On the Maximal Extensions of Monotone Operators and Criteria for Maximality

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Abstract

Within a nonzero, real Banach space we study the problem of characterising a maximal extension of a monotone operator in terms of minimality properties of representative functions that are bounded by the Penot and Fitzpatrick functions. We single out a property of this space of representative functions that enable a very compact treatment of maximality and pre-maximality issues.

This Paper is Dedicated to the memory of Charles Pearce.

Introduction

Monotone operators have found wide application in areas of optimization, control theory and the theory of partial differential equations. Many applications require maximality of a sum of monotone operators. Thus early on in the development of this theory the issue of when a sum of monotone operators is maximal was asked and was eventually resolved for reflexive Banach spaces by Rockafellar [14]. In this paper we single out a fundamental property of the space of representative functions that enables such a theorem to be efficiently derived. This leads to a particularly compact way of discussing issues of maximality and pre-maximality, at least in reflexive spaces.

Suppose X is a Banach space and X^* its dual. We may view $X\times X^*$ paired with $X^* \times X$ using the coupling $\langle (z,z^*),(x^*,x)\rangle = \langle z,x^*\rangle + \langle x,z^*\rangle$ and the norm $\|(z,z^*)\|^2 = \|z\|^2 + \|z^*\|^2$. Denote by by $PC(X\times X^*)$ the set of all proper convex functions $f : X \times X^* \to \mathbb{R}_+ := (-\infty,+\infty]$. When we pair with the topologies $s(X)\times\sigma(X^*,X)$ (product of the strong and weak*) with $\sigma(X^*,X)\times s(X)$ (product of the weak* and strong) the associated conjugation operation of a proper closed convex function $f \in \Gamma_{s\times w^*}(X\times X^*)$ is denoted by $f^* \in \Gamma_{w^*\times s}(X^*\times X)$ and given by

$$f^*(z^*,z) := \sup_{(x,x^*)} \{\langle (x,x^*),(z^*,z)\rangle - f(x,x^*)\}. \tag{1}$$

Definition 1 *We call a proper convex function $\mathcal{H}_T \in PC(X\times X^*)$ a* representative function *of a monotone mapping T on X when $\mathcal{H}_T(y,y^*) \geq \langle y,y^*\rangle$ for all $(y,y^*) \in X\times X^*$ with $\mathcal{H}_T(y,y^*) = \langle y,y^*\rangle$ if $y^* \in T(y)$. If T is not specified we say a proper convex function f is* representative *when $f(y,y^*) \geq \langle y,y^*\rangle$ for all $(y,y^*) \in X\times X^*$ and then say that it* represents *(the monotone set) $M_f := \{(x,x^*) \in X\times X^* \mid f(x,x^*) = \langle x,x^*\rangle\}$.*

Define the 'transpose' operator †: $(x^*,x) \leftrightarrow (x,x^*)$, and $c_T(\cdot,\cdot) := \delta_T(\cdot,\cdot) + \langle\cdot,\cdot\rangle$, where δ_T denotes the indicator function of the graph of T. Fitzpatrick [9] showed that $\mathcal{F}_T := c_T^{*\dagger}$ and $\mathcal{P}_T := \mathcal{F}_T^{*\dagger}$ induce functions that represent T, when T is maximal monotone. In [6] it is shown that $\mathcal{P}_T$ is representative when T is just monotone, while $\mathcal{F}_T$ may fail to be so when T is not maximal.

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It has been noted by a number of authors (see [6, 13, 5]) that when T is maximal, the largest representative function is $\mathcal{P}_T$ and the smallest is $\mathcal{F}_T$ ($\mathcal{P}_T \geq \mathcal{F}_T$ pointwise, see [12, Proposition 3]).

Recall that when f is a representative function then f^* is also a representative function, [4] i.e.

$$\left.\begin{array}{l} T : X \rightrightarrows X^* \text{ maximal and} \\ f \text{ representative for } T \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} f(x, x^*) \geq \langle x, x^* \rangle \quad \text{and} \\ f^*(x^*, x) \geq \langle x^*, x \rangle, \quad \forall (x, x^*) \in X \times X^* \end{array} \tag{2}$$

When X is reflexive the converse of (2) was first shown in [4]. It has been an open question as to whether the converse of (2) holds in nonreflexive spaces, but recently in [8] a counterexample is given in a non-reflexive Banach space where T is not maximal. In this example the representative function used does not have a "bigger conjugate". One might still conjecture that the converse is still true if one imposes the additional requirement that $f^{*\dagger} \geq f$. We will show otherwise. Denote $[\mathcal{F}_T, \mathcal{P}_T] := \{f \in PC(X \times X^*) \mid \mathcal{F}_T \leq f \leq \mathcal{P}_T\}$ and $bR(T) := \left\{f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle\right\}$.

When $\mathcal{F}_T$ is representative (i.e. T is "almost maximal", see [7, Proposition 2]) then $\mathcal{F}_T$ represents a maximal extension $T^\mu := M_{\mathcal{F}_T}^{\leq}$ of T. The related class of monotone operators are those that have a unique maximal extension. This class was studied in [11] and there termed "pre-maximal" and correspond to operators T such that T^μ is maximal. Clearly these classes are closely related with almost maximal operators being a subset of pre-maximal operators. We will show that the problem of characterising representative functions that describe maximal extensions is closely related to minimality with respect to the pointwise partial ordering of representative functions. We say that an element f in a partially ordered set $[S, \leq]$ is *minimal* iff $g \in S$ with $g \leq f$ implies $g = f$. When T is maximal, $\mathcal{F}_T$ is minimal in the class of representative functions (with bigger conjugates). When T is almost maximal, $\mathcal{F}_T$ is still minimal in $[\mathcal{F}_T, \mathcal{P}_T]$ (under the usual partial ordering of functions), but this is not necessarily so when we have pre-maximality. When T is not maximal, when does there exist such a minimal element and what does it represent?

Clearly, when g and f are representative and $g \geq f$ then $M_g \subseteq M_f$ with M_f monotone. The the search for a maximal extension of a monotone set M_g can be recast as a search for a "minimal" representative function. It is this study we undertake in this paper. We see that when T is not maximal and f represents a maximal extension of T then this representative function lies inside the interval $[\mathcal{F}_T, \mathcal{P}_T]$. Suppose this function is f; then as $f \geq \langle \cdot, \cdot \rangle$, we have for all $g \in [\mathcal{F}_T, \mathcal{P}_T]$ with $g \geq f$ also representative. Indeed, minimal representative functions in $R(T) := \{f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f \geq \langle \cdot, \cdot \rangle\}$ under the partial order $\leq$ have a special role and characterise maximal extensions. This was first observed and studied in [12] in the context of a reflexive Banach space, where it was shown that such minimal elements are Fitzpatrick functions of maximal monotone operators. It is not immediately clear that within an arbitrary Banach space *all* such minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ are Fitzpatrick functions. One can easily show that this deceptively simple property has a considerable number of consequences, in particular one has immediately that all minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ represent maximal monotone operators. We shall call BMLS (**B**urachik, **M**artínez-**L**egaz, **S**vaiter) spaces to be those Banach spaces X for which (for all monotone T on X) all minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ are Fitzpatrick functions (note: all reflexive spaces are BMLS spaces). Then we show the following are all equivalent to X being a BMLS space:

1. The space X has the property that for every monotone T, every minimal element f in $[R(T), \leq]$ represents a maximal monotone set M_f.

2. The space X has the property that for every monotone T, every $f \in bR(T)$ represent a maximal monotone set M_f (the subclass $bR(T)$ of representative functions with "bigger conjugates" was first identified by Simons [15]).

Thus this property is a potent tool for establishing maximality.

In a BMLS space (without any *a priori* assumption of reflexivity) one can show the standard certificates for maximality without reference to any of the usual duality machinery central to other approaches. Indeed the order-reversal property of conjugation is one of the main tools used.

