Deriving the convex hull of a polynomial partitioning set through lifting and projection*

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Abstract

Relaxations of the bilinear term, $x_1x_2 = x_3$, play a central role in constructing relaxations of factorable functions. This is because they can be used directly to relax products of functions with known relaxations. In this paper, we provide a compact, closed-form description of the convex hull of this and other more general bivariate monomial terms (which have similar applications in relaxation constructions) in the space of the original variables assuming that the variables and the monomial are restricted to lie in a hyperrectangle. This description is obtained as an intersection of convex hulls of related packing, $x_1x_2^{b_2} \leq x_3$, and covering, $x_1^{b_1}x_2^{b_2} \geq x_3$, sets, where b_1 and b_2 are constants greater than or equal to one. The convex hull of each packing/covering set is first obtained as an intersection of semi-infinite families of linear inequalities, each derived using lifting techniques. Then, each family is projected into a few linear/nonlinear inequalities which are fully characterized in the space of the original problem variables.

1 Introduction

The problem of constructing convex relaxations of nonconvex programs is central to global optimization. McCormick [9] proposes a scheme to construct such convex relaxations for factorable programs, *i.e.*, optimization problems whose objective and constraint functions are defined recursively through sums and products of a collection of univariate functions. This scheme requires that convex and concave relaxations of each univariate function in the collection are available, and that a convex relaxation of the bilinear constraint $x_3 = x_1x_2$ can be constructed over any hyperrectangle. Given these basic postulates, McCormick's scheme produces relaxations that can be utilized inside of branch-and-bound algorithms to obtain ϵ-optimal solutions to factorable programs. In order for the branch-and-bound algorithm to converge, the feasible region must be compact and the partitioning scheme must be exhaustive. Further, the relaxation scheme must produce convex relaxations that converge to the original problem when the bounds on the variables collapse to a point.

In [9], McCormick proposes a polyhedral relaxation for $x_3 = x_1x_2$ which depends on lower and upper bounds on the variables x_1 and x_2, and which is shown in [1] to be convex hull of the set when x_3 is not bounded. Sherali [14] develops a general characterization of the convex envelope of multilinear functions over a unit hypercube and special discrete sets by applying the reformulation-linearization techniques of Sherali and Adams [15]. Meyer and Floudas [10] generalize the results of [9] to develop explicit envelopes for trilinear terms. Jach et al. [5] describe a technique to construct

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convex envelopes of indefinite (n-1)-convex functions and apply this technique to obtain convex envelopes of certain two-dimensional functions. Meyer and Floudas [11] provide the concave envelope for a multilinear form with positive coefficients and non-negative variables. More recently, [8] investigates the strength of relaxations of multilinear functions, and compares the concave and convex envelopes of these functions with the relaxations that are obtained with standard McCormick's relaxations. Moreover, the authors extend some results of [11] to situations where lower bounds on variables are not positive. In [4], it is shown that the natural relaxations of multilinear convex envelopes obtained using duality are more computationally efficient than traditional methods. Locatelli and Schoen [7] develop techniques to derive the convex envelope of bilinear functions where the generating set of the envelope is the set of edges of the underlying polytope. In [19], the authors provide techniques to develop convex envelopes of general polyhedral functions, which include multilinear functions as a special case. When the restriction of the function to the corners of a hypercube is submodular, they provide a closed-form expression for the corresponding convex envelope. In [6], the authors develop the convex envelope of $x_1x_2^{b_2}$, where b_2 is a constant greater than or equal to one.

In all the above cases, x_3 is assumed to be unbounded. Notwithstanding, if finite bounds on x_3 are available, exploiting them can help improve the quality of the relaxation. In this paper, we derive convex hulls for the following sets:

$$\begin{aligned} S^{\geq} &= \Big\{(x_1,x_2,x_3) \in [l_1,u_1]\times[l_2,u_2]\times[l_3,u_3] \;\Big|\; x_1^{b_1}x_2^{b_2} \geq x_3\Big\} \\ S^{\leq} &= \Big\{(x_1,x_2,x_3) \in [l_1,u_1]\times[l_2,u_2]\times[l_3,u_3] \;\Big|\; x_1x_2^{b_2} \leq x_3\Big\} \\ S^{=} &= \Big\{(x_1,x_2,x_3) \in [l_1,u_1]\times[l_2,u_2]\times[l_3,u_3] \;\Big|\; x_1x_2^{b_2} = x_3\Big\}, \end{aligned}$$

