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On QPCCs, QCQPs and Completely Positive Programs

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Abstract This paper studies several classes of nonconvex optimization problems defined over convex cones, establishing connections between them and demonstrating that they can be equivalently formulated as convex completely positive programs. The problems being studied include: a quadratically constrained quadratic program (QCQP), a quadratic program with complementarity constraints (QPCC), and rank constrained semidefinite programs. Our results do not make any boundedness assumptions on the feasible regions of the various problems considered. The first stage in the reformulation is to cast the problem as a conic QCQP with just one nonconvex constraint $q(x) \leq 0$, where $q(x)$ is nonnegative over the (convex) conic and linear constraints, so the problem fails the Slater constraint qualification. A quadratic program with (linear) complementarity constraints (or QPCC) has such a structure; we prove the converse, namely that any conic QCQP satisfying a constraint qualification can be expressed as an equivalent conic QPCC. The second stage of the reformulation lifts the problem to a completely positive program, and exploits and generalizes a result of Burer. We also show that a Frank-Wolfe type result holds for a subclass of this class of QCQPs. Further, we derive necessary and

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sufficient optimality conditions for nonlinear programs where the only nonconvex constraint is a quadratic constraint of the structure considered elsewhere in the paper.

Keywords QCQP · QPCC · Completely Positive Representation · rank-constrained SDP · Local Optimality

1 Introduction

The quadratically constrained quadratic program, abbreviated QCQP, is a constrained optimization problem whose objective and constraint functions are all quadratic. In this paper, we also allow conic convex constraints on the variables. Recent works [3,9,8,21,24,42,48] on the QCQP have developed algorithms of various kinds for solving such a program. Of particular relevance to our work herein are the most recent papers [3,21,24] that lift a QCQP satisfying a boundedness assumption to an equivalent completely positive program [23], which is a problem of topical interest. Also relevant to this paper is the observation that many of the early studies of the QCQP addressed the issue of existence of optimal solutions [12,47,57] via the well-known Frank-Wolfe theorem originally proved for a quadratic program [35].

Quadratic programs with (linear) complementarity constraints (QPCCs) are instances of QCQPs and can be written using a single quadratic constraint to capture the complementarity restriction. Such a complementarity constraint renders the QPCC a nonconvex disjunctive program even if the objective function is convex. A notable feature of a QPCC is that the quadratic constraint cannot be satisfied strictly. We focus on QCQPs that possess this property; namely, they may have several convex quadratic constraints but they have just one nonconvex quadratic constraint $q(x) \leq 0$, with $q(x) \geq 0$ for any x that satisfies the linear and conic constraints (see problem (4) below). We show that any QCQP can be expressed in this form, generalizing a result for bounded QCQPs in [24]. Thus, the considered class of QCQPs is broad.

Playing an analogous role to that of a quadratic program in the class of nonlinear programs, the class of QPCCs [40,26] is a subclass of the class of mathematical programs with complementarity constraints (MPCCs) [45,49]. With increasingly many documented applications in diverse engineering fields, the MPCC provides a broad framework for the treatment of such problems as bilevel programs, inverse optimization, Stackelberg-Nash games, and piecewise programming; see [51]. Being a recent entry to the optimization field, the QPCC and its special case of a linear program with (linear) complementarity constraints (LPCC) have recently been studied in [6,38,39] wherein a logical Benders scheme [27] was proposed for their global resolution. We generalize the definition of QPCC in this paper to replace the nonnegative orthant by a convex cone. Yoshise [59] surveys research on complementarity problems over symmetric cones. Ding, Sun and Ye [32] investigate MPCCs over the semidefinite cone, and in particular they show that a semidefinite program with a rank constraint can be cast as an equivalent LPCC over the SDP cone.

This paper addresses several topics associated with the QPCC and QCQP: existence of an optimal solution to a QCQP, the formulation of a QCQP as a QPCC, the local optimality conditions of a class of quadratically constrained nonlinear programs failing constraint qualifications, and the formulation of a QCQP as a completely positive program. These problems are defined in Section 2, where we show that any conic QCQP has an equivalent conic QPCC provided the conic and linear constraints satisfy a constraint qualification.

Proved for a convex polynomial program as early as in the 1977 book [11] and reproved subsequently in [57, 47], the existence of an optimal solution to a convex QCQP over the nonnegative orthant is fully resolved via the classical Frank-Wolfe theorem, which states that such a minimization program, if feasible, has an optimal solution if and only if the objective function of the program is bounded below on the feasible set. Results for some classes of nonconvex QCQPs have been derived by Luo and Tseng [47]. We show in Section 3 that a Frank-Wolfe result holds for an additional class of nonconvex QCQPs.

We show in Section 4 that a QCQP of the type considered in this paper, if the objective matrix is copositive, can be lifted to an equivalent completely positive program. This is a convex programming problem, albeit one defined over a cone that is hard to work with [31]. An introduction of basic concepts and a summary of recent developments in copositive programming can be found in a survey paper [33], a book chapter [23], and a doctoral dissertation [29]. Computational approaches for solving completely positive programs can be found in [14, 18, 19, 22, 28, 55, 58]. Our work extends the recent papers [21, 23, 24] that address the completely positive representations of binary nonconvex quadratic programs, certain types of quadratically constrained quadratic programs, and a number of other NP-hard problems. Distinct from the earlier work, we make no boundedness assumptions on the variables. It follows from our results that any conic QCQP can be reformulated as an equivalent conic convex optimization problem, regardless of whether the feasible region of the QCQP is unbounded or bounded, and regardless of the convexity of the objective function or any of the constraints. This reformulation requires a single lifting, perhaps preceded by some manipulations as detailed in Section 2.

The results of Section 4 are specialized to binary quadratic programs, QPCCs, SDPs with rank constraints, and general QCQPs in Section 5. In particular, we show that any conic QCQP is equivalent to a convex completely positive program, if we first reformulate it as a QCQP with a linear objective function and just the one nonconvex constraint that doesn't have a Slater point. We also show in this section that a rank-sparsity decomposition problem is equivalent to a convex completely positive program.

Extending the QCQP, in Section 6 we consider a class of nonlinear programs with a single nonconvex constraint, namely $q(x) \leq 0$ with $q(x)$ quadratic and nonnegative over the linear and conic constraints. We show that checking the local optimality of this class of problems is equivalent to checking the global optimality of a mathematical program with a linearized objective function subject to the same constraints plus an imposed linear constraint. Checking the latter optimality condition is still hard; in the simplest case when the non-quadratic constraints are all linear, the linearized condition is equivalent to an LPCC.

In summary, this paper contains a wealth of new results pertaining to the three classes of problems appearing in the title; see Figure 1. Individually, these results are not particularly deep; collectively, they add significant insights to the problems. Most importantly, our study touches on a class of nonconvex programs failing the Slater constraint qualification and suggests that such problems have an underlying piecewise structure and can be converted to convex programs by a single lifting of their domain of definition.

2 Two Classes of Quadratic Problems

We begin with the formal definitions of the classes of QPCCs and QCQPs. Specifically, given a symmetric matrix $Q^0 \in \mathbb{R}^{n\times n}$, where $n \triangleq \bar{n}+2m$, a vector $c^0 \in \mathbb{R}^n$, a matrix $A \in \mathbb{R}^{k\times n}$, two closed convex cones $\bar{\mathcal{K}} \subseteq \mathbb{R}^{\bar{n}}$ and $\mathcal{K}^1 \subseteq \mathbb{R}^m$, and a nonzero vector $b \in \mathbb{R}^k$, the QPCC is the minimization problem:

$$\begin{array}{ll} \underset{x \triangleq (\bar{x}, x^1, x^2)}{\text{minimize}} & (c^0)^T x + \frac{1}{2}\, x^T Q^0 x \\ \text{subject to } Ax = b & \text{and} \quad \langle\, x^1,\, x^2 \rangle = 0 \\ \text{with} & \bar{x} \in \bar{\mathcal{K}},\, x^1 \in \mathcal{K}^1,\, x^2 \in \mathcal{K}^{1^*} \end{array} \tag{1}$$

where $\mathcal{K}^{1^*}$ is the dual cone to $\mathcal{K}^1$. QPCCs with $\bar{\mathcal{K}} \times \mathcal{K}^1 \times \mathcal{K}^{1^*} = \mathbb{R}^n_+$ are discussed in [6,7], for example. Mathematical programs with complementarity constraints with other cones are considered in [32,59], for example.

We may state the general QCQP as:

$$\begin{array}{ll} \underset{\tilde{x} \in \tilde{\mathcal{K}}}{\text{minimize}} & f_0(\tilde{x}) \triangleq (\, g^0\,)^T \tilde{x} + \frac{1}{2}\, \tilde{x}^T M^{\,0} \tilde{x} \\ \text{subject to } H\tilde{x} = p & \\ \text{and} & f_i(\tilde{x}) \triangleq \nu_i + (\, g^i\,)^T \tilde{x} + \frac{1}{2}\, \tilde{x}^T M^{\,i} \tilde{x} \le 0, \quad i = 1, \ldots, I, \end{array} \tag{2}$$

for some closed convex cone $\tilde{\mathcal{K}} \subseteq \mathbb{R}^{\tilde{n}}$ and for some nonnegative integer I, where $\nu_i \in \mathbb{R}$, $g^i \in \mathbb{R}^{\tilde{n}}$, and $M^{\,i} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ for $i = 0, 1, \ldots, I$, and each $M^{\,i}$ is a symmetric matrix. A QPCC is an example of a QCQP since the bilinear equation $\langle x^1, x^2 \rangle = 0$ in the QPCC is equivalent to the quadratic inequality $\langle x^1, x^2 \rangle \le 0$ due to the conic constraints.