Theorem 2 (Maximality via Auto-conjugates) *Let $T : X \rightrightarrows X^*$ be a monotone operator. Suppose*

X is BMLS*. If there exists $q \in \Gamma(X \times X^*)$ with $q^{*\dagger} = q$ that represents T (i.e. $M_q = T$), then T is maximal and $M_q = M_{q^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial q(x, x^*)\} = T$.*

Theorem 3 (Maximality via bigger conjugates) *Let $T : X \rightrightarrows X^*$ be a monotone operator, with X a* BMLS *space. If there exists f, $f^{*\dagger} \in PC(X \times X^*)$ with $f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle$ that represents T (i.e. $M_f = T$) then T is maximal and $M_f = M_{f^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial f(x, x^*)\} = T$.*

Theorem 4 (Maximality via representative functions) *Let $T : X \rightrightarrows X^*$ be a monotone operator, with X a* BMLS *space. If there exists f, $f^{*\dagger} \in PC(Z \times Z^*)$ with $f^{*\dagger} \geq \langle \cdot, \cdot \rangle$ and $f \geq \langle \cdot, \cdot \rangle$ that represents T (in that $M_{f^{*\dagger}} = T$) then T is maximal and $M_{f^{**}} = M_{f^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial f^{**}(x, x^*)\} = T$.*

All reflexive spaces are BMLS spaces [12]. It is an open question as to whether exist any non-reflexive BMLS spaces. One might ask whether it is possible to improve the last result to claim that T is maximal and $M_f = M_{f^{*\dagger}} = T$. A recent counter-example [8, Example 6.2] implies that no non-reflexive space can exhibit such a result. In [8, Example 6.2] they exhibit a representative f function with a conjugate $f^{*\dagger}$ which is also representative for which M_f is not maximal. We note that for this example $f \gneqq f^{*\dagger} = f^{**}$ and hence f does not have bigger conjugate. In particular $f = \delta_{\{0\} \times e_\perp}$ (where $e \in X^{**} \backslash X$ and $e_\perp := \{x^* \mid \langle x^*, e \rangle = 0\}$) and $f^{*\dagger} = \delta_{\{0\} \times X^*}$. Thus $f^{*\dagger} = \delta_{\{0\} \times X^*} = f^{**}$ represents a maximal skew-symmetric monotone operator, and does not provide a contradiction to the theorem on maximality via representative functions, as above. Any such a counter-example would be of interest as we have framed this paper in such as way as to provide equivalences wherever possible. Thus a counter-example to one of these results provides a counter-example to them all. Moreover, this would also implies some quite nonintuitive breakings of partial orderings of representative functions as compared with the graph-inclusions associated with the corresponding monotone operators (see Proposition 19).

1 Fixed-Points under Conjugation

We will occasionally require the existence of fixed-points under conjugation during this development. In a reflexive space Bauschke [3] has provided an explicit representation of an autoconjugate representative function of a given maximal monotone operator. In [10] it is shown that this formula fails to give such a representation for certain linear, skew-symmetric monotone operators. Thus outside of reflexive space no explicit formula for an auto-conjugate representative function is known. Despite this, such auto-conjugate representative functions do often exist. The following is a minor variation on the results of [13]. The proof is provided in an appendix for completeness.

Proposition 5 *Let X be a general Banach space. Let $f \in PC(X \times X^*)$ be a proper convex function such that $f^{*\dagger} \geq f$, and, if f is not closed, we assume there is $r \in \Gamma(X \times X^*)$ such that $f^{*\dagger} \geq r \geq r^{*\dagger} \geq f$. Then there exists a closed proper convex function $q \in \Gamma(X \times X^*)$ such that $q^* = q^\dagger$ and $f^{*\dagger} \geq q \geq f$.*

Proof. See Appendix A. ■

Remark 6 *As we begin with $\mathcal{F}_T \leq \mathcal{P}_T$, both proper closed and convex, there exists an autoconjugate $q \in [\mathcal{F}_T, \mathcal{P}_T]$.*

Note that q must be representative since by using the Fenchel inequality

$$q(x, x^*) = \frac{1}{2}(q(x, x^*) + q^*(x^*, x)) \geq \langle x, x^* \rangle.$$

Representability with convex functions with bigger conjugates is critical to the study of monotone operators, as was first demonstrated by Simons [15].

Proposition 7 *Let $f : X \times X^* \to \overline{\mathbb{R}}$ be a representative function for which $(f^*)^\dagger \geq f$ with $f^* \in PC(X^*, X)$. Assume also, if f is not closed, that there exists $r \in \Gamma(X \times X^*)$ for which $f^{*\dagger} \geq r \geq r^{*\dagger} \geq f$. Then there exists an auto–conjugate $q^* = q^\dagger$ with $f^{*\dagger} \geq q \geq f$ and so q represents M_f.*

Proof. As $f^* \geq f$ the existence of q is assured by Proposition 5. As $f^{*\dagger} \geq q \geq f$ it follows that

$$M_{f^{*\dagger}} \subseteq M_q \subseteq M_f = M_{f^{*\dagger}},$$

where the last equality follows from [15, Lemma 19.12]. Thus equality ensues. ■

We note that there now exist some explicit constructions of auto–conjugates and a survey of these may be found in [3] and [2] which are set mainly in reflexive spaces.

2 On Maximal Extensions of T

We know that when we do not have a maximal monotone operator $T : X \rightrightarrows X^*$ then $\mathcal{F}_T$ is not representative. Thus we are presented with the problem of finding a maximal extension. We begin this study by collecting some known results regarding partial-orders and inclusions.

Lemma 8 *If $f \in PC(X \times X^*)$ is representative, and either: (i) f is closed, or (ii) f satisfies $f \leq f^{*\dagger}$, then*

$$T \subseteq M_f \quad \Longrightarrow \quad \mathcal{F}_T \leq f \leq \mathcal{P}_T.$$

Proof. For any representative f, [9, Theorem 2.4] yields $T \subseteq M_f \subseteq M_{f^{*\dagger}}$, so that

$$f \leq c_T, \quad f^{*\dagger} \leq c_T \quad \text{and hence} \quad \mathcal{F}_T \leq f^{*\dagger}, \quad \mathcal{F}_T \leq f\,.$$

If f is closed, these yield $f = f^{**} \leq \mathcal{F}_T^{*\dagger} = \mathcal{P}_T$ giving $\mathcal{F}_T \leq f \leq \mathcal{P}_T$. If, instead, have $f \leq f^{*\dagger}$, then as $\mathcal{F}_T \leq f$, which implies $f^{*\dagger} \leq \mathcal{P}_T$ we obtain $\mathcal{F}_T \leq f \leq f^{*\dagger} \leq \mathcal{P}_T$. ■

The representativeness of $\mathcal{F}_T$ (i.e. the "almost" maximality of T, see [7, Proposition 2]) has been recognised as an important step towards establishing maximality. When T is not almost–maximal we will show that we need to search for minimal elements inside $[[\mathcal{F}_T, \mathcal{P}_T], \leq]$ to obtain a maximal extension of T.

Denote

$$M_f^{\leq} := \{(x, x^*) \in X \times X^* \mid f(x, x^*) \leq \langle x, x^* \rangle\}$$

and note that when $f \geq \langle \cdot, \cdot \rangle$ we have $M_f^{\leq} = M_f$.

Lemma 9 ([15, Lemma 19.12]) *Suppose $f : Z \times Z^* \to \overline{\mathbb{R}}$ is a proper convex function.*

1. *If $f^{*\dagger} \geq f$ then $M_{f^{*\dagger}}^{\leq} \subseteq M_f^{\leq}$. If $(x, x^*) \in M_{f^{*\dagger}}^{\leq}$ then $(x, x^*) \in \partial f^*(x^*, x) = (\partial f)^{-1}(x^*, x)$. Conversely when $(x^*, x) \in \partial f(x, x^*) = (\partial f^*)^{-1}(x, x^*)$ then $(x, x^*) \in M_f^{\leq}$. If, in addition, $f \geq \langle \cdot, \cdot \rangle$ then we have*

$$\begin{aligned} M_f = M_{f^{*\dagger}} &= \{(x, x^*) \mid (x^*, x) \in \partial f(x, x^*)\} \\ &= \{(x, x^*) \mid (x, x^*) \in \partial f^*(x^*, x)\}\,. \end{aligned} \tag{3}$$

2. *If $f \geq \langle \cdot, \cdot \rangle$ then M_f is a monotone set.*

3. *When $T : X \rightrightarrows X^*$ is monotone we have $\mathcal{F}_T(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in T$.*

Proof. See the appendix for proof. ■

If T is not maximal, $\mathcal{F}_T$ is not representative, while $\mathcal{P}_T$ does represent $M_{\mathcal{P}_T}$. When $\mathcal{F}_T$ is not representative we study

$$T^{\mu} := M_{\mathcal{F}_T}^{\leq} = \{(x, x^*) \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall (y, y^*) \in T\}\,.$$

We now recall some results from [11]. One always has $T^{\mu\mu\mu} = T^{\mu}$ and if T is monotone then $T \subseteq T^{\mu}$. In [11] T^{μ} is shown to be a polarity and as a consequence $A \subseteq B$ implies $A^{\mu} \supseteq B^{\mu}$, $T \subseteq T^{\mu\mu}$ and $(A \cup B)^{\mu} = A^{\mu} \cap B^{\mu}$. We shall call $T^{\mu\mu}$ the monotonic closure of T.

Proposition 10 ([11, Proposition 22]) *The following are equivalent:*

1. $T : X \rightrightarrows X^*$ *is monotone,*
2. $T \subseteq T^\mu$,
3. $T^{\mu\mu} \subseteq T^\mu$,
4. $T^{\mu\mu}$ *is monotone.*

Moreover, we have T maximal monotone iff $T = T^\mu$ (or T monotone and $T \supseteq T^\mu$).