where $l_1, l_2, l_3 \in \mathbb{R}_>$, $u_1, u_2, u_3 \in \mathbb{R}_>$, $l_i \leq u_i$ for $i = 1, 2, 3$, and $b_1, b_2 \geq 1$. Observe that b_1 and b_2 are not restricted to be integers. Therefore, the function on the left-hand-side of the defining inequalities is not necessarily a polynomial. We say that $S^{\leq}$ is the *packing relaxation* of $S^{=}$. When $b_1 = 1$, we say that $S^{\geq}$ is the *covering relaxation* of $S^{=}$. It is easy to see that the sets $S^{\geq}$, $S^{\leq}$ and $S^{=}$ are typically not convex. For instance, consider the situation where $b_1 = b_2 = 1$. Let $l_1 = l_2 = l_3 = 1$, $u_1 = u_2 = 2$, and $u_3 = 4$. Then, consider $S^{\geq}$ or $S^{=}$ along with the points $(1, 1, 1)$ and $(2, 2, 4)$. Observe that these points belong to $S^{\geq}$ and $S^{=}$, while their convex combination with equal weight $(1.5, 1.5, 2.5)$ does not, since $1.5^2 < 2.5$. Similarly, consider $S^{\leq}$ along with the points $(1, 2, 2)$ and $(2, 1, 2)$. Observe that these points belong to $S^{\leq}$ while their convex combination with equal weight $(1.5, 1.5, 2)$ does not, since $1.5^2 > 2$. To see that the bounds on x_3 can have an impact on the relaxation, consider the sets with modified bounds $l_3 = 2$ and $u_3 = 2.5$. Any relaxation scheme that ignores the bounds on x_3 will include $(1.5, 1.5, x_3)$ in the relaxation of $S^{\geq}$ or $S^{=}$ for all $x_3 \in [l_3, u_3]$. However, since $x_3 = 2.5$ defines a face of the above sets, $S^{\geq}$ is already convex on this face, and $1.5^2 < 2.5$, it follows that $(1.5, 1.5, 2.5)$ does not belong to the convex hull of $S^{\geq}$ or to the convex hull of $S^{=}$.

Throughout the paper, we assume, for notational convenience that $l > 0$. However, with minor modifications, our results also apply to the case when the lower bound is zero. Observe that since the set is compact, its convex hull is compact as well. Then, consider an $l \in \mathbb{R}^3_{\geq}$ for which some coordinates are zero and a decreasing monotone sequence $\{l^k\}$, where each $l^k \in \mathbb{R}^3_>$ and $l^k \to l$ as $k \to \infty$. Let $S^{\geq}(a) = \{x \in (a, u) \mid x_1^{b_1}x_2^{b_2} \geq x_3\}$. Then, the convex hull of the $S^{\geq}$ is obtained as a limit of the convex hulls of $S^{\geq}(l^k)$.

The techniques we develop also apply to the more general case involving variables that may take negative values. The convex hull descriptions, however, require modifications. When $l \not> 0$, we can define $x_1' = x_1 - l_1$, $x_2' = x_2 - l_2$, and $x_3' = x_3 - l_2x_1 - l_1x_2 + l_1l_2$. Then, consider the set in the space of x_1', x_2', and x_3' variables. For $S^{\geq}$, the resulting set is defined by $x_1'x_2' \geq x_3'$, where bounds on each variable are implied from the bounds on x_1, x_2, and x_3. This set is contained in a linear transformation of $S^{\geq}$, where bounds on x_3 have been relaxed. Since McCormick's relaxation is valid for $S^{\geq}$ with no bounds on x_3, the convex hull of the set in the space of x_1', x_2', and x_3' variables is contained inside the McCormick's relaxation.

