constraint (2.3), then $(x, q = y, Q = 0)$ is feasible for the ARCID (2.9). There is even equivalence between the ARCID model and the RO model if the estimation error is "too large". For all other scenarios where the information is inexact, but the estimation error is not "too large", the ARCID may result in lower worst case costs than the corresponding RO or ARO model. The RC and the ARC model may even yield infeasible solutions in situations where the ARCID produces good solutions as we shall see from our numerical results in Section 5.

3   The adjustable robust counterpart based on inexact data

In this section we derive robust counterparts with the techniques presented in Ben-Tal et al. (2012). In the following theorem we derive the adjustable robust counterpart with decision rules based on inexact data (ARCID).

Theorem 3.1. *Let $\mathcal{U}$ be as defined in (2.11) and let $f(a, x)$ be a closed concave functions in a such that the regularity condition (2.12) is satisfied. Then (x, q, Q) satisfies the ARCID (2.10) if and only if there exist $v^0, v^1 \in \mathbb{R}^m$ that satisfy*

$$d^\top q + \delta^*(v^0 | \mathcal{U}_0) + \delta^*(Q^\top d + v^1 | \mathcal{U}_1) + \delta^*(v^1 | \widehat{\mathcal{Z}}) - f_*(v^0 + v^1, x) \leq 0. \tag{3.1}$$

Proof. The proof follows by applying Theorem A.1 to (2.10). Hence, we need to determine $\delta^*((s^0, s^1) | \mathcal{U})$ and the concave conjugate of $(f(a, x) + d^\top(q + Q\hat{a}))$ with respect to a and $\hat{a}$. Note

---

[2]This statement also needs some extra regularity conditions: $\mathcal{U}_1 = \mathcal{U}_0$ and $0 \in \widehat{\mathcal{Z}}$.

that $\delta((a, \hat{a})|\mathcal{U}) = \delta(a|\mathcal{U}_0) + \delta(\hat{a}|\mathcal{U}_1) + \delta(a - \hat{a}|\widehat{\mathcal{Z}})$. Define the function $h(a, \hat{a}) = \delta(a - \hat{a}|\widehat{\mathcal{Z}})$. By Lemma A.2 we have that the conjugate of the function $h(a, \hat{a})$ is given by

$$h^*(v^1, r^1) = \begin{cases} \delta^*(v^1|\widehat{\mathcal{Z}}) & \text{if } v^1 + r^1 = 0 \\ \infty & \text{otherwise.} \end{cases}$$

The convex conjugate of the sum function $\delta((a, \hat{a})|\mathcal{U}) = \delta(a|\mathcal{U}_0) + \delta(\hat{a}|\mathcal{U}_1) + \delta(a - \hat{a}|\widehat{\mathcal{Z}})$ with respect to both variables a and $\hat{a}$ is therefore by Lemma A.3 equal to

$$\delta^*((s^0, s^1)|\mathcal{U}) = \min_{v^0, v^1, r^0, r^1} \{\delta^*(v^0|\mathcal{U}_0) + \delta^*(r^0|\mathcal{U}_1) + h^*(v^1, r^1) \mid v^0 + v^1 = s^0, r^0 + r^1 = s^1\}$$

$$= \min_{v^0, v^1, r^0, r^1} \{\delta^*(v^0|\mathcal{U}_0) + \delta^*(r^0|\mathcal{U}_1) + \delta^*(v^1|\widehat{\mathcal{Z}}) \mid v^0 + v^1 = s^0, r^0 + r^1 = s^1, v^1 + r^1 = 0\}.$$

The concave conjugate of the sum function $g(a, \hat{a}, x, q, Q) := f(a, x) + d^\top(q + Q\hat{a})$ with respect to the variables a and $\hat{a}$ can be found by Lemma A.4 and the fact that the function is separable in a and $\hat{a}$. Doing so, one easily derives that the concave conjugate is equal to the sum of the conjugates of $f(a, x)$ and $d^\top(q + Q\hat{a}))$, which is given by:

$$g_*(s^0, s^1, x, q, Q) = \begin{cases} -d^\top q + f_*(s^0, x) & \text{if } s^1 = Q^\top d \\ -\infty & \text{otherwise.} \end{cases}$$

Using Theorem A.1 we have that (x, q, Q) is feasible for constraint (2.10) if there exists $s^0 \in \mathbb{R}^m$ that satisfies

$$\min_{v^0, v^1, r^0, r^1} \Big\{ d^\top q - f_*(s^0, x) + \delta^*(v^0|\mathcal{U}_0) + \delta^*(r^0|\mathcal{U}_1) + \delta^*(v^1|\widehat{\mathcal{Z}})$$
$$\mid v^0 + v^1 = s^0, \ r^0 + r^1 = Q^\top d, \ v^1 + r^1 = 0 \Big\} \le 0. \tag{3.2}$$

We can also write (3.2) without the "min". Then (x, q, Q) is feasible for (2.10) if there exist $v^0, v^1, r^0, r^1, s^0 \in \mathbb{R}^m$ that satisfy the system of constraints

$$\begin{cases} d^\top q + \delta^*(v^0|\mathcal{U}_0) + \delta^*(r^0|\mathcal{U}_1) + \delta^*(v^1|\widehat{\mathcal{Z}}) - f_*(s^0, x) \le 0 \\ v^0 + v^1 = s^0, \ r^0 + r^1 = Q^\top d, \ v^1 + r^1 = 0. \end{cases} \tag{3.3}$$

The theorem then follows by eliminating the equality constraints in (3.3). $\qquad\square$

Note that for (3.1) computations involving $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{Z}$, and f are mutually independent. We can use all results on the support functions and conjugate functions presented in Ben-Tal et al. (2012) for the sets $\mathcal{U}_0$ and $\mathcal{U}_1$, but also for the region $\widehat{\mathcal{Z}}$, which specifies the set for estimation errors. To compare the ARCID to the ARC we also present the counterpart of the ARC, where all information in the decision rules has to be exact in the next lemma.

Lemma 3.2. *Let $\mathcal{U}_0$ be a closed convex set and let $f(a,x)$ be a closed concave functions in a such that the regularity condition (2.2) is satisfied. Then (x,q,Q) satisfies the ARC (2.4) if and only if there exists $v \in \mathbb{R}^m$ such that*

$$d^\top q + \delta^*(v + Q^\top d|\mathcal{U}_0) - f_*(v,x) \leq 0. \tag{3.4}$$

$\square$

The proof follows directly from the proof in Theorem 3.1 by taking $\mathcal{U}_1 = \mathcal{U}_0$ and $\mathcal{Z} = \{0\}$. Critically important is that the resulting ARCID constraint (3.1) is essentially of the same complexity as the ARC constraint (3.4) and requires only few extra variables or constraints.

As already noted, a solution that is feasible to the ARCID is also feasible for the corresponding ARC (without estimation error). Furthermore, their optimal values coincide if the estimates are exact, i.e., $\widehat{\mathcal{Z}} = \{0\}$ or $\hat{a} = a$ for all possible realizations of a. So (3.1) and (3.4) are equivalent for $\widehat{\mathcal{Z}} = \{0\}$. Similarly, it can be shown that the RC and the ARCID model coincide if $\mathcal{U}_0 \subset \{\widehat{\mathcal{Z}} + \tilde{a}\}$ for some $\tilde{a} \in \mathcal{U}_1$ (the region specifying the estimation error is very large). Therefore, in that case (3.1) is equivalent to

$$\delta^*(v|\mathcal{U}_0) - f_*(v,x) + d^\top y \leq 0,$$

which is again the tractable robust counterpart of the nonadjustable constraint (2.3).

We apply Theorem 3.1 to a fully linear constraint with an affine decision rule based on inexact data to show its practical use. For examples involving nonlinear functions f and the calculation of support functions δ^* for some commonly used uncertainty sets, see Appendix B.