Proposition 11 ([11, Proposition 25]) *Let $T : X \rightrightarrows X^*$ be a monotone operator. Then $\mathcal{F}_{T^\mu}$ is representative and*

$$M_{\mathcal{F}_{T^\mu}} = M^{\leqq}_{\mathcal{F}_{T^\mu}} = T^{\mu\mu}.$$

We will; extend the notion of "almost maximal" (see [7, Proposition 2]) to the context of an extension of a general monotone operator.

Remark 12 *We will work with the subset of $[\mathcal{F}_T, \mathcal{P}_T]$ consisting of representative functions with bigger conjugates. These functions are potentially not closed but the desire to find minimal elements has the effect of seeking a closed function (indeed a Fitzpatrick function). Let f be representative and suppose that its closure g is also representative. Then clearly $g \leq f$ (pointwise) and so in the case when $g \neq f$ we see that f could not have been minimal. Thus minimal functions should be expected to be closed with respect to all topologies (or closure operations with respect to a convergence) that are compatible with the duality pairing, such as the strong topology or the closure with respect to bounded $s \times w^*$-convergent nets. Furthermore a minimal element with respect to the set of proper convex, representative functions will also be minimal with respect to smaller subclasses to which it also belongs (i.e. closed, proper convex representative functions). Our main tool for obtaining the existence of minimal elements is Zorn's Lemma. Thus the proofs are not constructive. The use of proper convex functions (rather than closed ones) aids the use of this technique. The analysis would be greatly simplified if one could claim the existence of $s \times w^*$-closed minimal elements but this cannot be claimed a priori in a non-reflexive space.*

Lemma 13 *Let $T \subseteq X \times X^*$.*

A. *Denote*

$$b(T) := \left\{ f \in PC(X \times X^*) \mid f^{*\dagger} \geq f \text{ and } f^*(x^*, x) \leq \langle x, x^* \rangle \text{ for all } (x, x^*) \in T \right\}.$$

For all $f \in b(T)$ we have $M^{\leqq}_{f^{*\dagger}}$ a monotone set. Thus $b(T) \neq \emptyset$ iff T is a monotone set, in which case $\mathcal{F}_T \in b(T)$, and $\mathcal{F}_T \leq f$ for any $f \in b(T)$. Suppose f is minimal in $[b(T), \leq]$.

1. *Then $f = \mathcal{F}_{M^{\leqq}_{(f^*)^\dagger}}$ and $M^{\leqq}_f = \left(M^{\leqq}_{(f^*)^\dagger}\right)^\mu$. Thus $\mathcal{F}_{M^{\leqq}_f} = \mathcal{F}_{\left(M^{\leqq}_{(f^*)^\dagger}\right)^\mu}$ is representative.*
2. *Additionally, $M^{\leqq}_f$ is monotone iff $\left(M^{\leqq}_{(f^*)^\dagger}\right)^{\mu\mu} = \left(M^{\leqq}_f\right)^{\mu\mu}$ (or equivalently $\left(M^{\leqq}_{(f^*)^\dagger}\right)^\mu = \left(M^{\leqq}_f\right)^\mu$).*
3. *If T is monotone then $f = \mathcal{F}_T$ is the unique minimal element of $[b(T), \leq]$.*

B. *Suppose T is monotone. Denote*

$$bR(T) := \left\{ f \in PC(X \times X^*) \mid (f^*)^\dagger \geq f \geq \langle \cdot, \cdot \rangle,\ \mathcal{F}_T \leq f \leq \mathcal{P}_T \right\}.$$

1. *Then $bR(T) \subseteq b(T)$, and for all $f \in bR(T)$ we have $f \geq \mathcal{F}_{M_f}$.*
2. *Let f be minimal in $[bR(T), \leq]$. If $\mathcal{F}_{M_{(f^*)^\dagger}} (= \mathcal{F}_{M_f})$ is representative, then $f = \mathcal{F}_{M_{(f^*)^\dagger}} = \mathcal{F}_{M_f}$ and $M_f = (M_f)^\mu$ is maximal monotone. Moreover it follows that $\mathcal{F}_{(M_f)^\mu}$ has a bigger conjugate.*

Proof. Part A: Let $f \in b(T)$. Observing that $f^*(x^*, x) \leq \langle x, x^* \rangle$ for all $(x, x^*) \in T$ we have $M^{\leq}_{(f^*)^\dagger} \supseteq T$. By Lemma 9, $M^{\leq}_{f} \supseteq M^{\leq}_{(f^*)^\dagger} (\supseteq T$ by the definition of $b(T))$. Then

$$f \leq f^{*\dagger} \leq \langle \cdot, \cdot \rangle \quad \text{on} \quad M^{\leq}_{(f^*)^\dagger} \supseteq T,$$

so

$$f \leq f^{*\dagger} \leq \langle \cdot, \cdot \rangle + \delta_{M^{\leq}_{(f^*)^\dagger}} \leq \langle \cdot, \cdot \rangle + \delta_T,$$

$$\text{implying} \quad f^{*\dagger} \geq f \geq \left(\langle \cdot, \cdot \rangle + \delta_{M^{\leq}_{(f^*)^\dagger}} \right)^{*\dagger} = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \geq \mathcal{F}_T. \tag{4}$$

Note that (4) gives $M^{\leq}_{(f^*)^\dagger} \subseteq M^{\leq}_{f} \subseteq M^{\leq}_{\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}} = \left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu} \subseteq M^{\leq}_{\mathcal{F}_T} = T^{\mu}$ and $M^{\leq}_{(f^*)^\dagger} \subseteq \left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu}$. The latter containment and Proposition 10 imply $M^{\leq}_{(f^*)^\dagger}$ is monotone and hence T is monotone. Thus $b(T) \neq \emptyset$ then T is a monotone set. When T is monotone then $\mathcal{F}_T \in b(T) \neq \emptyset$, and $\mathcal{F}_T \leq f$, as seen from (4).

Thus we have $\left(\mathcal{F}^*_{M^{\leq}_{(f^*)^\dagger}} \right)^{\dagger} = \mathcal{P}_{M^{\leq}_{(f^*)^\dagger}} = \langle \cdot, \cdot \rangle$ on $M^{\leq}_{(f^*)^\dagger} \supseteq T$, as $\mathcal{P}_{M^{\leq}_{(f^*)^\dagger}}$ is representative for the monotone set $M^{\leq}_{(f^*)^\dagger}$, see [6, Proposition 8.1.7]. As we have

$$\mathcal{F}_T \leq \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \leq f \leq f^{*\dagger} \leq \left(\mathcal{F}^*_{M^{\leq}_{(f^*)^\dagger}} \right)^{\dagger} = \mathcal{P}_{M^{\leq}_{(f^*)^\dagger}} \leq \mathcal{P}_T,$$

it follows that $\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \in b(T)$. When f is minimal then $f = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$. Consequently

$$M^{\leq}_{f} = M^{\leq}_{\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}} = \left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu} \tag{5}$$

establishing 1.

For 2 we note that if $\left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu\mu} = \left(M^{\leq}_{f} \right)^{\mu\mu}$ it follows from (5) that

$$\left(M^{\leq}_{f} \right)^{\mu} = \left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu\mu} = \left(M^{\leq}_{f} \right)^{\mu\mu}$$

and Proposition 10 implies $M^{\leq}_{f}$ is a monotone set. On the other hand, when $M^{\leq}_{f}$ is monotone Proposition 10 implies $M^{\leq}_{f} \subseteq \left(M^{\leq}_{f} \right)^{\mu}$. Using (5) gives

$$\left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu} \subseteq \left(M^{\leq}_{f} \right)^{\mu}, \quad \text{and hence} \quad \left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu\mu} \supseteq \left(M^{\leq}_{f} \right)^{\mu\mu}.$$

For the reverse inclusion, from the first part of Lemma 9 gives $M^{\leq}_{(f^*)^\dagger} \subseteq M^{\leq}_{f}$, so $\left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu\mu} \subseteq \left(M^{\leq}_{f} \right)^{\mu\mu}$ by polarity. Thus equality ensues. Clearly, the relation $\left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu\mu} = \left(M^{\leq}_{f} \right)^{\mu\mu}$ is equivalent to $\left(M^{\leq}_{(f^*)^\dagger} \right)^{\mu} = \left(M^{\leq}_{f} \right)^{\mu}$ (as $A^{\mu\mu\mu} = A^{\mu}$).

For the last part assume T a monotone operator. Then by Lemma 9 (part 3) we have $(\mathcal{F}^*_T)^\dagger = \mathcal{P}_T = \langle \cdot, \cdot \rangle$ on T and so $\mathcal{F}_T \in b(T)$. Let $f \in b(T)$ be minimal. Then as $f \geq \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \geq \mathcal{F}_T$ we have $f = \mathcal{F}_T$.

