

# OSGA: A fast subgradient algorithm with optimal complexity

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**Abstract.** This paper presents an algorithm for approximately minimizing a convex function in simple, not necessarily bounded convex domains, assuming only that function values and subgradients are available. No global information about the objective function is needed apart from a strong convexity parameter (which can be put to zero if only convexity is known). The worst case number of iterations needed to achieve a given accuracy is independent of the dimension (which may be infinite) and – apart from a constant factor – best possible under a variety of smoothness assumptions on the objective function.

**Keywords:** complexity bound, convex optimization, optimal subgradient method, large-scale optimization, Nesterov’s optimal method, nonsmooth optimization, optimal first-order method, smooth optimization, strongly convex

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## 1 Introduction

In the recent years, first order methods for convex optimization have become prominent again as they are able to solve large-scale problems in millions of variables (often arising from applications to image processing, compressed sensing, or machine learning), where matrix-based interior point methods cannot even perform a single iteration. (However, a matrix-free interior point method by FOUNTOLAKIS et al. [12] works well in some large compressed sensing problems.)

In 1983, NEMIROVSKY & YUDIN [20] proved lower bounds on the complexity of first order methods (measured in the number of subgradient calls needed to achieve a given accuracy) for convex optimization under various regularity assumptions for the objective functions. (See NESTEROV [22, Sections 2.1.2 and 3.2.1] for a simplified account.) They constructed convex, piecewise linear functions in dimensions  $n > k$ , where no first order method can have function values more accurate than  $O(k^{-1/2})$  after  $k$  subgradient evaluations. This implies the need for at least  $O(\varepsilon^{-2})$  subgradient evaluations in the worst case if  $f$  is a nondifferentiable but Lipschitz continuous convex function. They also constructed convex

quadratic functions in dimensions  $n \geq 2k$  where no first order method can have function values more accurate than  $O(k^{-2})$  after  $k$  gradient evaluations. This implies the need for at least  $O(\varepsilon^{-1/2})$  gradient evaluations in the worst case if  $f$  is an arbitrarily often differentiable convex function. However in case of strongly convex functions with Lipschitz continuous gradients, the known lower bounds on the complexity allow a dimension-independent linear rate of convergence  $O(q^k)$  with  $0 < q < 1$ .

Algorithms by NESTEROV [22, 23, 26] (dating back in the unconstrained, not strongly convex case to 1983 [21]), achieve the optimal complexity order in all three cases. These algorithms need as input the knowledge of global parameters – a global Lipschitz constant for the objective functions in the nonsmooth case, a global Lipschitz constant for the gradient in the smooth case, and an explicit constant of strong convexity in the strongly convex case. Later many variants were described (see, e.g., AUSLENDER & TEBoulLE [5], LAN et al. [18]), some of which are adaptive in the sense that they estimate all required constants during the execution of the algorithm. BECK & TEBoulLE [8] developed an adaptive proximal point algorithm called FISTA, popular in image restauration applications. Like all proximal point based methods, the algorithm needs more information about the objective function than just subgradients, but delivers in return a higher speed of convergence. TSENG [28] gives a common uniform derivation of several variants of fast first order algorithms based on proximal points. BECKER et al. [9] (among other things) add adaptive features to Tseng’s class of algorithms, making them virtually independent of global (and hence often pessimistic) Lipschitz information. Other such adaptive algorithms include GONZAGA et al. [13, 14] and MENG & CHEN [19]. DEVOLDER et al. [11] show that both the nonsmooth case and the smooth case can be understood in a common way in terms of inexact gradient methods.

If the Lipschitz constant is very large, the methods with optimal complexity for the smooth case are initially much slower than the methods that have an optimal complexity for the nonsmooth case. This counterintuitive situation was remedied by LAN [17], who provides an algorithm that needs (expensive auxiliary computations but) no knowledge about the function except convexity and has the optimal complexity, both in the nonsmooth case and in the smooth case, without having to know whether or not the function is smooth. However, its worst case behavior on strongly convex problem is unknown. Similarly, if the constant of strong convexity is very tiny, the methods with optimal complexity for the strongly convex case are initially much slower than the methods that do not rely on strong convexity. Prior to the present work, no algorithm was known with optimal complexity both for the general nonsmooth case and for the strongly convex case.

**Content.** In this paper, we derive an algorithm for approximating a solution  $\hat{x} \in C$  of the convex optimization problem

$$f(\hat{x}) = \hat{f} := \min_{x \in C} f(x) \tag{1}$$

using first order information (function values  $f$  and subgradients  $g$ ) only. Here  $f : C \rightarrow \mathbb{R}$  is a convex function defined on a nonempty, convex subset  $C$  of a vector space  $V$  with bilinear pairing  $\langle h, z \rangle$  defined for  $z \in V$  and  $h$  in the dual space  $V^*$ . The minimum in (1) exists if there is a point  $x^0 \in C$  such that the level set  $\{x \in C \mid f(x) \leq f(x_0)\}$  is bounded.

Our method is based on monotonically reducing bounds on the error  $f(x_b) - \hat{f}$  of the function value of the currently best point  $x_b$ . These bounds are derived from suitable linear relaxations

and inequalities obtained with the help of a prox function. The solvability of an auxiliary optimization subproblem involving the prox function is assumed. In many cases, this auxiliary subproblem has a cheap, closed form solution; this is shown here for the unconstrained case with a quadratic prox function, and in AHOOKHOSH & NEUMAIER [3, 4] for more general cases involving simple, practically important convex sets  $C$ .

The OSGA algorithm presented here provides a fully adaptive alternative to current optimal first order methods. If no strong convexity is assumed, it shares the uniformity, the freeness of global parameters, and the optimal complexity properties of Lan’s method, but has a far simpler structure and derivation. Beyond that, it also gives the optimal complexity in the strongly convex case, though it needs in this case – like all other known methods with provable optimal linear convergence rate – the knowledge of an explicit constant of strong convexity. Furthermore – like Nesterov’s  $O(k^{-2})$  algorithm from [23] for the smooth case, but unlike his linearly convergent algorithm for the strongly convex case, scheme (2.2.19) in NESTEROV [22], the algorithm derived here does not evaluate  $f$  and  $g$  outside their domain. The method for analyzing the complexity of OSGA is also new; neither Tseng’s complexity analysis nor Nesterov’s estimating sequences are applicable to OSGA.

The OSGA algorithm can be used in place of Nesterov’s optimal algorithms for smooth convex optimization and its variants whenever the latter are traditionally employed. Thus it may be used as the smooth solver with methods for solving nonsmooth convex problems via smoothing (NESTEROV [23]), and for solving very large linear programs (see, e.g., AYBAT & IYENGAR [7], CHEN & BURER [10], BU et al. [15], NESTEROV [24, 25], RICHTARIK [27])

Numerical results are reported in AHOOKHOSH [1]; see also AHOOKHOSH & NEUMAIER [2].

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## 2 The OSGA algorithm

In this section we motivate and formulate the new algorithm.

In the following,  $V$  denotes a Banach space with norm  $\|\cdot\|$ , and  $V^*$  is the dual Banach space with the dual norm  $\|\cdot\|^*$ .  $C$  is a closed, convex subset of  $V$ . The objective function  $f : C \rightarrow \mathbb{R}$  is assumed to be convex, and  $g(x)$  denotes a particular computable subgradient of  $f$  at  $x \in C$ .