Corollary 3.3. *Consider the linear constraint*

$$a^\top x + d^\top y(\hat{a}) \leq 0, \qquad \forall (a,\hat{a}) \in \mathcal{U}, \tag{3.5}$$

with $x \in \mathbb{R}^m$ the nonadjustable variable, $d \in \mathbb{R}^p$, $\mathcal{U}$ a closed and convex uncertainty set as in (2.11) such that $ri(\mathcal{U}) \neq \emptyset$, $y(\hat{a}) \in \mathbb{R}^p$ and affine decision rule: $y(\hat{a}) = q + Q\hat{a}$ with $q \in \mathbb{R}^p$ and $Q \in \mathbb{R}^{p \times m}$. Then (x,q,Q) satisfies (3.5) if and only if there exist $v^1 \in \mathbb{R}^m$ such that

$$d^\top q + \delta^*(x - v^1|\mathcal{U}_0) + \delta^*(Q^\top d + v^1|\mathcal{U}_1) + \delta^*(v^1|\widehat{\mathcal{Z}}) \leq 0. \tag{3.6}$$

Proof. To use Theorem 3.1, we define the function $f(a,x) = a^\top x$. Note that

$$\mathrm{ri}(\mathrm{dom}\, f(.,x)) = \mathbb{R}^m,$$

so assumption (2.12) is satisfied because $\mathrm{ri}(\mathcal{U}) \neq \emptyset$. The conjugate function, with respect to the

first argument, is:

$$f_*(v, x) = \begin{cases} 0 & \text{if } v = x \\ -\infty & \text{otherwise.} \end{cases}$$

Substituting f_* into (3.1), we obtain the affinily adjustable robust counterparts with decision rules based on inexact data (AARCID):

$$\begin{cases} d^\top q + \delta^*(v^0 | \mathcal{U}_0) + \delta^*(Q^\top d + v^1 | \mathcal{U}_1) + \delta^*(v^1 | \hat{\mathcal{Z}}) \leq 0 \\ v^0 + v^1 = x. \end{cases}$$

The result then follows by eliminating the variable v^0. $\qquad\square$

As noted in Section 2.2, all results can be readily extended to uncertainty sets as in (2.6) and decision rules based on primitive uncertain parameters. The following corollary illustrates this extension for a fully linear function with affine decision rules.

Corollary 3.4. *Consider the affine constraint*

$$(a^0 + A\zeta)^\top x + d^\top y(\hat{\zeta}) \leq 0, \qquad \forall (\zeta, \hat{\zeta}) \in \mathcal{V}, \tag{3.7}$$

with $a^0 \in \mathbb{R}^m$ the nominal value, $A \in \mathbb{R}^{m \times L}$, $\zeta \in \mathbb{R}^L$ the primitive uncertainty, $x \in \mathbb{R}^m$ the nonadjustable variable, $d \in \mathbb{R}^p$, $\mathcal{V} = \{(\zeta, \hat{\zeta}) : \zeta \in \mathcal{V}_0, \hat{\zeta} \in \mathcal{V}_1, \zeta - \hat{\zeta} \in \widehat{\mathcal{W}}\}$, where $\mathcal{V}_0$, $\mathcal{V}_1$ and $\widehat{\mathcal{W}}$ are closed and convex such that $ri(\mathcal{V}) \neq \emptyset$, $y(\hat{\zeta}) \in \mathbb{R}^p$ the adjustable variable and affine decision rule: $y(\hat{\zeta}) = r + R\hat{\zeta}$ with $r \in \mathbb{R}^L$ and $R \in \mathbb{R}^{L \times m}$. Then (x, q, Q) satisfies (3.7) if and only if there exists $v^1 \in \mathbb{R}^m$ such that

$$(a^0)^\top x + d^\top r + \delta^*(A^\top x - v^1 | \mathcal{V}_0) + \delta^*(R^\top d + v^1 | \mathcal{V}_1) + \delta^*(v^1 | \widehat{\mathcal{W}}) \leq 0. \tag{3.8}$$

$\qquad\square$

For the AARCID constraint (3.8) we only had to introduce m variables more than in the corresponding AARC constraint (with $\mathcal{W} = \{0\}$). Furthermore, the AARC constraint can be infeasible if the information is inexact (all situations for which $\widehat{\mathcal{W}} \neq \{0\}$).

In many practical cases the estimates of an uncertain parameter become more accurate over time and decisions are made based on estimates with decreasing estimation error as time passes by. This is for instance the case in our numerical example in Section 4. The nonadjustable version of the constraints we encounter are of the form:

$$f(a, x) + \sum_{k=1}^{K} (d^k)^\top y^k(\hat{a}^k) \leq 0, \qquad \forall (a, \hat{a}^1, \dots, \hat{a}^K) \in \mathcal{U}, \tag{3.9}$$

where $x \in \mathbb{R}^n$, $\hat{a}^k \in \mathbb{R}^m$ the estimate for $a \in \mathbb{R}^m$ in the k-th period and $y^k(\hat{a}^k) \in \mathbb{R}^p$ adjustable for all $k \in \{1, \ldots, K\}$. The uncertain parameter and its estimates $(a, \hat{a}^1, \ldots, \hat{a}^K)$ are an element of the uncertainty set $\mathcal{U}$ which is given by

$$\mathcal{U} = \left\{ (a, \hat{a}^1, \ldots, \hat{a}^K) : a \in \mathcal{U}_0, \ \hat{a}^k \in \mathcal{U}_k, \ (a - \hat{a}^k) \in \widehat{\mathcal{Z}}_k, \quad \forall k \in \{1, \ldots, K\} \right\}, \tag{3.10}$$

where $\mathcal{U}_0$ is the uncertainty set for the parameter a and $\mathcal{U}_k$ the set that specifies the "estimation range" of the k-th estimate $\hat{a}^k$. The set $\widehat{\mathcal{Z}}_k$ specifies the error of the true parameter a in the k-th period. Normally we have decreasing estimation error over time which would be naturally modeled by the decreasing sequence of sets $\widehat{Z}_1 \supseteq \widehat{\mathcal{Z}}_2 \supseteq \cdots \supseteq \widehat{\mathcal{Z}}_K \supseteq \{0\}$. We then have K estimates $\hat{a}^1, \ldots, \hat{a}^K$ of the parameter a and in period k we decide on the value of the adjustable variables y^k via an affine decision rule depending on the estimate in period k:

$$y^k(\hat{a}^k) = s^k + S_k \hat{a}^k,$$

where $s^k \in \mathbb{R}^p$ and $S_k \in \mathbb{R}^{p \times m}$ are the new nonadjustable variables. With this decision rule (3.9) can be written as:

$$\max_{(a, \widehat{a}^1, \ldots, \widehat{a}^K) \in \mathcal{U}} \left\{ f(a, x) + \sum_{k=1}^{K} (d^k)^\top \left(s^k + S_k \hat{a}^k \right) \right\} \leq 0. \tag{3.11}$$

Since the dimension of the uncertainty set has changed, the regularity condition has to be written in the following way:

$$\mathrm{ri}(\mathcal{U}) \cap \{\mathrm{ri}(\mathrm{dom}\, f(., x)) \times R^m \times \cdots \times R^m\} \neq \emptyset, \ \forall x \in \mathbb{R}^n, \tag{3.12}$$

To cover this setting, Theorem 3.1 can be extended in the following way.

