The continuous knapsack set

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Abstract

We study the convex hull of the continuous knapsack set which consists of a single inequality constraint with n non-negative integer and m non-negative bounded continuous variables. When $n = 1$, this set is a slight generalization of the single arc flow set studied by Magnanti, Mirchandani, and Vachani (1993). We first show that in any facet-defining inequality, the number of distinct non-zero coefficients of the continuous variables is bounded by $2^n - n$. Our next result is to show that when $n = 2$, this upper bound is actually 1. This implies that when $n = 2$, the coefficients of the continuous variables in any facet-defining inequality are either 0 or 1 after scaling, and that all the facets can be obtained from facets of continuous knapsack sets with $m = 1$. The convex hull of the sets with $n = 2$ and $m = 1$ is then shown to be given by facets of either two-variable pure-integer knapsack sets or continuous knapsack sets with $n = 2$ and $m = 1$ in which the continuous variable is unbounded. The convex hull of these two sets has been completely described by Agra and Constantino (2006). Finally we show (via an example) that when $n = 3$, the non-zero coefficients of the continuous variables can take different values.

1 Introduction

In this paper we study the set $S = S_{LP} \cap (\mathbb{R}^m \times \mathbb{Z}^n)$ where

$$S_{LP} = \Big\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \;:\; \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} c_j y_j \geq b,\; u \geq x \geq 0,\; y \geq 0\Big\}$$

and $u, c, b > 0$ and rational. Throughout the paper, we refer to points in S_{LP} that are y-integral as integral points. The case where $n = 1$, known as the *single arc flow set*, was first studied by Magnanti, Mirchandani and Vachani [6]. They gave an explicit characterization of the convex hull of S via *residual capacity inequalities* (see Section 1.1). As discussed in [6], valid inequalities for set S yield cutting planes for network design problems where each variable y_j corresponds to a different facility that can be installed on an arc.

We study the properties of facet-defining inequalities for the case $n \geq 2$ and characterize the convex hull of S when $n = 2$. More precisely, we first prove in Section 2 that in any facet-defining inequality, the number of distinct non-zero coefficients of the continuous variables is at most $2^n - n$. We then study the case when $n = 2$ in detail, and show in Section 3 that all non-trivial facet-defining inequalities for *conv*(S) are of the form:

$$\sum_{i \in I} x_i + \gamma_1 y_1 + \gamma_2 y_2 \geq \beta$$

where $I \subseteq \{1, \ldots, m\}$ and $w + \gamma_1 y_1 + \gamma_2 y_2 \geq \beta$ is facet-defining for the set

$$Q(b', u') = conv\Big\{(w, y) \in \mathbb{R} \times \mathbb{Z}^2 \ : \ w + c_1 y_1 + c_2 y_2 \geq b', u' \ \geq w \geq 0, \ y \geq 0\Big\},$$

where $b' = b - \sum_{i \notin I} u_i$ and $u' = \sum_{i \in I} u_i$.

In other words, when $n = 2$, all facets of *conv*(S) can be obtained from three-variable relaxations. Throughout, $Q(b, u)$ will denote the special case of *conv*(S) when $m = 1$ and $n = 2$. In other words, $Q(b, u)$ is the convex hull of nonnegative $(w, y_1, y_2) \in \mathbb{R}^3$, with y_1, y_2 integral and $w \leq u$, satisfying $w + c_1 y_1 + c_2 y_2 \geq b$.

We then analyze the facial structure of $Q(b, u)$ in Section 4. We show that non-trivial facet-defining inequalities either have a zero coefficient for the w variable and therefore are facet-defining for

$$conv\Big\{(w, y) \in \mathbb{R} \times \mathbb{Z}^2 \ : \ c_1 y_1 + c_2 y_2 \geq b - u, \ y \geq 0\Big\},$$

or they can be obtained from a relaxation in which the upper bound is dropped:

$$Q(b, \infty) = conv\Big\{(w, y) \in \mathbb{R} \times \mathbb{Z}^2 \ : \ w + c_1 y_1 + c_2 y_2 \geq b, w \geq 0, \ y \geq 0\Big\}.$$

For $Q(b, \infty)$ for any $b > 0$, Agra and Constantino [2] gave a polynomial-time algorithm to enumerate all facet-defining inequalities. Note that S has an exponential number of relaxations of type $Q(b, u)$, one for each $I \subseteq \{1, \ldots, m\}$, and therefore our result does not lead directly to a polynomial-time separation algorithm for S. However, given a fixed objective function, we show that optimization over S can be carried out by solving m three variable problems, and thus the separation problem is also polynomially solvable using the ellipsoid algorithm.

Finally, in Section 5, we show that our results cannot be generalized to $n \geq 3$. In particular, we present a facet of a particular set with $n = 3$ such that the non-zero coefficients of the continuous variables in the associated inequality take different values.

1.1 Related Literature

Magnanti, Mirchandani and Vachani [6] studied the *single arc design problem* as a subproblem of the *network loading problem*. In particular, they studied the set

$$\Big\{(x, y) \in \mathbb{R}^m \times \mathbb{Z} : \sum_{i=1}^{m} x_i \leq cy, \quad u \geq x \geq 0, \ \ y \geq 0\Big\}.$$

The network loading problem is the problem of choosing arc capacities of minimum cost in a network so as to enable flows of different quantities (u_i) between m pairs of nodes, or equivalently to enable a multicommodity flow. On any arc, capacities must be chosen in integral multiples of a fixed constant (say c). The single arc design problem is the subproblem which enforces the fact that the flow through an arc for any commodity must be bounded above by the corresponding demand value (u_i), and the sum of flows is at most the chosen capacity (cy) on the arc.

By replacing each variable x_i by $u_i - x_i$, it is clear that the above set is equivalent to the set

$$\Big\{(x,y) \in \mathbb{R}^m \times \mathbb{Z} : \sum_{i=1}^{m} x_i + cy \geq b, \quad u \geq x \geq 0, \;\; y \geq 0\Big\},$$

where $b = \sum_{i=1}^{m} u_i$ (this is a special case of the set S described earlier, when $n = 1$). Magnanti et al. [6] showed that the *residual capacity inequalities* give the convex hull of solutions of this set, and Atamtürk and Rajan [4] give a polynomial-time separation algorithm for these inequalities.

When $n = 1$, Theorem 3.11 reduces to the result of Magnanti et al. More precisely, when $n = 1$, our result implies that every facet of S is of the form $\sum_{i \in I} x_i + \gamma y \geq \beta$ where $I \subseteq \{1, \ldots, m\}$ and $x + \gamma y \geq \beta$ is facet-defining for the two-variable set *conv*$\{(x,y) \in \mathbb{R} \times \mathbb{Z} : x + cy \geq b', u' \; \geq x \geq 0, \; y \geq 0\}$ where $b' = b - \sum_{i \notin I} u_i$ and $u' = \sum_{i \in I} u_i$. But the two-variable set has only one nontrivial facet-defining inequality, which is given either by the basic mixed-integer inequality, namely $x + (b' - \lfloor \frac{b'}{c} \rfloor c) y \geq (b' - \lfloor \frac{b'}{c} \rfloor) \lceil \frac{b'}{c} \rceil$ or by $y \geq \lceil (b' - u')/c \rceil$ when $I = \emptyset$. Thus our results yield an alternative proof of their result.

Atamtürk and Günlük [1] describe the set studied by Magnanti et al. as the *splittable flow arc set* which they studied as a subproblem of the *multicommodity network design problem*. In particular, they studied the more general version which is equivalent to S with $n = 1$ and arbitrary b, and used the results in Magnanti et al. to prove that the residual capacity inequalities still give the convex hull in this case.

Later Magnanti, Mirchandani, Vachani [7] stated that their results for $n = 1$ extend directly to give the convex hull for a special case of the single arc problem with two capacities (S with $n = 2$) in which all the data u_i and c_j are integer, $c_1 = 1$ and $c_2 > 1$. Note that in this case the associated capacity variable y_1 can be treated as continuous. Yaman [10] studied an extension of this version in which the capacity variables also impose an integer lower bound on the sum of the flows.

Wolsey and Yaman [9] study the continuous knapsack set with divisible capacities, i.e., $c_1 | \cdots | c_n$, and show that the coefficients of the continuous variables in any facet-defining inequality lie in $\{0, 1\}$ for all values of m and n.

2 Coefficients of continuous variables in facet-defining inequalities

In this section we consider a non-trivial facet-defining inequality

$$\sum_{i=1}^{m} \alpha_i x_i + \sum_{j=1}^{n} \gamma_j y_j \geq \beta$$

of *conv*(S) and study the properties of the vector α. Let $F = \{(x,y) \in S : \alpha x + \gamma y = \beta\}$ denote the set of points in S that satisfy this inequality as equality.

2.1 Basic properties of *conv*(S)

We start off with some basic polyhedral properties of *conv*(S). Let $e_i \in \mathbb{R}^m$ denote the unit vector with a one in the ith component and zeros elsewhere and $e \in \mathbb{R}^m$ denote the vector of all ones. We let $\bar{e}_i$ stand for the unit vector in $\mathbb{R}^n$ with a one in the ith component. We call the following inequalities that appear in the description of S_{LP}, *bound inequalities*: $x_i \geq 0$ and $x_i \leq u_i$ for $i = 1, \dots, m$; and $y_j \geq 0$ for $j = 1, \dots, n$. We call the inequality $ex + cy \geq b$, the *capacity inequality*. We also refer to all of these inequalities as *trivial inequalities* and the associated facets as trivial facets.

Lemma 2.1. *The following properties hold for conv*(S)*:*

(a) conv(S) *is full-dimensional and its recession cone is* $C = \{(0, y) \in \mathbb{R}^m \times \mathbb{R}^n : y \geq 0\}$.

(b) The bound inequalities $u_i \geq x_i$ *and* $x_i \geq 0$ *are facet-defining for* $i = 1, \dots, m$. *In addition, the bound inequalities* $y_j \geq 0$ *are facet-defining for* $j = 1, \dots, n$*, provided that* $n \geq 2$.

(c) A necessary condition for the capacity inequality to be facet-defining is $b \geq c_j$ *for all* $j \in N$. *A sufficient condition is* $\sum_{i \in M} u_i > b - \lfloor b/c_j \rfloor c_j$ *for all* $j \in N$ *and* $\sum_{i \in M} u_i > b - \lfloor b/c_k \rfloor c_k + c_k$ *for some* $k \in N$.

(d) If $\alpha x + \gamma y \geq \beta$ *is a non-trivial facet-defining inequality, then* $\alpha \geq 0$, $\gamma > 0$ *and* $\beta > 0$.

(e) All non-trivial facets are bounded.

Proof. (a) Consider the following $n + m + 1$ points in S: $w = (0, \lceil b/c_1 \rceil \bar{e}_1)$, $p_i = w + (u_i e_i, 0)$ for $i = 1, \dots, m$, and $q_j = w + (0, \bar{e}_j)$ for $j = 1, \dots, n$. These points are affinely independent as the points $p_i - w$ and $q_j - w$ are linearly independent. The recession cone of *conv*(S) is C as $c > 0$ and the continuous variables are bounded.

(b) For each $i = 1, \dots, m$, the $m + n$ points w, p_k for $k \neq i$ and q_j for $j = 1, \dots, n$ satisfy $x_i = 0$ and are affinely independent. Adding $(u_i e_i, 0)$ to each of these $n + m$ points, one obtains $m + n$ affinely independent points of S with $x_i = u_i$.

For $j = 2, \dots, n$ the $m + n$ points w, p_i for $i = 1, \dots, m$ and q_k for $k \neq j$ with $y_j = 0$ are affinely independent. As the choice of the coordinate corresponding to y_1 in the construction of w is arbitrary, $y_1 \geq 0$ is facet-defining as well.

(c) To see that the capacity inequality cannot be facet-defining if $b < c_j$ for some $j = 1, \dots n$, note that in this case the inequality cannot be tight if $y_j \geq 1$. Now assume that $b \geq c_j$ for all $j = 1, \dots n$. Let $\mu = \sum_{i=1}^m u_i$ and $\nu_j = b - \lfloor \frac{b}{c_j} \rfloor c_j$ for all j. The following points are affinely independent: $h_j = ((\nu_j/\mu)u, \lfloor b/c_j \rfloor \bar{e}_j)$ for $j \in N$, $f_i = h_1 + \epsilon(e_1 - e_i)$ for $i \in M \setminus \{1\}$ where $\epsilon > 0$ is a small enough number, and $r = (((\nu_k + c_k)/\mu)u, (\lfloor b/c_k \rfloor - 1)\bar{e}_k)$.