We're particularly interested in problems where (some of) the quadratic constraints have no Slater point in $\{\tilde{x} \in \mathcal{K} : H\tilde{x} = p\}$. Such problems include QPCCs, and also binary quadratic programs of the form

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n_+}{\text{minimize}} & (\, c^0\,)^T x + \frac{1}{2}\, x^T Q^{\,0} x \\ \text{subject to } Ax = b & \\ \text{and} & x_j \in \{0,1\} \text{ for } j \in B \subseteq \{1, \ldots, n\} \end{array} \tag{3}$$

(see §5.1 for more details).

Definition 1 Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a closed convex cone and let $\mathcal{M} = \{x \in \mathbb{R}^n : Ax = b\}$ with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. A QCQP of the form

$$\begin{aligned} &\underset{x\in\mathcal{K}\cap\mathcal{M}}{\text{minimize}} \quad q_0(x) \triangleq (c^0)^T x + \tfrac{1}{2} x^T Q^0 x \\ &\text{subject to } q(x) \triangleq \mathbf{h} + \mathbf{q}^T x + \tfrac{1}{2} x^T \mathbf{Q} x \le 0 \\ &\text{and} \qquad q_j(x) \triangleq h_j + (c^j)^T x + \tfrac{1}{2} x^T Q^j x \le 0, \quad j = 1, \dots, J, \end{aligned} \tag{4}$$

where $q(x)$ is a nonnegative function on $\mathcal{K} \cap \mathcal{M}$ and where the constraints $q_j(x) \le 0$, $j = 1 \dots, J$, are convex, is denoted as an *nSp-QCQP*. If $J = 0$, it is denoted as an *nSp0-QCQP*. Further, we define

$$\Gamma \triangleq \{ x \in \mathcal{K} \cap \mathcal{M} \mid q(x) \le 0 \}. \tag{5}$$

Multiple constraints arising from functions that are nonnegative throughout the set of interest can be aggregated into a single constraint, as we state in the following lemma, the proof of which is left to the reader:

Lemma 1 *Let $\mathcal{S} \subseteq \mathbb{R}^n$ and let $f_i(x) \ge 0 \ \forall x \in S$, $i = 1, \dots, m$. Define $f(x) := \sum_{i=1}^{I} f_i(x)$. Then*

$$\{x \in \mathcal{S} : f_i(x) \le 0, i = 1, \dots, m\} = \{x \in \mathcal{S} : f(x) \le 0\}.$$

It follows that there is no need to modify the definition of nSp-QCQP problems to allow multiple constraints that cannot be satisfied strictly.

We summarize the results of this paper in Figure 1. These results are all for the situation where $\mathcal{K}$ is a convex cone. The upward pointing arrows illustrate relationships where the lower problem can be regarded directly as an instance of the upper problem. The downward pointing arrows require proof and in some cases the cone is changed. The central problem is the nSp0-QCQP; some QCQPs are already in this form, while any other can be manipulated into this form. The manipulation also results in a convex objective function. Provided the objective is copositive on a certain subset of the recession cone of $\mathcal{K} \cap \mathcal{M}$, an nSp0-QCQP is equivalent to a convex completely positive program. The rest of the paper is devoted to the demonstration of the various results in this figure.

Burer and Dong [24] showed that any QCQP with a bounded feasible region has an equivalent representation in the form (4). In the following theorem, we show that this result holds even if the feasible region is unbounded, and we also show that the representation can be constructed with a convex objective. Thus, any QCQP can be written in the form (4).

Theorem 1 *A general QCQP of the form (2) with variables $\tilde{x} \in \mathbb{R}^{\tilde{n}}$ is equivalent to an nSp-QCQP of the form (4) with variables $x \in \mathbb{R}^{\tilde{n}+2}$ and with a convex objective function.*

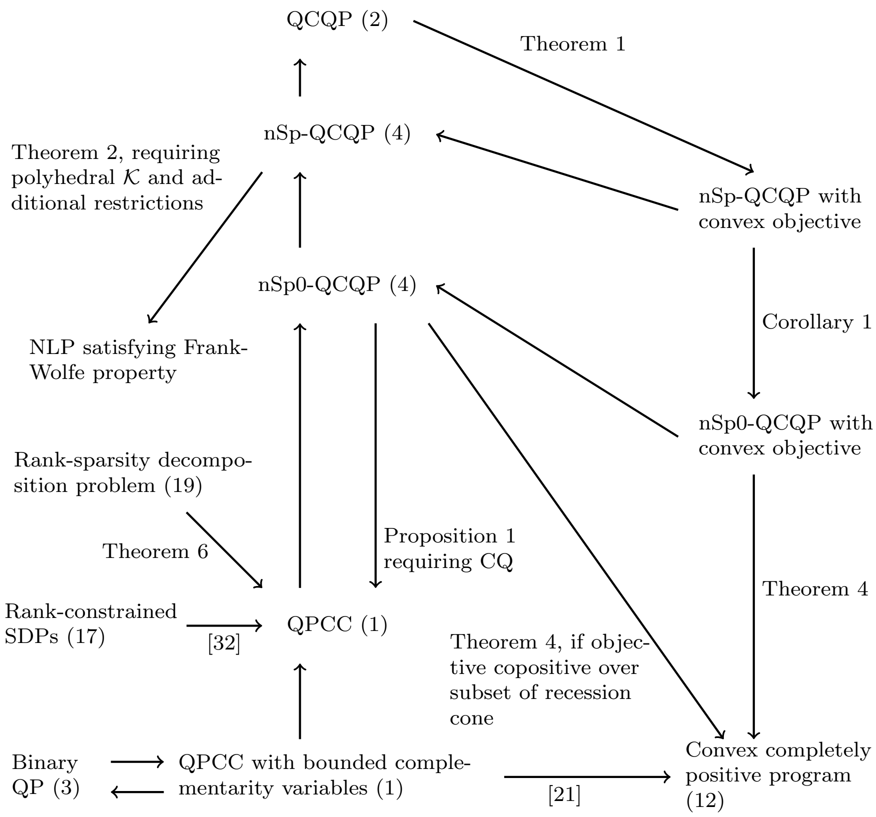

Fig. 1 Diagram of results. The notation "$p \to q$" means p is a subclass of q.

Proof First, we introduce two variables $s \geq 0$ and r and the constraint

$$\tilde{x}^T\tilde{x} + r^2 \leq s^2. \tag{6}$$

We take

$$\mathcal{K} = \{(r, s, \tilde{x}) \in \mathbb{R}^{\tilde{n}+2} : \tilde{x} \in \tilde{\mathcal{K}}, \tilde{x}^T\tilde{x} + r^2 \leq s^2, s \geq 0\}.$$

If each of the matrices M^i, $i = 0, 1, \ldots, I$ is positive semidefinite then the QCQP is already in the form (4), with $q(x)$ equal to the zero function. Otherwise, we impose the constraint

$$\tilde{q}(r, s, \tilde{x}) := (s^2 - r^2 - \tilde{x}^T\tilde{x}) \leq 0,$$

so $\tilde{q}(r, s, \tilde{x})$ is nonnegative on $\mathcal{K}$, and $\tilde{x}^T\tilde{x} = s^2 - r^2$ for any feasible solution. We also impose the additional linear constraint

$$s + r = 1 \tag{7}$$

so

$$s - r = s^2 - r^2 = \tilde{x}^T \tilde{x}.$$

Let

$$\lambda_i \leq \bar{\lambda}_i = \min \{0, \min\{e^i_j : e^i_j \text{ is an eigenvalue of } M^i\}\}. \tag{8}$$

for $i = 0, 1, \ldots, I$. The matrix $M^i - \lambda_i I$ is positive semidefinite and $f_i(\tilde{x})$ is equal to the function

$$q_i(r, s, \tilde{x}) := \nu_i + (g^i)^T \tilde{x} + \tfrac{1}{2}\lambda_i(s - r) + \tfrac{1}{2}\tilde{x}^T(M^i - \lambda_i I)\tilde{x} \tag{9}$$

since $\tilde{x}^T \tilde{x} = s - r$.

Combining the convex and nonconvex case, we define the parameter

$$\beta = \begin{cases} 1 & \text{if } \min_{i=0,1,\ldots,I} \{\lambda_i\} < 0 \\ 0 & \text{otherwise} \end{cases}$$

and define

$$q(r, s, \tilde{x}) := \beta(s^2 - r^2 - \tilde{x}^T \tilde{x}).$$

The following QCQP

$$\begin{aligned} \underset{(r,s,\tilde{x})\in\mathcal{K}}{\text{minimize}} \quad & q_0(r, s, \tilde{x}) \\ \text{subject to } & H\tilde{x} = p \\ & r + s = 1 \\ & q(r, s, \tilde{x}) \leq 0 \\ & q_i(r, s, \tilde{x}) \leq 0, \quad i = 1, \ldots, I \end{aligned}$$

is an nSp-QCQP with a convex objective function that is equivalent to the original QCQP. It is convex if the original QCQP is convex. □

A convex quadratic constraint can be represented as a second order cone constraint after adding two variables [2], so any nSp-QCQP could be represented as an equivalent nSp0-QCQP. Thus it follows from Theorem 1 that any QCQP can be represented as an equivalent nSp0-QCQP.