Theorem 3.5. *Let $\mathcal{U}$ be as defined in* (3.10) *and let $f(a, x)$ be a closed concave function in a such that the regularity condition* (3.12) *is satisfied. Then $x, s^1, \ldots s^K, S_1, \ldots S_K$ satisfy* (3.11) *if and only if there exist $v^0, \ldots, v^K \in \mathbb{R}^m$ that satisfy:*

$$\sum_{k=1}^{K} (d^k)^\top s^k + \delta^*(v^0|\mathcal{U}_0) + \sum_{k=1}^{K} \delta^*(S_k^\top d^k + v^k|\mathcal{U}_k) + \sum_{k=1}^{K} \delta^*(v^k|\widehat{\mathcal{Z}}_k) - f_* \left(\sum_{k=0}^{K} v^k, x \right) \leq 0. \tag{3.13}$$

$\square$

4   Test case: a multi-period inventory model

In this section we apply the ARCID approach on a numerical example to show its potential use in practical applications and its benefits over the ARC approach with decision rules based on exact

information. We focus on the test problem by Ben-Tal et al. (2004), the first paper where the ARC approach is applied.

4.1   The nominal model

We consider a single product inventory system, which is comprised of a warehouse and I factories. A planning horizon of T periods is used. In the model we use the following parameters and variables, using the same notation as in Ben-Tal et al. (2004):

Parameters

d_t	Demand for the product in time period t;
$P_i(t)$	Production capacity of factory i in time period t;
$c_i(t)$	Cost of producing one product unit at factory i in time period t;
V_{min}	Minimal allowed level of inventory at the warehouse;
V_{max}	Storage capacity of the warehouse;
Q_i	Cumulative production capacity of the i-th factory throughout the planning horizon.

Variables

$p_i(t)$	The amount of the product to be produced in factory i in period t;
$v(t)$	The amount of products in the warehouse at the beginning of period t ($v(1)$ is given).

We try to minimize the total production costs over all factories and the whole planning horizon. The restriction is that all demand in period t must be satisfied by units on stock in the warehouse or by the production in period t. If all the demand, and all other parameters, are certain in all periods $1, \ldots, T$, then the problem is modeled by the following linear program (Ben-Tal et al. (2004)

Section 5):

$$\min_{p_i(t),v(t),F} \quad F$$

$$\text{s.t.} \quad \sum_{t=1}^{T}\sum_{i=1}^{I} c_i(t)p_i(t) \leq F$$

$$0 \leq p_i(t) \leq P_i(t), \qquad i = 1,\ldots,I, t = 1,\ldots,T$$

$$\sum_{t=1}^{T} p_i(t) \leq Q_i, \qquad i = 1,\ldots,I$$

$$v(t+1) = v(t) + \sum_{i=1}^{I} p_i(t) - d_t, \qquad t = 1,\ldots,T$$

$$V_{min} \leq v(t) \leq V_{max}, \qquad t = 2,\ldots,T+1. $$

(4.1)

4.2 The affinily adjustable robust model based on inexact data

We assume that we can make decisions based on estimates of the realized demand scenario $d = (d_1,\ldots,d_T)$. We should specify our production policies for the factories before the planning periods starts, at period 0. When we specify these policies we only know that the demand in consecutive periods are independent and reside in a certain box region,

$$d_t \in \mathcal{V}_t = [d_t^* - \theta d_t^*, d_t^* + \theta d_t^*], \qquad t = 1,\ldots,T, \tag{4.2}$$

with given $0 < \theta \leq 1$, the level of uncertainty, and nominal demand d_t^* in period t. So far the model is exactly the same as in Ben-Tal et al. (2004) if we assume that we can estimate the demand d_t exactly in periods $r \in I_t$, where I_t is a given subset of $\{1,\ldots,T\}$. In Ben-Tal et al. (2004) different sets for I_t are used:

- $I_t = \{1,\ldots,t\}$, the information basis where demand from the past and the present is known exactly, for the future no extra information is known;

- $I_t = \{1,\ldots,t-1\}$, the information basis where all demand from the past is known exactly, there is no information about the present;

- $I_t = \{1,\ldots,t-4\}$, the information about the past is received with a four day delay. About other periods in the past ($t-3, t-2$ and $t-1$) there is no extra information at all.

Now we assume the decisions in period t are based on estimates $\widehat{d}_{r,t}$, made in period t, for the actual demand d_r in the period $r \in \{1,\ldots,t\}$. We assume that these estimates can in principle take any value that the demand d_r can take, so $\widehat{d}_{r,t} \in \mathcal{V}_r$ and that the estimation error $\widehat{d}_{r,t} - d_r$ lies

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in a box region:

$$\widehat{d}_{r,t} - d_r \in \widehat{\mathcal{Z}}_{r,t} = [-\rho_{r,t}d_t^*, \rho_{r,t}d_t^*], \tag{4.3}$$

where the parameter $\rho_{r,t}$ indicates the inaccuracy of the estimate $\widehat{d}_{r,t}$.

Note that if we have exact information for periods in the information basis, i.e. $\widehat{d}_{r,t} = d_t$ for all $r \in I_t$ and no extra information (besides $\widehat{d}_{r,t} \in \mathcal{V}_t$) for all periods outside the information basis, then we end up in the case of exact revealed information as considered by Ben-Tal et al. (2004). This situation can be can be modeled as a special case of our model by using the following values for $\rho_{r,t}$

$$\rho_{r,t} = \begin{cases} 0 & \text{if } r \in I_t \\ \theta & \text{otherwise,} \end{cases}$$

i.e. the estimation error equals zero for estimates on demand in periods that lie in the information set and it is θ, so very large, for periods outside this information basis.

The general situation with inexact data in between the two extreme scenarios where one either knows the demand exact, or not at all. For this we specify the information set in a more general way:

$$\hat{I}_t := \{r : \rho_{r,t} < \theta\}.$$

This definition of $\hat{I}_t$ is indeed a more general description. For large estimation errors ($\rho_{r,t} \geq \theta$) we could just as well decide on the variables beforehand, i.e., we have no extra information on the actual realizations compared to the information at time $t = 0$. We can therefore safely exclude all periods where the estimates are too noisy (the periods for which $r \notin \hat{I}_t$). Since we apply the AARCID method based on inexact data, we take affine decision rules based on *inexact* estimates:

$$p_i(t) = \pi_{i,t}^0 + \sum_{r \in \hat{I}_t} \pi_{i,t}^r \widehat{d}_{r,t}, \tag{4.4}$$

where the coefficients $\pi_{i,t}^r$ are the new nonadjustable variables in the model. For notational convenience we write the vector $\widehat{d}$ as the vector containing all the estimates $\widehat{d}_{r,t}$ for all $r \in \hat{I}_t$, $t = 1, \ldots, T$. The uncertainty set can now be written as:

$$\mathcal{U} := \left\{ (d, \widehat{d}) : d_r, \widehat{d}_{r,t} \in \mathcal{V}_r, (\widehat{d}_{r,t} - d_r) \in \widehat{\mathcal{Z}}_{r,t}, \quad \forall r \in \hat{I}_t, \quad t = 1, \ldots, T \right\},$$

with $\mathcal{V}_t$ and $\widehat{\mathcal{Z}}_{r,t}$ as specified in respectively (4.2) and (4.3). The linear program in (4.1) becomes (after elimination of the v-variables) an uncertain Linear Programming problem in the variables

$\pi_{i,t}^s$ and F if we use the decision rule (4.4):

$$
\left.
\begin{aligned}
&\min_{\pi,F} \quad F \\
&\text{s.t.} \quad \sum_{t=1}^{T}\sum_{i=1}^{I} c_i(t)\left(\pi_{i,t}^0 + \sum_{r\in\hat{I}_t}\pi_{i,t}^r \widehat{d}_{r,t}\right) \leq F \\
&\qquad 0 \leq \pi_{i,t}^0 + \sum_{r\in\hat{I}_t}\pi_{i,t}^r\widehat{d}_{r,t} \leq P_i(t), \qquad i=1,\dots,I, t=1,\dots,T \\
&\qquad \sum_{t=1}^{T}\left(\pi_{i,t}^0 + \sum_{r\in\hat{I}_t}\pi_{i,t}^r\widehat{d}_{r,t}\right) \leq Q_i, \qquad i=1,\dots,I \\
&\qquad V_{min} \leq v(1) + \sum_{s=1}^{t}\sum_{i=1}^{I}\left(\pi_{i,s}^0 + \sum_{r\in\hat{I}_t}\pi_{i,s}^r\widehat{d}_{r,s}\right) - \sum_{s=1}^{t} d_s \leq V_{max}, \qquad t=1,\dots,T
\end{aligned}
\right\} \forall (d,\widehat{d})\in\mathcal{U}.
$$

$$(4.5)$$

The tractable robust counterpart of this problem with decision rules based on inexact data can be derived by Theorem 3.5 and is given in Appendix C.