Part B: By Lemma 9 we have $M_f = M_{f^{*\dagger}} = M^{\leq}_{(f^*)^\dagger}$, a monotone set, hence for all $f \in bR(T)$ we have $f \in b\left(M^{\leq}_{(f^*)^\dagger} \right) = b(M_f)$. Thus $f \geq \mathcal{F}_{M_f}$ (by part 3 applied to the monotone set M_f). Furthermore $f^{*\dagger} \leq \mathcal{P}_{M_f} = \langle \cdot, \cdot \rangle$ on $M_f \supseteq T$, giving $bR(T) \subseteq b(T)$.

From the inequalities (4) we have $f \geq \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$ and so when $\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$ is representative and f minimal we have $f = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \in bR(T)$. As $f = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$ implies $M^{\leq}_f = \left(M^{\leq}_{(f^*)^\dagger}\right)^\mu = (M_f)^\mu$. Proposition 10 gives maximality of M_f. ■

As noted earlier, if $T \subseteq A$ are monotone sets then $\mathcal{F}_T \leq \mathcal{F}_A$. To date the behaviour of an arbitrary representative function for A in relation to $\mathcal{F}_T$ has not been addressed.

Corollary 14 *Suppose $T \subseteq X \times X^*$ is monotone. Then $\mathcal{F}_T \in b(T)$, and $f \geq \mathcal{F}_T$ for all $f \in b(T)$. Thus $\mathcal{F}_T$ is the unique minimal element of $[b(T), \leq]$.*

Remark 15 *If $g \in bR(T)$ does not represent a maximal monotone set M_g, then for an arbitrary $h \in b(T)$ for which $M^{\leq}_{h^{*\dagger}} \supseteq M_g$ we have $h \in b(M_g)$. Applying Corollary 14 (on the monotone set M_g) we deduce*

$$\mathcal{F}_{M_g} \leq h, \tag{6}$$

as in Lemma 8.

When dealing with pre-maximal monotone operators we have:

Proposition 16 ([11], Prop. 22, and 36.) *Let $T : X \rightrightarrows X^*$ be monotone. Denote by $\mathcal{M}(T)$ the set of all maximal monotone extensions of T. Then*

1. *$T^\mu = \bigcup_{B \in \mathcal{M}(T)} B$ and $T^{\mu\mu} = \bigcap_{B \in \mathcal{M}(T)} B$.*
2. *The following are equivalent:*
   - *(a) T has a unique maximal monotone extension (i.e. T is pre-maximal),*
   - *(b) $T^\mu = T^{\mu\mu}$,*
   - *(c) T^μ is monotone,*
   - *(d) T^μ is maximal monotone,*
   - *(e) $T^{\mu\mu}$ is maximal monotone.*

   Moreover, it follows that T^μ (or equivalently $T^{\mu\mu}$) is the unique maximal extension of T.

The following is well known.

Proposition 17 *Let M be a maximal monotone extension of T. Then $\mathcal{F}_M$ is a minimal element of $[bR(T), \leq]$. Hence also, $\mathcal{F}_M$ is the unique minimal element of $bR(M)$.*

Proof. Clearly $\mathcal{F}_M$ is in $bR(M)$. For the minimality, suppose $g \leq \mathcal{F}_M$ where $g \in bR(T)$. Then M_g is a monotone extension of $M_{\mathcal{F}_M} = M$, so $M_g = M$ by maximality. From part B of Lemma 13 follows that $g \geq \mathcal{F}_{M_g} = \mathcal{F}_M$ and so $g = \mathcal{F}_M$ and $\mathcal{F}_M$ is thus minimal. ■

The following is trivial but surprising.

Lemma 18 *Let $T : X \rightrightarrows X^*$ be a monotone operator and let $k, h \in bR(T)$ and suppose we have $h \leq k$ then $M_k = M_h \supseteq T$.*

Proof. Note that as $h \leq k$ and $h, k \in bR(T)$ we have

$$h \leq k \leq k^{*\dagger} \leq h^{*\dagger}$$

and so

$$M_h \supseteq M_k \supseteq M_{k^{*\dagger}} \supseteq M_{h^{*\dagger}} = M_h$$

where the last equality follows from Lemma 9, giving $M_h = M_k$. ■

The issue as to whether $\mathcal{F}_{M_f} = f$ for any minimal element of $bR(T)$ is clearly of interest as it implies the identity $(M_f)^\mu = M_f$ and hence maximality of M_f. By $\operatorname{co} f$ we denote the convex function whose epigraph corresponds to $\operatorname{co} \operatorname{epi} f$ and by $\operatorname{co}\{f, g\}$ we denote the convex function $\operatorname{co}\{\min\{f, g\}\}$. Clearly $\operatorname{epi} \operatorname{co}\{\min\{f, g\}\} = \operatorname{co}\{\operatorname{epi} f \cup \operatorname{epi} g\}$.

Proposition 19 *Suppose f is minimal in $bR(T)$. Then following are all equivalent to the condition $f = \mathcal{F}_{M_f}$.*

1. *M_f is maximal monotone.*
2. *$\mathcal{F}_{M_f} \in bR(T)$.*
3. *For any maximal monotone extension $M \supseteq M_f$, the function $\max\{\operatorname{co}\{f, \mathcal{F}_M\}, \langle\cdot,\cdot\rangle\}$ is convex.*
4. *For any maximal monotone extension $M \supseteq M_f$, have that $\mathcal{F}_{M_f} \le \mathcal{F}_M \le f$.*
5. *For all $k \in bR(T)$ (with $k \ge f$) and any maximal monotone extension $M \supseteq M_k$, have $\mathcal{F}_{M_k} \le \mathcal{F}_M \le k$.*
6. *For any $g \in R(T)$ with $g \ge f$ (and hence $M_g \subseteq M_f$) the function $\operatorname{co}\{\mathcal{F}_{M_f}, g\}$ is representative.*

Proof. If M_f is maximal, then $\mathcal{F}_{M_f} \in bR(T)$. If $\mathcal{F}_{M_f} \in bR(T)$ then $\mathcal{F}_{M_f} \le f$, and hence $f = \mathcal{F}_{M_f}$ by minimality of f. If $f = \mathcal{F}_{M_f}$ then $M_f = (M_f)^{\mu}$. This establishes that parts 1 and 2 are equivalent to the condition $f = \mathcal{F}_{M_f}$.

(2 implies 3): We consider $\max\{\operatorname{co}\{f, \mathcal{F}_{M_f}\}, \langle\cdot,\cdot\rangle\} = \{g, \langle\cdot,\cdot\rangle\}$ where $g := \operatorname{co}\{f, \mathcal{F}_{M_f}\}$. From the equivalences already verified above, M_f is maximal and $f = \mathcal{F}_{M_f}$, so $g = \operatorname{co}\{f, \mathcal{F}_{M_f}\} = \mathcal{F}_{M_f} \in bR(T)$. Thus we have $\max\{\operatorname{co}\{f, \mathcal{F}_{M_f}\}, \langle\cdot,\cdot\rangle\} = g$ because $g \ge \langle\cdot,\cdot\rangle$. Hence $\max\{\operatorname{co}\{f, \mathcal{F}_{M_f}\}, \langle\cdot,\cdot\rangle\}$ is convex. Now suppose $M \supseteq M_f$ is any maximal extension but as part 1 is equivalent to 2 we have $M = M_f$ as M_f is maximal and hence $\max\{\operatorname{co}\{f, \mathcal{F}_M\}, \langle\cdot,\cdot\rangle\} = \max\{\operatorname{co}\{f, \mathcal{F}_{M_f}\}, \langle\cdot,\cdot\rangle\} = g$ is convex, yielding part 3.

(3 implies 4): Let $M \supseteq M_f$ be a maximal extension, and that $g := \max\{\operatorname{co}\{f, \mathcal{F}_M\}, \langle\cdot,\cdot\rangle\}$ is convex. Now, $g \ge \langle\cdot,\cdot\rangle$, and $g \le f$, $g \le \mathcal{F}_M$, so that $g \le f \le f^{*\dagger} \le g^{*\dagger}$. Thus $g \in bR(M_g)$, so $M_g = M_{g^{*\dagger}}$ by Lemma 9. Further, $g \le \mathcal{F}_M$ implies $M_g \supseteq M_{\mathcal{F}_M} = M$, so by maximality of the latter, follows $M_g = M$. Then, as $g^{*\dagger} = \langle\cdot,\cdot\rangle$ on $M_{g^{*\dagger}} = M$, we have that $g^{*\dagger} \le \langle\cdot,\cdot\rangle + \delta_M$, so $g \ge \mathcal{F}_M$. This yields that $f \ge g \ge \mathcal{F}_M \ge \mathcal{F}_{M_f}$, as required.