**The basic idea.** The method is based on monotonically reducing bounds on the error  $f(x_b) - \hat{f}$  of the function value of the currently best point  $x_b$ . These bounds are derived from suitable linear relaxations

$$f(z) \geq \gamma + \langle h, z \rangle \quad \text{for all } z \in C \quad (2)$$

(where  $\gamma \in \mathbb{R}$  and  $h \in V^*$ ) with the help of a continuously differentiable **prox function**  $Q : C \rightarrow \mathbb{R}$  satisfying

$$Q_0 := \inf_{z \in C} Q(z) > 0, \quad (3)$$

$$Q(z) \geq Q(x) + \langle g_Q(x), z - x \rangle + \frac{1}{2} \|z - x\|^2 \quad \text{for all } x, z \in C, \quad (4)$$

where  $g_Q(x)$  denotes the gradient of  $Q$  at  $x \in C$ . (Thus  $Q$  is strongly convex with strong convexity parameter  $\sigma = 1$ . Choosing  $\sigma = 1$  simplified the formulas, and is no restriction of generality, as we may always rescale a prox function to enforce  $\sigma = 1$ .) We require that, for each  $\gamma \in \mathbb{R}$  and  $h \in V^*$ ,

$$E(\gamma, h) := - \inf_{z \in C} \frac{\gamma + \langle h, z \rangle}{Q(z)} \quad (5)$$

is attained at some  $z = U(\gamma, h) \in C$ . This requirement implies that, for arbitrary  $\gamma_b \in \mathbb{R}$  and  $h \in V^*$ ,

$$\gamma_b + \langle h, z \rangle \geq -E(\gamma_b, h)Q(z) \quad \text{for all } z \in C.$$

From (2) for  $z = \hat{x}$  and (3), we find that

$$\gamma_b = \gamma - f(x_b), \quad E(\gamma_b, h) \leq \eta \quad \Rightarrow \quad 0 \leq f(x_b) - \hat{f} \leq \eta Q(\hat{x}). \quad (6)$$

Typically,

$$u := U(\gamma_b, h), \quad \bar{\eta} := E(\gamma_b, h) \quad (7)$$

are computed together; clearly one may update a given  $\eta$  by  $\bar{\eta}$ , thus improving the bound.

Note that the form of the auxiliary optimization problem (5) is forced by this argument. Although this is a nonconvex optimization problem, it is shown in AHOOKHOSH & NEUMAIER [3, 4] that there are many important cases where  $E(\gamma, h)$  and  $U(\gamma, h)$  are cheap to compute. In particular, we shall show in Section 5 that this is the case when  $C = V$  and the prox function is quadratic.

If an upper bound for  $Q(\hat{x})$  is known or assumed, the bound (6) translates into a computable error estimate for the minimal function value. But even in the absence of such an upper bound, we can solve the optimization problem (1) to a target accuracy

$$0 \leq f(x_b) - \hat{f} \leq \varepsilon Q(\hat{x}) \quad (8)$$

if we manage to decrease the error factor  $\eta$  from its initial value until  $\eta \leq \varepsilon$  for some target tolerance  $\varepsilon > 0$ . This will be achieved by Algorithm 2.4 defined below. We shall prove for this algorithm complexity bounds on the number of iterations that are independent of the dimension of  $V$  (which may be infinite), and – apart from a constant factor – best possible under a variety of assumptions on the objective function.

**Constructing linear relaxations.** The convexity of  $f$  implies for  $x, z \in C$  the bound

$$f(z) \geq f(x) + \langle g(x), z - x \rangle, \quad (9)$$

where  $g(x)$  denotes a subgradient of  $f$  at  $x \in C$ . Therefore (2) always holds with

$$\gamma = f(x_b) - \langle g(x_b), x_b \rangle, \quad h = g(x_b).$$

We can find more general relaxations of the form (2) by accumulating past information. Indeed, if (2) holds,  $\alpha \in [0, 1]$ , and  $x \in C$  then (2) remains valid when we substitute

$$\bar{\gamma} := \gamma + \alpha(f(x) - \langle g(x), x \rangle - \gamma),$$

$$\bar{h} := h + \alpha(g(x) - h)$$

in place of  $\gamma, h$ , as by (9),

$$\begin{aligned} f(z) &= (1 - \alpha)f(z) + \alpha f(z) \\ &\geq (1 - \alpha)(\gamma + \langle h, z \rangle) + \alpha(f(x) + \langle g(x), z - x \rangle) \\ &= (1 - \alpha)\gamma + \alpha(f(x) - \langle g(x), x \rangle) + \langle (1 - \alpha)h + \alpha g(x), z \rangle \\ &= \bar{\gamma} + \langle \bar{h}, z \rangle. \end{aligned}$$

For appropriate choices of  $x$  and  $\alpha$ , this may give much improved error bounds. We discuss suitable choices for  $x$  later.

**Step size selection.** The **step size parameter**  $\alpha$  controls the fraction of the new information (9) incorporated into the new relaxation. It is chosen with the hope for a reduction factor of approximately  $1 - \alpha$  in the current error factor  $\eta$ , and must therefore be adapted to the actual progress made.

First we note that in practice,  $Q(\hat{x})$  is unknown; hence the numerical value of  $\eta$  is meaningless in itself. However, quotients of  $\eta$  at different iterations have a meaning, quantifying the amount of progress made.

In the following, we use bars to denote quantities tentatively modified in the current iteration, but they replace the current values of these quantities only if an acceptance criterion is met that we now motivate. We measure progress in terms of the quantity

$$R := \frac{\eta - \bar{\eta}}{\lambda\alpha\eta}, \quad (10)$$

where  $\lambda \in ]0, 1[$  is a fixed number. A value  $R \geq 1$  indicates that we made sufficient progress in that

$$\bar{\eta} = (1 - \lambda R\alpha)\eta \quad (11)$$

was reduced at least by a fraction  $\lambda$  of the designed improvement of  $\eta$  by  $\alpha\eta$ ; thus the step size is acceptable or may even be increased if  $R > 1$ . On the other hand, if  $R < 1$ , the step size must be reduced significantly to improve the chance of reaching the design goal. Introducing a maximal step size  $\alpha_{\max} \in ]0, 1[$  and two parameters with  $0 < \kappa' \leq \kappa$  to control the amount of increase or decrease in  $\alpha$ , we update the step size according to

$$\bar{\alpha} := \begin{cases} \alpha e^{-\kappa} & \text{if } R < 1, \\ \min(\alpha e^{\kappa'(R-1)}, \alpha_{\max}) & \text{if } R \geq 1. \end{cases} \quad (12)$$

Since updating the linear relaxation and  $u$  makes sense only when  $\eta$  was improved, we obtain the following update scheme.