4.3 Data set from Ben-Tal et al. (2004)

We take the same data set as in the illustrative example by Ben-Tal et al. (2004, p.370-371):

"There are $I = 3$ factories producing a seasonal product, and one warehouse. The decisions concerning production are made every two weeks, and we are planning production for 48 weeks, thus the time horizon is $T = 24$ periods. The nominal demand d^* is seasonal, reaching its maximum in winter, specifically,

$$
d_t^* = 1000\left(1 + \tfrac{1}{2}\sin\left(\tfrac{\pi(t-1)}{12}\right)\right), \quad t=1,\dots,24.
$$

We assume that the uncertainty level θ is 20%, i.e., $d_t \in [0.8d_t^*, 1.2d_t^*]$, as shown on Figure 2.
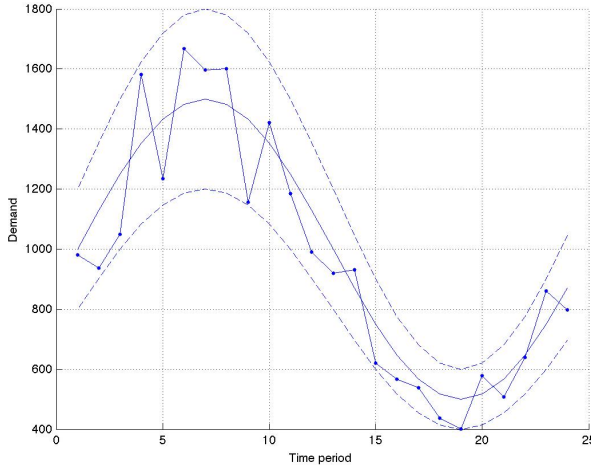
- nominal demand (solid)
- "demand tube" - nominal demand ±20% (dashed)
- a sample realization of actual demand (dotted)
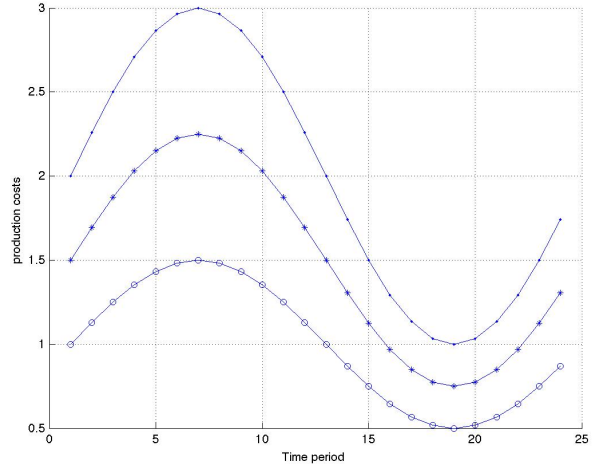
Figure 2: Demand.

Figure 3: Costs.

The production cost per unit of the product depend on the factory and on time and follow the same seasonal pattern as the demand, i.e., rise in winter and fall in summer. The production cost for a factory i at a period t is given by:

$$c_i(t) = \alpha_i \left(1 + \tfrac{1}{2} \sin \left(\tfrac{\pi(t-1)}{12} \right) \right), \quad t = 1, \ldots, 24.$$

$$\alpha_1 = 1$$

$$\alpha_2 = 1.5$$

$$\alpha_3 = 2$$

The maximal production capacity of each one of the factories at each two-weeks period is $P_i(t) = 567$ units, and the integral production capacity of each one of the factories for a year is $Q_i = 13600$. The inventory at the warehouse should be no less than 500 units, and cannot exceed 2000 units."

5 Numerical results for the test cases

Ben-Tal et al. (2004) conduct two series of experiments based on the data given in Section 4.3. In the first series of experiments they modify the parameter θ to analyze the influence of demand uncertainty on the total production cost. In the second series of experiments they change the information basis I_t, the (exact) information that is used in the decision rule. Note that Ben-Tal et al. (2004) deal with the case where in period t all demand from the periods in the information set I_t is known *exactly*. For instance, if the information set is equal to $I_t = \{1, \ldots, t-1\}$, then in

period t we can base our production decision rule on the exact values of the demand realizations in periods $1, \ldots, t-1$, and use no information on the demand in periods after $t-1$. We extend these experiments to include inexact data in some periods to show the benefits of the ARCID model over the ARC model.

Just as in Ben-Tal et al. (2004), we test the management policies by simulating 100 demand trajectories, $d = (d_1, \ldots, d_T)$. For every simulation the demand trajectory is randomly generated with d_t uniformly distributed in $[(1-\theta)d_t^*, (1+\theta)d_t^*]$, where 20% ($\theta = 0.2$) is the chosen uncertainty level. The uncertainty level of the demand is set to 20% in all experiments, as this seems to be the most restrictive level of uncertainty and is the same level that has been used by Ben-Tal et al. (2004). For higher uncertainty levels like 30%, even the nominal model (4.1) is no longer feasible for the maximal demand pattern with $d_t = (1+\theta)d_t^*$ (without uncertainty) because of the bounds on production imposed by $P_i(t)$ and Q_i. In line with the experiments performed by Ben-Tal et al., 2004, we compute the *average* costs for our solutions over 100 simulated demand trajectories. The costs resulting from this randomization could differ slightly if one uses another set of random generated trajectories. Alternatively, one could just as well have put the costs for the nominal demand here which equals the expected costs of a random demand tracjectory. All solutions are obtained by the commercial solver Gurobi Optimization (2013) programmed in the YALMIP language (Löfberg, 2004) in MATLAB$^{©}$ on a 64-bit Windows 7 PC with a 2.8 Ghz Intel Core i7 processor and 4GB of RAM.

5.1 Experiments with decision rules using inexact data on demand

Similar to Ben-Tal et al. (2004), we saved the demand trajectories to compute the so-called costs of the Ideal setting, the utopian world where the entire demand trajectory is known beforehand. The Ideal setting is used to benchmark the performance of the ARCID solution. In the Ideal setting one sets the policy only for one sample demand realization, so the solution does not have to be feasible for all possible demand trajectories. Hence, the costs in the Ideal setting are obviously a lower bound of the costs for the ARCID solutions. For the Ideal setting the worst case is the demand trajectory with the highest demand: $d_t = (1+\theta)d_t^*$ for all t. The worst case costs in the Ideal setting can be easily solved and turned out to be 44,199 and the mean costs, the average over the 100 simulated demand trajectories, 33,729.

In our model, the demand from the past periods is not known exact, as in Ben-Tal et al. (2004), but we assume to have inexact estimates for some past and present periods. Several cases are investigated, for instance those where the delay for receiving the exact demand information is even more than 2 periods, i.e., the *exact* demand is known after 3, 4 or more periods. These cases were infeasible in the ARC model, see Ben-Tal et al. (2004).

In the experiments, the influence of the estimation error $\rho_{r,t}$ on the total production costs is tested.