(4 implies 5): Let $k \ge f$ and $M \supseteq M_k$ any maximal extension. Apply Lemma 18 to deduce that $M_k = M_f$ and hence M extends M_f. Applying the inequalities from part (4) we have $\mathcal{F}_{M_k} = \mathcal{F}_{M_f} \le \mathcal{F}_M \le f \le k$, establishing part 5.

(5 implies 1): Assume part 5 with $k = f$, so when $M \supseteq M_f$ is any maximal extension, have $\mathcal{F}_{M_f} \le \mathcal{F}_M \le f$. As f is minimal in $bR(T)$ (and $\mathcal{F}_M \in bR(T)$ since M is maximal) it follows that $f = \mathcal{F}_M$ and so $M_f = M$, maximal monotone, and we have part 1. Thus (1) to (5) are equivalent.

(5 implies 6): As $M_g \subseteq M_f$ then since part (5) implies part (1), M_f is maximal so M_f is a maximal extension of M_g. Applying part (5) we have $\mathcal{F}_{M_g} \le \mathcal{F}_{M_f} \le g$, implying that $\operatorname{co}\{\mathcal{F}_{M_f}, g\} = \mathcal{F}_{M_f}$, the latter being representative.

(6 implies 3): Taking $g = f$ in part (6), then $\operatorname{co}\{\mathcal{F}_{M_f}, g\} = \operatorname{co}\{\mathcal{F}_{M_f}, f\}$ is representative. Hence $\operatorname{co}\{\mathcal{F}_M, f\} \ge \langle\cdot,\cdot\rangle$, implying $\max\{\operatorname{co}\{f, \mathcal{F}_M\}, \langle\cdot,\cdot\rangle\} = \operatorname{co}\{\mathcal{F}_{M_f}, f\}$, which is convex. ■

Remark 20 *It is clear from the proof of Proposition 19 that part 5 could be replaced by the proposition that: For all $k \ge f$ then* there exists *a maximal monotone extension $M \supseteq M_k$ with $\mathcal{F}_{M_k} \le \mathcal{F}_M \le k$. Clearly part 5 implies this. Conversely we may apply this for $k = f$ in the same way we have argued that part 5 implies part 1.*

Definition 21 (The space of all representative functions for the monotone set T)

$$R(T) := \{f \in PC(X \times X^*) \mid f \ge \langle\cdot,\cdot\rangle, T \subseteq M_f\}.$$

Minimal elements in $[R(T), \le]$ always exist.

The following was first observed in [12, Theorem 5].

Lemma 22 *Let $g \in R(T)$ be minimal in $[R(T), \le]$, with T monotone. Then $g \in bR(T)$, $g \ge \mathcal{F}_{M_g}$, and g is minimal in $[bR(T), \le]$.*

Proof. By a simple adaptation of the proof of [12, Theorem 5], follows that $g \leq g^{*\dagger}$, and as $g \geq \langle \cdot, \cdot \rangle$, Lemma 9 gives $M_g = M_{g^{*\dagger}}$, and since then $g^{*\dagger} = \langle \cdot, \cdot \rangle$ on M_g, we have that $g^{*\dagger} \leq \langle \cdot, \cdot \rangle + \delta_{M_g}$, so $g \geq g^{**} \geq \mathcal{F}_{M_g}$ ($\geq \mathcal{F}_T$ since $M_g \supseteq T$). Thus $g \in bR(T)$, and since $bR(T) \subseteq R(T)$, it follows that g is also minimal for $[bR(T), \leq]$. ■

Since we do not demand closedness from the outset, the existence of minimal elements can be deduced via Zorn's Lemma.

Lemma 23 *For T monotone, and $f \in R(T)$, let*

$$L_f(T) := \{g \in bR(T) \mid g \leq f\}. \tag{7}$$

Then $L_f(T)$ is nonempty, there exists a minimal element of $[L_f(T), \leq]$, and this element is also minimal for $[bR(T), \leq]$.

Proof. For the nonemptiness, consider the set $R_f(T) := \{g \in R(T) \mid g \leq f\}$. Since $f \in R_f(T)$, the latter is nonempty. Any *totally-ordered* subset S of $[R_f(T), \leq]$ has a lower bound in $R_f(T)$, given by the pointwise infimum of the members of S. To see this, if $h := \inf_{g \in S} g$, then h is convex, since its epigraph is the union of a nested family of convex sets (nested in the sense that for any pair of functions from S, one epigraph must contain the other) and therefore itself convex. Also, since $\langle \cdot, \cdot \rangle \leq g \leq f$ and $T \subseteq M_g$ for all $g \in S \subseteq R_f(T)$ it follows that $\langle \cdot, \cdot \rangle \leq h \leq f$ and $T \subseteq M_g \subseteq M_h$, so $h \in R_f(T)$ as claimed. Zorn's Lemma then yields a minimal element $k \in R_f(T)$. Then by Lemma 22, $k \in bR(T)$, hence $k \in L_f(T)$, and the latter is nonempty.

For the remaining assertions we repeat this argument, but now on $L_f(T)$. Indeed, if S totally ordered in $[L_f(T), \leq]$, again defining $h := \inf_{g \in S} g$, yields $\langle \cdot, \cdot \rangle \leq h \leq f$ and $h \leq g \leq g^{*\dagger} \leq h^{*\dagger}$, giving $h \in bR(T)$ and so $h \in L_f(T)$. Thus every chain in $[L_f(T), \leq]$ has a lower bound in $L_f(T)$ and so by Zorn's Lemma there exists a minimal element we denote (again) by $h \in L_f(T)$. Now suppose $g \in bR(T)$ and $g \leq h$. From transitivity of the partial order follows $g \leq h \leq f$ and hence $g \in L_f(T)$. By minimality of h in $[L_f(T), \leq]$ we have $g = h$ and hence h is minimal in $[bR(T), \leq]$. ■

Any representative function f for which there is a criterion for maximality of M_f (for example [1]) will yield $\mathcal{F}_{M_f}$ as the minimal element of $L_f(T)$. Consequently by Lemma 18 all $g \geq \mathcal{F}_{M_f}$ will represent the same maximal set M_f. Maximality now becomes endemic on spaces where any of the equivalent statements in Proposition 19 hold (for any monotone T) for any minimal element of $bR(T)$. Let us call these BMLS spaces i.e. BMLS spaces possess the property that every minimal element in $bR(T)$ is a Fitzpatrick function. From the work of Burachik, Martínez-Legaz and Svaiter [4, 12] we know that BMLS spaces include all reflexive spaces.

Theorem 24 *The space X is a* BMLS *space if and only if (for every monotone T) all $f \in bR(T)$ represent maximal monotone operators.*

Proof. By Lemma 23, for any $g \in bR(T)$ we have $g \geq h$ for some minimal h in $[bR(T), \leq]$. When X is an BMLS space then $h = \mathcal{F}_{M_h}$ for a maximal M_h. By Lemma 18 it follows that $M_g = M_h$ and as M_h is maximal so is M_g. Conversely, suppose $bR(T)$ only represent maximal monotone operators. Then Proposition 19 part 1 holds and X is then a BMLS space. ■

3 Detecting Maximal Extensions in a BMLS space

We now develop practical criteria for detecting the maximality of a monotone operator by identifying it with its maximal extension. This approach allows us to totally avoid the use of duality theorems of the Fenchel type but instead use the simple fact that conjugation is order-reversing. Bringing this together we have the following.

Theorem 25 *Let $T : X \rightrightarrows X^*$ be a monotone operator.*

1. *For every maximal extension $\widetilde{T}$ of T there exists a representative function p with $\mathcal{F}_T \leq p \leq p^{*\dagger} \leq \mathcal{P}_T$ such that $p = \mathcal{F}_{\widetilde{T}}$, and there does not exist any other representative function $g \in PC(X \times X^*)$*

with $g \neq p$ and $g \leq p$ (i.e. p is a minimal element of $[bR(T), \leq]$). Moreover,

$$M_p = \{(x, x^*) \mid (x^*, x) \in \partial p(x, x^*)\}. \tag{8}$$

2. *If $q \in bR(T)$ is such that $q = q^{*\dagger} \in PC(X \times X^*)$ (i.e. is auto-conjugate), then, whenever X is a* BMLS *space, M_q is a maximal monotone extension of T. Conversely (for general X) an auto-conjugate representative function exists for every maximal monotone extension $\widetilde{T}$ of T.*

3. *Suppose X is a* BMLS *space. If $f \in R(T)$ and both f, $f^{*\dagger}$ are representative, then $M_{f^{*\dagger}} = M_{f^{**}}$ is maximal monotone.*

Proof. To begin, we note the following: applying Lemma 8 we know that as $T \subseteq \widetilde{T}$ we have $\mathcal{F}_T \leq \mathcal{F}_{\widetilde{T}} \leq \mathcal{P}_{\widetilde{T}} \leq \mathcal{P}_T$, so $\mathcal{P}_{\widetilde{T}} \in \left\{r \in \Gamma(X \times X^*) \mid \mathcal{P}_T \geq r \geq r^{*\dagger} \geq \mathcal{F}_T\right\} \neq \emptyset$, and Proposition 5 yields q with $\mathcal{F}_T \leq q = q^{*\dagger} \leq \mathcal{P}_T$, so $q \geq \langle \cdot, \cdot \rangle$ by Fenchel inequality.