(d) Again let $F = \{(x, y) \in S : \alpha x + \gamma y = \beta\}$. For any $i = 1, \dots, m$, there is a point $(x, y) \in F$ with $x_i < u_i$, and for some small $\epsilon > 0$, we have $(x, y) + (\epsilon e_i, 0) \in S$. This implies that $\alpha \geq 0$. In addition, as the recession cone of *conv*(S) is C, we have $\gamma \geq 0$. As the inequality is not implied by the non-negativity constraints, $\beta > 0$. Finally, if $\gamma_i = 0$ for some i in $\{1, \ldots, n\}$, then $\alpha x + \gamma y \geq \beta$ is violated by $(0, \lceil b/c_i \rceil \bar{e}_i) \in S$. Therefore $\gamma > 0$.

(e) As $x \leq u$ for all $(x, y) \in S$ and $\gamma > 0$ for all non-trivial facet-defining inequalities, the claim holds. □

Lemma 2.2. *Let $(\hat{x}, \hat{y})$ be an extreme point of S. Then $u_j > \hat{x}_j > 0$ for at most one $j \in \{1, \ldots, m\}$. Furthermore, if $\hat{x} \neq 0$, then $e\hat{x} + c\hat{y} = b$.*

Proof. Suppose there exist two indices i and j such that $u_i > \hat{x}_i > 0$ and $u_j > \hat{x}_j > 0$.. For any positive $\epsilon \leq \min\{u_i - \hat{x}_i, \hat{x}_i, u_j - \hat{x}_j, \hat{x}_j\}$ the points $(\hat{x} + \epsilon e_i - \epsilon e_j, \hat{y})$ and $(\hat{x} - \epsilon e_i + \epsilon e_j, \hat{y})$ belong to S and $(\hat{x}, \hat{y})$ is a convex combination of these points, contradicting the extremeness of $(\hat{x}, \hat{y})$. Furthermore, if $u_i > \hat{x}_i > 0$ for some index i, and $e\hat{x} + c\hat{y} > b$, then for any ϵ such that $0 < \epsilon \leq \min\{u_i - \hat{x}_i, \hat{x}_i, e\hat{x} + c\hat{y} - b\}$, the two points obtained from $(\hat{x}, \hat{y})$ by increasing and decreasing the ith component of $\hat{x}$ by ϵ are contained in S, and $(\hat{x}, \hat{y})$ is a convex combination of these points. □

2.2 Facets of *conv*(S) obtainable from relaxations

We next study the conditions under which a non-trivial facet-defining inequality $\alpha x + \gamma y \geq \beta$ can be obtained from a lower-dimensional relaxation of S. Remember that F denotes the set of points in S that satisfy this inequality as equality. We next present two observations that lead to the main result of this section. First we consider the case when some of the entries of α are zero.

Lemma 2.3. *If $\alpha_m = 0$, then $\sum_{i=1}^{m-1} \alpha_i x_i + \sum_{j=1}^{n} \gamma_j y_j \geq \beta$ is facet-defining for the convex hull of*

$$S' = \left\{(x, y) \in \mathbb{R}^{m-1} \times \mathbb{Z}^n \ : \ \sum_{i=1}^{m-1} x_i + \sum_{j=1}^{n} c_j y_j \geq b - u_m, \ y \geq 0, \ u' \geq x \geq 0\right\}$$

where $u'_i = u_i$ for $i = 1, \ldots, m-1$.

Proof. First notice that *conv*(S') is obtained by deleting x_m from the set of points *conv*$(S) \cap X_m$ where $X_m = \{(x, y) \in \mathbb{R}^{n+m} : x_m = u_m\}$. Therefore, if $\alpha x + \gamma y \geq \beta$ is valid for *conv*(S), and consequently for *conv*$(S) \cap X_m$, then $\sum_{i=1}^{m-1} \alpha_i x_i + \sum_{j=1}^{n} \gamma_j y_j \geq \beta$ is valid for *conv*(S'). We next argue that the inequality is facet-defining for *conv*(S').

Let $p^k = (x^k, y^k) \in \mathbb{R}^m \times \mathbb{Z}^n$ be a collection of $m + n$ affinely independent points in F, which exist as *conv*(F) is a facet of *conv*(S). Let $\hat{p}^k = (\hat{x}^k, y^k) \in \mathbb{R}^{m-1} \times \mathbb{Z}^n$ be obtained from p^k by deleting the last entry of x^k for all $k = 1, \ldots, m + n$. Also let $\hat{\alpha} \in \mathbb{R}^{m-1}$ be obtained from α by deleting the last entry. Notice that $\sum_{i=1}^{m-1} \hat{x}_i^k = \sum_{i=1}^{m} x_i^k - x_m \geq \sum_{i=1}^{m} x_i^k - u_m$ and therefore $\hat{p}^k \in S'$ for all $k = 1, \ldots, m + n$. Furthermore, $\hat{\alpha}\hat{x}^k = \alpha x^k$, and consequently $\hat{\alpha}\hat{x}^k + \gamma y^k = \beta$ for all $k = 1, \ldots, m+n$. As the affine rank of $\{\hat{p}^1, \ldots, \hat{p}^{m+n}\}$ is one less than the affine rank of $\{p^1, \ldots, p^{m+n}\}$, we conclude that the claim is true. □

Applying this observation repeatedly, we make the following observation when $\alpha = 0$.

Corollary 2.4. *If $\alpha = 0$, then $\gamma y \geq \beta$ is facet-defining for the convex hull of $S' = conv\{y \in \mathbb{Z}^n : \sum_{j=1}^{n} c_j y_j \geq b - \sum_{i=1}^{m} u_i,\ y \geq 0\}$. In addition, if $\sum_{i=1}^{m} u_i \geq b$, then the facet is one of the non-negativity facets associated with y.*

We next consider the case when some of the entries of the coefficient vector α are the same.

Lemma 2.5. *If $\alpha_{m-1} = \alpha_m$, then $\sum_{i=1}^{m-1} \alpha_i x_i + \sum_{j=1}^{n} \gamma_j y_j \geq \beta$ is facet-defining for the convex hull of*

$$S'' = \left\{(x, y) \in \mathbb{R}^{m-1} \times \mathbb{Z}^n \ : \ \sum_{i=1}^{m-1} x_i + \sum_{j=1}^{n} c_i y_i \geq b,\ y \geq 0,\ u' \geq x \geq 0\right\}$$

where $u'_i = u_i$ for $i = 1, \dots, m-2$ and $u'_{m-1} = u_{m-1} + u_m$.

Proof. The proof is very similar to the proof of Lemma 2.3. First we observe that $\sum_{i=1}^{n-1} \alpha_i x_i + \sum_{j=1}^{n} \gamma_j y_j \geq \beta$ is valid for S'' provided that $\alpha x + \gamma y \geq \beta$ is valid for S. Then, we modify the points p^i defined in Lemma 2.3 by combining the last two entries of the continuous variables. The resulting (lower dimensional) points are in S'' and have the desired affine rank to conclude the proof. □

Theorem 2.6. *Assume that $\alpha_i \in \{0, \hat{\alpha}_1, \dots, \hat{\alpha}_t\}$ for all $i = 1, \dots, m$ where $\hat{\alpha}_1, \dots, \hat{\alpha}_t$ are distinct positive numbers. Then $\sum_{k=1}^{t} \hat{\alpha}_k w_k + \sum_{j=1}^{n} \gamma_j y_j \geq \beta$ is facet-defining for the convex hull of*

$$\hat{S} = \left\{(w, y) \in \mathbb{R}^{t} \times \mathbb{Z}^n \ : \ \sum_{k=1}^{t} w_k + \sum_{j=1}^{n} c_j y_j \geq b - \theta,\ y \geq 0,\ \hat{u} \geq \hat{x} \geq 0\right\}$$

where $\hat{u}_k = \sum_{k:\alpha_i = \hat{\alpha}_k} u_i$ for $k = 1, \dots, t$ and $\theta = \sum_{k:\alpha_i = 0} u_i$.

Proof. Applying Lemma 2.3 and Lemma 2.5 repeatedly proves the claim. □

Note that each w_k variable in the set $\hat{S}$ essentially stands for the sum of the x_i variables that have the coefficient $\hat{\alpha}_k$ in the facet-defining inequality. For example, w_1 in $\hat{S}$ stands for $\sum_{i:\alpha_i = \hat{\alpha}_1} x_i$ and has an upper bound equal to the sum of the upper bounds of the associated x_i variables. To demonstrate this observation further, we consider the following set

$$\begin{aligned} S = \{(x, y) \in \mathbb{R}^4 \times \mathbb{Z}^3 \quad : \quad & x_1 + x_2 + x_3 + x_4 + 5y_1 + 13y_2 + 22y_3 \geq 98, \\ & 1 \geq x_1 \geq 0, 1 \geq x_2 \geq 0, 2 \geq x_3 \geq 0, 3 \geq x_4 \geq 0, y \geq 0\}. \end{aligned}$$

It can be checked that the inequality

$$x_2 + x_3 + 2x_4 + 5y_1 + 13y_2 + 21y_3 \geq 94 \tag{1}$$

is valid for S by extending the proof of Proposition 5.4. Furthermore, it is facet-defining for $conv(S)$ as the following linearly independent points in S satisfy the inequality as equality: $([1, 0, 0, 0][2, 0, 4])$, $([1, 0, 0, 0][1, 2, 3])$, $([1, 1/3, 2/3, 0][6, 0, 3])$, $([1, 1, 2, 1][1, 0, 4])$, $([1, 1, 2, 0][3, 1, 3)]$, $([1, 1, 0, 0][6, 0, 3])$, and $([0, 0, 0, 0][2, 0, 4])$.

Note that in inequality (1), the coefficient of variable x_1 is zero and variables x_2 and x_3 have the same coefficient. Consequently, by Lemma 2.6,

$$w_1 + 2w_2 + 5y_1 + 13y_2 + 21y_3 \geq 94 \tag{2}$$

is facet-defining for the convex hull of the set

$$\hat{S} = \{(w, y) \in \mathbb{R}^2 \times \mathbb{Z}^3 : w_1 + w_2 + 5y_1 + 13y_2 + 22y_3 \geq 97,\ 3 \geq w_1, w_2 \geq 0, y \geq 0\}$$

where $\theta = 1$, $\hat{u}_1 = u_2 + u_3$ and $\hat{u}_2 = u_4$. Notice that variable w_1 in $\hat{S}$ stands for $x_2 + x_3$ and w_2 stands for x_4 . The fact that inequality (2) is facet-defining for *conv*$(\hat{S})$ is shown in Proposition 5.4.

We also note that the reverse of Theorem 2.6 is not true in the sense that a facet-defining inequality for *conv*$(\hat{S})$ does not necessarily lead to a facet-defining inequality for *conv*(S). To see this consider the sets

$$S = \{(x, y) \in \mathbb{R}^2 \times \mathbb{Z}^2 : x_1 + x_2 + 5y_1 + 10y_2 \geq 25,\ 2 \geq x_1 \geq 0, 3 \geq x_2 \geq 0, y \geq 0\}$$

and

$$\hat{S} = \{(w, y) \in \mathbb{R} \times \mathbb{Z}^2 : w + 5y_1 + 10y_2 \geq 25,\ 5 \geq w \geq 0, y \geq 0\}$$

where $\hat{S}$ is obtained by introducing variable w to stand for $x_1 + x_2$. It is easy to check that inequality $w + 5y_1 + 5y_2 \geq 15$ is facet-defining for *conv*$(\hat{S})$, whereas the corresponding inequality $x_1 + x_2 + 5y_1 + 5y_2 \geq 15$ is not facet-defining for *conv*(S).

2.3 Bounding the number of distinct coefficients in facet-defining inequalities

We next consider a facet-defining inequality $\alpha x + \gamma y \geq \beta$ such that $\alpha > 0$ and all of the entries of α are distinct (α may have a single component). Remember that F denotes the set of points in S that satisfy this inequality as equality. We start off with a technical observation that we use later.