An estimation error of 0% for the demand in period $t-1$ means that $\rho_{t-1,t} = 0$ (exact information). An estimation uncertainty of 10% for the demand in period $t-4$ means that $\rho_{t-4,t} = 0.1\theta$ and so forth. Table 1 presents some cases with different percentages of estimation uncertainty.

We draw the estimates on demand from a uniform distribution as well, using the same simulated actual demand trajectories across all cases. In every period t we know for the estimate $\widehat{d}_{r,t}$ on the simulated demand in period r that $\widehat{d}_{r,t} - d_r \in [-\rho_{r,t}d_r^*, \rho_{r,t}d_r^*]$, where the value d_r is taken from the earlier simulated demand patterns. Furthermore, $\widehat{d}_{r,t}$ resides in the box region $[(1-\theta)d_r^*, (1+\theta)d_r^*]$. The estimates are therefore uniformly drawn from the region:

$$[d_r - \rho_{r,t}d_r^*, d_r + \rho_{r,t}d_r^*] \cap [(1-\theta)d_r^*, (1+\theta)d_r^*],$$

which can also be written as

$$[\max(d_r - \rho_{r,t}d_r^*, (1-\theta)d_r^*), \min(d_r + \rho_{r,t}d_r^*, (1+\theta)d_r^*)].$$

Table 1: The influence of the estimation errors on the mean costs and worst case costs (WC) in the ARCID model.

| Cases | Demand estimation error $\rho_{r,t}$ (in %) | | | | | | | | | | Costs | |
	$\rho_{1,t},\ldots,\rho_{t-9,t}$	$\rho_{t-8,t}$	$\rho_{t-7,t}$	$\rho_{t-6,t}$	$\rho_{t-5,t}$	$\rho_{t-4,t}$	$\rho_{t-3,t}$	$\rho_{t-2,t}$	$\rho_{t-1,t}$	$\rho_{t,t}$	Mean	WC
1	0	0	0	0	0	0	0	0	0	0	34,528	44,199
2	0	0	0	0	0	0	0	0	0	1	35,166	44,207
3	0	0	0	0	0	0	0	0	0	5	35,096	44,239
4	0	0	0	0	0	0	0	0	0	10	34,994	44,268
5	0	0	0	0	0	0	0	0	0	20	34,905	44,273
6	0	0	0	0	0	0	0	0	0	-	34,905	44,273
7	0	0	0	0	0	0	0	0	1	-	36,120	44,310
8	0	0	0	0	0	0	0	0	5	-	36,139	44,438
9	0	0	0	0	0	0	0	0	10	-	35,894	44,554
10	0	0	0	0	0	0	0	0	20	-	35,557	44,582
11	0	0	0	0	0	0	0	0	-	-	35,558	44,582
12	0	0	0	0	0	0	0	1	-	-	36,498	44,638
13	0	0	0	0	0	0	0	5	-	-	36,595	44,955
14	0	0	0	0	0	0	0	10	-	-	Infeasible	Infeasible
15	0	0	0	0	0	0	0	20	-	-	Infeasible	Infeasible
16	0	0	0	0	0	0	0	-	-	-	Infeasible	Infeasible
17	0	1	1	1	1	1	1	1	1	-	36,498	44,364
18	0	5	5	5	5	5	5	5	5	-	36,584	44,764
19	0	10	10	10	10	10	10	10	10	-	36,686	45,429
20	0	0	0	0	0	0	0	1	5	-	36,673	44,474
21	0	0	0	0	0	0	1	5	10	-	36,693	44,883
22	0	0	0	0	0	5	5	10	10	-	36,667	45,220
23	0	0	0	5	5	5	10	10	10	-	36,673	45,326

The cases in Table 1 can be explained in the following way:

- Case 1, 6, 11 and 16 are equivalent to the uncertainty sets from Ben-Tal et al. (2004) with

exact revealed information and the information sets being respectively $\{1, \ldots, t\}$, $\{1, \ldots, t-1\}$, $\{1, \ldots, t-2\}$, $\{1, \ldots, t-3\}$.

- For Cases 2-5 we assume that all demand from the past is known exactly. For the present period we have a good estimate on the demand that gives extra information compared to the information known at the start of the planning period ($t = 0$).

- The Cases 6-23 assume to have no additional knowledge about the present. Furthermore, the exact demand from previous periods is received with a certain delay, but there are already estimates on the demand available before this information is received.

To compare the solutions in different cases we have to take into account that there could be multiple optimal solutions. These solutions all give the same worst case costs, but could perform differently on individual demand trajectories and therefore also result in different mean costs. To overcome this problem, we used the two step approach that has been given in Iancu and Trichakis (2013) and de Ruiter (2013). In this two step approach one first minimizes the worst case costs as usual in robust optimization. To choose one solution among the set of robustly optimal solutions that performs good on average, a second step is introduced. In this second step we add a constraint that the worst case costs do not exceed the optimal worst case costs and we replace the objective by the costs attained for the nominal demand. If in the second step the costs are minimized for the nominal demand, then one obtains the costs that are best for the mean.

The mean costs in Table 1 show a strange pattern among the different cases at first sight. For instance, Case 2 produces higher mean costs than Case 6, but the estimation error is much less! This phenomenon can be explained in the following way. In the two step approach we first search for a solution with minimal worst case costs F^* and then we search among all solutions with worst case costs F^* for the solution that minimizes the nominal demand trajectory. Any additional, more exact information is used to decrease the worst case costs, possibly at the cost of the average behavior.

5.2 Comparison with affinily adjustable robust model based on exact data

For each case we compare the WC costs and feasibility of the AARCID to the costs and feasibility resulting from the AARC approach, where one is only allowed to use the estimates that are exact (estimates with an estimation error of 0%). Hence, for the AARC solutions we only included the exact estimates, those corresponding with $\rho_{r,s} = 0$, in the decision rule. The results are given in Table 2.

Table 2: WC costs of the AARC model and the AARCID model for each case.

| Cases	Worst case costs		Cases	Worst case costs	
	AARC	AARCID		AARC	AARCID
1	44,199	44,199	13	Infeasible	44,955
2	44,273	44,207	14	Infeasible	Infeasible
3	44,273	44,239	15	Infeasible	Infeasible
4	44,273	44,268	16	Infeasible	Infeasible
5	44,273	44,273	17	Infeasible	44,364
6	44,273	44,273	18	Infeasible	44,764
7	44,582	44,310	19	Infeasible	45,429
8	44,582	44,438	20	Infeasible	44,474
9	44,582	44,554	21	Infeasible	44,883
10	44,582	44,582	22	Infeasible	45,220
11	44,582	44,582	23	Infeasible	45,326
12	Infeasible	44,638			

The Cases 1, 6, 11 and 16 only deal with exact estimates. In Section 3 we already derived theoretically that the AARCID and the AARC are equivalent in those cases, which is also reflected in their worst case costs. That also explains why Case 16, which was infeasible in the AARC, see Ben-Tal et al. (2004), is still infeasible for the AARCID. There are many situations, namely in Case 12, 13, and Cases 17 up to 23, where the AARCID use the extra inexact data to produce feasible solutions whereas the AARC is infeasible.