Corollary 1 *A conic QCQP of the form (2) with variables $\tilde{x} \in \mathbb{R}^{\tilde{n}}$ is equivalent to an nSp0-QCQP of the form (4) with variables $x \in \mathbb{R}^{\tilde{n}+2+2I}$ and with a convex objective function.*

The results in §4 and §5 are derived for nSp0-QCQP problems. Some problems such as QPCCs and binary quadratic programs are already in this form, and by Corollary 1 any QCQP can be reformulated as an nSp0-QCQP, so this is a broad class of problems. We choose to work with explicit convex quadratic constraints and the form nSp-QCQP in some of our results below, because this may be a more natural representation for some problems and for some algorithms. Some of our results (particularly in §3) require that $\mathcal{K}$ be a polyhedral cone, which again requires that convex quadratic constraints be represented explicitly and not incorporated into the cone. For similar reasons, we don't require the objective function to be convex in the definition of an nSp-QCQP.

Any conic QCQP can be represented as an equivalent conic QPCC, provided the conic and linear constraints satisfy a property involving their normal cones. Let $\mathcal{N}_S(x)$ denote the normal cone to the convex set S at the point $x \in S$. If H_1 and H_2 are convex sets that satisfy a constraint qualification then the normal cone intersection formula states that

$$\mathcal{N}_{H_1 \cap H_2}(x) \,=\, \mathcal{N}_{H_1}(x) \,+\, \mathcal{N}_{H_2}(x) \tag{10}$$

for each $x \in H_1 \cap H_2$. Appropriate constraint qualifications are discussed in, for example, [10,16,17,20]. For example, the formula holds if H_1 and H_2 are both polyhedral or if H_2 is polyhedral and there exists an

$$\bar{x} \,\in\, \mathrm{int}(H_1) \,\cap\, H_2,$$

or if H_2 is polyhedral and $H_1 = H_3 \cap H_4$ with H_4 polyhedral and there exists an

$$\bar{x} \,\in\, \mathrm{int}(H_3) \,\cap\, H_4 \,\cap\, H_2.$$

Proposition 1 *An nSp0-QCQP (4) satisfying (10) with $H_1 = \mathcal{K}$ and $H_2 = \mathcal{M}$ for every $x \in \mathcal{K} \cap \mathcal{M}$ can be expressed as an equivalent QPCC of the form (1).*

Proof Since $q(x) \geq 0 \ \forall x \in \mathcal{K} \cap \mathcal{M}$, it follows that, if $\Gamma \neq \emptyset$, then

$$\Gamma \,=\, \left[\operatorname*{argmin}_{x \in \mathcal{K} \cap \mathcal{M}} q(x)\right] \,=\, \{\, x \,\in\, \mathcal{K} \cap \mathcal{M} \,\mid\, q(x) \,=\, 0 \,\}\,.$$

Under the assumption that (10) holds, the condition that $-\nabla q(x)$ be in the normal cone to $\mathcal{K} \cap \mathcal{M}$ at a point $x \in \mathcal{K} \cap \mathcal{M}$ can be expressed as:

$$\begin{aligned} &\mathcal{K} \,\ni\, x \,\perp\, \mathbf{q} + \mathbf{Q}x + A^T\lambda \,\in\, \mathcal{K}^* \\ &0 \,=\, Ax - b, \end{aligned}$$

since $-\mathcal{N}_{\mathcal{K}}(x) = \{w \in K^* : x^T w = 0\}$. Note further that if these conditions hold then a simple algebraic manipulation gives $q(x) = \frac{1}{2}\left(\mathbf{q}^T x - b^T \lambda\right) + \mathbf{h}$. Since Γ consists of local minimizers of $q(x)$ with value 0, we obtain an equivalent QPCC

$$\begin{aligned} \underset{(\lambda,x,w)}{\text{minimize}} \quad & (c^0)^T x + \tfrac{1}{2}\, x^T Q^0 x \\ \text{subject to} \quad & Ax \,=\, b \\ & \mathbf{q} + \mathbf{Q}x + A^T\lambda - w \,=\, 0 \\ & \tfrac{1}{2}\left(\mathbf{q}^T x - b^T \lambda\right) + \mathbf{h} \,=\, 0 \\ & x^T w \,=\, 0 \\ \text{and} \quad & \lambda \,\in\, \mathbb{R}^k,\ x \,\in\, \mathcal{K},\ w \,\in\, \mathcal{K}^* \end{aligned}$$

of the required form. □

The following corollary follows directly from Corollary 1 and Proposition 1.

Corollary 2 *Any conic QCQP (2) with either int*($\tilde{\mathcal{K}}$)$\cap\{\tilde{x} : H\tilde{x} = p\}$ *nonempty or* $\tilde{\mathcal{K}}$ *polyhedral can be reformulated as an equivalent conic QPCC.*

Proof An equivalent nSp0-QCQP is constructed as in Theorem 1 and Corollary 1, with $\lambda_i = \bar{\lambda}_i - \epsilon$ for some $\epsilon > 0$ for $i = 1, \ldots, I$. Let $\tilde{x} \in \tilde{\mathcal{K}}$ satisfy $H\tilde{x} = p$ and set

$$\tilde{s} = \tfrac{1}{2} + \tfrac{1}{2}\tilde{x}^T\tilde{x} + \Xi, \qquad \tilde{r} = \tfrac{1}{2} - \tfrac{1}{2}\tilde{x}^T\tilde{x} - \Xi,$$

for some $\Xi > 0$. The triple $(\tilde{s}, \tilde{r}, \tilde{x})$ satisfies (6) strictly and (7), and it satisfies $q_i(\tilde{s}, \tilde{r}, \tilde{x}) < 0$ for $i = 1, \ldots, I$ for Ξ sufficiently large, even if $f_i(\tilde{x}) > 0$. If $\tilde{x} \in \text{int}(\tilde{\mathcal{K}}) \cap \mathcal{M}$ or if $\tilde{\mathcal{K}}$ is polyhedral then the condition of Proposition 1 holds so the result follows. □

3 Frank-Wolfe Attainment Results

A Frank-Wolfe (FW) result states that a minimization problem attains a finite minimum objective value if and only if it is feasible and the objective function is bounded below on the feasible set. There has been extensive work [4,5,13, 50] published on the existence of optimal solutions to nonconvex programs in general. The sharpest Frank-Wolfe type existence results for a feasible QCQP with a nonconvex (quadratic) objective are obtained in [47] and summarized as follows:

> the QCQP attains its minimum if the objective function is bounded below on the feasible set and either:
> (a) [47, Theorem 2] there is at most one nonlinear (but convex) quadratic constraint, or
> (b) [47, Theorem 3] the objective function is quasiconvex on the feasible region and all the constraints are convex.

The authors in the latter reference also give an example of a QCQP with two convex quadratic constraints satisfying a Slater condition and a nonconvex objective function bounded below on the feasible set for which the infimum objective value is not attained. Since a Frank-Wolfe type existence result holds for a QPCC, it is natural to ask whether there is a class of QCQPs, broader than the class of QPCCs, for which a Frank-Wolfe type existence result holds. It turns out that the answer to this question is affirmative for certain nSp-QCQPs; interestingly, for nSp0-QCQPs, those which are solvable will have rational optimal solutions if the data are rational numbers to begin with. We first prove a second corollary to Proposition 1 when $\mathcal{K}$ is a polyhedral cone.

Corollary 3 *Assume* $\mathcal{K}$ *is a polyhedral cone. The set* Γ *consists of a union of a finite number of polyhedra.*

Proof Without loss of generality, we can write $\mathcal{K}$ as a finitely constrained cone and its dual cone as a finitely generated cone, so

$$\mathcal{K} = \{x \in \mathbb{R}^n : Hx \geq 0\} \quad \text{and} \quad \mathcal{K}^* = \{w \in \mathbb{R}^n : w = H^T\pi, \, \pi \in \mathbb{R}^p_+\}$$

where $H \in \mathbb{R}^{p\times n}$. Then $\Gamma = \bigcup_\alpha P^{\alpha}$, with

$$P^\alpha \triangleq \left\{ (x,w,y,\pi,\lambda) \in \mathbb{R}^{2n+2p+m} : \begin{array}{l} Ax = b \\ Hx - y = 0 \\ \mathbf{q} + \mathbf{Q}x + A^T\lambda - w = 0 \\ H^T\pi - w = 0 \\ \frac{1}{2}\left(\mathbf{q}^T x - b^T\lambda\right) + \mathbf{h} = 0 \\ y_\alpha = 0, \, y \geq 0 \\ \pi_{\bar{\alpha}} = 0, \, \pi \geq 0 \end{array} \right\}$$

for all subsets $\alpha \subseteq \{1,2,\ldots,p\}$ with complement $\bar{\alpha}$. □

It follows that the feasible region of (4) can be expressed as $\bigcup_\alpha \mathcal{C}^{\alpha}$, where

$$\mathcal{C}^{\alpha} \triangleq \{x \mid q_j(x) \leq 0 \text{ for } j = 1,\ldots,J, \, (x,w,y,\pi,\lambda) \in P^\alpha \text{ for some } w,y,\pi,\lambda\}$$

is the intersection of the zero-level sets of the convex quadratic functions $q_j(x)$, $j = 1,\ldots,J$, over the projection of the polyhedron P^{α}. Based on this piecewise representation, we can prove the following Frank-Wolfe attainment results.