Lemma 2.7. *Assume that $\alpha > 0$ has all distinct coefficients. If $(x^1, \hat{y}), (x^2, \hat{y}) \in F$ then $x^1 = x^2$.*

Proof. Clearly $\alpha x^1 + \gamma \hat{y} = \alpha x^2 + \gamma \hat{y} = \beta$ and therefore $\alpha x^1 = \alpha x^2$. Assume $x^1 \neq x^2$. If $\alpha \in \mathbb{R}$, then the result trivially follows. Otherwise there must exist two indices i and j such that $x_i^1 \neq x_i^2$ and $x_j^1 \neq x_j^2$ and therefore $\hat{x} = \frac{1}{2}x^1 + \frac{1}{2}x^2$ has $u_i > \hat{x}_i > 0$ and $u_j > \hat{x}_j > 0$. Note that $(\hat{x}, \hat{y}) \in F \subseteq S$. Now assume $\alpha_i > \alpha_j$ and notice that for some small $\epsilon > 0$ a new point x' obtained by reducing $\hat{x}_i$ by ϵ and increasing $\hat{x}_j$ by ϵ gives $(x', \hat{y}) \in S$. This point $(x', \hat{y})$, however, violates the facet-defining inequality as $\alpha x' < \alpha \hat{x}$, a contradiction. □

Lemma 2.8. *Assume that $\alpha > 0$ has all distinct coefficients. Then F contains a subset of $m + n$ affinely independent points $\{(x^i, y^i) : i = 1, \ldots, m+n\}$ such that conv$(y^1, \ldots, y^{m+n})$ has the following properties: (i) it is full-dimensional, (ii) its vertices are precisely $y^1, \ldots, y^{m+n}$, and (iii) it contains no other integer points.*

Proof. As *conv*$(F) \subseteq \mathbb{R}^{m+n}$ has dimension $m+n-1$, F contains $m+n$ affinely independent points. Among all such sets of points, let $L = \{(x^i, y^i) : i = 1, \dots, m+n\}$ stand for the one with the property that *conv*(L_y), where $L_y = \{y^1, \dots, y^{m+n}\}$, has fewest possible integral points.

(i) If *conv*(L_y) is not full-dimensional, then there exists $0 \neq \gamma' \in \mathbb{R}^n, \beta' \in \mathbb{R}$ such that $\gamma' y^i = \beta'$ for $i = 1, \dots, m+n$. But this means that points in F satisfy the equation $\gamma' y = \beta'$, which contradicts the fact that $\alpha x + \gamma y = \beta$ is uniquely defined up to multiplication by a scalar and $\alpha > 0$.

(ii) Suppose $y^k \in L_y$ is not a vertex of *conv*(L_y) for some k. Then $y^k = \sum_{i=1}^{m+n} \mu_i y^i$ for some $\mu_i \geq 0$ with $\sum_{i=1}^{m+n} \mu_i = 1$ and $\mu_k = 0$. Let $\bar{x} = \sum_{i=1}^{m+n} \mu_i x^i$. Then $(\bar{x}, y^k) \in F$. Lemma 2.7 implies that $\bar{x} = x^k$, and therefore (x^k, y^k) is a convex combination of other points in L, which contradicts the affine independence of points in L. Therefore the vertices of *conv*(L_y) are precisely the points $y^1, \dots, y^{m+n}$.

(iii) Suppose *conv*(L_y) contains an integer point $\bar{y}$ not contained in L_y. By definition, $\bar{y} = \sum_{i=1}^{m+n} \mu_i y^i$ for some $\mu_i \geq 0$ with $\sum_{i=1}^{m+n} \mu_i = 1$; also μ has at least two components strictly between 0 and 1 as $\bar{y} \notin L_y$. Let $\bar{x} = \sum_{i=1}^{m+n} \mu_i x^i$. Then $(\bar{x}, \bar{y}) \in F$. As the points in L are affinely independent, for some index $j \in \{1, \dots, m+n\}$ with $\mu_j > 0$, the set $L' = L \cup \{(\bar{x}, \bar{y})\} \setminus \{(x^j, y^j)\}$ is affinely independent. L' also has the property that the convex hull of L'_y is strictly contained in the convex hull of L_y and has fewer integral points (it does not contain y^j). This contradicts the definition of L. □

We next bound the number of distinct values of the entries of α when S has an arbitrary number of continuous variables. By Theorem 2.6, one needs to consider the case when α has distinct non-zero coefficients.

Theorem 2.9. *If $\alpha x + \gamma y \geq \beta$ is a facet-defining inequality for conv(S) then α has at most $2^n - n$ distinct non-zero entries.*

Proof. As the claim holds for trivial facet-defining inequalities, we only consider non-trivial inequalities. First assume that $\alpha > 0$ and has all distinct coefficients and let $L_y \subseteq \mathbb{R}^n$ be defined as in the proof of Lemma 2.8. Suppose L_y contains more than 2^n integer points. Then it contains at least two distinct points, say y^k and y^l, with the same odd/even parity (that is, for all i: y^k_i is odd if and only if y^l_i is odd). Consequently, $\hat{y} = (y^k + y^l)/2 \in$ *conv*$(L_y) \cap \mathbb{Z}^n$ which contradicts Lemma 2.8. Therefore when $\alpha > 0$ and has all distinct coefficients $m + n \leq 2^n$. Combining Theorem 2.6 with this observation completes the proof. □

Corollary 2.10. *If $n = 2$, then all facet-defining inequalities for conv(S) can be obtained from relaxations of the form $\hat{S}$ presented in Theorem 2.6 that have 2 continuous variables.*

2.4 Bounding the number of distinct coefficients in disjunctive cuts

We next derive an upper bound on the number of distinct positive coefficients of continuous variables in facet-defining inequalities of disjunctive relaxations of S. More precisely, we consider a disjunctive cut $\alpha x + \gamma y \geq \beta$ for *conv*(S) that can be derived using the $|K|$-term disjunction $D = \cup_{k \in K} D^k$ where

$$D^k = \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ A^k y \geq d^k\right\}$$

for $k \in K$. We assume that $\mathbb{R}^m \times \mathbb{Z}^n \subseteq D$ and therefore $\alpha x + \gamma y \geq \beta$ is a valid inequality for *conv*(S). Furthermore, we assume that $\alpha x + \gamma y \geq \beta$ defines a facet of the disjunctive relaxation of *conv*(S)

$$Q = \mathit{conv}(\cup_{k \in K}(D^k \cap S_{LP}))$$

that is distinct from the bound constraints on the x variables. Note that $\alpha_i \geq 0$ for all i (as the related facet must contain a point with $x_i < u_i$ and if $\alpha_i < 0$, this point can be perturbed to obtain a new point in Q violating the inequality). Clearly some of the sets $D^k \cap S_{LP}$ can be empty. Without loss of generality, assume that $D^k \cap S_{LP} \neq \emptyset$ for $k \in \bar{K} = \{1, \ldots, |\bar{K}|\}$ and $D^k \cap S_{LP} = \emptyset$ for $k > |\bar{K}|$.

As the inequality $\alpha x + \gamma y \geq \beta$ is valid for

$$D^k \cap S_{LP} = \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ ex + cy \geq b, \ A^k y \geq d^k, \ y \geq 0, \ u \geq x \geq 0\right\}$$

for $k \leq |\bar{K}|$, there exist nonnegative multipliers θ^k (associated with $ex + cy \geq b$), η^k (associated with $A^k y \geq d^k$), and λ^k (associated with $-x \geq -u$) that yield the valid inequality

$$(\theta^k e - \lambda^k)x + (\theta^k c + \eta^k A^k)y \ \geq \ \theta^k b + \eta^k d^k - \lambda^k u$$

for $D^k \cap S_{LP}$ where $\alpha \geq (\theta^k e - \lambda^k)$, $\gamma \geq (\eta^k A^k + \theta^k c)$ and $\beta \leq (\theta^k b + \eta^k d^k - \lambda^k u)$. Furthermore, as $\alpha x + \gamma y \geq \beta$ is facet-defining for Q by assumption, we have (here e_j is a unit vector in $\mathbb{R}^n$ with a one in the j component)

$$\alpha_i = \max_{k \in \bar{K}}\{\theta^k - \lambda_i^k\}, \ \ \gamma_j = \max_{k \in \bar{K}}\{\theta^k c_j + \eta^k A^k e_j\}, \ \text{ and } \ \beta = \min_{k \in \bar{K}}\{\theta^k b + \eta^k d^k - \lambda^k u).$$

Note that if $\hat{\theta} = \min_{k \in \bar{K}}\{\theta^k\} > 0$, then decreasing all entries of the vector θ by $\hat{\theta}$ yields a stronger valid inequality for Q. This is not possible as $\alpha x + \gamma y \geq \beta$ is facet-defining for Q. Consequently, we conclude that $\min_{k \in \bar{K}}\{\theta^k\} = 0$. Without loss of generality, we assume that $\theta^{|\bar{K}|} \geq \theta^{|\bar{K}|-1} \geq \ldots \geq \theta^1 = 0$.

Using the multipliers θ and η (but not λ) it is easy to see that for all $k \in \bar{K}$ the inequality $\theta^k ex + \gamma y \geq \beta^k$, where $\beta^k = \theta^k b + \eta^k d^k$, is valid for $D^k \cap S_{LP}$. Consequently, for all $k \in \bar{K}$ we can define the following relaxation of the set $D^k \cap S_{LP}$:

$$W^k = \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ \theta^k ex + \gamma y \geq \beta^k, \ \ u \geq x \geq 0\right\} \supseteq D^k \cap S_{LP}.$$

Notice that $\alpha x + \gamma y \geq \beta$ is valid for each W^k and consequently, it is valid for $W = \mathit{conv}(\cup_{k \in \bar{K}} W^k)$. Furthermore, as W is a relaxation of Q, the inequality $\alpha x + \gamma y \geq \beta$ is facet-defining for W.

Theorem 2.11. *Given a t-term disjunction and a facet-defining inequality $\alpha x + \gamma y \geq \beta$ for the associated disjunctive relaxation, the vector α has at most $2(t-1)$ distinct non-zero coefficients.*

Proof. Using the notation introduced in the the preceding discussion, $\alpha x + \gamma y \geq \beta$ is valid for W^k for all $k \in \bar{K}$, and consequently there exists a non-negative vector $\lambda^k \in \mathbb{R}^m$ for each $k \in \bar{K}$ such that $\alpha_i \geq \theta^k - \lambda_i^k$ and $\beta \leq \beta_k - \lambda^k u$. As $\alpha x + \gamma y \geq \beta$ is facet-defining for the associated disjunctive relaxation, $\alpha_i = \max_{k\in\bar{K}}\{\theta^k - \lambda_i^k\}$ and $\beta = \min_{k\in\bar{K}}\{\beta_k - \lambda^k u\}$. Clearly, $\lambda_i^k \geq \theta^k - \alpha_i$ and $\lambda_i^k \geq 0$ implying $\lambda_i^k \geq (\theta^k - \alpha_i)^+$ for all $i = 1, \ldots, m$ and $k \in \bar{K}$. Without loss of generality, we also assume that $\lambda_i^k = (\theta^k - \alpha_i)^+$ for all $i = 1, \ldots, m$ and $k \in \bar{K}$.