For the cases where both the AARC and the AARCID model are feasible, we notice that there is only a minor improvement in the worst case costs. If both the AARC and the AARCID are feasible, then the question might rise whether we can neglect the estimation error and just apply the AARC model from Ben-Tal et al. (2004). In contrast to the AARC that we used to obtain the results in Table 2, now take the information set for the AARC that includes all (estimated) demands that have an estimation error less than 100%. Hence, all estimation errors strictly between 0% and 100% are neglected and the corresponding demand estimates are used as if they were exact. The only constraints that are different in the AARCID model, see (AARCID-BT) in Appendix C, compared to the AARC model are those involving the minimum and maximum allowed inventory level. For each case we check for how many demand trajectories, out of the 100 simulated realizations, the inventory level is lower than the minimimum inventory level V_{min} of 500 at some point in the planning period. We also give the percentage of demand trajectories where the inventory level exceeds the maximum inventory level V_{max} of 2000 in some periods. The results are given in Table 3. In Cases 1, 6, 11 and 16 there are no violations, since these are equivalent to the AARC based on exact information as we argued in Section 5.1. The ARC solution in Case 12 does not violate the constraints as well, but this is probably due to the fact that the only estimation error unequal to 0% or 100 is the one in period $t - 2$ and that this estimation uncertainty is only 1%. From

Table 3: Percentage of demand trajectories that violate the minimum required inventory level (V_{min}) and maximum allowed inventory level (V_{max}) when neglecting estimation errors.

Cases	Percentage of demand trajectories that violate the bounds		Cases	Percentage of demand trajectories that violate the bounds	
	V_{min}	V_{max}		V_{min}	V_{max}
1	0	0	13	0	2
2	69	47	14	2	5
3	70	44	15	5	11
4	64	55	16	Infeasible	Infeasible
5	80	38	17	2	2
6	0	0	18	18	15
7	4	3	19	26	20
8	12	11	20	14	7
9	30	17	21	27	15
10	42	38	22	30	19
11	0	0	23	26	15
12	0	0			

Table 3 we also see that constraints are violated more often when the estimation uncertainty is in the recent periods t and $t-1$. For example Case 3, which has only 5% estimation uncertainty in period t, the solution violated the minimum required inventory level 70 out of 100 times and for 44 simulated demand trajectories the stock level exceeded maximum allowed inventory level. The inventory levels for 100 trajectories of Case 3 are depicted in Figure 4 for both the AARCID and the AARC that neglects the estimation errors.
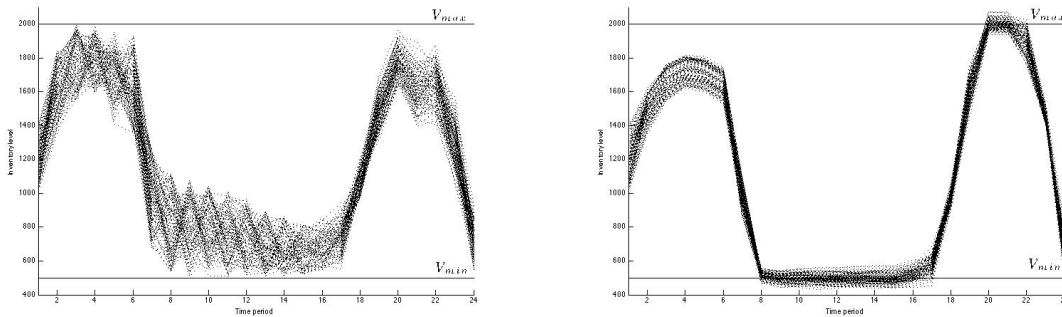
(a) AARCID solution taking estimation errors into account.

(b) AARC solution neglecting the estimation errors.

- horizontal lines: min. required inventory level (V_{min}) and max. allowed inventory level (V_{max})

- dotted lines: Sample realizations of demand trajectories

Figure 4: Bounds on inventory level are violated if estimation error is neglected in Case 3.

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6 Conclusions and further research

In this paper we extend the adjustable robust counterpart (ARC) method to the (ARCID) method in which the decision rules are based on *inexact* revealed data. A small simple example showed that ARCID generates a significantly lower worst case cost than the corresponding ARC solution. Moreover, the results from our production problem demonstrated that ARCID outperforms ARC on feasibility as well: 9 out of 12 cases that were infeasible for the ARC solution, were feasible for the ARCID solution. It is evident that neglecting to take into account the inexact nature of the revealed data may have severe consequences. For example, in our numerical testing of the production application, the inventory level dropped below the allowed minimum in up to 80% of the simulated demand trajectories.

The use of the ARCID method is thus well justified in particular so since the resulting optimization problem that need to be solved maintain a comparable tractability status to that of the ARC method.

References

Ben-Tal, A., L. El Ghaoui, and A. Nemirovski (2009). *Robust Optimization*. Princeton Series in Applied Mathematics. Princeton University Press.

Ben-Tal, A., D. den Hertog, and J.-Ph. Vial (2012). Deriving robust counterparts of nonlinear uncertain inequalities. *CentER Discussion Paper* 2012-53. Tilburg University.

Ben-Tal, A., A. Goryashko, E. Guslitzer, and A. Nemirovski (2004). Adjustable robust solutions of uncertain linear programs. *Mathematical Programming* 99 (2), pp. 351–376.

Bertsimas, D., D. A. Iancu, and P. A. Parrilo (2011a). A hierarchy of near-optimal policies for multistage adaptive optimization. *Automatic Control, IEEE Transactions on* 56 (12), pp. 2809–2824.

Bertsimas, D., D. B. Brown, and C. Caramanis (2011b). Theory and applications of robust optimization. *SIAM Review* 53 (3), pp. 464–501.

Beyer, H.-G. and B. Sendhoff (2007). Robust optimization–a comprehensive survey. *Computer Methods in Applied Mechanics and Engineering* 196 (33 - 34), pp. 3190 –3218.

De Ruiter, F. J. C. T. (2013). Adjustable Robust Optimization Based on Inexact Information. MSc thesis. Tilburg School of Economics and Management.

DeHoratius, N. and A. Raman (2008). Inventory record inaccuracy: an empirical analysis. *Management Science* 54 (4), pp. 627–641.

Goh, J. and M. Sim (2010). Distributionally robust optimization and its tractable approximations. *Operations Research* 58 (4-Part-1), pp. 902–917.

Gurobi Optimization, Inc. (2013). *Gurobi Optimizer Reference Manual*.

Haug, A., F. Zachariassen, and D. van Liempd (2011). The costs of poor data quality. *Journal of Industrial Engineering and Management* 4 (2), pp. 168–193.

Iancu, D. and N. Trichakis (2013). Pareto efficiency in robust optimization. *To appear in Management Science.*

Löfberg, J. (2004). YALMIP–a toolbox for modeling and optimization in MATLAB. *Proceedings of the CACSD Conference.* Taipei, Taiwan.

Redman, T. C. (1998). The impact of poor data quality on the typical enterprise. *Communications of the ACM* 41 (2), pp. 79–82.

Rockafellar, R. T. (1970). *Convex Analysis.* Princeton University Press.

Soffer, P. (2010). Mirror, mirror on the wall, can I count on you at all? Exploring data inaccuracy in business processes. *Enterprise, Business-Process and Information Systems Modeling.* Springer, pp. 14–25.

Vawdrey, D. K., R. M. Gardner, R. S. Evans, J. F. Orme Jr, T. P. Clemmer, L. Greenway, and F. A. Drews (2007). Assessing data quality in manual entry of ventilator settings. *Journal of the American Medical Informatics Association* 14 (3), pp. 295–303.