Theorem 2 *Assume the cone $\mathcal{K}$ is polyhedral. Then the FW attainment result holds for the nSp-QCQP (4) if either*

1. $J \leq 1$ *or*
2. $q_0(x)$ *is a quasiconvex function on* $\mathcal{K} \cap \mathcal{M}$.

Proof From the piecewise representation, problem (4) is equivalent to a finite union of quadratically constrained quadratic programs

$$\min_\alpha \underset{x\in\mathcal{C}^{\alpha}}{\text{minimize}}\, q_0(x).$$

The two cases then follow from [47]. In particular, we have

1. If $J \leq 1$ then each of the individual QCQPs has a single quadratic constraint that is convex. The result follows from [47, Theorem 2].
2. If the objective function $q_0(x)$ is quasiconvex on $\mathcal{K}\cap\mathcal{M}$ then each of the individual QCQPs has a quasiconvex objective function and convex quadratic constraints. The result follows from [47, Theorem 3].

□

nSp0-QCQPs are nonconvex programs that fail the Slater constraint qualification; yet, without the convex quadratic constraints, these QCQPs have an interesting property when $\mathcal{K}$ is polyhedral that we highlight in the following result.

Proposition 2 *Suppose $\mathcal{K}$ is a polyhedral cone. If an nSp0-QCQP (4) has an optimal solution, then it has a rational optimal solution, provided that the input data are all rational.*

Proof By the proof of Theorem 2, it follows that the nSp0-QCQP (4) has an optimal solution that is an optimal solution of a (possibly nonconvex) quadratic program. Thus, it suffices to show that if the input data of a solvable quadratic program are all rational, then the program has a rational optimal solution. The set of stationary points of a quadratic program is a finite union of polyhedra, each polyhedron defined by a particular complementarity selection in the KKT conditions. The objective function value is constant on each piece [46], so the set of global minimizers is a union of polyhedra given by rational data. Any nonempty polyhedron must contain at least one rational point, provided the data of its linear constraints are rational numbers. □

Remark 1 Proposition 2 does not hold for the more general class of QCQPs that have a quadratic inequality constraint with a Slater point. The scalar problem: minimize x subject to $x^2 \leq 2$ provides a simple counterexample that illustrates the failure of the proposition under the Slater assumption. □

4 Completely Positive Representation of nSp0-QCQPs

We consider nSp0-QCQP problems in this section, that is, problems of the form given in (4) with $J = 0$, with $q(x) \geq 0$ for all $x \in \mathcal{K} \cap \mathcal{M}$, so there is no Slater point for the constraint $q(x) \leq 0$. Note that a QPCC is an nSp0-QCQP. In addition, a binary quadratic program (3) is an nSp0-QCQP if the binary restrictions are represented by the quadratic inequality $\sum_{j \in B} x_j - x_j^2 \leq 0$, provided $0 \leq x_j \leq 1 \, \forall j \in B$ for all $x \in \mathcal{K} \cap \mathcal{M}$. Further, as noted in Corollary 1, any QCQP can be represented as an equivalent nSp0-QCQP. In this section, we investigate a completely positive relaxation of (4).

A completely positive program is a linear optimization problem in matrix variables in the form of:

$$\begin{array}{ll} \underset{X \in \mathcal{S}^{1+n}}{\text{minimize}} & \langle A_0, X \rangle \\ \text{subject to} & \langle A_i, X \rangle \;=\; b_i, \quad i = 1, \ldots, \ell \\ \text{and} & X \;\in\; \mathcal{CP}_{1+n}(\mathcal{K}), \end{array}$$

where $\mathcal{S}^{1+n}$ is the space of symmetric $(1+n) \times (1+n)$ matrices and $\mathcal{CP}_{1+n}(\mathcal{K})$ is the cone of completely positive matrices over $\mathcal{K}$,

$$\mathcal{CP}_{1+n}(\mathcal{K}) \triangleq \text{conv}\left\{ M \in \mathcal{S}^{1+n} \mid M = xx^T, \, x \in \mathbb{R}_+ \times \mathcal{K} \right\},$$

whose dual $\mathcal{COP}_{1+n}(\mathcal{K})$ is the cone of copositive matrices over $\mathcal{K}$,

$$\mathcal{COP}_{1+n}(\mathcal{K}) \triangleq \left\{ M \in \mathcal{S}^{1+n} \mid x^T M x \geq 0, \ \forall x \in \mathbb{R}_+ \times \mathcal{K} \right\}.$$

We also use the notation $\mathcal{CP}_{1+n} := \mathcal{CP}_{1+n}(\mathbb{R}^n_+)$ and $\mathcal{COP}_{1+n} := \mathcal{COP}_{1+n}(\mathbb{R}^n_+)$.

An nSp0-QCQP (4) is equivalent to the following problem:

$$\begin{aligned} &\underset{x \in \mathcal{K}}{\text{minimize}} \quad (c^0)^T x + \tfrac{1}{2} \langle Q^0, X \rangle \\ &\text{subject to } A x = b, \\ &\qquad\qquad q(x) \leq 0 \quad \text{and} \quad X = x x^T. \end{aligned} \tag{11}$$

By relaxing the rank-1 constraint on the matrix X we get the following completely positive program:

$$\begin{aligned} &\underset{x,\,X}{\text{minimize}} \quad (c^0)^T x + \tfrac{1}{2} \langle Q^0, X \rangle \\ &\text{subject to } A x = b \quad \text{and} \quad \text{diag}(AXA^T) = b \circ b \\ &\qquad\qquad \mathbf{h} + \mathbf{q}^T x + \tfrac{1}{2} \langle \mathbf{Q}, X \rangle = 0 \\ &\text{and} \qquad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{CP}_{1+n}(\mathcal{K}), \end{aligned} \tag{12}$$

where $u \circ v$ denotes the Hadamard product between the vectors u and v. Note that we have exploited the assumption regarding the lack of a Slater point in this formulation.

Burer [21] addressed the copositive representations of a binary or continuous nonconvex QCQP of the form (4), where $\mathcal{K} = \mathbb{R}^n_+$ and $x \in \mathcal{K} \cap \mathcal{M}$ implies $0 \leq x_i \leq 1$ for the binary variables x_i and with restrictions on the quadratic constraints. He proved that a completely positive relaxation is equivalent to the original QCQP. Burer [23] showed that the results of [21] extend to the situation of a general convex cone $\mathcal{K}$, and showed that the completely positive relaxation is equivalent to the original problem under the assumptions that $d^T \mathbf{Q} d = 0$ for all d in the recession cone of $\mathcal{K} \cap \mathcal{M}$ and that $q(x)$ is bounded above and below on $\{x \in \mathcal{K} : Ax = b\}$. These two assumptions are not satisfied by either a general QPCC of the form (1) or by the QCQP constructed in Corollary 1. Burer and Dong [24] extended the results of [21] to QCQP's with no restrictions on the quadratic constraints, but under the assumption that $\mathcal{K} \cap \mathcal{M}$ is bounded. Dickinson et al. [30] have developed similar results for more general sets, under certain assumptions.

As a special instance of a QCQP albeit with bilinear quadratic constraints, can a QPCC be cast as a completely positive program? In what follows, we prove a more general result. In particular, the equivalence will be established between an nSp0-QCQP whose objective function is copositive on a particular subset of the recession cone of $\mathcal{K} \cap \mathcal{M}$, and its completely positive representation. Our proof borrows from that in [23].

In general, a QPCC with a nonconvex objective function is not guaranteed to have an equivalent copositive representation without using the construction of Theorem 1, except under limited conditions such as bounded complementarity variables, as we show in the following example.