We next argue that if $\alpha_i, \alpha_j \notin \{\theta^1, \ldots, \theta^{|\bar{K}|}\}$, then $\max\{\alpha_i, \alpha_j\} > \theta^l > \min\{\alpha_i, \alpha_j\}$ for some l. As $\alpha_i = \max_{k\in\bar{K}}\{\theta^k - \lambda_i^k\} \leq \theta^{|\bar{K}|}$ and $\theta^1 = 0$ we have $\theta^1 \leq \alpha_i \leq \theta^{|K|}$ for all i. Assume that there exists distinct v, w together with an index k_1 such that $\theta^{k_1+1} > \alpha_v, \alpha_w > \theta^{k_1}$. Clearly, $\lambda_v^k = \lambda_w^k = 0$ when $k \leq k_1$ and $\lambda_v^k, \lambda_w^k > 0$, otherwise. Let $\epsilon > 0$ be sufficiently small and define:

$$\alpha(\epsilon) = \begin{cases} \alpha_i & i \notin \{u, v\} \\ \alpha_v - \epsilon/u_v & i = v \\ \alpha_w + \epsilon/u_w & i = w \end{cases} \qquad \alpha(-\epsilon) = \begin{cases} \alpha_i & i \notin \{u, v\} \\ \alpha_v + \epsilon/u_v & i = v \\ \alpha_w - \epsilon/u_w & i = w \end{cases}$$

and similarly,

$$\lambda_i^k(\epsilon) = \begin{cases} \lambda_i^k & i \notin \{u, v\} \text{ and } k \in \bar{K} \\ \lambda_i^k = 0 & i \in \{u, v\} \text{ and } k \leq k_1 \\ \lambda_v^k + \epsilon/u_v & i = v \text{ and } k \geq k_1 + 1 \\ \lambda_w^k - \epsilon/u_w & i = w \text{ and } k \geq k_1 + 1 \end{cases} \qquad \lambda_i^k(-\epsilon) = \begin{cases} \lambda_i^k & i \notin \{u, v\} \text{ and } k \in \bar{K} \\ \lambda_i^k = 0 & i \in \{u, v\} \text{ and } k \leq k_1 \\ \lambda_v^k - \epsilon/u_v & i = v \text{ and } k \geq k_1 + 1 \\ \lambda_w^k + \epsilon/u_w & i = w \text{ and } k \geq k_1 + 1. \end{cases}$$

Notice that for all k, we have $\lambda^k(\epsilon)u = \lambda^k(-\epsilon)u = \lambda^k u$. Consequently, $\beta = \min_{k\in\bar{K}}\{\beta_k - \lambda^k(\epsilon)u\} = \min_{k\in\bar{K}}\{\beta_k - \lambda^k(-\epsilon)u\}$. Furthermore, as $\theta^{k_1+1} > \alpha_v, \alpha_w > \theta^{k_1}$ by assumption and $\epsilon > 0$ is small enough, we also have $\alpha_i(\epsilon) = \max_{k\in\bar{K}}\{\theta^k - \lambda_i^k(\epsilon)\}$ and $\alpha_i(-\epsilon) = \max_{k\in\bar{K}}\{\theta^k - \lambda_i^k(-\epsilon)\}$ for all i. Therefore, both $\alpha(\epsilon)x + \gamma y \geq \beta$ and $\alpha(-\epsilon)x + \gamma y \geq \beta$ are valid inequalities for *conv*$(\cup_{k\in\bar{K}} W^k)$. But in this case $\alpha x + \gamma y \geq \beta$ cannot be facet-defining as $\alpha = (\alpha(\epsilon) + \alpha(-\epsilon))/2$. We can therefore conclude that if $\alpha_u, \alpha_w \notin \{\theta^1, \ldots, \theta^{|K|}\}$, then $\max\{\alpha_u, \alpha_w\} > \theta^l > \min\{\alpha_u, \alpha_w\}$ for some l. Consequently, there can only be at most one α_i that lies between two consecutive θ entries. As $\theta^1 = 0$, we conclude that the vector α has at most $2(t-1)$ distinct non-zero coefficients. □

Note that any facet-defining inequality for S can be generated as a disjunctive cut from a disjunction with at most 2^n-terms [3]. Consequently, Theorem 2.11 implies that if $\alpha x + \gamma y \geq \beta$ is facet-defining for *conv*(S) then α has at most $2(2^n - 1)$ distinct non-zero entries. This bound is weaker than that given by Theorem 2.9, however Theorem 2.11 still leads to useful observations:

Corollary 2.12. *If D is a split disjunction, then α has at most 2 distinct coefficients.*

3 Characterizing facet-defining inequalities when $n = 2$

In this section we show Theorem 3.11, namely that if $\alpha x + \gamma y \geq \beta$ is a nontrivial facet-defining inequality for *conv*(S) with $n = 2$ (i.e., $y \in \mathbb{R}^2$), then all nonzero components of α are equal. The proof is by contradiction; we assume Theorem 3.11 is not true and consider a minimal counterexample to Theorem 3.11 with $n = 2$, i.e., a set S with $n = 2$ and a facet-defining inequality $\alpha x + \gamma y \geq \beta$ such that S has as few variables as possible. We can assume that $\alpha > 0$ and its coefficients are all distinct. The reason for this assumption is the following. Let S and $\alpha x + \gamma y \geq \beta$ form a minimal counterexample – i.e., it is facet-defining for *conv*(S) and the nonzero coefficients of α are not all equal, and $m+n$ is as small as possible. If a component of α is zero or a pair of components of α are nonzero but equal, we can apply either Lemma 2.3 or Lemma 2.5 and obtain a facet-defining inequality of a set S' with fewer variables than S, but with the facet-defining inequality having the same set of nonzero α values as before. This would contradict the assumption of minimality of S. Therefore, in a minimal counterexample, $\alpha > 0$ and its coefficients are all distinct. In this case, Corollary 2.10 implies that $m \leq 2$.

If $m = 1$ and α has a single component, there is nothing to prove. So we make the following assumption.

Assumption 3.1. *Suppose $n = 2$. If S and $\alpha x + \gamma y \geq \beta$ form a minimal counterexample to Theorem 3.11, then $m = 2$, and $0 < \alpha_1 < \alpha_2$.*

We start off by proving some properties of integral points contained in nontrivial facets of *conv*(S) for arbitrary n, and then focus on the case $n = 2$ and $m = 2$.

3.1 Properties of integral points on facets of *conv*(S)

We next study properties of integral points on facets of *conv*(S) for general n. Throughout we assume that $\alpha x + \gamma y \geq \beta$ is a nontrivial facet-defining inequality for *conv*(S) and F is the set of points in S lying on the corresponding facet.

Lemma 3.2. *Let $(x, y) \in F$ and let $x_j > 0$ for some $j \in \{1, \ldots, m\}$. Then for every index $i \neq j$ with $\alpha_i < \alpha_j$, we have $x_i = u_i$. Furthermore, if $0 < x_i < u_i$ for $i = 1, \ldots, m$, then α has all its coefficients equal.*

Proof. If there is some index $i \neq j$ with $\alpha_i < \alpha_j$ such that $x_i < u_i$, then letting $\epsilon = \min\{u_i - x_i, x_j\}$, we see that $(x, y) + \epsilon e_i - \epsilon e_j \in S$ but violates $\alpha x + \gamma y \geq b$.

For the second part of the Lemma, assume $0 < x_i < u_i$ for $i = 1, \ldots, m$. If $\alpha_k \neq \alpha_l$ for any pair of indices $k, l \in \{1, \ldots, m\}$, then either $\alpha_k < \alpha_l$ or $\alpha_k > \alpha_l$, and the first part of the Lemma implies, respectively, that $x_k = u_k$ or $x_l = u_l$, a contradiction. ∎

Lemma 3.3. *Assume $\alpha > 0$ has all distinct coefficients. If $(x, y) \in F$ with $x \neq 0$, then $ex + cy = b$. Therefore F contains a point (x, y) with $x = 0$.*

Proof. Let $(x, y) \in F$ with $x \neq 0$. Suppose $ex + cy = b + \epsilon$ for some $\epsilon > 0$. By definition, $x_i > 0$ for some $i \in \{1, \ldots, m\}$; then $(x - \min\{x_i, \epsilon\}e_i, y) \in S$ but violates $\alpha x + \gamma y \geq \beta$ (as $\alpha > 0$), a contradiction to the fact that this inequality is valid for *conv*(S). If the second part of the Lemma is not true, then each point in F satisfies $ex + cy = b$ by the first part of the Lemma. As *conv*(S) is full-dimensional, this means that $\alpha x + \gamma y \geq \beta$ is a scalar multiple of $ex + cy \geq b$, which contradicts the nontriviality of $\alpha x + \gamma y \geq \beta$. ∎

Lemma 3.4. *Assume $\alpha > 0$ has all distinct coefficients. Let $(\hat{x}, \hat{y}), (\bar{x}, \bar{y}) \in F$. If $\alpha\hat{x} \leq \alpha\bar{x}$, then $\hat{x} \leq \bar{x}$. Therefore $\alpha\hat{x} = \alpha\bar{x}$ if and only if $\hat{x} = \bar{x}$.*

Proof. Let the conditions of the Lemma be true, but assume $\alpha\hat{x} \leq \alpha\bar{x}$ and $\hat{x}_j > \bar{x}_j$ for some index $j \in \{1, \ldots, m\}$. The fact that $\hat{x}_j > 0$ implies (by Lemma 3.2) that $\hat{x}_i = u_i \geq \bar{x}_i$ for all $i \neq j$ with $\alpha_i < \alpha_j$. The fact that $\bar{x}_j < u_j$ implies (by Lemma 3.2) that $\bar{x}_i = 0 \leq \hat{x}_i$ for all $i \neq j$ with $\alpha_i > \alpha_j$. Then $\hat{x}_i \geq \bar{x}_i$ for $i = 1, \ldots, m$ (as all coefficients of α are distinct), and $\hat{x}_j > \bar{x}_j$ which implies that $\alpha\hat{x} > \alpha\bar{x}$ (as $\alpha > 0$), a contradiction. The second part of the Lemma follows trivially from the first. ∎

Note that Lemma 3.4 implies Lemma 2.7.

3.2 Properties of S when $n = 2, m = 2$.

We now focus on the case $n = 2$ and $m = 2$ and assume that there exists a nontrivial facet-defining inequality $\alpha x + \gamma y \geq \beta$ with $\alpha_2 > \alpha_1 > 0$. By Theorem 2.1 we have $\gamma > 0$ and $\beta > 0$. We define F to be the set of integral points in S satisfying $\alpha x + \gamma y = \beta$. Let $L = \{(x^i, y^i) : i = 1, \ldots, 4\} \subset \mathbb{R}^2 \times \mathbb{R}^2$ be a set of affinely independent points in F which has the properties in Lemma 2.8. We refer to these points as $p^1, \ldots, p^4$. As $\beta > 0$, these points are also linearly independent. As before, let $L_y = \{y^1, \ldots, y^4\}$, and let $Q = conv(L_y)$.

Lemma 3.5. *Q is a parallelogram.*

Proof. Lemma 2.8 implies that $Q \subseteq \mathbb{R}^2$ is full-dimensional, has four vertices (namely $y^1, \ldots, y^4$), and contains no other integer points. In $\mathbb{R}^2$, such a set can only be a parallelogram. To see this, let the vertices of the quadrilateral Q be $y^1, \ldots, y^4$ in clockwise order, with the interior angles (in degrees) between the edges defining these vertices equal to $\theta_1, \ldots, \theta_4$. If Q is not a parallelogram, we can assume, without loss of generality, that $\theta_1 + \theta_2 > 180$. Furthermore, we can assume that either $\theta_4 + \theta_1 \geq 180$ or $\theta_3 + \theta_2 \geq 180$. In the first case, $y^4 + y^2 - y^1$ is contained in Q (and distinct from y^3), and in the second case $y^3 + y^1 - y^2$ is contained in Q (and distinct from y^4), a contradiction. Therefore Q is a parallelogram. ∎

We next assume that the points in L are sorted by nondecreasing values of γy^i, i.e., $\gamma y^1 \leq \gamma y^2 \leq \gamma y^3 \leq \gamma y^4$, and therefore by nonincreasing values of αx^i. Lemma 3.4 implies that $x^i \geq x^j$ for $j > i$. If $y^i = y^k$, then Lemma 2.7 implies that $x^i = x^k$ which contradicts the distinctness of p^i and p^k, so we can assume all y^is are distinct.

Lemma 3.6. *If $\gamma y^1 = \gamma y^2$ and $\gamma y^3 = \gamma y^4$, then the points $p^i (i = 1, \ldots, 4)$ are linearly dependent.*

Proof. If the conditions of the Lemma are satisfied, then Lemma 3.4 implies that $x^1 = x^2$ and $x^3 = x^4$. Next observe that the y^is are all distinct. As $0 \neq \gamma \in \mathbb{R}^2$, and $0 \neq y^2 - y^1$ and $0 \neq y^4 - y^3$ are orthogonal to γ, we infer that $y^2 - y^1$ is a scalar multiple of $y^4 - y^3$ (from the fact that these vectors lie in $\mathbb{R}^2$), and therefore $y^2 - y^1 - \mu(y^4 - y^3) = 0$ for some nonzero scalar μ. As $x^2 - x^1 = 0$ and $x^4 - x^3 = 0$, it follows that $p^2 - p^1 - \mu(p^4 - p^3) = 0$. ∎

As the points in L are linearly independent, we conclude that the conditions of Lemma 3.6 cannot hold. Using the fact that γy^i is non-decreasing and either $\gamma y^1 < \gamma y^2$ or $\gamma y^3 < \gamma y^4$, leads to the following observation.