Appendices

A   Results on Fenchel duality and conjugate functions

We first present some results concerning (nonadjustable) robust counterparts from Ben-Tal et al. (2012). The nonadjustable constraints are of the type:

$$f(a, x) \leq 0 \qquad \forall a \in \mathcal{U}_0, \tag{A.1}$$

which all satisfy the following regularity condition:

$$\text{ri}(\mathcal{U}_0) \cap \text{ri}(\text{dom}\, f(., x)) \neq \emptyset, \ \forall x. \tag{A.2}$$

The uncertainty set $\mathcal{U}_0$ could for instance be of the form:

$$\mathcal{U}_0 = \{a = a^0 + A\zeta \mid \zeta \in \mathcal{V}_0\}, \tag{A.3}$$

where $A \in \mathbb{R}^{m \times L}$, $\mathcal{V}_0 \subset \mathbb{R}^L$ is a nonempty, convex and closed set, a^0 the *nominal value* and ζ the vector of *primitive uncertainties.* Ben-Tal et al. (2012) then derived the robust counterpart by Fenchel duality:

Theorem A.1. *Let $f(a, x)$ be a closed concave function in $a \in \mathbb{R}^m$ for every $x \in \mathbb{R}^n$ and let $\mathcal{U}_0$*

be a closed convex uncertainty set such that the regularity condition (A.2) is satisfied. Then $x \in \mathbb{R}^n$ satisfies (A.1) if and only if $\exists v \in \mathbb{R}^m$ such that x and v satisfy

$$\delta^*(v|\mathcal{U}_0) - f_*(v, x) \leq 0. \tag{A.4}$$

Furthermore, if $\mathcal{U}_0$ is as in (A.3), then (A.4) is equivalent to

$$(a^0)^\top v + \delta^*(A^\top v|\mathcal{V}_0) - f_*(v, x) \leq 0. \tag{A.5}$$

$\square$

To derive the tractable robust counterpart for constraints with decision rules based on inexact information we need the following lemma which is useful in determining the convex conjugate of functions that has the difference of two variables as input.

Lemma A.2. *Let $h(a, \hat{a})$ be such that $h(a, \hat{a}) = \hat{h}(a - \hat{a})$ for some closed convex function $\hat{h}$. Then we have*

$$(h)^*(v^0, v^1) = \begin{cases} (\hat{h})^*(v^0) & \text{if } v^0 + v^1 = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Proof. The conjugate of $h(a, \hat{a})$ with respect to both its arguments is given by

$$
\begin{aligned}
(h)^*(v^0, v^1) &= \sup_{a,\hat{a}}\{(a)^\top v^0 + (\hat{a})^\top v^1 - \hat{h}(a - \hat{a})\} \\
&= \sup_{a,\hat{a}}\{(a - \hat{a})^\top v^0 + (\hat{a})^\top(v^0 + v^1) - \hat{h}(a - \hat{a})\}. \tag{A.6}
\end{aligned}
$$

Now suppose $v^0 + v^1 \neq 0$ and assume w.l.o.g. that $\exists i$ such that $(v^0 + v^1)_i > 0$. Then we take $a = \hat{a}$ to obtain

$$\sup_{a,\hat{a}}\{(a - \hat{a})^\top v^0 + (\hat{a})^\top(v^0 + v^1) - \hat{h}(a - \hat{a})\} \geq (\hat{a})^\top(v^0 + v^1) - \hat{h}(0)$$

$$\to \infty \text{ if } \hat{a}_i \to \infty.$$

Therefore, we derive from (A.6) the following expression:

$$
\begin{aligned}
\sup_{a,\hat{a}}\{(a - \hat{a})^\top v^0 + (\hat{a})^\top(v^0 + v^1) - \hat{h}(a - \hat{a})\} &= \begin{cases} \sup_{a,\hat{a}}\{(a - \hat{a})^\top v^0 - \hat{h}(a - \hat{a})\} & \text{if } v^0 + v^1 = 0 \\ \infty & \text{otherwise} \end{cases} \\
&= \begin{cases} (\hat{h})^*(v^0) & \text{if } v^0 + v^1 = 0 \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

$\square$

Lemmas A.3 and A.4 stated below can be found in Rockafellar (1970, Chapter 16) and are useful for determining the conjugates of functions that are the sum or the minimum of a set of functions.

Lemma A.3 (Convex conjugate of the sum of functions). *Let $f_1, \ldots, f_K$ be proper convex functions such that $\bigcap_{k=1}^{K} ri(dom(f_k)) \neq \emptyset$. Then*

$$(f_1 + \cdots + f_K)^*(y) = \inf_{\{v^1, \ldots, v^K\}} \{f_1^*(v^1) + \cdots + f_K^*(v^K) \mid v^1 + \cdots + v^K = y \}.$$

$\square$

Lemma A.4 (Concave conjugate of sum function). *Let $g_1, \ldots, g_k$ be proper concave functions such that $\bigcap_{k=1}^{K} ri(dom(g_k)) \neq \emptyset$. Then*

$$(g_1 + \cdots + g_K)_*(y) = \sup_{\{v^1, \ldots, v^K\}} \{(g_1)_*(v^1) + \cdots + (g_K)_*(v^K) \mid v^1 + \cdots + v^K = y \}.$$

$\square$

B Examples of conjugate functions and support functions

The concave conjugate of $f(a, x)$ with respect to a is given by

$$f_*(v, x) = \inf_a \{v^\top a - f(a, x)\}$$

and the support function of a set $\mathcal{U}$ is given by

$$\delta^*(v|S) = \sup_{x \in \mathcal{U}} v^\top x.$$

In the tables below we give the results of these derivations for some different choices of f and uncertainty regions $\mathcal{U}$. For a large variety of other examples we refer the reader to Ben-Tal et al. (2012).

Table 4: Examples of functions and their concave conjugates.

Function	$f(a, x)$		$f_*(v, x)$
linear in a	$a^\top f(x)$		$\begin{cases} 0 & \text{if } v = f(x) \\ -\infty & \text{otherwise} \end{cases}$
quadratic in a	$b - \frac{1}{2} \sum_{i=1}^{n} (a^\top Q_i a) x_i$	$(Q_i \succ 0)$	$b - \min_{s^1, \ldots, s^n} \left\{ \frac{1}{2} \sum_{i=1}^{n} \frac{(s^i)^\top Q_i^{-1} s^i}{x_i} \mid \sum_{i=1}^{n} s^i = v \right\}$

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Table 5: Examples of uncertainty sets and their support functions.

Uncertainty set	$\mathcal{U}$	$\delta^*(v\|\mathcal{U})$
box	$\{\zeta : \|\|\zeta\|\|_\infty \leq \alpha\}$	$\alpha\|\|v\|\|_1$
ball	$\{\zeta : \|\|\zeta\|\|_2 \leq \alpha\}$	$\alpha\|\|v\|\|_2$
polyhydral	$\{\zeta : b - B\zeta \geq 0\}$	$\begin{cases} b^\top z & \text{if } B^\top z = v, z \geq 0 \\ \infty & \text{otherwise} \end{cases}$

C The tractable robust counterpart based on inexact data

The uncertain LP (4.5) can be rewritten to

$$
\begin{aligned}
\min_{\pi,F} \quad & F \\
\text{s.t.} \quad & \left. \begin{aligned}
& \sum_{t=1}^{T}\sum_{i=1}^{I} c_i(t)\pi_{i,t}^0 + \sum_{t=1}^{T}\sum_{r\in\hat{I}_t}\left(\sum_{i=1}^{I} c_i(t)\pi_{i,t}^r\right)\widehat{d}_{r,t} - F \leq 0 \\
& \pi_{i,t}^0 + \sum_{r\in\hat{I}_t}\pi_{i,t}^r\widehat{d}_{r,t} \leq P_i(t), \qquad i=1,\ldots,I, t=1,\ldots,T \\
& -\pi_{i,t}^0 - \sum_{r\in\hat{I}_t}\pi_{i,t}^r\widehat{d}_{r,t} \leq 0, \qquad i=1,\ldots,I, t=1,\ldots,T \\
& \sum_{t=1}^{T}\pi_{i,t}^0 + \sum_{t=1}^{T}\sum_{r\in\hat{I}_t}\pi_{i,t}^r\widehat{d}_{r,t} \leq Q_i, \qquad i=1,\ldots,I \\
& \sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^0 + \sum_{r=1}^{t}\sum_{s\leq t, r\in\hat{I}_s}\left(\sum_{i=1}^{I}\pi_{i,s}^r\right)\widehat{d}_{r,s} - \sum_{r=1}^{t}d_r \leq V_{max} - v(1), \qquad t=1,\ldots,T \\
& -\sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^0 - \sum_{r=1}^{t}\sum_{s\leq t, r\in\hat{I}_s}\left(\sum_{i=1}^{I}\pi_{i,s}^r\right)\widehat{d}_{r,s} + \sum_{r=1}^{t}d_r \leq v(1) - V_{min}, \qquad t=1,\ldots,T
\end{aligned} \right\} \quad \forall(d,\widehat{d})\in\mathcal{U},
\end{aligned}
$$