### 2.1 Algorithm. (Update scheme)

global tuning parameters:  $\lambda \in ]0, e^{-\kappa}]; \quad \alpha_{\max} \in ]0, 1[; \quad 0 < \kappa' \leq \kappa;$

input:  $\alpha, \eta, \bar{h}, \bar{\gamma}, \bar{\eta}, \bar{u};$

output:  $\alpha, h, \gamma, \eta, u;$

$R = (\eta - \bar{\eta})/(\lambda\alpha\eta);$

if  $R < 1$ ,  $\bar{\alpha} = \alpha e^{-\kappa};$

else  $\bar{\alpha} = \min(\alpha e^{\kappa'(R-1)}, \alpha_{\max});$

end;

$\alpha = \bar{\alpha};$

if  $\bar{\eta} < \eta,$

$h = \bar{h}; \gamma = \bar{\gamma}; \eta = \bar{\eta}; u = \bar{u};$

end;

If  $\alpha_{\min}$  denotes the smallest actually occurring step size (which is not known in advance), we have global linear convergence with a convergence factor of  $1 - e^{-\kappa}\alpha_{\min}$ . However,  $\alpha_{\min}$  and hence this global rate of convergence may depend on the target tolerance  $\varepsilon$ ; thus the convergence speed in the limit  $\varepsilon \rightarrow 0$  may be linear or sublinear depending on the properties of the specific function minimized.

**Strongly convex relaxations.** If  $f$  is strongly convex, we may know a number  $\mu > 0$  such that  $f - \mu Q$  is still convex. In this case, we have in place of (9) the stronger inequality

$$f(z) - \mu Q(z) \geq f(x) - \mu Q(x) + \langle g(x) - \mu g_Q(x), z - x \rangle \quad \text{for } x, z \in C. \quad (13)$$

In the following, we only assume that  $\mu \geq 0$ , thus covering the case of linear relaxations, too.

(13) allows us to construct strongly convex relaxations of the form

$$f(z) \geq \gamma + \langle h, z \rangle + \mu Q(z) \quad \text{for all } z \in C. \quad (14)$$

For example, (14) always holds with

$$h = g(x_b) - \mu g_Q(x_b), \quad \gamma = f(x_b) - \mu Q(x_b) - \langle h, x_b \rangle.$$

Again more general relaxations of the form (14) are found by accumulating past information.

**2.2 Proposition.** *Suppose that  $x \in C$ ,  $\alpha \in [0, 1]$ , and let*

$$\bar{h} = h + \alpha(g - h), \quad \bar{\gamma} = \gamma + \alpha \left( f(x) - \mu Q(x) - \langle g, x \rangle - \gamma \right),$$

where

$$g = g(x) - \mu g_Q(x).$$

If (14) holds and  $f - \mu Q$  is convex then (14) also holds with  $\bar{\gamma}$  and  $\bar{h}$  in place of  $\gamma$  and  $h$ .

*Proof.* By (13) and the assumptions,

$$\begin{aligned} f(z) - \mu Q(z) &= (1 - \alpha)(f(z) - \mu Q(z)) + \alpha(f(z) - \mu Q(z)) \\ &\geq (1 - \alpha)(\gamma + \langle h, z \rangle) \\ &\quad + \alpha \left( f(x) - \mu Q(x) + \langle g(x), z - x \rangle - \mu \langle g_Q(x), z - x \rangle \right) \\ &= \bar{\gamma} + \langle \bar{h}, z \rangle. \end{aligned}$$

□

The relaxations (14) lead to the following error bound.

**2.3 Proposition.** *Let*

$$\gamma_b := \gamma - f(x_b), \quad \eta := E(\gamma_b, h) - \mu.$$

*Then (14) implies*

$$0 \leq f(x_b) - \hat{f} \leq \eta Q(\hat{x}). \tag{15}$$

*Proof.* By definition of  $E(\gamma_b, h) = \eta + \mu$  and (14), we have

$$-(\eta + \mu)Q(z) \leq \gamma_b + \langle h, z \rangle = \gamma - f(x_b) + \langle h, z \rangle \leq f(z) - f(x_b) - \mu Q(z).$$

for all  $z \in C$ . Substituting  $z = \hat{x}$  gives (15). □

Note that for  $\mu = 0$ , we simply recover the previous results for general convex functions.

**An optimal subgradient algorithm.** For a nonsmooth convex function, the subgradient at a point does not always determine a direction of descent. However, we may hope to find better points by moving from the best point  $x_b$  into the direction of the point (7) used to determine our error bound. We formulate on this basis the following algorithm, for which optimal complexity bounds will be proved in Section 4.

## 2.4 Algorithm. (Optimal subgradient algorithm, OSGA)

global tuning parameters:  $\lambda, \alpha_{\max} \in ]0, 1[$ ;  $0 < \kappa' \leq \kappa$ ;

input parameters:  $\mu \geq 0$ ;  $\varepsilon > 0$ ;  $f_{\text{target}}$ ;

output:  $x_b$ ;

assumptions:  $f - \mu Q$  is convex;

begin

choose  $x_b$ ; stop if  $f(x_b) \leq f_{\text{target}}$ ;

$h = g(x_b) - \mu g_Q(x_b)$ ;  $\gamma = f(x_b) - \mu Q(x_b) - \langle h, x_b \rangle$ ;

$\gamma_b = \gamma - f(x_b)$ ;  $u = U(\gamma_b, h)$ ;  $\eta = E(\gamma_b, h) - \mu$ ;

$\alpha = \alpha_{\max}$ ;

while 1,

$x = x_b + \alpha(u - x_b)$ ;  $g = g(x) - \mu g_Q(x)$ ;

$\bar{h} = h + \alpha(g - h)$ ;  $\bar{\gamma} = \gamma + \alpha(f(x) - \mu Q(x) - \langle g, x \rangle - \gamma)$ ;

$x'_b = \operatorname{argmin}_{z \in \{x_b, x\}} f(z)$ ;

$\gamma'_b = \bar{\gamma} - f(x'_b)$ ;  $u' = U(\gamma'_b, \bar{h})$ ;  $x' = x_b + \alpha(u' - x_b)$ ;

choose  $\bar{x}_b$  with  $f(\bar{x}_b) \leq \min(f(x'_b), f(x'))$ ;

$\bar{\gamma}_b = \bar{\gamma} - f(\bar{x}_b)$ ;  $\bar{u} = U(\bar{\gamma}_b, \bar{h})$ ;  $\bar{\eta} = E(\bar{\gamma}_b, \bar{h}) - \mu$ ;

$x_b = \bar{x}_b$ ;

stop if some user-defined test is passed;

update  $\alpha, h, \gamma, \eta, u$  by Algorithm 2.1;

end;

end;

Note that the strong convexity parameter  $\mu$  needs to be specified to use the algorithm. If  $\mu$  is unknown, one may always put  $\mu = 0$  (ignoring possible strong convexity), at the cost of possibly slower worst case asymptotic convergence. (Techniques like those used in JUDITSKY & NESTEROV [16] or GONZAGA & KARAS [13] for choosing  $\mu$  adaptively can probably be applied to the above algorithm to remove the dependence on having to know  $\mu$ . However, [16] requires an explicit knowledge of a Lipschitz constant for the gradient, while [13] proves only sublinear convergence. It is not yet clear how to avoid both problems.)

The analysis of the algorithm will be independent of the choice of  $\bar{x}_b$  allowed in Algorithm 2.4. The simplest choice is  $\bar{x}_b = \operatorname{argmin}_{z \in \{x'_b, x'\}} f(z)$ . If the best function value  $f(x_b)$  is stored and

updated, each iteration then requires the computation of two function values  $f(x)$  and  $f(x')$  and one subgradient  $g(x)$ .