Corollary 3.7. *For the points in L we either have $\gamma y^1 < \gamma y^2$ or $\gamma y^3 < \gamma y^4$ and therefore $\gamma y^1 < \gamma y^4$.*

Lemma 3.8. *The points y^1 and y^4 form opposite corners of the parallelogram Q.*

Proof. By Corollary 3.7 we have $\gamma y^1 < \gamma y^4$. Assume the result is not true and that y^1 and y^4 define adjacent corners of Q. Then the other adjacent corner of Q to y^1 is defined by either y^3 (in which case $y^4 - y^1 = y^2 - y^3$) or by y^2 (in which case $y^4 - y^1 = y^3 - y^2$). In the first case we have $0 < \gamma(y^4 - y^1) = \gamma(y^2 - y^3)$ which contradicts the fact that $\gamma y^2 \leq \gamma y^3$. Therefore $y^4 - y^1 = y^3 - y^2$ and $0 < \gamma(y^4 - y^1) = \gamma(y^3 - y^2)$. This combined with $\gamma y^1 \leq \gamma y^2 \leq \gamma y^3 \leq \gamma y^4$ implies that $\gamma y^1 = \gamma y^2$ and $\gamma y^3 = \gamma y^4$. Lemma 3.6 then implies that $p^1, \ldots, p^4$ are linearly dependent, a contradiction. ■

Lemma 3.9. *The point $x^1 > 0$ with $x^1_1 = u_1$ and $x^4 = 0$. Further, one can assume that $cy^4 > b$.*

Proof. For $i = 1, 2$, there is a point $(\bar{x}, \bar{y}) \in L$ with $\bar{x}_i < u_i$ and a point (x', y') with $x'_i > 0$. This follows directly from the fact that *conv*(S) is full-dimensional and $\alpha x + \gamma y \geq \beta$ is not equal to $x_i \leq u_i$ or $x_i \geq 0$ for $i = 1, 2$. As $x^1 \geq \cdots \geq x^4$ by Lemma 3.4, we have $x^1_2 > 0$, and from Lemma 3.2, the first part of the result follows. If $x^4 \neq 0$, then Lemma 3.4 implies that $x^1, \ldots, x^4 \neq 0$ which contradicts Lemma 3.3. Therefore x^4 must be 0.

If $ex^4 + cy^4 = cy^4 = b$, then $ex^k + cy^k > b$ for some $k < 4$ (again, because we are dealing with a nontrivial facet-defining inequality). But Lemma 3.3 implies that $x^k = 0 = x^4 \Rightarrow \gamma y^k = \gamma y^4 = \beta$. Therefore, if we switch the points (x^k, y^k) and (x^4, y^4), L is still sorted by nondecreasing values of γy^i and $cy^4 > b$ and $x^4 = 0$. ■

Combining these observations, we next show that a nontrivial facet-defining inequality $\alpha x + \gamma y \geq \beta$ with $\alpha_2 > \alpha_1 > 0$ cannot exist.

Theorem 3.10. *Let $m = 2, n = 2$ and consider a nontrivial facet-defining inequality $\alpha x + \gamma y \geq \beta$. If $\alpha > 0$, then $\alpha_1 = \alpha_2$.*

Proof. Suppose the coefficients of α are distinct and assume that $0 < \alpha_1 < \alpha_2$. We know that y^1 and y^4 form nonadjacent corners of Q, and y^2 and y^3 form the remaining corners of Q. Therefore

$$y^4 - y^3 = y^2 - y^1. \tag{3}$$

From the equations $\alpha x^i + \gamma y^i = \beta$ for $i = 1, \ldots, 4$, we conclude that

$$\alpha(x^3 - x^4) + \gamma(y^3 - y^4) = 0 = \alpha(x^1 - x^2) + \gamma(y^1 - y^2).$$

Using equation (3) we can conclude that

$$\alpha(x^3 - x^4) = \alpha(x^1 - x^2). \tag{4}$$

Furthermore, given that $0 < \gamma(y^2 - y^1) = \gamma(y^4 - y^3)$, from Corollary 3.7, we have $\alpha x^3 > \alpha x^4$ and $\alpha x^1 > \alpha x^2$. Therefore $x^1, x^2, x^3 \neq 0$ as $x^4 = 0$ by Lemma 3.9. Using Lemma 3.3 we have

$$ex^i + cy^i = b \text{ for } i = 1, \ldots, 3,$$

and consequently, (x_4, y_4) cannot lie on the capacity constraint, implying

$$cy^4 = ex^4 + cy^4 > b.$$

From this we can conclude that

$$e(x^3 - x^4) + c(y^3 - y^4) < 0 = e(x^1 - x^2) + c(y^1 - y^2)$$

Again using equation (3) we infer that

$$e(x^3 - x^4) < e(x^1 - x^2). \tag{5}$$

We now have two cases.

Case 1: $x_1^2 = u_1$. Scale α such that $\alpha_1 < 1$ and $\alpha_2 = 1$. Recall that $x_1^1 = u_1$ and $x_2^1 > 0$ (by Lemma 3.9). As $x^1 - x^2 \neq 0$, we conclude that $x^1 - x^2$ is nonzero only in the second coordinate. As $x^3 \neq 0 = x^4$, $x_1^3 > 0$ by Lemma 3.2, and therefore $x^3 - x^4$ is definitely nonzero in the first coordinate. Therefore $\alpha(x^1 - x^2) = e(x^1 - x^2)$ and $\alpha(x^3 - x^4) < e(x^3 - x^4)$. But this combined with (5) and (4) leads to a contradiction.

Case 2: $x_1^2 < u_1$. Scale α such that $\alpha_1 = 1$ and $\alpha_2 > 1$. Lemma 3.2 implies that $x_2^2 = 0$. As $x_2^1 > 0$, $\alpha(x^1 - x^2) > e(x^1 - x^2)$. Furthermore $x^3 \leq x^2$ which implies that $x_1^3 < u_1$ and therefore $x_2^3 = 0$ by Lemma 3.2. Therefore $\alpha(x^3 - x^4) = e(x^3 - x^4)$. Combining this fact and $\alpha(x^1 - x^2) > e(x^1 - x^2)$ with (5) and (4) leads to a contradiction.

Therefore, we have shown that α_1 and α_2 must be the same. ∎

Let $M = \{1, \ldots, m\}$, and for any $I \subseteq M$, let $u(I) = \sum_{i \in I} u_i$. Combining Corollary 2.4 with Theorems 2.6 and 3.10 leads to the following result:

Theorem 3.11. *When $n = 2$, all non-trivial facet-defining inequalities for conv(S) are either of the form*

$$\gamma_1 y_1 + \gamma_2 y_2 \geq \beta$$

where $\gamma_1 y_1 + \gamma_2 y_2 \geq \beta$ is facet-defining for the set

$$CG^* = conv\Big\{y \in \mathbb{Z}^2 \ : \ c_1 y_1 + c_2 y_2 \geq b - u(M), y \geq 0\Big\},$$

or of the form

$$\sum_{i \in I} x_i + \gamma_1 y_1 + \gamma_2 y_2 \geq \beta$$

where $\emptyset \neq I \subseteq M$ and $w + \gamma_1 y_j + \gamma_2 y_2 \geq \beta$ is facet-defining for the set

$$Q(b - u(M \setminus I), u(I)) = conv\Big\{(w, y) \in \mathbb{R} \times \mathbb{Z}^2 : w + c_1 y_1 + c_2 y_2 \geq b - u(M \setminus I), u(I) \ \geq w \geq 0, \ y \geq 0\Big\}.$$

We note that, for any $\emptyset \neq I \subseteq M$, replacing w by $\sum_{i \in I} x_i$ in a valid inequality for $Q(b-u(M \setminus I), u(I))$ yields a valid inequality for S. In addition, CG^* is essentially the face of $Q(b-u(M \setminus I), u(I))$ defined by setting w to $u(I)$. Consequently, inequalities defining CG^* can be obtained from facet-defining inequalities of $Q(b-u(M \setminus I), u(I))$. Therefore, when $n = 2$, all nontrivial facet-defining inequalities of *conv*(S) are obtainable from facets of 3-variable mixed-integer sets of the form $Q(b', u')$, and we next study sets of this form.

4 The structure of $Q(b, u)$

Given rational numbers $c_1, c_2 > 0$, recall that the set

$$Q(b,u) = conv(\{(w,y) \in \mathbb{R}^1 \times \mathbb{Z}^2 : w + c_1 y_1 + c_2 y_2 \geq b, u \geq w \geq 0, y \geq 0\})$$

where b and u are positive rational numbers (u can also be infinity). The main result we will prove in this section is that

$$Q(b,u) = Q(b,\infty) \cap \{(w,y) \in \mathbb{R}^3 : w \leq u\} \cap (\mathbb{R} \times P_{\geq}(b-u))$$

where

$$P_{\geq}(b-u) = conv(\{y \in \mathbb{Z}^2 : c_1 y_1 + c_2 y_2 \geq b-u,\ y \geq 0\}).$$

In other words, we will show that every nontrivial facet-defining inequality for $Q(b,u)$ either defines a facet of $Q(b,\infty)$ (i.e., u plays no role) or the coefficient of the x variable is zero and the inequality (when treated as an inequality on the variables y_1 and y_2) defines a facet of $P_{\geq}(b-u)$. The latter set corresponds to the convex hull of integer points in $Q(b,u)$ with $w = u$. We will also show how to obtain vertices of $Q(b,u)$ from vertices of some simpler sets in Theorem 4.5.

Agra and Constantino [2] gave a polynomial-time algorithm to enumerate the facets of $P_{\geq}(b)$ for any b (and therefore for $P_{\geq}(b-u)$), and then extended their algorithm to obtain all facets of $Q(b,\infty)$. Our result implies that one can thus use their algorithm to get all nontrivial facets of $Q(b,u)$.

To analyze the facets of $Q(b,u)$, we study lower-dimensional sets, defined in the space of the integer variables only. Accordingly, in addition to $P_{\geq}(b-u)$, we also define

$$P_{\leq}(b) = conv(\{y \in \mathbb{Z}^2 : c_1 y_1 + c_2 y_2 \leq b,\ y \geq 0\}),$$

$$P(b-u,b) = conv(\{y \in \mathbb{Z}^2 : b-u \leq c_1 y_1 + c_2 y_2 \leq b,\ y \geq 0\}).$$

We next study the integral points of $Q(b,u)$ that lie on the capacity constraint. The convex hull of the projection of these points on the y-coordinates gives the set $P(b-u,b)$, the convex hull of all integer points between two parallel hyperplanes.

4.1 Characterization of $P(b-u,b)$

Let $c \in \mathbb{R}^2$ with $c > 0$, and consider real numbers $b, u > 0$ with $b - u \geq 0$.

Theorem 4.1. $P(b-u,b) = P_{\leq}(b) \cap P_{\geq}(b-u)$.

Proof. For ease of notation, we let $Q_0 = P(b-u,b), Q_1 = P_{\leq}(b)$ and $Q_2 = P_{\geq}(b-u)$. Furthermore, let

$$\begin{aligned} P_0 &= \{y \in \mathbb{R}^2 : b-u \leq cy \leq b,\ y \geq 0\}, \\ P_1 &= \{y \in \mathbb{R}^2 : cy \leq b,\ y \geq 0\}, \\ P_2 &= \{y \in \mathbb{R}^2 : cy \geq b-u,\ y \geq 0\}. \end{aligned}$$

Therefore $Q_i = conv(P_i \cap \mathbb{Z}^2)$ for $i = 0, 1, 2$. Note that $P_0 = P_1 \cap P_2$ and therefore $Q_0 \subseteq Q_1 \cap Q_2$. Q_2 is a full-dimensional anti-blocking polyhedron with facets defined by inequalities of the form $y_1 \geq 0$ or $y_2 \geq 0$ or $\hat{c}y \geq \hat{\gamma}$ for $\hat{c} > 0$ and $\hat{\gamma} > 0$. Similarly Q_1 is a blocking polyhedron; if full-dimensional, its facets are defined by inequalities of the form $y_1 \geq 0$ or $y_2 \geq 0$ or $\hat{c}y \leq \hat{\gamma}$ for $\hat{c} \geq 0$ and $\hat{\gamma} > 0$.