(Part 1): By Proposition 17, $p := \mathcal{F}_{\widetilde{T}}$ is minimal in $bR(T)$. Lemma 9 then yields (8).

(Part 2): First we show that (for any Banach space) such auto–conjugate representative functions always exist. Observe that if $f \in bR(T)$ is a representative of a maximal extension $\widetilde{T}$ of T then $f \geq \mathcal{F}_{M_f} = \mathcal{F}_{\widetilde{T}}$ and we have

$$\mathcal{F}_T \leq \mathcal{F}_{\widetilde{T}} \leq f \leq f^{*\dagger} \leq \mathcal{F}_{\widetilde{T}}^{*\dagger} = \mathcal{P}_{\widetilde{T}} \leq \mathcal{P}_T.$$

Applying Proposition 5 to the pair $\mathcal{F}_{\widetilde{T}}$ and $\mathcal{P}_{\widetilde{T}}$ there exists $\mathcal{F}_{\widetilde{T}} \leq q \leq \mathcal{P}_{\widetilde{T}}$ such that $q = q^{*\dagger}$. From Lemma 18 we get $M_q = \widetilde{T}$. Thus the autoconjugate $q \in bR(T)$ represents $\widetilde{T}$.

Now suppose instead that X is BMLS, and that $\mathcal{F}_T \leq q \leq \mathcal{P}_T$ with $q = q^{*\dagger}$. Then $q \in bR(T)$, and by Theorem 24, M_q is maximal, and hence a maximal extension of T.

(Part 3): By [12, Theorem 5 and 6] one may deduce that for any $f \in R(T)$ there always exists a minimal element $h \leq \min\left\{f, f^{*\dagger}\right\}$ in $R(T)$ such that $h \leq h^{*\dagger}$ and $T \subseteq M_f \subseteq M_h$. Thus, by Lemma 22, $h \in bR(T)$ and is also minimal in $bR(T)$. The BMLS property now implies maximality of M_h, so that $\mathcal{F}_{M_h} \in bR(T)$, and then $h = \mathcal{F}_{M_h}$ by minimality of h. We see now that h is closed, so that $h = h^{**} \leq f^{**} \leq f$, implying that $f^{**} \in R(T)$ also. As $h \leq f^{*\dagger}$ we have $f^{**} \leq h^{*\dagger}$ and so $M_h = M_{h^{*\dagger}} \subseteq M_{f^{**}}$, implying $M_{f^{**}}$ is maximal and $M_h = M_{f^{**}}$. Similarly $h \leq f$ implies $f^{*\dagger} \leq h^{*\dagger}$ and so $M_h = M_{h^{*\dagger}} \subseteq M_{f^{*\dagger}}$. The maximality of M_h gives $M_h = M_{f^{*\dagger}}$. ∎

In [15] Lemma 21.9 it is shown that in reflexive, real Banach spaces, an autoconjugate function (i.e. $q = q^{*\dagger}$) with $\mathcal{P}_T \geq q \geq \mathcal{F}_T$, represents a maximal monotone set which is an extension of T. This phenomenon was further studied by Bauschke in [3], once again in reflexive spaces. Bauschke showed that in reflexive Banach space any autoconjugate representative function represents a maximal monotone operator. We will extend [15, Lemma 21.9] and [3, Theorem 5.7] first to the case of a BMLS space.

Theorem 26 (Maximality via Auto-conjugates) *Let $T : X \rightrightarrows X^*$ be a monotone operator. Suppose X is* BLMS*. If there exists $q \in \Gamma(X \times X^*)$ with $q^{*\dagger} = q$ that represents T (i.e. $M_q = T$), then T is maximal and $M_q = M_{q^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial q(x, x^*)\} = T$.*

Proof. Here $M_q = T$ is its own maximal extension and hence T is maximal. ∎

The next result follows directly from Theorem 24 but it is interesting that they follow directly from Theorem 26.

Theorem 27 (Maximality via bigger conjugates) *Let $T : X \rightrightarrows X^*$ be a monotone operator, with X a* BMLS *space. If there exists f, $f^{*\dagger} \in PC(X \times X^*)$ with $f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle$ that represents T (i.e. $M_f = T$) then T is maximal and $M_f = M_{f^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial f(x, x^*)\} = T$.*

Proof. By Proposition 7 there exists an auto–conjugate q with $f^{*\dagger} \geq q \geq f$ and $M_q = M_f = T$ by Lemma 18. Hence T is maximal. ∎

Theorem 28 (Maximality via representative functions) *Let $T : X \rightrightarrows X^*$ be a monotone operator, with X a* BMLS *space. If there exists f, $f^{*\dagger} \in PC(X \times X^*)$ with $f^{*\dagger} \geq \langle \cdot, \cdot \rangle$ and $f \geq \langle \cdot, \cdot \rangle$ that represents T (in that $M_{f^{*\dagger}} = T$) then T is maximal and $M_{f^{**}} = M_{f^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial f^{**}(x, x^*)\} = T$.*

Proof. By Theorem 25 part 3 when $M_{f^{*\dagger}} = T$ then T is maximal. ■

4 On Pre-Maximality

Pre-maximality may also be studied via representative functions and minimality. We embark on this next. Maximality of M_f is related to the minimality of $\mathcal{F}_{M_f}$ in $bR(T)$, while pre-maximality is related to the minimality of $\mathcal{F}_{(M_f)^\mu}$.

Let $R(T)$ denote the space of representative functions of the monotone operator T, that is, $R(T) := \{f \in PC(X \times X^*) \mid f \geq \langle\cdot,\cdot\rangle,\ T \subseteq M_f\}$. Minimal elements of $[R(T), \leq]$ always exist. It would be of interest to know if a minimal element of the subset of $s \times w^*$-closed representative functions exists. We will return to this issue later.

Lemma 29 *Let $f \in R(T)$. Then*

$$\mathcal{F}_{M_f} \leq \mathcal{F}_{(M_f)^{\mu\mu}} \leq \mathcal{F}_{(M_f)^\mu} \leq \mathcal{P}_{(M_f)^{\mu\mu}} \leq \mathcal{P}_{M_f}. \tag{9}$$

Proof. The first two inequalities are a consequence of $M_f \subseteq (M_f)^{\mu\mu} \subseteq (M_f)^\mu$. Now note that for all $(x_i, x_i^*) \in (M_f)^{\mu\mu}$ and $\sum_i \lambda_i = 1$ with $\lambda_i \geq 0$ we have

$$\langle x_i, x_i^* \rangle = \mathcal{F}_{(M_f)^\mu}(x_i, x_i^*)$$

since $\mathcal{F}_{(M_f)^\mu}$ is a representative function of $(M_f)^{\mu\mu}$. Thus by the convexity of $\mathcal{F}_{(M_f)^\mu}$,

$$\sum_i \lambda_i \langle x_i, x_i^* \rangle = \sum_i \lambda_i \mathcal{F}_{(M_f)^\mu}(x_i, x_i^*) \geq \mathcal{F}_{(M_f)^\mu}\left(\sum_i \lambda_i (x_i, x_i^*)\right) = \mathcal{F}_{(M_f)^\mu}(x, x^*)$$

for any $\sum_i \lambda_i (x_i, x_i^*) = (x, x^*)$. Hence, for all $(x, x^*) \in \mathrm{co}\,(M_f)^{\mu\mu} = \mathrm{dom}(\mathrm{co}(\langle\cdot,\cdot\rangle + \delta_{(M_f)^{\mu\mu}}))$,

$$\inf\left\{\sum_i \lambda_i \langle x_i, x_i^* \rangle \mid (x_i, x_i^*) \in (M_f)^{\mu\mu},\ \sum_i \lambda_i (x_i, x_i^*) = (x, x^*),\ \sum_i \lambda_i = 1 \text{ and } \lambda_i \geq 0\right\} \geq \mathcal{F}_{(M_f)^\mu}(x, x^*)$$

and on taking a closure we have $\mathcal{P}_{(M_f)^{\mu\mu}} \geq \mathcal{F}_{(M_f)^\mu}$. Finally $\mathcal{P}_{(M_f)^{\mu\mu}} \leq \mathcal{P}_{M_f}$ because $M_f \subseteq (M_f)^{\mu\mu}$. ■

Lemma 30 *For $f \in R(T)$, we have $\mathcal{F}_{M_f} \leq f$, and there exists $g \in bR(T)$ minimal for $[R(T), \leq]$ for which $\mathcal{F}_{M_f} \leq \mathcal{F}_{M_g} \leq g \leq f$.*