However, the algorithm allows the incorporation of heuristics to look for improved function values before deciding on the choice of  $\bar{x}_b$ . This may involve additional function evaluations at points selected by a line search procedure (see, e.g., BECK & TEBOLLE [8]), a bundle optimization (see, e.g., LAN [17]), or a local quadratic approximation (see, e.g., YU et al. [29]).

Numerical results are reported in AHOOKHOSH [1]; see also AHOOKHOSH & NEUMAIER [2].



### 3 Inequalities for the error factor

The possibility to get worst case complexity bounds rests on the establishment of a strong upper bound on the error factor  $\eta$ . This bound depends on global information about the function  $f$ ; while not necessary for executing the algorithm itself, it is needed for the analysis. Depending on the properties of  $f$ , global information of different strength can be used, resulting in inequalities of corresponding strength. The key steps in the analysis rely on the following lower bound for the term  $\gamma + \langle h, z \rangle$ .

**3.1 Proposition.** *Let  $v = U(\gamma, h)$ . Then*

$$\gamma + \langle h, v \rangle = -E(\gamma, h)Q(v). \quad (16)$$

Moreover, if  $E(\gamma, h) \geq 0$  then for all  $z \in C$ ,

$$\gamma + \langle h, z \rangle \geq E(\gamma, h) \left( \frac{1}{2} \|z - v\|^2 - Q(z) \right), \quad (17)$$

$$E(\gamma, h)(Q(z) - Q(v)) + \langle h, z - v \rangle \geq 0. \quad (18)$$

*Proof.* By definition of  $E(\gamma, h)$ , the function  $\phi : C \rightarrow \mathbb{R}$  defined by

$$\phi(z) := \gamma + \langle h, z \rangle + E(\gamma, h)Q(z)$$

is nonnegative and vanishes for  $z = v := U(\gamma, h)$ . This implies (16). Writing  $g_\phi(z)$  for the gradient of  $\phi$  at  $z \in C$ , strong convexity (4) of  $Q$  implies

$$\phi(z) - \phi(v) - \langle g_\phi(v), z - v \rangle = E(\gamma, h) \left( Q(z) - Q(v) - \langle g_Q(v), z - v \rangle \right) \geq \frac{E(\gamma, h)}{2} \|z - v\|^2.$$

But

$$\langle g_\phi(v), z - v \rangle = \lim_{\alpha \downarrow 0} \frac{\phi(v + \alpha(z - v)) - \phi(v)}{\alpha} \geq 0$$

since  $\phi(v) = 0$ . This proves (17). If we eliminate  $\gamma$  using (16) and delete the norm term, we obtain (18).  $\square$

**3.2 Theorem.** *In Algorithm 2.4, the error factors are related by*

$$\bar{\eta} - (1 - \alpha)\eta \leq \frac{\alpha^2 \|g(x)\|_*^2}{2(1 - \alpha)(\eta + \mu)Q_0}, \quad (19)$$

where  $\|\cdot\|_*$  denotes the norm dual to  $\|\cdot\|$ .

*Proof.* We first establish some inequalities needed for the later estimation. By convexity of  $Q$  and the definition of  $\bar{h}$ ,

$$\begin{aligned} \alpha\mu \left( Q(\bar{u}) - Q(x) + \langle g_Q(x), x \rangle \right) &\geq \alpha\mu \langle g_Q(x), \bar{u} \rangle = \langle h - \bar{h} + \alpha(g(x) - h), \bar{u} \rangle \\ &= (1 - \alpha) \langle h, \bar{u} \rangle + \langle \alpha g(x) - \bar{h}, \bar{u} \rangle. \end{aligned}$$

By definition of  $x$ , we have

$$(1 - \alpha)(x_b - x) = -\alpha(u - x).$$

Hence (13) (with  $\mu = 0$ ) implies

$$(1 - \alpha)(f(x_b) - f(x)) \geq (1 - \alpha)\langle g(x), x_b - x \rangle = -\alpha\langle g(x), u - x \rangle.$$

By definition of  $\bar{\gamma}$ , we conclude from these two inequalities that

$$\begin{aligned} \bar{\gamma} - f(x) + \alpha\mu Q(\bar{u}) &= (1 - \alpha)(\gamma - f(x)) - \alpha\langle g(x), x \rangle + \alpha\mu(Q(\bar{u}) - Q(x) + \langle g_Q(x), x \rangle) \\ &\geq (1 - \alpha)\left(\gamma - f(x) + \langle h, \bar{u} \rangle\right) + \alpha\langle g(x), \bar{u} - x \rangle - \langle \bar{h}, \bar{u} \rangle \\ &\geq (1 - \alpha)\left(\gamma - f(x_b) + \langle h, \bar{u} \rangle\right) + \alpha\langle g(x), \bar{u} - u \rangle - \langle \bar{h}, \bar{u} \rangle. \end{aligned}$$

Using this, (16) (with  $\bar{\gamma}_b = \bar{\gamma} - f(\bar{x}_b)$  in place of  $\gamma$  and  $\bar{h}$  in place of  $h$ ), and  $E(\bar{\gamma}_b, \bar{h}) = \bar{\eta} + \mu$  now gives

$$\begin{aligned} (\bar{\eta} + \mu - \alpha\mu)Q(\bar{u}) &= f(\bar{x}_b) - \bar{\gamma} - \langle \bar{h}, \bar{u} \rangle - \alpha\mu Q(\bar{u}) \\ &\leq f(\bar{x}_b) - f(x) - \alpha\langle g(x), \bar{u} - u \rangle - (1 - \alpha)\left(\gamma - f(x_b) + \langle h, \bar{u} \rangle\right). \end{aligned} \tag{20}$$

Using (17) (with  $\gamma_b = \gamma - f(x_b)$  in place of  $\gamma$ ) and  $\eta + \mu = E(\gamma_b, h)$ , we find

$$(\eta + \mu)Q(\bar{u}) \geq f(x_b) - \gamma - \langle h, \bar{u} \rangle + \frac{\eta + \mu}{2}\|\bar{u} - u\|^2. \tag{21}$$

Now (20) and (21) imply

$$\begin{aligned} (\bar{\eta} - (1 - \alpha)\eta)Q(\bar{u}) &= (\bar{\eta} + \mu - \alpha\mu)Q(\bar{u}) - (1 - \alpha)(\eta + \mu)Q(\bar{u}) \\ &\leq f(\bar{x}_b) - f(x) - (1 - \alpha)\left(\gamma - f(x_b) + \langle h, \bar{u} \rangle\right) \\ &\quad - \alpha\langle g(x), \bar{u} - u \rangle \\ &\quad - (1 - \alpha)\left(f(x_b) - \gamma - \langle h, \bar{u} \rangle + \frac{\eta + \mu}{2}\|\bar{u} - u\|^2\right) \\ &= f(\bar{x}_b) - f(x) + \bar{S}, \end{aligned}$$

where

$$\begin{aligned} \bar{S} &:= -\alpha\langle g(x), \bar{u} - u \rangle - \frac{(1 - \alpha)(\eta + \mu)}{2}\|\bar{u} - u\|^2 \\ &\leq \alpha\|g(x)\|_*\|\bar{u} - u\| - \frac{(1 - \alpha)(\eta + \mu)}{2}\|\bar{u} - u\|^2 \\ &= \frac{\alpha^2\|g(x)\|_*^2 - (\alpha\|g(x)\|_* + (1 - \alpha)(\eta + \mu)\|\bar{u} - u\|)^2}{2(1 - \alpha)(\eta + \mu)} \leq \frac{\alpha^2\|g(x)\|_*^2}{2(1 - \alpha)(\eta + \mu)}. \end{aligned} \tag{22}$$