Assume that $Q_1 \cap Q_2 \not\subseteq Q_0$. Then $Q_1 \cap Q_2$ must have an extreme point that is neither an extreme point of Q_1 nor an extreme point of Q_2; otherwise $Q_1 \cap Q_2$ would be integral and thus would be contained in Q_0. We can assume Q_1 is full-dimensional, for if it were not, then $Q_1 \subseteq \{y \in \mathbb{R}^2 : y_1 = 0\} \cup \{y \in \mathbb{R}^2 : y_2 = 0\}$, and in that case, $Q_1 \cap Q_2$ is easily seen to be an integral polyhedron.

Let v be a vertex of $Q_1 \cap Q_2$ that is not an extreme point of Q_1 or Q_2. As $Q_1, Q_2 \subset \mathbb{R}^2$, we can assume $v = f_1 \cap f_2$, where (i) $f_1 = conv(p_1, p_2)$ is a facet of Q_1 and p_1 and p_2 are extreme points of Q_1, and (ii) $f_2 = conv(q_1, q_2)$ is a facet of Q_2 and q_1 and q_2 are extreme points of Q_2. Let $c^1 y \leq \gamma^1$ be the inequality defining f_1, and let $c^2 y \geq \gamma^2$ be the inequality defining f_2; these inequalities are unique (up to multiplication by a scalar) as Q_1, Q_2 are full-dimensional.

As v is not equal to any of the endpoints of f_1 or f_2, it must lie in the relative interior of each facet. Furthermore, one of the end points of f_1, say p_2, must strictly satisfy $c^2 y \geq \gamma^2$. Then p_1 strictly violates $c^2 y \geq \gamma^2$. This implies that $p_1 \in \mathbb{R}^2_+ \setminus P_2$ as $c^2 y \geq \gamma^2$ is valid for all integral points in P_2. As f_2 is entirely contained in P_2 and intersects $f_1 = conv(p_1, p_2)$, this means that p_2 must be contained in P_2; thus $p_2 \in P_1 \cap P_2$. Similarly, we can assume q_1 strictly violates $c^1 y \leq \gamma^1$ and therefore $q_1 \in \mathbb{R}^2_+ \setminus P_1$. Furthermore, q_2 strictly satisfies $c^1 y \leq \gamma^1$ and also belongs to P_1, otherwise f_2 would not intersect f_1 which is contained in P_1. Therefore, $q_2 \in P_1 \cap P_2$. In other words, we have

$$c^1 q_2 < \gamma^1, \qquad c^1 q_1 > \gamma^1, \tag{6}$$

$$c^2 p_2 > \gamma^2, \qquad c^2 p_1 < \gamma^2, \tag{7}$$

$$cp_1 < b-u, \qquad b-u \leq cp_2 \leq b, \tag{8}$$

$$cq_1 > b, \qquad b-u \leq cq_2 \leq b. \tag{9}$$

It is clear that the lines $c^1 y = \gamma^1$ and $c^2 y = \gamma^2$ have different slopes as the points p_1, p_2, q_1, q_2 and v are not collinear. Then $c^1_2/c^1_1 \neq c^2_2/c^2_1$ (here c^1_1 stands for the first component of c^1, c^1_2 for the second component,

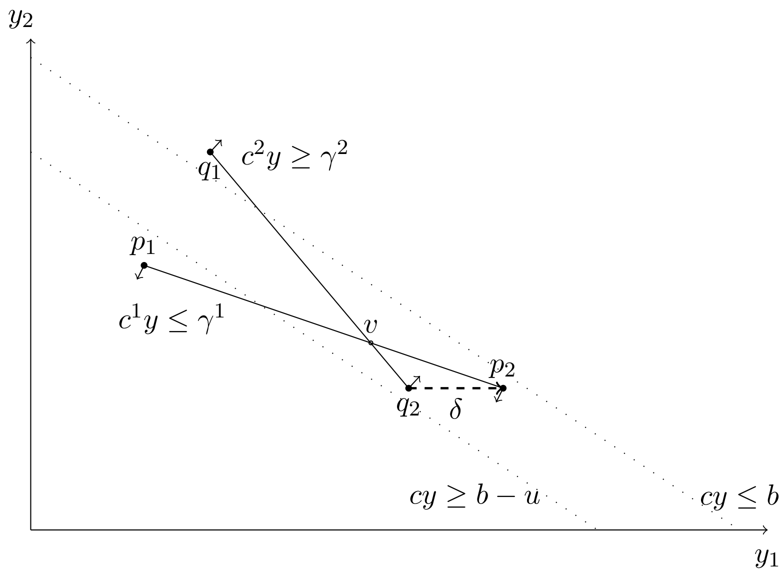

Figure 1: Facets f_1 and f_2 and their intersection v

etc.). Note that $c^2 > 0$ and so c_2^2/c_1^2 is a positive number. As for c^1, c_1^1 may equal zero, in which case we will think of c_2^1/c_1^1 as the 'number' ∞ and greater than any positive number. We can assume, without loss of generality, that $c_2^1/c_1^1 > c_2^2/c_1^2$; if $c_2^1/c_1^1 < c_2^2/c_1^2$, we can switch the coefficients of c and construct an instance with the desired relationship of slopes of the lines $c^1y = \gamma^1$ and $c^2y = \gamma^2$. See Figure 1 for a depiction of p_1, p_2, q_1, q_2 and v.

Let $\delta = p_2 - q_2 \in \mathbb{Z}^2$. As $c^1q_2 < \gamma^1$ and $c^1p_2 = \gamma^1$, we have $c^1\delta = c^1(p_2 - q_2) > 0$. Similarly, as $c^2p_2 > \gamma^2$ and $c^2q_2 = \gamma^2$, we have $c^2\delta > 0$. Note that as $c^1 \geq 0$, $c^1\delta > 0$ implies that at least one component of δ is positive. Also, as $cp_2 \leq b$ and $cq_2 \geq b - u$, we have

$$c\delta \leq u. \tag{10}$$

Case 1: $\delta_1 \geq 0$. Clearly $p_1 + \delta \in \mathbb{Z}^2$. Then $c^1(p_1 + \delta) > \gamma^1$ as $c^1\delta > 0$. Further, $cp_1 < b - u$ and (10) together imply that $c(p_1 + \delta) < b$. Finally, as the second component of p_1 is greater than the second component of q_2 (because of the relationship between the slopes of the lines $c^1y = \gamma^1$ and $c^2y = \gamma^2$) and $q_2 + \delta = p^2 \in \mathbb{R}_+^2$, we have $p_1 + \delta \in \mathbb{R}_+^2$. In other words, $p_1 + \delta$ is an integral point in P_1 which violates $c^1y \leq \gamma^1$ a contradiction to the fact that this inequality is valid for all integral points in P_1.

Case 2: $\delta_1 < 0, \delta_2 > 0$. First $q_1 - \delta \in \mathbb{Z}^2$. Next, $c^2(q_1 - \delta) < \gamma^2$ as $c^2\delta > 0$. Further, $cq_1 > b$ and (10) imply that $c(q_1 - \delta) > b - u$. Finally, as the second component of q_1 is greater than the second component of p_2 and $p_2 - \delta = q_2 \in \mathbb{R}_+^2$, we have $q_1 - \delta \in \mathbb{R}_+^2$. In other words, $q_1 - \delta$ is an integral point in P_2 which violates $c^2y \geq \gamma^2$. This contradicts the fact that $c^2y \geq \gamma^2$ is valid for all integral points in P_2. ■

We note that a closely related result appears in an unpublished manuscript of Basu, Bonami, Conforti and Cornuéjols where the authors study integer programs with two variables where one of the variables has both an upper and a lower bound. Also note that in the proof of Theorem 4.1 we also showed the following fact.

Corollary 4.2. *A point y^* is an extreme point of $P(b-u,b)$ if and only if either (i) y^* is an extreme point of $P_{\leq}(b)$ and $cy^* \geq b-u$, or, (ii) y^* is an extreme point of $P_{\geq}(b-u)$ and $cy^* \leq b$.*

4.2 The facets of $Q(b,u)$

We need the following easy lemma before we prove the main result of this section in Theorem 4.4.

Lemma 4.3. *Let $y^1, y^2, y^3 \in \mathbb{Z}^2$ be three distinct points such that conv(y^1, y^2, y^3) contains no other integer point and let y^2, y^3 lie on the line $\gamma y = \beta$ where the coefficients of $\gamma \in \mathbb{Z}^2$ are coprime integers. Then $\beta - 1 \leq \gamma y^1 \leq \beta + 1$.*

Proof. As γ, y^1, y^2 are integral, so is β. We can also assume, without loss of generality, that $\gamma y^1 \leq \beta$ (by multiplying γ, β by -1 if necessary). Also, there is nothing to prove if $\gamma y^1 = \beta$, so we assume γy^1 is an integer less than β. As γ has coprime coefficients, there is a 2×2 unimodular matrix U such that $\bar{\gamma} = \gamma U = (1,0)$. Consider the points $\bar{y}^i = U^{-1}y^i - U^{-1}y^3$ for $i = 1,\ldots,3$. The mapping $U^{-1}y - U^{-1}y^3$ is a linear, one-to-one, invertible mapping which maps $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\mathbb{Z}^2$ to $\mathbb{Z}^2$. Therefore *conv*$(\bar{y}^1, \bar{y}^2, \bar{y}^3)$ contains no integer points besides $\bar{y}^1, \bar{y}^2, \bar{y}^3$. Furthermore,

$$\gamma U U^{-1} y^j = \gamma y^j \Rightarrow \bar{\gamma}\bar{y}^j = \gamma y^j - \gamma y^3 = 0 \text{ for } j = 2,3 \text{ and } \bar{\gamma}\bar{y}^1 < 0.$$

Therefore $\bar{y}^3 = (0,0)$, the first component of $\bar{y}^2$ is zero, i.e., $\bar{y}^2_1 = 0$ and $\bar{y}^1_1 < 0$. As *conv*$(\bar{y}^2, \bar{y}^3)$ contains no integer points other than $\bar{y}^2, \bar{y}^3$, it follows that $\bar{y}^2$ is either $(0,1)$ or $(0,-1)$. In either case, if $\bar{y}^1_1 \leq -2$ then *conv*$(\bar{y}^1, \bar{y}^2, \bar{y}^3)$ has an integer point besides $\bar{y}^1, \bar{y}^2, \bar{y}^3$. Therefore $\bar{y}^1_1 = \bar{\gamma}\bar{y}^1 = -1$ and $\gamma y^1 = \beta - 1$. ∎

Theorem 4.4.

$$Q(b,u) = Q(b,\infty) \cap \{(w,y) \in \mathbb{R}\times\mathbb{R}^2 : w \leq u\} \cap (\mathbb{R} \times P_{\geq}(b-u)). \tag{11}$$

Proof. $Q(b,u)$ is a subset of each of the three sets on the right-hand side of (11), and therefore $Q(b,u)$ is a subset of their intersection. To prove the reverse inclusion, we next show that each facet-defining inequality of $Q(b,u)$ is a valid inequality for one of the three right-hand-side sets.

Any trivial facet-defining inequality for $Q(b,u)$ is either valid for $Q(b,\infty)$ or for $\{(w,y) \in \mathbb{R}\times\mathbb{R}^2 : w \leq u\}$. Therefore let $\alpha w + \gamma y \geq \beta$ define a nontrivial facet F of $Q(b,u)$. Scale the inequality so that γ is integral and the components of γ are coprime. As $Q(b,u)$ is a special case of the set *conv*(S) studied in Lemma 2.1, we can conclude that $Q(b,u)$ is full-dimensional and $\alpha \geq 0, \gamma > 0$ and $\beta > 0$. Let $\alpha = 0$. Then Lemma 2.3 implies that $\gamma y \geq \beta$ is facet-defining for $P_{\geq}(b-u)$ (this is exactly the set S' in Lemma 2.3). Therefore $0w + \gamma y \geq \beta$ is facet-defining for $\mathbb{R}\times P_{\geq}(b-u)$. We henceforth assume that $\alpha > 0$. We will show that under this condition $\alpha w + \gamma y \geq \beta$ defines a facet of $Q(b,\infty)$.