Proof. By the Zorn Lemma, there is a minimal $g \in [R(T), \leq]$ for which $g \leq f$. By Lemma 22, $g \in bR(T)$ and $g \geq \mathcal{F}_{M_g}$. Since $g \leq f$, we have $T \subseteq M_f \subseteq M_g$, and hence that $\mathcal{F}_{M_f} \leq \mathcal{F}_{M_g} \leq g \leq f$. ■

Lemma 31 *Let $f \in R(T)$, and let M any maximal monotone extension of M_f. Then $\mathcal{F}_M \in bR(T)$, and is minimal in $[R(T), \leq]$.*

Proof. Clearly $\mathcal{F}_M \in R(T)$, from the maximality. Suppose $h \in R(T)$ with $h \leq \mathcal{F}_M$. There is a minimal $g \in R(T)$ with $g \leq h$, and it satisfies $g \geq \mathcal{F}_{M_g}$ (by Lemma 22). Since $M = M_{\mathcal{F}_M} \subseteq M_h \subseteq M_g$ and hence $g \geq \mathcal{F}_{M_g} \geq \mathcal{F}_M \geq h \geq g$, it follows that $h = \mathcal{F}_M$, establishing the minimality of $\mathcal{F}_M$ in $[R(T), \leq]$. ■

Note that when M_f is pre-maximal, then $(M_f)^\mu$ is a maximal extension of M_f and hence by Lemma 31 have $\mathcal{F}_{(M_f)^\mu} \in bR(T)$ and minimal in $[R(T), \leq]$. Then $\mathcal{F}_{(M_f)^\mu} \geq \mathcal{F}_{M_f}$, where $\mathcal{F}_{M_f}$ and $\mathcal{F}_{(M_f)^\mu}$ represent the same maximal monotone set (in that $M_{\mathcal{F}_{(M_f)^\mu}} = (M_f)^{\mu\mu} = (M_f)^\mu = M^{\leq}_{\mathcal{F}_{M_f}}$) but we do not know that $\mathcal{F}_{M_f}$ is representative. Indeed in a BMLS space, representativity of $\mathcal{F}_{M_f}$ implies maximality of M_f (by Proposition 19).

We revisit pre-maximality and then link this to maximality (of operators) via minimality of representative functions.

Theorem 32 *For $f \in R(T)$, let*

$$bR_f(T) := \left\{ g \in R(T) \mid g \text{ minimal in } [R(T), \leq], \text{ with } g \geq \mathcal{F}_{M_f} \right\}.$$

1. *Denote by h (or h_f to indicate role of f) the function defined by*

$$h^* := \sup_{g \in bR_f(T)} g^* \quad \textit{and} \quad h = h^{**}. \tag{10}$$

Then,

(a) $h^{*\dagger} \geq h$,

(b)

$$f \geq h = \overline{\text{co}} \left(\inf_{g \in bR_f(T)} g^{**} \right) \geq \mathcal{F}_{M_f}, \quad \textit{and} \tag{11}$$

(c) *we have*

$$M_{h^{*\dagger}} \subseteq (M_f)^{\mu\mu} \subseteq (M_f)^{\mu} \subseteq M_h^{\leq}, \tag{12}$$

with $M_{h^{*\dagger}} = (M_f)^{\mu\mu}$ if X is a BMLS *space.*

2. *When $h = h_f$ in (10) is representative, then M_f is pre-maximal, with $h = \mathcal{F}_{(M_f)^\mu}$. Conversely, whenever M_f is pre-maximal we have $(M_f)^\mu = M_h^{\leq}$, and when X is a* BMLS *space, have $h = \mathcal{F}_{(M_f)^\mu}$ (and hence h is representative).*

Proof. Observe that $bR_f(T) \neq \emptyset$, for if M a maximal monotone extension of M_f, then $\mathcal{F}_M$ is minimal in $R(T)$ by Lemma 30, so $\mathcal{F}_M \in bR_f(T)$.

(1): Note that for any maximal extension $M \supseteq M_f$ we have $\mathcal{F}_M \in bR_f(T)$ by Lemma 31, giving

$$h^{*\dagger} = \sup_{g \in bR_f(T)} g^{*\dagger} \geq \sup \{\mathcal{P}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\}. \tag{13}$$

Observe that if X has the BMLS property, then there is an *equality* in (13) (for if $g \in bR_f(T)$, then $g \in bR(T)$ and is minimal in $bR(T)$. If X is BMLS, then M_g maximal, so $g = \mathcal{F}_{M_g}$ by Proposition 19, and since $\mathcal{F}_{M_g} \geq \mathcal{F}_{M_f}$ we get $M_f \subseteq M_g$ so $g^{*\dagger} \leq$ rhs of (13)) and so

$$\begin{aligned} M_{h^{*\dagger}} &= \bigcap \{M_{\mathcal{P}_M} \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &= \bigcap \{M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} = (M_f)^{\mu\mu}. \end{aligned}$$

Otherwise an inclusion ($\subseteq$) holds. Now consider

$$h := h^{**} = \left(\sup_{g \in bR_f(T)} g^* \right)^* = \left(\left(\text{co} \inf_{g \in bR_f(T)} g \right)^* \right)^* = \overline{\text{co}} \left(\inf_{g \in bR_f(T)} g^{**} \right). \tag{14}$$

We have

$$\begin{aligned} M_h^{\leq} &\supseteq \bigcup_{g \in bR_f(T)} M_g = \bigcup_{g \in bR_f(T)} M_{g^{*\dagger}} \supseteq \bigcup \{M_{\mathcal{P}_M} \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &= \bigcup \{M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} = (M_f)^\mu. \end{aligned} \tag{15}$$

As $(M_f)^{\mu\mu} \subseteq (M_f)^\mu$ for any monotone set M_f (recall $f \in R(T)$) this gives part (1c).

For the inequalities in (11), to see that $h \leq f$, note that by Lemma 30, there is $g \in bR_f(T)$ with $g \leq f$ so that $h \leq g^{**} \leq g \leq f$. Also, $h \geq \mathcal{F}_{M_f}$ since $g \geq \mathcal{F}_{M_f}$ (and hence $g^{**} \geq \mathcal{F}_{M_f}$) for each $g \in bR_f(T)$.

One may show $h^{*\dagger} \geq h$ by direct calculation; as $g^{*\dagger} \geq g$ for each $g \in bR_f(T)$ we have

$$h^{*\dagger} = \sup_{g \in bR_f(T)} g^{*\dagger} \geq \inf_{g \in bR_f(T)} g \geq \overline{\text{co}} \left(\inf_{g \in bR_f(T)} g^{**} \right) = h.$$

(2):($\Longrightarrow$) When h is representative we have by Lemma 9 that

$$M_h = M_{h^{*\dagger}} \subseteq (M_f)^{\mu\mu} \subseteq (M_f)^{\mu} \subseteq M_h^{\leq} = M_h,$$

forcing $M_h = (M_f)^{\mu\mu} = (M_f)^{\mu}$. By Proposition 16, M_f is pre-maximal, so $M_h = (M_f)^{\mu}$ is maximal monotone. By Lemma 31, $\mathcal{F}_{(M_f)^{\mu}} \in bR_f(T)$, so that $\mathcal{F}_{(M_f)^{\mu}} = (\mathcal{F}_{(M_f)^{\mu}})^{**} \geq h$. But as $\mathcal{F}_{(M_f)^{\mu}}$ is minimal in $[R(T), \leq]$ and h is representative (so that $h \in R(T)$) we have $h = \mathcal{F}_{(M_f)^{\mu}}$.

(2):($\Longleftarrow$) Assume M_f is pre-maximal. We now argue that $\mathcal{F}_{(M_f)^{\mu}} \leq h^{*\dagger}$ and so $\mathcal{P}_{(M_f)^{\mu}} \geq h$. Using (13) we have

$$\begin{aligned} h^{*\dagger} &:= \sup_{g \in bR_f(T)} g^{*\dagger} \geq \sup\{\mathcal{P}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &\geq \sup\{\mathcal{F}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &= \sup\{\mathcal{F}_M \mid M \supseteq (M_f)^{\mu\mu} \text{ and } M \text{ maximal monotone}\} \\ &= \mathcal{F}_{\cup\{M \text{ max mon. } \mid M \supseteq (M_f)^{\mu\mu}\}} = \mathcal{F}_{((M_f)^{\mu\mu})^{\mu}} = \mathcal{F}_{(M_f)^{\mu}}. \end{aligned}$$

Thus $\mathcal{P}_{(M_f)^{\mu}} \geq h \geq \mathcal{F}_{M_f}$ and as $(M_f)^{\mu}$ is maximal we have

$$(M_f)^{\mu} = (M_f)^{\mu\mu} \subseteq M_h^{\leq} \subseteq (M_f)^{\mu}, \tag{16}$$

giving $M_h^{\leq} = (M_f)^{\mu}$.