If  $\bar{\eta} \leq (1 - \alpha)\eta$  then (19) holds trivially. Thus we assume that  $\bar{\eta} > (1 - \alpha)\eta$ . Then

$$(\bar{\eta} - (1 - \alpha)\eta)Q_0 \leq (\bar{\eta} - (1 - \alpha)\eta)Q(\bar{u}) \leq f(\bar{x}_b) - f(x) + \bar{S}. \tag{23}$$

Since  $f(\bar{x}_b) \leq f(x)$ , we conclude again that (19) holds. Thus (19) holds generally.  $\square$

Note that the arguments used in this proof did not make use of  $x'$ ; thus (19) even holds when one sets  $x' = x$  in the algorithm, saving some work.

**3.3 Theorem.** *If  $f$  has Lipschitz continuous gradients with Lipschitz constant  $L$  then, in Algorithm 2.4,*

$$\bar{\eta} > (1 - \alpha)\eta \quad \Rightarrow \quad (1 - \alpha)(\eta + \mu) < \alpha^2 L. \quad (24)$$

*Proof.* The proof follows the general line of the preceding proof, but now we must consider the information provided by  $x'$ .

Since  $E$  is monotone decreasing in its first argument and  $f(x'_b) \geq f(\bar{x}_b)$ , the hypothesis of (24) implies that

$$\eta' := E(\bar{\gamma} - f(x'_b), \bar{h}) - \mu \geq E(\bar{\gamma} - f(\bar{x}_b), \bar{h}) - \mu = \bar{\eta} > (1 - \alpha)\eta.$$

By convexity of  $Q$  and the definition of  $\bar{h}$ ,

$$\begin{aligned} \alpha\mu \left( Q(u') - Q(x) + \langle g_Q(x), x \rangle \right) &\geq \alpha\mu \langle g_Q(x), u' \rangle = \langle h - \bar{h} + \alpha(g(x) - h), u' \rangle \\ &= (1 - \alpha) \langle h, u' \rangle + \langle \alpha g(x) - \bar{h}, u' \rangle. \end{aligned}$$

By definition of  $x$ , we have

$$(1 - \alpha)(x_b - x) = -\alpha(u - x).$$

Hence (13) (with  $\mu = 0$ ) implies

$$(1 - \alpha)(f(x_b) - f(x)) \geq (1 - \alpha) \langle g(x), x_b - x \rangle = -\alpha \langle g(x), u - x \rangle.$$

By definition of  $\bar{\gamma}$ , we conclude from the last two inequalities that

$$\begin{aligned} \bar{\gamma} - f(x) + \alpha\mu Q(u') &= (1 - \alpha)(\gamma - f(x)) - \alpha \langle g(x), x \rangle + \alpha\mu \left( Q(u') - Q(x) + \langle g_Q(x), x \rangle \right) \\ &\geq (1 - \alpha) \left( \gamma - f(x) + \langle h, u' \rangle \right) + \alpha \langle g(x), u' - x \rangle - \langle \bar{h}, u' \rangle \\ &\geq (1 - \alpha) \left( \gamma - f(x_b) + \langle h, u' \rangle \right) + \alpha \langle g(x), u' - u \rangle - \langle \bar{h}, u' \rangle. \end{aligned}$$

Using this, (16) (with  $\gamma'_b = \bar{\gamma} - f(x'_b)$  in place of  $\gamma$  and  $\bar{h}$  in place of  $h$ ), and  $E(\gamma'_b, \bar{h}) = \eta' + \mu$  now gives

$$\begin{aligned} (\eta' + \mu - \alpha\mu)Q(u') &= f(x'_b) - \bar{\gamma} - \langle \bar{h}, u' \rangle - \alpha\mu Q(u') \\ &\leq f(x'_b) - f(x) - \alpha \langle g(x), u' - u \rangle \\ &\quad - (1 - \alpha) \left( \gamma - f(x'_b) + \langle h, u' \rangle \right). \end{aligned} \quad (25)$$

Using (17) (with  $\gamma_b = \gamma - f(x_b)$  in place of  $\gamma$ ) and  $\eta + \mu = E(\gamma_b, h)$ , we find

$$(\eta + \mu)Q(u') \geq f(x_b) - \gamma - \langle h, u' \rangle + \frac{\eta + \mu}{2} \|u' - u\|^2. \quad (26)$$

Now (25) and (26) imply

$$\begin{aligned}
(\eta' - (1 - \alpha)\eta)Q(u') &= (\eta' + \mu - \alpha\mu)Q(u') - (1 - \alpha)(\eta + \mu)Q(u') \\
&\leq f(x'_b) - f(x) - (1 - \alpha)\left(\gamma - f(x_b) + \langle h, u' \rangle\right) \\
&\quad - \alpha\langle g(x), u' - u \rangle \\
&\quad - (1 - \alpha)\left(f(x_b) - \gamma - \langle h, u' \rangle + \frac{\eta + \mu}{2}\|u' - u\|^2\right) \\
&= f(x'_b) - f(x) + S',
\end{aligned}$$

where

$$S' := -\alpha\langle g(x), u' - u \rangle - \frac{(1 - \alpha)(\eta + \mu)}{2}\|u' - u\|^2,$$

giving

$$(\eta' - (1 - \alpha)\eta)Q_0 \leq f(x'_b) - f(x) + S'.$$

Now

$$\begin{aligned}
f(x'_b) &\leq f(x') \leq f(x) + \langle g(x), x' - x \rangle + \frac{L}{2}\|x' - x\|^2 \\
&= f(x) + \alpha\langle g(x), u' - u \rangle + \frac{\alpha^2 L}{2}\|u' - u\|^2,
\end{aligned} \tag{27}$$

so that under the hypothesis of (24)

$$0 < (\eta' - (1 - \alpha)\eta)Q_0 \leq \frac{\alpha^2 L - (1 - \alpha)(\eta + \mu)}{2}\|u' - u\|^2.$$

Thus  $\alpha^2 L - (1 - \alpha)(\eta + \mu) > 0$ , and the conclusion of (24) holds.  $\square$

## 4 Bounds for the number of iterations

We now use the inequalities from Theorem 3.2 and Theorem 3.3 to derive bounds for the number of iterations. The weakest global assumption, mere convexity, leads to the weakest bounds and guarantees sublinear convergence only, while the strongest global assumption, strong convexity and Lipschitz continuous gradients, leads to the strongest bounds guaranteeing  $R$ -linear convergence. Our main result shows that, asymptotically as  $\varepsilon \rightarrow 0$ , the number of iterations needed by the OSGA algorithm matches the lower bounds on the complexity derived by NEMIROVSKI & YUDIN [20], apart from constant factors:

**4.1 Theorem.** *Suppose that  $f - \mu Q$  is convex. Then:*

(i) (**Nonsmooth complexity bound**)

*If the points generated by Algorithm 2.4 stay in a bounded region of the interior of  $C$ , or if  $f$  is Lipschitz continuous in  $C$ , the total number of iterations needed to reach a point with  $f(x) \leq f(\hat{x}) + \varepsilon$  is at most  $O((\varepsilon^2 + \mu\varepsilon)^{-1})$ . Thus the asymptotic worst case complexity is  $O(\varepsilon^{-2})$  when  $\mu = 0$  and  $O(\varepsilon^{-1})$  when  $\mu > 0$ .*

(ii) (**Smooth complexity bound**)

If  $f$  has Lipschitz continuous gradients with Lipschitz constant  $L$ , the total number of iterations needed by Algorithm 2.4 to reach a point with  $f(x) \leq f(\hat{x}) + \varepsilon$  is at most  $O(\varepsilon^{-1/2})$  if  $\mu = 0$ , and at most  $O(|\log \varepsilon| \sqrt{L/\mu})$  if  $\mu > 0$ .

In particular, if  $f$  is strongly convex and differentiable with Lipschitz continuous gradients,  $\mu > 0$  holds with arbitrary quadratic prox functions, and we get a complexity bound similar to that achieved by the preconditioned conjugate gradient method for linear systems; cf. AXELSSON & LINDSKOG [6].

Note that (24) generalizes to other situations by replacing (27) with a weaker smoothness property of the form

$$f(z) \leq f(x) + \langle g(x), z - x \rangle + \phi(\|z - x\|) \quad (28)$$

with  $\phi$  convex and monotone increasing. For example, this holds with  $\phi(t) = L_1 t$  if  $f$  has subgradients with bounded variation, and with  $\phi(t) = L_s t^{s+1}$  if  $f$  has Hölder continuous gradients with exponent  $s \in ]0, 1[$ , and with linear combinations thereof in the composite case considered by LAN [17]. Imitating the analysis below of the two cases stated in the theorem then gives corresponding complexity bounds matching those obtained by Lan.

Theorem 4.1 follows from the two propositions below covering the different cases, giving in each case explicit upper bounds on the number  $K_\mu(\alpha, \eta)$  of further iterations needed to complete the algorithm from a point where the values of  $\alpha$  and  $\eta$  given as arguments of  $K_\mu$  were achieved. We write  $\alpha_0$  and  $\eta_0$  for the initial values of  $\alpha$  and  $\eta$ . Only the dependence on  $\mu$ ,  $\alpha$ , and  $\eta$  is made explicit.

**4.2 Proposition.** *Suppose that the dual norm of the subgradients  $g(x)$  encountered during the iteration remains bounded by the constant  $c_0$ . Let  $c_1 > 0$ , and define*

$$c_1 := \frac{c_0^2}{2Q_0}, \quad c_2 := \max \left( \frac{c_1}{(1 - e^{-\kappa})(1 - \alpha_{\max})}, \frac{\eta_0(\eta_0 + \mu)}{\alpha_0} \right), \quad c_3 = \frac{c_2}{2\lambda}.$$

(i) *In each iteration,*

$$\eta(\eta + \mu) \leq \alpha c_2. \quad (29)$$

(ii) *The algorithm stops after at most*

$$K_\mu(\alpha, \eta) := 1 + \kappa^{-1} \log \frac{c_2 \alpha}{\varepsilon(\varepsilon + \mu)} + \frac{c_3}{\varepsilon(\varepsilon + \mu)} - \frac{c_3}{\eta(\eta + \mu)} \quad (30)$$

*further iterations.*

*In particular, (i) and (ii) hold when the iterates stay in a bounded region of the interior of  $C$ , or when  $f$  is Lipschitz continuous in  $C$ .*

Note that any convex function is Lipschitz continuous in any closed and bounded domain inside its support. Hence if the iterates stay in a bounded region  $R$  of the interior of  $C$ ,  $\|g\|$  is bounded by the Lipschitz constant of  $f$  in the closure of the region  $R$ .

*Proof.* (i) Condition (29) holds initially, and is preserved in each update unless  $\alpha$  is reduced. But then  $R < 1$ , hence  $\bar{\eta} \geq (1 - \lambda\alpha)\eta$ . Thus Theorem 3.2 implies

$$(1 - \lambda)\alpha\eta \leq \bar{\eta} - (1 - \alpha)\eta \leq \frac{\alpha^2 c_1}{(1 - \alpha)(\eta + \mu)}.$$

This implies

$$(1 - \lambda)(1 - \alpha)\eta(\eta + \mu) \leq \alpha c_1,$$

and since  $\lambda \leq e^{-\kappa} < 1$ ,

$$\bar{\eta}(\bar{\eta} + \mu) \leq \eta(\eta + \mu) \leq \frac{\alpha c_1}{(1 - \lambda)(1 - \alpha)} \leq \alpha c_2.$$

Thus (29) holds again after the reduction, and hence always.

(ii) As the algorithm stops once  $\eta \leq \varepsilon$ , (29) implies that in each iteration  $c_2\alpha \geq \varepsilon(\varepsilon + \mu)$ . As  $\alpha$  is reduced only when  $R < 1$ , and then by a fixed factor  $e^{-\kappa}$ , this cannot happen more than  $\kappa^{-1} \log \frac{c_2\alpha}{\varepsilon(\varepsilon + \mu)}$  times in turn. Thus after some number of  $\alpha$ -reductions we must always have another step with  $R \geq 1$ . By (11), this gives a reduction of  $\eta$  by a factor of at least  $1 - \lambda\alpha$ . But this implies that the stopping criterion  $\eta \leq \varepsilon$  is eventually reached. Therefore the algorithm stops eventually. Since  $R \geq 0$ , (12) implies  $\bar{\alpha} \leq \alpha e^{\kappa(R-1)}$ . Therefore

$$\log(\alpha/\bar{\alpha}) \geq \kappa(1 - R). \quad (31)$$

Now (31), (11), and (29) imply

$$\begin{aligned} K_\mu(\alpha, \eta) - K_\mu(\bar{\alpha}, \bar{\eta}) &= \frac{\log(\alpha/\bar{\alpha})}{\kappa} + \frac{c_3}{\bar{\eta}(\bar{\eta} + \mu)} - \frac{c_3}{\eta(\eta + \mu)} \\ &\geq 1 - R + \frac{c_3}{\bar{\eta}(\bar{\eta} + \mu)} - \frac{c_3}{\eta(\eta + \mu)} \\ &= 1 - R + \frac{c_3}{(1 - \lambda R\alpha)\eta((1 - \lambda R\alpha)\eta + \mu)} - \frac{c_3}{\eta(\eta + \mu)} \\ &\geq 1 - R + c_3 \frac{(\eta + \mu) - (1 - \lambda R\alpha)((1 - \lambda R\alpha)\eta + \mu)}{(1 - \lambda R\alpha)\eta(\eta + \mu)((1 - \lambda R\alpha)\eta + \mu)} \\ &= 1 - R + c_3 \frac{\lambda R\alpha((2 - \lambda R\alpha)\eta + \mu)}{(1 - \lambda R\alpha)\eta(\eta + \mu)((1 - \lambda R\alpha)\eta + \mu)} \\ &\geq 1 - R + c_3 \frac{2\lambda R\alpha}{\eta(\eta + \mu)} = 1 - R + \frac{2c_3\lambda R}{c_2} = 1. \end{aligned}$$