(C.1)

where $\mathcal{U} := \left\{(d,\widehat{d}) : d_r, \widehat{d}_{r,t} \in \mathcal{V}_r, (d_{r,t} - d_r) \in \widehat{\mathcal{Z}}_{r,t}, \quad \forall r \in \hat{I}_t \quad t=1,\ldots,T\right\}$. For deriving the robust counterpart (without uncertainty), we observe that the first four constraints only involve the estimate $\widehat{d}_{r,t}$ and not the actual demand parameter d_r. Hence, those constraints should hold for all $\widehat{d}_{r,t} \in \mathcal{V}_t$. Note that this is the same uncertainty set as for the uncertain parameters d_t, so we could just as well write d_t instead of $\widehat{d}_{r,s}$ in those constraints. Therefore, we can apply the same basic techniques as presented in Ben-Tal et al. (2004) or Ben-Tal et al. (2009) to find their deterministic counterparts.

The derivation of the constraints involving V_{max} and V_{min} do involve both d_t and its estimates $\widehat{d}_{r,s}$. We can find the following equivalent system for any constraint involving V_{max} after introduc-

ing new variables τ_t^r, $\lambda_{s,t}^r$, $\mu_{s,t}^r$, ν_t^r $\omega_{s,t}^r$, η_t^r and ξ_t^r:

$$\sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^0 + \sum_{r=1}^{t}\sum_{s\leq t, r\in\hat{I}_s}\left(\sum_{i=1}^{I}\pi_{i,s}^r\right)\widehat{d}_{r,s} - \sum_{r=1}^{t}d_r \leq V_{max} - v(1) \qquad t = 1,\ldots,T \qquad \forall(d,\widehat{d})\in\mathcal{U}$$

(C.2)

$\Leftrightarrow$

$$\begin{cases} \sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^0 + \sum_{r=1}^{t}\xi_t^r d_r^* + \theta\sum_{r=1}^{t}\eta_t^r d_r^* + \sum_{r=1}^{t}\sum_{s\leq t, r\in I_s}\rho_{r,s}\omega_{s,t}^r d_r^* \leq V_{max} - v(1), 1\leq t\leq T \\ \tau_t^r + \sum_{s\leq t, r\in I_s}\lambda_{s,t}^r = -1,\ \eta_t^r = \nu_t^r + \sum_{s\leq t, r\in I_s}\mu_{s,t}^r, \quad 1\leq r\leq t\leq T \\ \xi_t^r = \sum_{s\leq t, r\in I_s}\sum_{i=1}^{I}\pi_{i,s}^r - 1,\ -\nu_t^r \leq \tau_t^r \leq \nu_t^r, \quad 1\leq r\leq t\leq T \\ -\mu_{s,t}^r \leq \sum_{i=1}^{I}\pi_{i,s}^r + \lambda_{s,t}^r \leq \mu_{s,t}^r,\ -\omega_{s,t}^r \leq \lambda_{s,t}^r \leq \omega_{s,t}^r, \quad r\in I_s, 1\leq s\leq t\leq T. \end{cases}$$

(C.3)

This equivalence is obtained by writing (C.2) in the form as in (2.9) and then applying Theorem 3.5. With this set of constraints we can convert the uncertain LP (C.1) into an equivalent LP, the

AARCID version of the model in Ben-Tal et al. (2004) denoted by (AARCID-BT):

$$\min_{\pi,F,\alpha,\beta,\gamma,\delta,\zeta,\xi,\eta,\mu,\nu} \quad F$$

$s.t.$

$$\sum_{i=1}^{I}\sum_{t:r\in\hat{I}_t} c_i(t)\pi_{i,t}^r = \alpha_r, \ -\beta_r \leq \alpha_r \leq \beta_r, \quad 1 \leq r \leq T, \quad \sum_{t=1}^{T}\sum_{i=1}^{I} c_i(t)\pi_{i,t}^0 + \sum_{r=1}^{T}\alpha_r d_r^* + \theta\sum_{r=1}^{T}\beta_r d_r^* \leq F;$$

$$-\gamma_{i,t}^r \leq \pi_{i,t}^r \leq \gamma_{i,t}^r, \qquad 1 \leq i \leq I, \qquad 1 \leq r,t \leq T;$$

$$\pi_{i,t}^0 + \sum_{r\in\hat{I}_t}\pi_{i,t}^r d_r^* - \theta\sum_{r\in\hat{I}_t}\gamma_{i,t}^r d_r^* \geq 0, \ \pi_{i,t}^0 + \sum_{r\in\hat{I}_t}\pi_{i,t}^r d_r^* + \theta\sum_{r\in\hat{I}_t}\gamma_{i,t}^r d_r^* \leq P_i(t), \qquad 1 \leq i \leq I, \ 1 \leq t \leq T,$$

$$\sum_{t:r\in\hat{I}_t}\pi_{i,t}^r = \delta_i^r, -\zeta_i^r \leq \delta_i^r \leq -\zeta_i^r, \quad 1 \leq r \leq T, \quad \sum_{t=1}^{T}\pi_{i,t}^0 + \sum_{r=1}^{T}\delta_i^r d_r^* + \theta\sum_{r=1}^{T}\zeta_i^r d_r^* \leq Q_i, \qquad 1 \leq i \leq I$$

$$\tau_t^r + \sum_{s\leq t, r\in\hat{I}_t}\lambda_{s,t}^r = -1, \ \xi_t^r = \sum_{s\leq t, r\in\hat{I}_t}\sum_{i=1}^{I}\pi_{i,s}^r - 1, \qquad 1 \leq r \leq t \leq T$$

$$-\mu_{s,t}^r \leq \sum_{i=1}^{I}\pi_{i,s}^r + \lambda_{s,t}^r \leq \mu_{s,t}^r, \ -\omega_{s,t}^r \leq \lambda_{s,t}^r \leq \omega_{s,t}^r, \quad s : r \in I_s, \qquad 1 \leq r \leq t \leq T$$

$$-\nu_t^r \leq \tau_t^r \leq \nu_t^r, \ \eta_t^r = \nu_t^r + \sum_{s=1}^{t}\mu_{s,t}^r, \qquad 1 \leq r \leq t \leq T$$

$$\sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^0 + \sum_{r=1}^{t}\xi_t^r d_r^* + \theta\sum_{r=1}^{t}\eta_t^r d_r^* + \sum_{r=1}^{t}\sum_{s\leq t, r\in\hat{I}_t}\rho_{r,s}\omega_{s,t}^r d_r^* \leq V_{max} - v(1), \qquad 1 \leq t \leq T$$

$$-\sum_{s=1}^{t}\sum_{i=1}^{I}\pi_{i,s}^0 - \sum_{r=1}^{t}\xi_t^r d_r^* + \theta\sum_{r=1}^{t}\eta_t^r d_r^* + \sum_{r=1}^{t}\sum_{s\leq t, r\in\hat{I}_t}\rho_{r,s}\omega_{s,t}^r d_r^* \leq v(1) - V_{min}, \qquad 1 \leq t \leq T.$$

$$\text{(AARCID-BT)}$$