To show that h is representative, we assume now the BMLS condition. By Lemma 31, $\mathcal{F}_{(M_f)^{\mu}} \in bR(T)$ and is minimal in $[R(T), \leq]$, so that $\mathcal{F}_{(M_f)^{\mu}} \in bR_f(T)$ and hence $h \leq \mathcal{F}_{(M_f)^{\mu}}$ from (14). When BMLS holds, (13) consists of equalities, and gives

$$h^{*\dagger} = \sup\{\mathcal{P}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \leq \mathcal{P}_{\cap\{M \text{ max mon } \mid M \supseteq M_f\}} = \mathcal{P}_{(M_f)^{\mu\mu}} = \mathcal{P}_{(M_f)^{\mu}}$$

which yields that $h \geq \mathcal{F}_{(M_f)^{\mu}}$. Hence $h = \mathcal{F}_{(M_f)^{\mu}}$ is now representative. ■

Corollary 33 *Let X be a BMLS space. Then $f \in R(T)$ has M_f is pre-maximal if and only if:*

$$h_f = \mathcal{F}_{(M_f)^{\mu}} \text{ and is the unique minimal element of } bR(T) \text{ with } f \geq h \geq \mathcal{F}_{M_f}.$$

Proof. ($\Longrightarrow$) This follows from Theorem 32 part 2, which ensures uniqueness and Lemma 30, which ensures there exists an minimal element $g \in bR_f(T)$ with $f \geq g \geq \mathcal{F}_{M_f}$. Indeed by Theorem 32 part 2, $h = \mathcal{F}_{(M_f)^{\mu}}$ representative when M_f is pre-maximal. As $h = \mathcal{F}_{(M_f)^{\mu}} \in bR_f(T)$ then (11) implies $g \geq \mathcal{F}_{(M_f)^{\mu}}$ for all $g \in bR_f(T)$. But as g is minimal and $\mathcal{F}_{(M_f)^{\mu}} \in bR_f(T)$ we have $g = \mathcal{F}_{(M_f)^{\mu}}$ and uniqueness ensues.

($\Longleftarrow$) Immediate from part (2)($\Longrightarrow$) of the Theorem. ■

5 Appendix A

Proof of Proposition 5. Let $\mathcal{R}$ be the set of closed proper convex functions $r \in \Gamma(Z \times Z^*)$ for which $p \geq r \geq (r^*)^{\dagger} \geq f$ (where $p := (f^*)^{\dagger}$). This set is nonempty by definition, as $p \in \mathcal{R}$ when f is closed.

Now let $(r_i)_{i \in I} \in \mathcal{R}$ be a totally (downwardly) ordered family and let $r := \overline{(\inf_{i \in I} r_i)} \in \Gamma(Z \times Z^*)$. As $p \geq r_i$ we have $p \geq \inf_{i \in I} r_i$ and on taking closures

$$p \geq r = \overline{\left(\inf_{i \in I} r_i\right)}.$$

Also $r^* = \left(\overline{\inf_{i \in I} r_i}\right)^* = \sup_{i \in I} r_i^* \geq f^{\dagger}$. Now $r_i^* \leq r_i$ for all i, and as $\{r_i\}_{i \in I}$ is downwardly ordered we have $\{r_i^*\}_{i \in I}$ upwardly ordered, so $\sup_{i \in I} r_i^* = \lim_i r_i^* \leq \lim_i r_i^{\dagger} = \inf_{i \in I} r_i^{\dagger}$ and

$$r = \left(\inf_{i \in I} r_i\right)^{**} = \left(\sup_{i \in I} r_i^*\right)^* \geq \left(\inf_{i \in I} r_i^{\dagger}\right)^* = \left(\overline{\inf_{i \in I} r_i}\right)^{*\dagger} = (r^*)^{\dagger}.$$

Consequently $p \geq r \geq (r^*)^\dagger \geq f$ and $r \in \mathcal{R}$. Thus there exists a lower bound in $\mathcal{R}$ to every totally–ordered chain. By Zorn's Lemma $\mathcal{R}$ has a minimal element q. We now show $q^* = q^\dagger$. Let $t = \frac{1}{2}\left(q + (q^*)^\dagger\right) \leq q$. Then $t^* \geq q^* \geq f^\dagger$, and

$$\begin{aligned} t^*(x^*, x) &= \frac{1}{2}\left(q + (q^*)^\dagger\right)^* (2(x^*, x)) = \frac{1}{2}\overline{(q^* \Box q^\dagger)}\,(2(x^*, x)) \\ &\leq \frac{1}{2}\left(q^* \Box q^\dagger\right)(2(x^*, x)) \leq \frac{1}{2}\left(q(x^*, x) + (q^*)^\dagger(x^*, x)\right) = t^\dagger(x^*, x). \end{aligned}$$

Thus $t \in \mathcal{R}$, and $t \leq q$, implying $t = q$ by minimality of q, and so $q = (q^*)^\dagger$. ■

Proof of Lemma 9 part 1. Clearly $M^{\leq}_{f^{*\dagger}} \subseteq M^{\leq}_f$ whenever $f^{*\dagger} \geq f$. The fact that $M^{\leq}_f \subseteq M_{f^{*\dagger}}$ follows via [15, Lemma 19.12] when one assumes $f \geq \langle \cdot, \cdot \rangle$, we show this for completeness. Let $(x, x^*) \in M^{\leq}_f$ and $\lambda \in [0, 1]$ and $\mu := 1 - \lambda \in [0, 1]$. Then

$$\begin{aligned} &\lambda^2 \langle y, y^* \rangle + \lambda\mu\langle (x, x^*), (y, y^*)\rangle + \mu^2 \langle x, x^* \rangle \\ = \ &\frac{1}{2}\langle \lambda(y, y^*) + \mu(y, y^*), \lambda(y, y^*) + \mu(x, x^*)\rangle = \langle \lambda y + \mu x, \lambda y^* + \mu x^* \rangle \\ \leq \ &f(\lambda(y, y^*) + \mu(x, x^*)) \leq \lambda f(y, y^*) + \mu f(x, x^*) \leq \lambda f(y, y^*) + \mu\langle x, x^* \rangle \end{aligned}$$

so

$$\begin{aligned} \lambda\mu\langle (x, x^*), (y, y^*)\rangle - \lambda f(y, y^*) &\leq \left(\mu - \mu^2\right)\langle x, x^* \rangle - \lambda^2 \langle y, y^* \rangle \\ &= \mu\lambda\langle x, x^* \rangle - \lambda^2 \langle y, y^* \rangle. \end{aligned}$$

Dividing by λ and letting $\lambda \to 0$ we have $\langle (x, x^*), (y, y^*)\rangle - f(y, y^*) \leq \langle x, x^* \rangle$ for all (y, y^*) giving $f^*(x^*, x) \leq \langle x, x^* \rangle$ and so $(x, x^*) \in M^{\leq}_{(f^*)^\dagger}$. Thus $M^{\leq}_f \subseteq M^{\leq}_{(f^*)^\dagger}$ with equality ensuing.

Now suppose $(x, x^*) \in M^{\leq}_{f^{*\dagger}} \subseteq M^{\leq}_f$. Then

$$\begin{aligned} &\langle x, x^* \rangle \geq f(x, x^*) \quad \text{and} \quad \langle x, x^* \rangle \geq f^*(x^*, x) \\ \text{so} \qquad &2\langle x, x^* \rangle \geq f(x, x^*) + f^*(x^*, x) \geq \langle (x, x^*), (x^*, x)\rangle = 2\langle x, x^* \rangle \end{aligned}$$

using the Fenchel inequality, so $f(x, x^*) + f^*(x^*, x) = \langle (x, x^*), (x^*, x)\rangle$. Hence $(x^*, x) \in \partial f(x, x^*)$ (and by $\partial f^*(x^*, x) = (\partial f)^{-1}(x^*, x)$ we have the last identity in (3)). Conversely when $(x^*, x) \in \partial f(x, x^*)$ we have

$$f(x, x^*) + f^*(x^*, x) = \langle (x, x^*), (x, x^*)\rangle,$$

which combined with $f^{*\dagger} \geq f$ gives $2f(x, x^*) \leq 2\langle x, x^* \rangle$ and $(x, x^*) \in M^{\leq}_f = M_f$ when $f \geq \langle \cdot, \cdot \rangle$.

The assertion 2 is well known (see [15, Lemma 19.8]) and the last assertion 3 follows from the definition of the Fitzpatrick function

$$\mathcal{F}_T(x, x^*) = \langle x, x^* \rangle - \inf_{(y, y^*) \in T} \langle x - y, x^* - y^* \rangle = \langle x, x^* \rangle$$

because when $(x, x^*) \in T$ we have $\inf_{(y,y^*)\in T}\langle x - y, x^* - y^* \rangle = 0$ (as T is monotone implies $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(y, y^*) \in T$). ■

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