This implies the complexity bound by reverse induction, since immediately before the last iteration,  $c_2\alpha \geq \varepsilon(\varepsilon + \mu)$  and  $\eta > \varepsilon$ , hence  $K_\mu(\alpha, \eta) \geq 1$ .  $\square$

**4.3 Proposition.** *Suppose that  $f$  has Lipschitz continuous gradients with Lipschitz constant  $L$ , and put*

$$c_4 = \max\left(\frac{\eta_0 + \mu}{\alpha_0^2}, \frac{e^{2\kappa}L}{1 - \alpha_{\max}}\right), \quad c_5 = \frac{4c_4}{\lambda^2}, \quad c_6 = \sqrt{\frac{c_4}{\mu}}, \quad c_7 = \frac{c_6}{\lambda}.$$

(i) In each iteration

$$\eta + \mu \leq \alpha^2 c_4, \quad (32)$$

(ii) The algorithm stops after at most  $K_\mu(\alpha, \eta)$  further iterations. Here

$$K_0(\alpha, \eta) := 1 + \kappa^{-1} \log \left( \alpha \sqrt{\frac{c_4}{\varepsilon}} \right) + \sqrt{\frac{c_5}{\varepsilon}} - \sqrt{\frac{c_5}{\eta}}, \quad (33)$$

$$K_\mu(\alpha, \eta) := 1 + \frac{\log(c_6 \alpha)}{\kappa} + c_7 \log \frac{\eta}{\varepsilon} \quad \text{for } \mu > 0. \quad (34)$$

*Proof.* (i) (32) holds initially, and is preserved in each update unless  $\alpha$  is reduced. But then  $R < 1$ , hence by Theorem 3.3,  $(1 - \alpha)(\eta + \mu) = \alpha^2 L$  before the reduction. Therefore

$$\bar{\eta} + \mu \leq \eta + \mu \leq \frac{\alpha^2 L}{1 - \alpha} \leq \frac{\alpha^2 L}{1 - \alpha_{\max}} \leq \alpha^2 e^{-2\kappa} c_4 \leq \bar{\alpha}^2 c_4.$$

Thus (32) holds again after the reduction, and hence always. As in the previous proof, we find that the algorithm stops eventually, and (31) holds.

(ii) If  $\mu = 0$  then (11), (31), and (32) imply

$$\begin{aligned} K_0(\alpha, \eta) - K_0(\bar{\alpha}, \bar{\eta}) &= \frac{\log(\alpha/\bar{\alpha})}{\kappa} + \sqrt{\frac{c_5}{(1 - \lambda R \alpha)\eta}} - \sqrt{\frac{c_5}{\bar{\eta}}} \\ &\geq 1 - R + \left(1 - \sqrt{1 - \lambda R \alpha}\right) \sqrt{\frac{c_5}{(1 - \lambda R \alpha)\eta}} \\ &\geq 1 - R + \frac{\lambda R \alpha}{2} \sqrt{\frac{c_5}{\eta}} \geq 1 - R + \frac{\lambda R}{2} \sqrt{\frac{c_5}{c_4}} = 1. \end{aligned}$$

This implies the complexity bound by reverse induction, since immediately before the last iteration,  $\alpha \geq \sqrt{\varepsilon/c_4}$ , hence  $K_0(\alpha, \eta) \geq 1$ .

(iii) If  $\mu > 0$  then (32) shows that always  $c_6 \alpha \geq 1$ , hence  $\bar{\eta} = (1 - \lambda R \alpha)\eta \leq (1 - R/c_7)\eta$ . Therefore

$$K_\mu(\alpha, \eta) - K_\mu(\bar{\alpha}, \bar{\eta}) = \frac{\log(\alpha/\bar{\alpha})}{\kappa} + c_7 \log \frac{\eta}{\bar{\eta}} \geq 1 - R + c_7 \log \frac{1}{1 - R/c_7} \geq 1,$$

and the result follows as before.  $\square$

*Proof of Theorem 4.1.*

(i) We apply Proposition 4.2(ii) to the first iteration, and note that  $K_0(\alpha, \eta) = O(e^{-2})$  and  $K_\mu(\alpha, \eta) = O(e^{-1})$  if  $\mu > 0$ .

(i) We apply Proposition 4.3(ii) to the first iteration, and note that  $K_0(\alpha, \eta) = O(e^{-1/2})$  and  $K_\mu(\alpha, \eta) = O(\log \varepsilon^{-1})$  if  $\mu > 0$ .

## 5 Quadratic prox functions for unconstrained problems

To use the algorithm in practice, we need prox functions for which  $E(\gamma, h)$  and  $U(\gamma, h)$  can be evaluated easily. We first derive the optimality conditions for the associated auxiliary optimization problem (5).

**5.1 Proposition.** *If the function  $E_{\gamma, h} : C \rightarrow \mathbb{R}$  defined by*

$$E_{\gamma, h}(z) := -\frac{\gamma + \langle h, z \rangle}{Q(z)}$$

*attains its supremum  $E(\gamma, h)$  at  $z = u$ , the point  $u = U(\gamma, h)$  satisfies*

$$E(\gamma, h)Q(u) = -\gamma - \langle h, u \rangle, \quad (35)$$

$$\langle E(\gamma, h)g_Q(u) + h, z - u \rangle \geq 0 \quad \text{for all } z \in C. \quad (36)$$

*Proof.* (35) holds since  $E(\gamma, h) = E_{\gamma, h}(u)$ . To get (36), we differentiate the identity

$$E_{\gamma, h}(z)Q(z) = -\gamma - \langle h, z \rangle$$

and obtain

$$\frac{\partial E_{\gamma, h}(z)}{\partial z}Q(z) + E_{\gamma, h}(z)g_Q(z) = -h.$$

Now (36) follows from  $Q(z) > 0$  and the first order optimality condition

$$\left\langle \frac{-\partial E_{\gamma, h}(u)}{\partial z}, z - u \right\rangle \geq 0 \quad \text{for all } z \in C$$

for a minimum of  $-E_{\gamma, h}(z)$  at  $z = u$ . □

Simple domains  $C$  and prox functions for which the conditions (35) and (36) can be solved easily are discussed in AHOOKHOSH & NEUMAIER [3, 4].

In the remainder, we only discuss the simplest case, where the original optimization problem is unconstrained (so that  $C = V$ ) and the norm on  $V$  is Euclidean,

$$\|z\| := \sqrt{\langle Bz, z \rangle},$$

where the **preconditioner**  $B$  is a symmetric and positive definite linear mapping  $B : V \rightarrow V^*$ . The associated dual norm on  $V^*$  is then given by

$$\|h\|_* := \|B^{-1}h\| = \sqrt{\langle h, B^{-1}h \rangle}.$$

Given the preconditioner, it is natural to consider the quadratic prox function

$$Q(z) := Q_0 + \frac{1}{2}\|z - z_0\|^2, \quad (37)$$



where  $Q_0$  is a positive number and  $z_0 \in V$ . The assumption of Proposition 5.1 is satisfied since as the quotient of a linear and a positive quadratic function,  $E_{\gamma,h}(z)$  takes positive and negative values and is arbitrarily small outside a ball of sufficiently large radius. Therefore the level sets for nonzero function values are compact, and the supremum of a continuous function on a compact set is attained. Similarly, the infimum is attained.