Example 1

$$\begin{array}{ll} \underset{x \geq 0}{\text{miminize}} & -x_2^2 \\ \text{subject to} & x_1 + x_2 - x_3 = 3 \\ & x_1 - x_4 \; = \; 2 \\ \text{and} & x_1 \; \perp \; x_2. \end{array}$$

The optimal objective value of the above QPCC is 0 and the only feasible ray is $d \triangleq (1, 0, 1, 1)$. Below is the completely positive representation of this QPCC:

$$\begin{array}{ll} \underset{x,\,X}{\text{minimize}} & -X_{2,2} \\ \text{subject to} & x_1 + x_2 - x_3 = 3 \\ & x_1 - x_4 \; = \; 2 \\ & X_{1,1} + 2X_{1,2} + X_{2,2} - 2X_{1,3} - 2X_{2,3} + X_{3,3} = 9 \\ & X_{1,1} + X_{4,4} - 2X_{1,4} = 4 \\ & X_{1,2} = 0 \\ \text{and} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{CP}_5. \end{array}$$

Denote $\bar{d} \triangleq (0, 1, 1, 0)$, a ray of $\mathcal{K} \cap \mathcal{M}$. It is easy to show that $\begin{pmatrix} 0 & 0 \\ 0 & \bar{d}\bar{d}^T \end{pmatrix}$ is a feasible ray of the above completely positive program, therefore this completely positive problem is unbounded below. The ray satisfies $\bar{d}_1 \perp \bar{d}_2$ and $A\bar{d} = 0$, which is why it gives a ray for the completely positive lifting. However, it is not a ray for the QPCC because every feasible solution has $x_1 > 0$ and hence $x_2 = 0$. □

The completely positive program (12) is not only a relaxation but also an equivalent form of the nSp0-QCQP under one assumption on the objective function matrix Q^0. To prove the equivalence, we define the following sets:

$$\begin{aligned}
\mathcal{L}_\infty &\triangleq \{\, d \in \mathcal{K} \mid A\, d = 0\} \\
L \quad &\triangleq \left\{\, d \in \mathcal{K} \mid Ad = 0 \text{ and } d^T\mathbf{Q}d = 0 \,\right\} \subseteq \mathcal{L}_\infty \\
\Gamma^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \mid x \in \Gamma \right\} \\
L^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & dd^T \end{pmatrix} \mid d \in L \right\} \\
\mathcal{L}_\infty^+ &\triangleq \operatorname{conv}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & dd^T \end{pmatrix} \mid d \in \mathcal{L}_\infty \right\} \\
\Sigma^+ &\triangleq \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \mid (x, X) \text{ feasible for (12)} \right\},
\end{aligned} \tag{13}$$

where Γ was defined in (5). Among the above sets, $\mathcal{L}_\infty$ is the recession cone of $\mathcal{K} \cap \mathcal{M}$, Γ^+ is isomorphic to the convex hull of the set of feasible solutions of program (11), and Σ^+ is the set of feasible solutions of the completely positive program (12). In general, the set Γ^+ might not be closed, but all the other listed sets are closed. It follows from the lack of a Slater point that the matrix $\mathbf{Q}$ is copositive on $\mathcal{L}_\infty$, as we prove in the following lemma.

Lemma 2 *If $q(x) \geq 0$ for all $x \in \mathcal{K} \cap \mathcal{M} \neq \emptyset$ then $d^T\mathbf{Q}d \geq 0$ for all $d \in \mathcal{L}_\infty$.*

Proof Let $\bar{d} \in \mathcal{L}_\infty$ with $\bar{d}^T\mathbf{Q}\bar{d} < 0$ and let $\bar{x} \in \mathcal{K} \cap \mathcal{M}$. Then $\bar{x} + \alpha\bar{d} \in \mathcal{K} \cap \mathcal{M}$ for all $\alpha \geq 0$, but $q(\bar{x} + \alpha\bar{d}) < 0$ for sufficiently large α. □

The following result is due to Burer [23].

Theorem 3 *(Theorem 8.3, [23]) Assume $d^T\mathbf{Q}d = 0$ for all $d \in \mathcal{L}_\infty$ and $q(x)$ is bounded above on $\mathcal{K} \cap \mathcal{M}$. Then*

$$\Gamma^+ + \mathcal{L}_\infty^+ = cl(\Gamma^+) = \Sigma^+.$$

Neither of the assumptions in Theorem 3 is valid in general for QPCCs (1) or for the nSp0-QCQP constructed in Corollary 1. As a result of the cone constraint $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{CP}_{1+n}(\mathcal{K})$, it is clear that $\Gamma^+ \subseteq \Sigma^+$ and L^+ is contained in the recession cone of Σ^+. Therefore $\Gamma^+ + L^+ \subseteq \Sigma^+$. We establish the converse inclusion in the proof of the following proposition. The difference between our result and that in [23] is that we show that directions $d \in \mathcal{L}_\infty$ with $d^T\mathbf{Q}d > 0$ cannot form part of the completely positive expansion of feasible solutions to (12). This removes the need for the first assumption of Theorem 3 for nSp0-QCQPs.

Proposition 3 *If $q(x) \geq 0$ for all $x \in \mathcal{K} \cap \mathcal{M}$, then $\Sigma^+ = \Gamma^+ + L^+$.*

Proof Assume (x, X) is feasible to (12). As shown in Proposition 8.2 in [23], we have two finite index sets J_+ and J_0 with

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{j \in J_+} v_j^2 \begin{pmatrix} 1 \\ \xi_j \end{pmatrix} \begin{pmatrix} 1 \\ \xi_j \end{pmatrix}^T + \sum_{j \in J_0} \begin{pmatrix} 0 \\ d_j \end{pmatrix} \begin{pmatrix} 0 \\ d_j \end{pmatrix}^T \in \mathcal{CP}_{1+n}(\mathcal{K}) \tag{14}$$

where

1. $\sum_{j \in J_+} v_j^2 = 1$ and $v_j \neq 0$ for all $j \in J_+$;
2. $\xi_j \in \mathcal{K} \cap \mathcal{M}$ for all $j \in J_+$;
3. $d_j \in \mathcal{L}_\infty$ for all $j \in J_0$.

From points 2 and 3, we have that $q(\xi_j) \geq 0$ for all $j \in J_+$, and $d_j^T \mathbf{Q} d_j \geq 0$ for all $j \in J_0$ from Lemma 2. In particular, we have

$$0 \leq \mathbf{h} + \mathbf{q}^T \xi_j + \frac{1}{2} \xi_j^T \mathbf{Q} \xi_j \quad \forall j \in J_+.$$

Multiplying by v_j^2 and adding over J_+, we obtain

$$\begin{aligned} 0 &\leq \mathbf{h} \sum_{j \in J_+} v_j^2 + \sum_{j \in J_+} v_j^2 \, \mathbf{q}^T \xi_j + \tfrac{1}{2} \sum_{j \in J_+} v_j^2 \xi_j^T \mathbf{Q} \xi_j \\ &= \mathbf{h} + \sum_j v_j^2 \, \mathbf{q}^T \xi_j + \tfrac{1}{2} \sum_{j \in J_+} v_j^2 \xi_j^T \mathbf{Q} \xi_j + \tfrac{1}{2} \sum_{j \in J_0} d_j^T \mathbf{Q} d_j - \tfrac{1}{2} \sum_{j \in J_0} d_j^T \mathbf{Q} d_j \\ &= -\tfrac{1}{2} \sum_{j \in J_0} d_j^T \mathbf{Q} d_j \\ &\leq 0 \end{aligned}$$

since $\sum_{j \in J_+} v_j^2 = 1$, $d_j^T \mathbf{Q} d_j \geq 0$ for all $j \in J_0$, and the matrix is feasible in (12). It follows that

$$\mathbf{h} + \mathbf{q}^T \xi_j + \frac{1}{2} \xi_j^T \mathbf{Q} \xi_j = 0 \quad \forall j \in J_+$$

$$\text{and} \quad d_j^T \mathbf{Q} d_j = 0 \quad \forall j \in J_0.$$

Therefore $\xi_j \in \Gamma$ for all $j \in J_+$ and $d_j \in L$ for all $j \in J_0$. Therefore, $\Sigma^+ \subseteq \Gamma^+ + L^+$. □

Based on Proposition 3, we can now establish the claimed equivalence between the nSp0-QCQP (4) formulated as (11) and the completely positive program (12). In contrast to [23], our result makes no assumptions regarding boundedness of the feasible region. We make a copositivity assumption regarding the objective function matrix, which ensures that no ray of $\mathcal{L}_\infty$ can lead to an improving ray for the completely positive program. (This assumption is satisfied for nSp0-QCQPs constructed as in Corollary 1.)

Theorem 4 *Assume Q^0 is copositive over L. The nSp0-QCQP (4) and the completely positive program (12) are equivalent in the sense that*

1. *The nSp0-QCQP (4) is feasible if and only if the completely positive program (12) is feasible.*
2. *Either the optimal values of the nSp0-QCQP (4) and the completely positive program (12) are finite and equal, or both of them are unbounded below.*
3. *Assume both the nSp0-QCQP (4) and the completely positive program (12) are bounded below, and $(\bar{x}, \bar{X})$ is optimal for the completely positive program, then $\bar{x}$ is in the convex hull of the set of optimal solutions of the nSp0-QCQP.*
4. *The optimal value of the nSp0-QCQP (4) is attained if and only if the same holds for the completely positive program (12).*

Proof 1. The first result follows immediately from (14), since $\xi_j \in \Gamma$ for each $j \in J_+$ from the proof of Proposition 3.

2. Denote the optimal values of the nSp0-QCQP (4) and the completely positive program (12) as Opt(4) and Opt(12) respectively. Since (4) is equivalent to (11) and since (12) is a relaxation of (11), it is immediate that Opt(4) $\geq$ Opt(12).

Since Q^0 is copositive over L,

$$\begin{aligned}\text{Opt}(12) \triangleq & \underset{Y \in \Sigma^+ = \Gamma^+ + L^+}{\text{minimize}} \left\langle \begin{pmatrix} 0 & \frac{1}{2}\,(c^0)^T \\ \frac{1}{2}\,c^0 & \frac{1}{2}\,Q^0 \end{pmatrix}, Y \right\rangle \\ \geq & \underset{Y \in \Gamma^+}{\text{minimize}} \left\langle \begin{pmatrix} 0 & \frac{1}{2}\,(c^0)^T \\ \frac{1}{2}\,c^0 & \frac{1}{2}\,Q^0 \end{pmatrix}, Y \right\rangle = \text{Opt}(4).\end{aligned}$$

Therefore, Opt(12) $\geq$ Opt(4). As a result, the two optimum objectives are equal.