Let $L = \{(w^1,y^1),(w^2,y^2),(w^3,y^3)\}$ be a subset of three affinely independent integral points in F such that *conv*(y^1,y^2,y^3) is full-dimensional and contains no other integer points. These points exist by

Lemma 2.8. Without loss of generality, assume that $w^1 \geq w^2 \geq w^3$. If $w^i > 0$ for $i = 1, 2, 3$ then (w^i, y^i) lies on the capacity inequality, contradicting the nontriviality of F, therefore $w^3 = 0$. If all three points in L satisfy $w^i = 0$, then F is defined by $w \geq 0$ contradicting the nontriviality of F, therefore $w^1 > 0$. We next consider two cases.

Case 1: Let $w^2 = 0$. As $w^3 = 0$ and $u \geq w^1 > 0$, we have $\gamma y^1 < \gamma y^2 = \gamma y^3 = \beta$. As γ is integral (by scaling) and so is y^2, β is an integer. By Lemma 2.8, *conv*(y^1, y^2, y^3) is a full-dimensional polyhedron in $\mathbb{R}^2$ containing no integer points other than y^1, y^2, y^3. Lemma 4.3 implies that $\gamma y^1 = \beta - 1$.

We will now show that $\alpha w + \gamma y \geq \beta$ is valid and facet-defining for $Q(b, \infty)$. Clearly, if this inequality is valid, then it is facet-defining as the inequality is tight for the points (w^i, y^i) for $i = 1, 2, 3$ which are contained in $Q(b, \infty)$. Suppose $\alpha w + \gamma y \geq \beta$ is not valid for $Q(b, \infty)$. Then there is a point in $Q(b, \infty)$ that violates this inequality and its w-coordinate is strictly larger than u. Therefore, for some $\hat{\alpha}$ satisfying $(\beta/u) \geq \hat{\alpha} > \alpha$, $\hat{\alpha} w + \gamma y \geq \beta$ is facet-defining for $Q(b, \infty)$. To see this first observe that setting $\hat{\alpha}$ to β/u, the resulting inequality is clearly satisfied by any point in $Q(b, \infty)$ with the w-coordinate larger than u, and is thus valid for $Q(b, \infty)$. By considering the finitely many vertices of $Q(b, \infty)$ with w-coordinate strictly greater than u, we can choose $\hat{\alpha}$ so that $\hat{\alpha} w + \gamma y \geq \beta$ is tight for at least one such vertex while remaining valid for $Q(b, \infty)$. Then, such a vertex along with the points (w^i, y^i) for $i = 2, 3$ shows that $\hat{\alpha} w + \gamma y \geq \beta$ defines a facet of $Q(b, \infty)$.

This inequality cannot be facet-defining for $Q(b, u)$ (as it would be implied by the valid inequalities $\alpha w + \gamma y \geq \beta$ and $w \geq 0$ for $Q(b, u)$). Among the integral points $(\hat{w}, \hat{y})$ with $\hat{w} > u$ in $Q(b, \infty)$ satisfying $\hat{\alpha}\hat{w} + \gamma\hat{y} = \beta$, let $(\hat{w}, \hat{y})$ be chosen so that $\hat{w}$ is as small as possible. To see that such a point $(\hat{w}, \hat{y})$ exists, note that there are only finitely many choices of $\hat{y}$ as $0 \leq c\hat{y} < b$ and $c > 0$ and $\hat{y} \geq 0$, and therefore finitely many choices of $\hat{w}$ as $\hat{w} + c\hat{y} = b$.

Let the convex hull of $\{\hat{y}, y^2, y^3\}$ be the triangle H, and let $\hat{F}$ stand for the set of integral points on the facet of $Q(b, \infty)$ defined by $\hat{\alpha} w + \gamma y \geq \beta$. Suppose H contains some integral point y' different from the three vertices of H. Then it equals $\lambda_1 \hat{y} + \lambda_2 y^2 + \lambda_3 y^3$ where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $0 \leq \lambda_1, \lambda_2, \lambda_3 < 1$. If $\lambda_1 = 0$, then y' is contained in the convex hull of y^2, y^3 which contradicts the assumption that this convex hull contains no other integer points besides y^2, y^3. Therefore $0 < \lambda_1 < 1$. Let $w' = (\beta - \gamma y')/\hat{\alpha}$. Then (w', y') satisfies $\hat{\alpha} w + \gamma y = \beta$ and $(w', y') = \lambda_1(\hat{w}, \hat{y}) + \lambda_2(w^2, y^2) + \lambda_3(w^3, y^3)$. In other words, (w', y') is a point in $\hat{F}$ with $u < w' < \hat{w}$ which contradicts our assumption on the minimality of $\hat{w}$. Therefore *conv*$(\hat{y}, y^2, y^3)$ contains no other integer points besides $\hat{y}, y^2, y^3$, and Lemma 4.3 implies that $\gamma\hat{y} = \beta - 1$ and $\hat{\alpha}\hat{w} = 1$. But then $1 = \hat{\alpha}\hat{w} > \alpha w^1 = 1$ as $w^1 \leq u < \hat{w}$ and $\alpha < \hat{\alpha}$. Thus we obtain a contradiction.

Case 2: Let $w^2 > 0$. Recall that $w^1 > 0$ and $w^3 = 0$. Let F_c denote the face of $Q(b, u)$ defined by the capacity inequality. As before, we can argue that (w^2, y^2) and (w^3, y^3) lie on F_c. Now consider the set of points in F_c satisfying $\alpha w + \gamma y \geq \beta$. As points on F_c satisfy $w = b - cy$, substituting for w in $\alpha w + \gamma y \geq \beta$, we get

$$\alpha(b - cy) + \gamma y \geq \beta$$

and therefore

$$(\gamma - \alpha c)y \geq \beta - \alpha b$$

as a valid inequality for $\{(w, y) \in F_c : \alpha w + \gamma y \geq \beta\}$. Let $\gamma' = \gamma - \alpha c$ and $\beta' = \beta - \alpha b$. Then

$$\alpha(w + cy \geq b) + (0w + \gamma' y \geq \beta') = (\alpha w + \gamma y \geq \beta), \tag{12}$$

where multiplying an inequality by a nonnegative number and adding two inequalities has the usual meaning.

As $\alpha w + \gamma y \geq \beta$ is valid for all points in $Q(b, u)$, all integral points $(w, y) \in F_c$ satisfy $\gamma' y \geq \beta'$; furthermore for such points we have $w + cy = b$ with $0 \leq w \leq u \Rightarrow b - u \leq cy \leq b$. Also, as the inequality $\alpha w + \gamma y \geq \beta$ is tight for the points $(w^1, y^1), (w^2, y^2)$ which also lie in F_c, we have $\gamma' y^1 = \gamma' y^2 = \beta'$. In other words, $\gamma' y \geq \beta'$ is both valid for $P(b-u, b)$ and facet-defining. By our previous results, $P(b-u, b) = P_{\geq}(b - u) \cap P_{\leq}(b)$. Therefore $\gamma' y \geq \beta'$ defines a facet of either $P_{\geq}(b - u)$ or of $P_{\leq}(b)$.

Case 2a: Let $\gamma' y \geq \beta'$ define a facet of $P_{\geq}(b - u)$. In other words, $\gamma' y \geq \beta'$ is valid for all integral $y \geq 0$ with $cy \geq b - u$. But any integral $(w, y) \in Q(b, u)$ satisfies $y \geq 0$ and $cy \geq b - u$. Therefore $0w + \gamma' y \geq \beta'$ is a valid inequality for $Q(b, u)$. But then by (12), $\alpha w + \gamma y \geq \beta$ is the sum of two distinct valid inequalities for $Q(b, u)$ and cannot define a facet of $Q(b, u)$, a contradiction.

Case 2b: Let $\gamma' y \geq \beta'$ define a facet of $P_{\leq}(b)$. If $\alpha w + \gamma y \geq \beta$ does not define a facet of $Q(b, \infty)$ then there exists an integral point $(\hat{w}, \hat{y}) \in Q(b, \infty) \setminus Q(b, u)$ with $\alpha \hat{w} + \gamma \hat{y} < \beta$ and $\hat{w} > u$ and $\hat{w} + c\hat{y} \geq b$. Therefore $\hat{w} \geq \max\{u, b - c\hat{y}\}$. If $u \geq b - c\hat{y}$, then setting $\hat{w}$ to u, we get a point which violates $\alpha w + \gamma y \geq \beta$ but belongs to $Q(b, u)$, a contradiction. So we can assume that setting $\hat{w}$ to $b - c\hat{y}$, we get an integral point $(\hat{w}, \hat{y}) \in Q(b, \infty) \setminus Q(b, u)$ which violates $\alpha w + \gamma y \geq \beta$ and lies on the capacity constraint. But then $\hat{y} \in P_{\leq}(b)$ and therefore $\gamma' \hat{y} \geq \beta'$. This, combined with $\hat{w} + c\hat{y} = b$ and (12) implies that $\alpha \hat{w} + \gamma \hat{y} \geq \beta$, a contradiction. ∎

4.3 The vertices of $Q(b, u)$

Agra and Constantino [2] show how to obtain the extreme point/extreme ray descriptions of $P_{\geq}(b), P_{\leq}(b)$, and $Q(b, \infty)$ of size polynomial in b (in addition to the facet descriptions of these sets, as described earlier).

Lemma 2.1 implies that the three unit vectors are the only extreme rays of $Q(b, u)$. We next describe how to obtain an extreme point characterization of $Q(b, u)$ in terms of the extreme points of $P_{\geq}(b), P_{\leq}(b)$, and $P_{\geq}(b - u)$. Let $EP(P)$ stand for the set of extreme points of the polytope P.

It is clear that any face of $Q(b, u)$ equals the convex hull of integral points on that face. Therefore, as $w \geq 0$ defines a face of $Q(b, u)$, we have

$$\{(w, y) \in Q(b, u) : w = 0\} = \textit{conv}(\{(0, y) \in \mathbb{R} \times \mathbb{Z}^2 : cy \geq b, y \geq 0\}) = \{0\} \times P_{\geq}(b).$$

Similarly, as $w \leq u$ defines a face of $Q(b, u)$, we have

$$\{(w, y) \in Q(b, u) : w = u\} = \textit{conv}(\{(u, y) \in \mathbb{R} \times \mathbb{Z}^2 : cy \geq b - u, y \geq 0\}) = \{u\} \times P_{\geq}(b - u).$$

Also $w + cy \geq b$ defines a face F_c of $Q(b, u)$, where

$$F_c = \textit{conv}(\{(w, y) \in \mathbb{R} \times \mathbb{Z}^2 : w + cy = b, u \geq w \geq 0, y \geq 0\}).$$

Therefore $proj_y(F_c) = conv(\{y \in \mathbb{Z}^2 : b - u \leq cy \leq b, y \geq 0\}) = P(b-u, b)$. In addition, for any $y^* \in P(b-u,b)$, setting $w^* = cy^* - b$, we get the unique point in F_c such its projection on the y-space is y^*. In other words, there is a one-to-one correspondence between the points of $P(b-u,b)$ and F_c, in particular, there is a one-to-one correspondence between the extreme points of these sets.

Lemma 2.2 implies that for every extreme point $(w^*, y^*) \in Q(b,u)$, $w^* \in \{0, b - cy^*, u\}$. In other words, every extreme point lies on the face of $Q(b,u)$ defined by $w \geq 0$ or $w \geq u$ or $w + cy \geq b$. This result and the preceding analysis yield the following result.

Theorem 4.5. *If* $(w^*, y^*) \in EP(Q(b,u))$ *then either*

(i) $w^* = 0$ *and* $y^* \in EP(P_{\geq}(b))$*, or*
(ii) $u > w^* > 0$ *and* $y^* \in EP(P(b-u,b))$*, or*
(iii) $w^* = u$ *and* $y^* \in EP(P_{\geq}(b-u))$.

Conversely,

(iv) if $y^* \in EP(P_{\geq}(b))$*, then* $(0, y^*) \in EP(Q(b,u))$,
(v) if $y^* \in EP(P(b-u,b))$*, then* $(b - cy^*, y^*) \in EP(Q(b,u))$,
(vi) if $y^* \in EP(P_{\geq}(b-u))$*, then* $(u, y^*) \in EP(Q(b,u))$.