Since  $C = V$  and  $g_Q(z) = B(z - z_0)$ , we conclude from the proposition that  $E(\gamma, h)B(u - z_0) + h = 0$ , where  $u = U(\gamma, h)$ , so that

$$U(\gamma, h) = z_0 - E(\gamma, h)^{-1}B^{-1}h. \quad (38)$$

Inserting this into (35) and writing  $e = E(\gamma, h)$ , we find

$$e\left(Q_0 + \frac{1}{2}\| -e^{-1}B^{-1}h\|^2\right) = eQ(u) = -\gamma - \langle h, z_0 - e^{-1}B^{-1}h \rangle,$$

which simplifies to the quadratic equation

$$Q_0e^2 + \beta e - \frac{1}{2}\|h\|_*^2 = 0, \quad \beta = \gamma + \langle h, z_0 \rangle.$$

The two solutions are the only stationary points, hence the solution with positive (negative) function value must be the unique maximizer (resp. minimizer). One easily checks that the maximizer is given by

$$E(\gamma, h) = \frac{-\beta + \sqrt{\beta^2 + 2Q_0\|h\|_*^2}}{2Q_0} = \frac{\|h\|_*^2}{\beta + \sqrt{\beta^2 + 2Q_0\|h\|_*^2}}. \quad (39)$$

(The first form is numerically stable when  $\beta \leq 0$ , the second when  $\beta > 0$ .)

A reasonable choice is to take the starting point of the iteration for  $z_0$ , and  $Q_0 \approx \frac{1}{2}\|\hat{x} - z_0\|^2$ , an order of magnitude guess.

## References

- [1] Ahookhosh, M.: Optimal subgradient algorithms with application to large-scale linear inverse problems, Manuscript, University of Vienna (2014).
- [2] M. Ahookhosh and A. Neumaier, High-dimensional convex optimization via optimal affine subgradient algorithms, in ROKS workshop (2013), 83–84.
- [3] M. Ahookhosh and A. Neumaier, Optimal subgradient methods for structured convex constrained optimization. I: theoretical results, Manuscript, University of Vienna (2014).
- [4] M. Ahookhosh and A. Neumaier, Optimal subgradient methods for structured convex constrained optimization. II: numerical results, Manuscript, University of Vienna (2014).
- [5] A. Auslender and M. Teboulle, Interior gradient and proximal methods for convex and conic optimization, SIAM J. Optimization 16 (2006), 697–725.

- [6] O. Axelsson and G. Lindskog, On the rate of convergence of the conjugate gradient method, *Numer. Math.* 48 (1986), 499–523.
- [7] N.S. Aybat and G. Iyengar, A first-order augmented Lagrangian method for compressed sensing, *SIAM J. Optim.*, 22(2) (2012), 429459.
- [8] A. Beck and M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM J. Imaging Sci.* 2 (2009), 183–202.
- [9] S.R. Becker, E.J. Candès, and M.C. Grant, Templates for convex cone problems with applications to sparse signal recovery, *Math. Programming Comput.* 3 (2011), 165–218.
- [10] J. Chen and S. Burer, A First-Order Smoothing Technique for a Class of Large-Scale Linear Programs, Manuscript (2011). [http://www.optimization-online.org/DB\\_FILE/2011/11/3233.pdf](http://www.optimization-online.org/DB_FILE/2011/11/3233.pdf)
- [11] O. Devolder, F. Glineur, and Y. Nesterov, First-order methods of smooth convex optimization with inexact oracle, *Mathematical Programming*, (2013) DOI 10.1007/s10107-013-0677-5.
- [12] K. Fountoulakis, J. Gondzio and P. Zhlobich, Matrix-free Interior Point Method for Compressed Sensing Problems, Manuscript (2012). arXiv:1208.5435
- [13] C.C. Gonzaga and E.W. Karas, Fine tuning Nesterov’s steepest descent algorithm for differentiable convex programming, *Math. Programming* 138 (2013), 141–166.
- [14] C.C. Gonzaga, E.W. Karas, and D.R. Rossetto, An optimal algorithm for constrained differentiable convex optimization, Manuscript (2011). [http://www.optimization-online.org/DB\\_FILE/2011/06/3053.pdf](http://www.optimization-online.org/DB_FILE/2011/06/3053.pdf)
- [15] M. Gu, L.-H. Lim and C.J. Wu, PARNES: A rapidly convergent algorithm for accurate recovery of sparse and approximately sparse signals, *Numer. Algor.* 64 (2013), 321–347.
- [16] A. Juditsky and Y. Nesterov, Primal-dual subgradient methods for minimizing uniformly convex functions, Manuscript (2010). <http://hal.archives-ouvertes.fr/docs/00/50/89/33/PDF/Strong-hal.pdf>
- [17] G. Lan, Bundle-level type methods uniformly optimal for smooth and nonsmooth convex optimization, *Mathematical Programming*, (2013) DOI 10.1007/s10107-013-0737-x.
- [18] G. Lan, Z. Lu and R.D.C. Monteiro, Primal-dual first-order methods with  $O(1/\varepsilon)$  iteration-complexity for cone programming, *Math. Programming* 126 (2011), 1–29.
- [19] X. Meng and H. Chen, Accelerating Nesterov’s method for strongly convex functions with Lipschitz gradient, Arxiv preprint arXiv:1109.6058 (2011).
- [20] A.S. Nemirovsky and D.B. Yudin, Problem complexity and method efficiency in optimization, Wiley, New York 1983.
- [21] Y. Nesterov, A method of solving a convex programming problem with convergence rate  $O(1/k^2)$  (in Russian), *Doklady AN SSSR* 269 (1983), 543-547. Engl. translation: *Soviet Math. Dokl.* 27 (1983), 372–376.

- [22] Y. Nesterov, *Introductory lectures on convex optimization: A basic course*, Kluwer, Dordrecht 2004.
- [23] Y. Nesterov, Smooth minimization of non-smooth functions, *Math. Programming* 103 (2005), 127–152.
- [24] Y. Nesterov, Rounding of convex sets and efficient gradient methods for linear programming problems, *Optim. Methods Softw.* 23 (2008), 109–128.
- [25] Y. Nesterov, Unconstrained convex minimization in relative scale, *Math. Operations Res.* 34 (2009), 180–193.
- [26] Y. Nesterov, Primal-dual subgradient methods for convex problems, *Math. Programming* 120 (2009), 221–259.
- [27] P. Richtarik, Improved algorithms for convex minimization in relative scale, *SIAM J. Optimization* 1 (2010), 1141–1167.
- [28] P. Tseng, On accelerated proximal gradient methods for convex-concave optimization, *Manuscript* (2008).
- [29] J. Yu, S.V.N. Vishvanathan, S. Günter and N.N. Schraudolph, A quasi-Newton approach to nonsmooth convex optimization problems in machine learning, *J. Machine Learning Res.* 11 (2010), 1145–1200.