3. Assume that $(\bar{x}, \bar{X})$ is optimal for the completely positive program (12). Then there exist $\bar{v}_j$, $\bar{\xi}_j$ for $j \in J_+$ and $\bar{d}_j$ for $j \in J_0$ of the form given in (14) where each $\bar{\xi}_j \in \Gamma$ and each $\bar{d}_j \in L$. It follows from the optimality of $(\bar{x}, \bar{X})$ that $\langle Q^0, \bar{d}_j \bar{d}_j^T \rangle = 0$ for all $j \in J_0$ and it is then straightforward to show that each $\bar{\xi}_j$ is optimal for (4). The vector $\bar{x}$ is a convex combination of these optimal $\bar{\xi}_j$.

4. The copositive lifting of any optimal solution to (4) immediately gives a solution to (12). The converse follows from the construction in the proof of part 3. □

In Example 1, the direction $\bar{d}$ is in the set L. The direction also satisfies $\bar{d}^T Q^0 \bar{d} < 0$, so Q^0 is not copositive over L. Hence this example does not satisfy the assumption of Theorem 4.

Our results require a copositivity assumption over the recession cone of the linear and conic constraints. Sturm and Zhang [56] discuss cones of nonnegative quadratic functions and impose a copositivity restriction on a more general set. More recently, the papers [43, 44] have addressed quadratically constrained quadratic programs by examining cones of nonnegative quadratic functions, based on the programs' KKT conditions that are lifted along with the feasible region of the QCQPs to define a certain cone of symmetric matrices. It would be interesting to investigate the detailed connections of the latter papers with our work; this investigation is left for a future study.

5 Specializations of Theorem 4

5.1 Binary quadratic programs

Burer [21] proved that a mixed binary nonconvex quadratic program (3) is equivalent to its copositive representation,

$$\begin{array}{lll} \text{minimize} & \frac{1}{2}\left\langle Q^0, X\right\rangle + {c^0}^T x & \\ \text{subject to} & Ax = b \quad \text{and} \quad \operatorname{diag}(AXA^T) \;=\; b \circ b & \\ & x_j = X_{jj}, \quad \forall j \in B, & \\ \text{and} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{CP}_{1+n} & \end{array} \tag{15}$$

under Burer's key assumption that the linear constraints imply $0 \leq x_j \leq 1$ $\forall j \in B$. The binary constraints in (3) can simply be represented as a single quadratic constraint

$$q(x) = \sum_{j \in B} (x_j - x_j^2) \;\leq\; 0.$$

Note that $q(x) \geq 0$ for all x satisfying the linear constraints, under Burer's key assumption. Since the linear constraints imply the binary variables are bounded and since the quadratic constraint only involves the binary variables, our two cones $\mathcal{L}_\infty$ and L are identical. It is also shown in [21] that if the objective function matrix Q^0 is not copositive over $\mathcal{L}_\infty$ then both the binary nonconvex quadratic program (3) and its completely positive relaxation (15) are unbounded. Thus problem (3) is either unbounded or satisfies the assumptions of Theorem 4, so the conclusions of the theorem hold for any Q^0.

5.2 Quadratic programs with complementarity constraints

A QPCC is in the nSp0-QCQP framework of (4), so the results of Theorem 4 apply directly. Our result requires that the objective function matrix Q^0 be copositive over L. Even if the QPCC has a bounded objective function value, Q^0 might not be copositive over L, as illustrated in Example 1. If Q^0 is not copositive over L and if an additional condition is satisfied then the QPCC is unbounded, as we show in the next lemma.

Lemma 3 *If there exists* $\hat{d} = (\bar{d}, \hat{d}^1, \hat{d}^2) \in L$ *with* $\hat{d}^T Q^0 \hat{d} < 0$ *and a feasible solution* $\hat{x} = (\bar{x}, \hat{x}^1, \hat{x}^2)$ *to a QPCC (1) satisfying* $\hat{x}^1 \perp \hat{d}^2$ *and* $\hat{x}^2 \perp \hat{d}^1$ *then the QPCC has an unbounded optimal value.*

Proof The point $\hat{x} + \alpha\hat{d}$ is feasible in (1) for any $\alpha \geq 0$. The objective function value of this set of points is unbounded below as $\alpha \to \infty$. □

Testing copositivity of Q^0 over L is equivalent to solving a homogenized version of (1), namely

$$\begin{array}{ll} \underset{d \triangleq (d^0, d^1, d^2)}{\text{minimize}} & \frac{1}{2}\, d^T Q^0 d \\ \text{subject to} & Ad = 0 \quad \text{and} \quad \langle\, d^1,\, d^2 \rangle = 0 \\ \text{with} & d^0 \in \bar{\mathcal{K}},\, d^1 \in \mathcal{K}^1,\, d^2 \in \mathcal{K}^{1^*}. \end{array} \tag{16}$$

If the conditions of Lemma 3 hold, the conclusions of Theorem 4 hold trivially, in that the QPCC and the completely positive program both have an unbounded optimal value. The lemma indicates that the only way to violate the conclusion of Theorem 4 is if there is an unbounded ray for (16), but this ray is not a feasible direction from any feasible point to the QPCC (1). If $\mathcal{K}$ is polyhedral, this condition can be restated as follows: the only way to violate the conclusion of Theorem 4 is if there is an unbounded ray for (16) for which the corresponding piece of the QPCC (1) is infeasible. This is the case with Example 1.

Algorithms for finding global optima to QPCCs can be found in [6], for example. These algorithms can be strengthened through the use of improved lower bounds coming from relaxations. It may be possible to obtain good bounds in reasonable computational time by examining relaxations of (12), as in [14, 22, 33] for example.

5.3 Semidefinite programs with rank constraints or objective

Ding et al. [32] contains several examples of problems of the form (1) where $\mathcal{K}^1$ is the positive semidefinite cone and the objective function is either linear or a convex quadratic function. For all these problems, we have an equivalence between (1) and the corresponding completely positive formulation (12), in the sense of Theorem 4. These problems include the rank constrained nearest correlation matrix problem, bilinear matrix inequality problems, and problems in the electric power market with uncertain data. In particular, it is shown in [32] that an SDP with an additional rank constraint is equivalent to an SDP problem with a complementarity constraint. That is, consider a problem of the form

$$\begin{array}{ll} \underset{X}{\text{minimize}} & q(X) \\ \text{subject to} & \langle A_i, X \rangle = b_i \quad i = 1, \ldots, k \\ & \text{rank}(X) \leq p \\ & X \in S^n_+ \end{array} \tag{17}$$

where S_+^n denotes the set of $n \times n$ symmetric positive semidefinite matrices, each A_i is a symmetric $n \times n$ matrix, $1 \le p < n$, and $q(X)$ is a copositive quadratic function over the set of positive semidefinite matrices satisfying the linear constraints. We allow $q(X)$ to be linear as a special case. This problem is then equivalent to the problem

$$\begin{array}{lll} \underset{X,W}{\text{minimize}} & q(X) & \\ \text{subject to} & \langle A_i, X\rangle = b_i & i = 1,\dots,k \\ & \text{trace}(W) = p & \\ & S_+^n \ni X \perp I - W \in S_+^n & \\ & W \in S_+^n. & \end{array} \tag{18}$$

The equivalence follows by noting that every eigenvalue of W must lie between 0 and 1 and so the rank of $I - W$ is at least $n - p$, so the rank of X can be no larger than p. Further, if X is a psd matrix of rank no larger than p then it can be factored as

$$X = [U_1\, U_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix},$$

where $[U_1, U_2]$ is an orthogonal $n \times n$ matrix, the rank of U_1 is p, and Λ_1 is a $p \times p$ nonnegative diagonal matrix; we can then obtain a feasible solution to (18) by taking $W = U_1 U_1^T$. Since the objective function satisfies the condition of Theorem 4, it follows that (18) is equivalent to its completely positive lifting. Thus we have the following theorem:

Theorem 5 *A rank constrained semidefinite program of the form (17) is equivalent to a convex completely positive program, in the sense of Theorem 4.*

The completely positive program is a lifting of (18), so it will have $O(n^4)$ variables.

A completely positive lifting can also be used to find an equivalent convex formulation of the rank-sparsity decomposition problem [25]. In this problem, it is desired to decompose a given matrix C as a sum of a sparse matrix A and a low rank matrix B, with A and B required to lie in polyhedral sets $\mathcal{A}$ and $\mathcal{B}$, respectively. This can be expressed as the problem

$$\underset{A,B}{\text{minimize}}\{\gamma||A||_0 + \text{rank}(B) \,:\, A + B = C,\, A \in \mathcal{A},\, B \in \mathcal{B}\}, \tag{19}$$

where $||A||_0$ counts the number of nonzero entries in A and γ is a positive parameter. The matrices A and B are $l \times q$ real matrices. A branch-and-bound approach for this problem has recently been proposed by Lee and Zou [41]. We can cast this as a conic complementarity problem by first exploiting the following lemma.