By Corollary 4.2, we know that y^* is an extreme point of $P(b-u,b)$ if and only if either (a) y^* is an extreme point of $P_{\leq}(b)$ and $cy^* \geq b-u$, or, (b) y^* is an extreme point of $P_{\geq}(b-u)$ and $cy^* \leq b$. Agra and Constantino [2] present an algorithm which can enumerate all extreme points of $P_{\geq}(b)$ and $P_{\geq}(b-u)$. Their algorithm runs in time bounded by a polynomial function of the encoding size of the data defining these sets. One can take these extreme points and check the conditions (a) and (b) above to get all extreme points of $P(b-u,b)$ in polynomial time. These facts, along with Theorem 4.5, yield the following result.

Corollary 4.6. *All extreme points of* $Q(b,u)$ *can be obtained in time bounded by a polynomial function of the input size.*

Finally, we note that an extreme point y^* of $P(b-u,b)$ can yield an extreme point of $Q(b,u)$ of the form (u, y^*) if $b - cy^* = u$. In this case, $y^* \in P_{\geq}(b-u)$ and therefore the point (u, y^*) can also be obtained from the extreme points of $P_{\geq}(b-u)$. Similarly, an extreme point y^* of $P(b-u,b)$ would yield the extreme point $(0, y^*)$ of $Q(b,u)$ if $b = cy^*$. In this case, $y^* \in P_{\geq}(b)$ and (u, y^*) can be obtained from the extreme points of $P_{\geq}(b)$.

5 Concluding Remarks

Using earlier results, we now give a complete characterization of the convex hull of S when $n = 2$ in terms of facets of three-variable sets. Let $M = \{1, \ldots, m\}$, and for any $I \subseteq M$, let $u(I) = \sum_{i \in I} u_i$.

Theorem 5.1. *When $n = 2$,*

$$\textit{conv}(S) = U \cap CG \cap \Big(\bigcap_{\emptyset \neq I \subseteq M} \{(x,y) \in \mathbb{R}^m \times \mathbb{Z}^2 \;:\; (\sum_{i \in I} x_i, y) \in P_I\} \Big),$$

where

$$\begin{aligned} CG &= \textit{conv}\{(x,y) \in \mathbb{R}^m \times \mathbb{Z}^2 : cy \geq b - u(M),\ y \geq 0\}, \\ U &= \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^2 :\ u \geq x\}, \\ P_I &= \textit{conv}\{(w_I, y) \in \mathbb{R}^1 \times \mathbb{Z}^2 :\ w_I + cy \geq b - u(M \setminus I),\ w_I, y \geq 0\}. \end{aligned}$$

Proof. By Theorem 3.11, *conv*(S) is given by intersecting trivial facet-defining inequalities (the capacity inequality and variable bounds) with facet-defining inequalities for CG and facet-defining inequalities arising from $Q(b - u(M \setminus I), u(I))$ for nonempty subsets I of M. Furthermore, by Theorem 4.4, every facet of $Q(b-u(M\setminus I), u(I))$ either defines a facet of $Q(b-u(M\setminus I), \infty)$, or is the upper bound constraint $w \leq u(I)$, or defines a facet of $CG^* = \{y \in \mathbb{Z}^2 : cy \geq b - u(M), y \geq 0\}$. Clearly $\mathbb{R}^m \times CG^* = CG$, and every facet-defining inequality of CG^* defines a facet of CG and conversely. In addition, the inequality $w \leq u(I)$, after replacing w by $\sum_{i \in I} x_i$ is implied by the constraints defining U. As $P_I = Q(b - u(M \setminus I), \infty)$, the result follows. □

For a fixed set $I \subseteq M$, it is possible to use the results of Agra and Constantino [2] to enumerate all facet-defining inequalities for P_I in polynomial-time. Thus it is possible to give a complete description of *conv*(S) with valid inequalities. As there is an exponential number of choices for the set I, this does not lead directly to a polynomial time separation algorithm for S.

In a similar fashion, when $n = 2$, one can give a complete description of *conv*(S) in terms of its extreme points and rays. Lemma 2.1(a) gives the recession cone of *conv*(S). Lemma 2.2 shows that every extreme point of *conv*(S) has the property that at most one x variable has value strictly between its bounds and the rest of the variables are at their upper or lower bounds. In other words, there exists an index $k \in M$, and sets T, T' partitioning $M \setminus \{k\}$ such that every such extreme point lies on the face of *conv*(S) defined by $x_i = u_i \ \forall i \in T$ and $x_i = 0 \ \forall i \in T'$. Clearly, such a face corresponds to $Q(b - u(M \setminus T), u_k)$ after projecting out the fixed variables. Therefore one can obtain such extreme points from Corollary 4.6.

For general $m \geq 1$, the optimization problem $\min\{px + qy : (x,y) \in S\}$ can be solved by solving at most m three-variable integer programs. Assume that the variables are indexed in such a way that $p_1 \leq \cdots \leq p_m$ and consider an optimal solution $(\bar{x}, \bar{y})$. $\bar{x}$ must be an optimal solution of the linear program $\min\{px : \sum_{i=1}^m x_i \geq b - c\bar{y}, 0 \leq x \leq u\}$. An optimal extreme point solution can be constructed greedily and has at most one variable strictly between its bounds. It follows that there is an (alternative)-optimal solution $(\tilde{x}, \bar{y})$ with $\tilde{x}_i = u_i$ for $i < k$, $\tilde{x}_i = 0$ for $i > k$, and $0 \leq \tilde{x}_k \leq u_k$ for some value of $k \in \{1, \ldots, m\}$. But then $(\tilde{x}, \bar{y})$ is also an optimal solution to

$$\sum_{i:i<k} p_i u_i + \min\{p_k x_k + qy : x_k + \sum_{j=1}^{2} c_j y_j \geq b - \sum_{i:i<k} u_i, 0 \leq x_k \leq u_k,\ y \in \mathbb{Z}^2_+\}.$$

Thus it suffices to solve the m three variable problems obtained as k varies from 1 to m. Each of these problems can be solved in polynomial-time using the results of Agra and Constantino [2].

It follows that the separation problem for *conv*(S) can be solved in polynomial time using the ellipsoid algorithm. However it would be preferable to have a more practical algorithm: a natural conjecture is that it suffices to order the variables such that $\frac{x_1^*}{u_1} \geq \cdots \geq \frac{x_m^*}{u_m}$ and then separate over the $m+1$ sets P_{T_i} where $T_i = \{i, i+1, \ldots, m\}$ and $i = 1, \ldots, m+1$. This provides a polynomial algorithm in the case of $n = 1$, see Atamturk and Rajan [4] and in a mixing set variant, see Di Summa and Wolsey [5].

With three or more integer variables, the sets

$$P_T^n = \textit{conv}((x_T, y) \in \mathbb{R}_+^1 \times \mathbb{Z}_+^n : x_T + cy \geq b - u(M \setminus T))$$

still lead to valid relaxations for *conv*(S), but the main results of Section 4 do not generalize. For an arbitrary number of integer variables, similar results to those of Section 4 hold when the coefficients are divisible (Wolsey and Yaman [9]); in particular similar relaxations give a complete description of *conv*(S).

For the remainder of this section, we assume that $P_{\geq}(b), P(b-u, b), P_{\leq}(b)$ and $Q(b, u)$ are defined in a similar fashion to the definitions inSection 4 when $y \in \mathbb{R}^3$. For example, for given coefficients c_1, c_2, c_3, $P_{\geq}(b) = \textit{conv}\{y \in \mathbb{Z}^3 : c_1 y_1 + c_2 y_2 + c_3 y_3 \geq b\}$.

Remark 5.2. *Theorem 4.1 does not generalize to the case* $n = 3$.

Consider the set

$$P(94, 97) = \textit{conv}\{y \in \mathbb{Z}_+^3 : 94 \leq 5y_1 + 13y_2 + 22y_3 \leq 97\}.$$

The point $(\frac{8}{3}, \frac{2}{3}, \frac{10}{3})$ lies in $P_{\leq}(97) \cap P_{\geq}(94)$ as it can be written as $1/3(1,0,4) + 1/3(1,2,3) + 1/3(6,0,3) \in P_{\leq}(97)$ and as $1/3(2,0,4) + 2/3(3,1,3) \in P_{\geq}(94)$. However it is cut off by the valid inequality (also facet-defining) $y_3 \leq 3$ of $P(94, 97)$. To see that all solutions of $P(94, 97)$ satisfy $y_3 \leq 3$, first note that $y_3 \leq 4$ is a valid inequality: $y_3 \leq 97/22$ follows from the nonnegativity of y and we can round down the right-hand-side of this inequality as y_3 is integral. Observe that if $(\bar{y}_1, \bar{y}_2, 4)$ is a non-negative integral point in $P(94, 97)$, then $5\bar{y}_1 + 13\bar{y}_2 \in [6, 9]$ which is not possible.

Remark 5.3. *Theorem 4.4 does not generalize in the case* $n = 3$.

Consider the set

$$Q(97, 3) = \textit{conv}\{(x, y) \in \mathbb{R}_+^1 \times \mathbb{Z}_+^3 : x + 5y_1 + 13y_2 + 22y_3 \geq 97, x \leq 3\}.$$

The point $(\frac{5}{3}, \frac{8}{3}, \frac{2}{3}, \frac{10}{3}) = 1/3(4,1,0,4) + 1/3(0,1,2,3) + 1/3(1,6,0,3) \in Q(97, \infty)$. From Remark 5.2, it lies in $\mathbb{R}_+^1 \times P_{\geq}(94)$ and clearly $0 \leq x \leq 3$. However it is cut off by the valid inequality (also facet-defining) $x + 5y_1 + 13y_2 + 21y_3 \geq 94$ of $Q(97, 3)$. To see validity, note that this inequality is valid when $y_3 \leq 3$ (simply add $-y_3 \geq -3$ to the capacity inequality) or when $y_3 \geq 5$ (as then $21y_3 \geq 105$ and all other variables are nonnegative). Finally, for points $(\bar{x}, \bar{y})$ in $Q(97, 3)$ with $\bar{y}_3 = 4$, $\bar{x} + 5\bar{y}_1 + 13\bar{y}_2 \geq 9$. Clearly $x + 5y_1 + 13y_2 \geq 9$ and $0 \leq x \leq 3$ together imply that $x + 5y_1 + 13y_2 \geq 10$ as there is no solution to $x + 5y_1 = 9$ with y_1 a non-negative integer and $x \in [0, 3]$. But then $\bar{x} + 5\bar{y}_1 + 13\bar{y}_2 + 21\bar{y}_3 \geq 94$.

Proposition 5.4. *The convex hull of the following set has a facet-defining inequality with distinct nonzero coefficients for the continuous variables:*

$$S = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{Z}^3_+ : x_1 + x_2 + 5y_1 + 13y_2 + 22y_3 \geq 97, x_1, x_2 \leq 3\}.$$

Proof. We will show that

$$x_1 + 2x_2 + 5y_1 + 13y_2 + 21y_3 \geq 94 \tag{13}$$

is facet-defining for S. As in the proof after Remark 5.3, it is easy to see that (13) is valid for points in S with $y_3 \leq 3$ or $y_3 \geq 5$. To see the validity when $y_3 = 4$, consider a point $(\bar{x}, \bar{y}) \in S$ with $\bar{y}_3 = 4$. Then $\bar{x}_1 + \bar{x}_2 + 5\bar{y}_1 + 13\bar{y}_2 \geq 9$. Therefore either $\bar{x}_1 + \bar{x}_2 + 5\bar{y}_1 + 13\bar{y}_2 = 9$ (and $y_2 = 0$ and $x_2 \geq 1$ as $0 \leq x_1 \leq 3$ and y_1 is integral) or $\bar{x}_1 + \bar{x}_2 + 5\bar{y}_1 + 13\bar{y}_2 \geq 10$. In either case, $\bar{x}_1 + 2\bar{x}_2 + 5\bar{y}_1 + 13\bar{y}_2 \geq 10$ and the validity of (13) follows when $y_3 = 4$. The linearly independent points $(0, 0, 2, 0, 4)$, $(0, 0, 1, 2, 3)$, $(1, 0, 6, 0, 3)$, $(3, 1, 1, 0, 4)$, $(3, 0, 3, 1, 3)$ are contained in S and are tight for (13) showing that (13) defines a facet. ■

Previously, the smallest instance with a nontrivial facet with distinct coefficients for the x variables of which we are aware had $n = 8$ integer variables [8].

Some questions remain open. Are there some more general conditions under which Theorem 4.1 holds? Theorem 4.1 is a necessary condition for Theorem 4.4 to hold. Are there cases in which it is also sufficient?

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