Lemma 4 *There exist matrices F and G so that the matrix*

$$M = \begin{bmatrix} F & B \\ B^T & G \end{bmatrix}$$

is positive semidefinite with rank equal to the rank of B.

Proof The result follows from taking the singular value decomposition of B, so $B = U\Sigma V^T$. The matrix Σ is diagonal with all diagonal entries positive. We then have

$$M = \begin{bmatrix} U\Sigma^{1/2} \\ V\Sigma^{1/2} \end{bmatrix} \begin{bmatrix} \Sigma^{1/2}U^T & \Sigma^{1/2}V^T \end{bmatrix}$$

which is positive semidefinite with the same rank as B. □

We can now use a construction similar to (18) to capture the rank of B using a semidefinite complementarity formulation. The $||A||_0$ term can be expressed using an LPCC formulation as described in [34]; in particular we define an $l \times q$ matrix ξ with each entry between 0 and 1 and we add the linear complementarity constraint $A_{ij}(1 - \xi_{ij}) = 0$, so we must have $\xi_{ij} = 1$ if $A_{ij} \neq 0$. Putting it all together, we can express problem (19) as a conic QPCC with a linear objective function, and with complementarity constraints over both the nonnegative orthant and over the positive semidefinite cone:

$$\begin{array}{ll} \underset{A,B,W,M,F,G,\xi}{\text{minimize}} & \gamma\,\langle E, \xi \rangle + \text{trace}(W) \\ \text{subject to} & A + B = C \\ & A \in \mathcal{A} \\ & B \in \mathcal{B} \\ & 0 \leq E - \xi \perp A \\ & \xi \geq 0 \\ & \begin{bmatrix} F & B \\ B^T & G \end{bmatrix} = M \\ & S_+^{l+q} \ni M \perp I - W \in S_+^{l+q} \\ & W \in S_+^{l+q} \end{array} \tag{20}$$

where E is the matrix of ones. It is worth noting that the constraints force trace(W) to be integral at optimality and thus equal to the rank of both M and B. It follows from Theorem 4 that a rank-sparsity decomposition problem is equivalent to a conic completely positive program, after first performing some manipulations to express it as a conic QPCC and then lifting it. We summarize this in the following theorem.

Theorem 6 *The rank-sparsity decomposition problem (19) is equivalent to a convex conic completely positive program.*

Many combinatorial optimization problems can be lifted to obtain semidefinite programming relaxations where feasible solutions to the original problem correspond to rank-one solutions to the SDP [36, 54]. It follows from Theorem 5 that completely positive relaxations can be constructed for these problems which give optimal solutions to the original combinatorial optimization problem. Stronger results have been proved by Bomze et al. [15] and by De Klerk and Pasechnik [28] in the cases of standard quadratic programming and the stability number of a graph, respectively. They showed that for these problems the SDP constraint can be replaced by a completely positive constraint, and the resulting completely positive program is equivalent to the original problem, without the need of a second lifting. These results were generalized in [21]. Much of the current interest in semidefinite programming was sparked by the relaxations of MaxCut and Satisfiability derived by Goemans and Williamson [37]; these problems can be represented as binary quadratic programs, so Burer's results [21] show that they have exact completely positive reformulations. Results for convex reformulations for SDP problems with rank requirements greater than unity were not previously known, as far as we are aware. Our results imply, for example, that finding the closest covariance matrix with bounded rank can be expressed as an equivalent completely positive program through the use of a single lifting.

5.4 General QCQPs

By Corollary 1, any QCQP can be represented as an nSp0-QCQP of the form (4) with a convex objective function. It follows that any QCQP can be manipulated in order to satisfy the assumption of Theorem 4, namely that Q^0 be copositive on $\text{conv}(L)$. Thus we have the following theorem.

Theorem 7 *A quadratically constrained quadratic program is equivalent to a convex conic optimization problem in the sense of Theorem 4.*

Note that this theorem makes no assumptions about the problem. In particular, it is not required that any of the quadratic functions be convex, or that the feasible region be bounded. If the QCQP has a bounded optimal value, but this value is not attained, then it follows from part 3 of Theorem 4 that the optimal value of the completely positive program is also not attained. Thus, a Frank-Wolfe result would not hold for the completely positive program in this setting.

Sensor network localization [1] is an example of a nonconvex quadratically constrained quadratic program that can be expressed in the form (4) without introducing the parameter λ and the variables r and s of Theorem 1. In this problem, the locations of some sensors and the distances between some pairs of sensors are given. It is then desired to determine the locations of all the sensors. If distances are known exactly, the model can be expressed as a quadratically constrained feasibility problem, with constraints of the form

$$||x^i - x^j||^2 \; - \; d_{ij}^2 \; = \; 0$$

where x^i and x^j are the locations of two sensors and d_{ij} is the distance between them. We can generalize this structure to a problem with a linear objective and quadratic equality constraints:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & p_i(x) = 0 \quad i = 1, \ldots, k \\ & Ax = b \end{array} \tag{21}$$

where each quadratic function $p_i(x)$ has a positive semidefinite Hessian. This is then equivalent to the problem

$$\begin{array}{rl} \underset{x\in\mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & p_i(x) \leq 0 \quad i = 1, \ldots, k \\ \sum_{i=1}^m p_i(x) \geq 0 & \\ & Ax = b. \end{array} \tag{22}$$

Each of the convex quadratic constraints $p_i(x) \leq 0$ can then be represented by a second order cone, so we directly obtain an nSp0-QCQP. Since the objective is linear, the assumption of Theorem 4 is vacuously satisfied. Thus, we can construct a convex completely positive problem that is equivalent to (21) without needing to use the construction of Theorem 1.

6 Local Optimality for QC-Problems Failing CQs

Although Theorem 2 has resolved the issue of solvability of the QCQP (4) satisfying the assumptions of this theorem, the question of how to characterize the optimality of a solution to such a problem is not addressed by this theorem, nor is it treated in the current literature, due to the lack of a suitable constraint qualification. In what follows we deal with this issue for the following generalization of an nSp0-QCQP:

$$\begin{array}{l} \underset{x\,\in\, C\,\cap\,\mathcal{K}\cap\mathcal{M}}{\text{minimize}} \ \theta(x) \\ \text{subject to} \ \ q(x) \triangleq \mathbf{h} + \mathbf{q}^T x + \frac{1}{2} x^T \mathbf{Q} x \leq 0, \end{array} \tag{23}$$

where $\mathcal{M} \triangleq \{x \in \mathbb{R}^n \mid Ax = b\}$, $\mathcal{K}$ is a convex cone, $q(x) \geq 0 \ \forall x \in \mathcal{K} \cap \mathcal{M}$, $\theta : \mathbb{R}^n \to \mathbb{R}$, and C is a closed convex set. For example, C can be the intersection of the level-sets $\{x \mid f_l(x) \leq 0\}$ for quasiconvex functions $f_l(x)$ for $l = 1, \ldots, L$. The feasible region of (23) is in general nonconvex. This formulation includes "convex" MPCCs, that is, convex programs with additional linear complementarity constraints.

Stationarity concepts such as the KKT conditions, M-stationarity and B-stationarity [52] examine linearizations of the problem. We show that problems involving gradients can be used to partially characterize the stationarity of a candidate point. In particular, we prove necessary conditions for local optimality of (23) in Theorem 8, and we prove a sufficient condition in Theorem 9.

Theorem 8 *Let $C \subseteq \mathbb{R}^n$ be closed convex and $\theta : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Let $S = \{ x \in C \cap \mathcal{K} \cap \mathcal{M} \mid q(x) = 0 \}$ where $q(x)$ is nonnegative over $\mathcal{K} \cap \mathcal{M}$. Let x^* be a feasible solution of (23). Consider the following statements:*

(a) *x^* is a locally optimal solution of (23);*

(b) x^* *is a globally optimal solution of (24):*

$$\begin{array}{ll} \underset{x \in S}{\textit{minimize}} \; (x - x^*)^T \nabla \theta(x^*) \\ \textit{subject to} \; (x - x^*)^T \nabla q(x^*) \leq 0; \end{array} \tag{24}$$

(c) x^* *is a locally optimal solution of the following:*

$$\underset{x \in S}{\textit{minimize}} \; (x - x^*)^T \nabla \theta(x^*). \tag{25}$$

It holds that (a) ⇒ (b). If $\mathcal{K}$ is a polyhedral cone then (b) ⇒ (c).

Proof (a) ⇒ (b). Assume by way of contradiction that x^* is not a globally optimal solution to (24). Then there exists $\bar{x}$ feasible to (24) such that $d^T \nabla \theta(x^*) < 0$, where $d \triangleq \bar{x} - x^*$. Note that

$$q(x^* + \lambda d) \,=\, q(x^*) + \lambda(\,\mathbf{q} + \mathbf{Q}x^*\,)^T d + \frac{\lambda^2}{2}\, d^T \mathbf{Q} d. \tag{26}$$

Since $q(x) \geq 0$ for all $x \in \mathcal{K} \cap \mathcal{M}$, we must have $(\mathbf{q} + \mathbf{Q}x^*)^T d \geq 0$; therefore from the constraint of (24) we have $(\mathbf{q} + \mathbf{Q}x^*)^T d = 0$. Since $q(x^*) = 0$ and $q(\bar{x}) = 0$ we must have $d^T \mathbf{Q} d = 0$. Therefore $x^* + \lambda d$ is feasible in (23) for all $\lambda \in [0, 1]$; hence $\theta(x^* + \lambda d) \geq \theta(x^*)$ for all $\lambda \geq 0$ sufficiently small, implying that $d^T \nabla \theta(x^*) \geq 0$. This is a contradiction.

(b) ⇒ (c) if $\mathcal{K}$ is polyhedral. We show the contrapositive. Assume x^* is not a locally optimal solution to (25). Then a sequence $\{x_k\} \subset S$ converging to x^* exists such that $(x_k - x^*)^T \nabla \theta(x^*) < 0$ for all k. By Corollary 3, we have

$$S \,=\, \{\, x \,\in\, C \,\cap\, \mathcal{K} \cap \mathcal{M} \,\mid\, q(x) \,=\, 0\,\} \,=\, C \,\cap\, \bigcup_{j=1}^{J} P^{\,j}\,.$$

There must exist a point x_{k_0} in the sequence such that x_{k_0} and x^* belong to the same piece $P^{j_0} \cap C$. Let $d \triangleq x_{k_0} - x^*$. Since $P^{j_0} \cap C$ is convex, $x^* + \lambda d \in S$ for all $\lambda \in [0, 1]$. From the same argument as in the previous part, we again obtain $(\mathbf{q} + \mathbf{Q}x^*)^T d = 0$, or $\nabla q(x^*)^T d = 0$, so $x^* + \lambda d$ is feasible in (24) for all $\lambda \in [0, 1]$. Therefore x^* is not a globally optimal solution of (24). □

A sufficient condition is given in the following theorem.

Theorem 9 *Assume the conditions of Theorem 8 hold. Assume in addition that $\mathcal{K}$ is a polyhedral cone and θ is convex on $C \cap \mathcal{K} \cap \mathcal{M}$. It follows that (b) ⇒ (a).*

Proof Assume by way of contradiction that x^* is not a local minimum to (23). There exists a sequence $\{x_k\}$ feasible to (23) converging to x^* such that $\theta(x_k) < \theta(x^*)$ for all k. As in the proof of (b) implying (c), there exists an x_{k_0} such that $x^* + \lambda d$ is feasible in (23) for all $\lambda \in [0, 1]$, where $d \triangleq x_{k_0} - x^*$. Thus $q(x^* + \lambda d) = 0$ for all such λ. The expansion (26) then implies that

$(\mathbf{q}+\mathbf{Q}x^*)^T d = 0$. Thus $x^* + \lambda d$ is feasible in (24) for all $\lambda \in [0, 1]$; in particular, so is x_{k_0}. As $\theta(x)$ is convex, we have

$$\theta(x_{k_0}) \geq \theta(x^*) + \nabla\theta(x^*)^T (x_{k_0} - x^*),$$

which implies

$$\nabla\theta(x^*)^T (x_{k_0} - x^*) \leq \theta(x_{k_0}) - \theta(x^*) < 0,$$

which means x^* is not a global optimum for (24). This completes the contrapositive proof. □

The example below shows that if x^* is a locally optimal solution to (23), it is not necessarily a *globally* optimal solution of (25); thus, to obtain a characterization of x^* as a globally optimal solution of a "linearized problem", it is essential that we add the extra constraint $\nabla q(x^*)^T (x - x^*) \leq 0$, yielding the problem (24). The example also shows that the implication "(c) ⇒ (b)" does not hold for the elements of the theorem, even when $\theta(x)$ is convex on $C \cap \mathcal{K} \cap \mathcal{M}$, so condition (c) is not sufficient to guarantee optimality.

Example 2 Consider the following simple 2-variable QPCC

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & (x_1 - 2)^2 + x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 3 \\ \text{and} & 0 \leq x_1 \perp x_2 \geq 0 \end{array}$$

so $q(x) = x_1 x_2$. The optimal solution to this "convex" QPCC is $x^* = (3, 0)$. The corresponding problem (25) is

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & 2(x_1 - 3) \\ \text{subject to} & x_1 + x_2 \geq 3 \\ \text{and} & 0 \leq x_1 \perp x_2 \geq 0, \end{array}$$

whose optimal solutions are all points of the form $(0, x_2)$ with $x_2 \geq 3$, none of these is the point $x^* = (3, 0)$. Adding the linearized complementarity constraint, we obtain the problem:

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & 2(x_1 - 3) \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & 0 \leq x_1 \perp x_2 \geq 0 \\ \text{and} & 3x_2 \leq 0, \end{array}$$

whose unique globally optimal solution is precisely x^*. □

In [52], four types of stationary points are defined for MPCCs, among which the so-called Bouligand or B-stationarity [45] yields the strongest conclusions; see also [53] where the concept of stationarity is generalized to nonlinear programs with "structurally nonconvex" feasible sets that include MPCCs. Checking B-stationarity is equivalent to solving an LPCC, therefore it is hard. To date, there is no clear understanding of the stationarity condition for a general nonconvex mathematical program that fails constraint qualifications (CQs). The problem (24) generalizes the idea of B-stationarity for an MPCC. Specifically consider the special case of (23) where $x \triangleq (\bar{x}, x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^{2m}_+$, $\mathcal{C} \cap \mathcal{K} \cap \mathcal{M}$ is a polyhedron, and $q(x) \triangleq (x^1)^T x^2$. For a feasible vector $x^* \triangleq (x^{*,0}, x^{*,1}, x^{*,2})$ of (23), define the 3 index sets pertaining to the complementarity condition: $0 \leq x^1 \perp x^2 \geq 0$:

$$\begin{aligned}
\alpha_* &\triangleq \{\, i \in 1, \ldots, m \mid x_i^{*,1} \,>\, 0 \,=\, x_i^{*,2} \,\} \\
\beta_* &\triangleq \{\, i \in 1, \ldots, m \mid x_i^{*,1} \,=\, 0 \,=\, x_i^{*,2} \,\} \\
\gamma_* &\triangleq \{\, i \in 1, \ldots, m \mid x_i^{*,1} \,=\, 0 \,<\, x_i^{*,2} \,\}.
\end{aligned}$$

These 3 index sets play a key role in the B-stationarity of the MPCC:

$$\begin{array}{ll}
\underset{x \triangleq (\bar{x}, x^1, x^2) \in C \cap \mathcal{K} \cap \mathcal{M}}{\text{minimize}} & \theta(x) \\
\text{subject to} & 0 \,\leq\, x^1 \,\perp\, x^2 \,\geq\, 0.
\end{array} \tag{27}$$

It is not difficult to show that with S denoting the feasible region of (27) and $q(x) \triangleq (x^1)^T x^2$, we have

$$\begin{aligned}
&\{\, x \,\in\, S \mid \nabla q(x^*)^T (x - x^*) \,\leq\, 0 \,\} \,= \\
&\quad \{\, x \,\in\, C \,\cap\, \mathcal{K} \cap \mathcal{M} \mid x_i^2 = 0 \;\forall\, i \in \alpha_*;\; x_i^1 = 0 \;\forall\, i \in \gamma_*;\; 0 \,\leq\, x_i^1 \,\perp\, x_i^2 \,\geq 0\}.
\end{aligned}$$

The above expression gives 2 structurally different representations of the same set; the left-hand representation expresses the set as defined by the closed set C intersected with the linearly-quadratically constrained set $\{x \in \mathcal{K} \cap \mathcal{M} \mid q(x) \leq 0,\, \nabla q(x^*)^T (x - x^*) \leq 0\}$, whereas the right-hand representation reveals the disjunctive structure of the latter set with reference to the given point x^* and shows that it is the union of finitely many closed convex sets. When in addition C is a polyhedron, then the problem (24) is a LPCC with a linear objective function. The upshot of this development is that in this case, the latter LPCC yields the optimality conditions of the quadratically constrained optimization problem (23) that fails the Slater constraint qualification.

7 Conclusions and Future Work

We have extended the literature to show that any conic QCQP (convex or nonconvex, with bounded or unbounded feasible region) is equivalent to a completely positive program, which is a convex optimization problem. This result applies in particular to semidefinite programs with a rank constraint.

For QPCCs and binary quadratic programs and some other classes of problems that can be represented directly as nSp0-QCQPs, the lifting can be performed without prior manipulations.

In addition, certain classes of nonconvex quadratically constrained quadratic programs are guaranteed to satisfy the Frank-Wolfe property. The nonconvex constraints in these problems violate the Slater constraint qualification. In deriving these results, we have exploited a relationship between these problems and quadratic programs with complementarity constraints. Further, we have related local optimality conditions for these QCQPs to stationarity conditions for mathematical programs with equilibrium constraints.

By combining Theorems 2 and 4, it follows that (12), which is a special copositive program, has the Frank-Wolfe property when $\mathcal{K}$ is a polyhedral cone, via the equivalence with the QCQP (4) failing the Slater condition with respect to its quadratic inequality constraints. A natural question to ask is whether there is a broader class of copositive programs for which the Frank-Wolfe result holds. At this time, we do not have an answer to this question.

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