The set S with $b_1 = b_2 = 1$ has been studied in the past. In particular, its convex hull is computed in [17] when at least one of x_1 or x_2 is unbounded. Further, [2] describes the convex hull of quadratic terms in two variables through extended reformulations that involve semi-definite constraints. In [3], the authors sketch a procedure to describe the convex hull of S with $b_1 = b_2 = 1$ through an infinite collection of linear inequalities. Instead, we provide, in this paper, a closed-form nonlinear convex hull description of S in the original space of variables x_1, x_2, and x_3. To the best of our knowledge, such a description has not been obtained before. The work presented in this article is a concrete illustration of the general technique [16] that can be used to construct convex hulls of disjunctive sets.

We seek a compact formulation of the convex hull in the original space of variables for a variety of reasons. First, the derived nonlinear inequalities can be used in factorable programming solvers. If polyhedral relaxations are sought, simple linearization strategies can be adopted. Second, we wish to expose the structure of the nonlinear inequalities that describe the convex hull of the above sets. Knowing the form of these inequalities explicitly may facilitate the exploration of new relaxation techniques for constraints involving polynomial functions. Although convex hull representations of certain special cases of the sets we study, *i.e.*, with $b_1 = b_2 = 1$, are known in a higher-dimensional space [2], the structural properties of the required inequalities in the original space are not known. For example, projections of spectrahedra (sets defined using semidefinite constraints) are in general not spectrahedra. Third, it was shown in [18] that the convex hull of bilinear inequalities with multiple terms on the left-hand-side can still be obtained relatively easily if the variables are unbounded and the right-hand-side is a constant. The current work relaxes these assumptions but treats the case with just one term on the left-hand-side. Interestingly, the convex hull descriptions obtained here are much more complex than the ones obtained in [18] although the sets treated here contain only three variables. Fourth, although many techniques exist for generating valid linear or lifted semidefinite constraints for nonlinear sets, techniques to generate convex nonlinear cuts in the space of the original variables are not widely explored.

In Section 2, we give preliminary results that help streamline the presentation of the paper. In particular, we argue that the convex hull of $S^{=}$ can be obtained as the intersection of the convex hulls of its packing and covering relaxations. In Section 3, we derive the convex hull of $S^{\geq}$. In Section 4, we derive the convex hull of $S^{\leq}$. To obtain the desired convex hulls, we first obtain a semi-infinite representation of the convex hull of the sets through lifting. We then project parametric coefficiencts from the families of the resulting linear inequalities into nonlinear inequalities in the original space. This procedure follows the approach we described in [12], where we obtained a nonlinear convex hull description for a specific bilinear covering example in three variables. We conclude the paper with remarks and directions for future research in Section 5.

2 Preliminary results

In the remainder of this paper, we use the notation $\text{conv}(T)$ to denote the convex hull of set T. For a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and a convex set $X \subseteq \mathbb{R}^n$, we denote by $\text{conv}_X f$, the convex envelope of the restriction of the function f to set X. Let $S^{\leq}$ and $S^{\geq}$ be the packing and covering relaxations of $S^=$, respectively. It is clear that $\text{conv}(S^=) \subseteq \text{conv}(S^\leq) \cap \text{conv}(S^\geq)$ since $S^= = S^\leq \cap S^\geq$, *i.e.*, a convex relaxation of $S^=$ can be obtained from the convex hulls of $S^\geq$ and $S^\leq$. For this particular set however, it can be shown that this relaxation is, in fact, the convex hull of $S^=$. This result follows from Proposition 2.1, which is proven in [16].

Proposition 2.1. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function, and let $X \subseteq \mathbb{R}^n$ be a convex set. Consider $T^= = \{x \in X \mid f(x) = 0\}$. Define $T^\geq = \{x \in X \mid f(x) \geq 0\}$, and $T^\leq = \{x \in X \mid f(x) \leq 0\}$. Then, $\text{conv}(T^=) = \text{conv}(T^\geq) \cap \text{conv}(T^\leq)$.* $\square$

Proposition 2.1 yields the following corollary.

Corollary 2.2. *Let $S^\leq$ and $S^\geq$ be the packing and covering relaxations of $S^=$, respectively. Then $\text{conv}(S^=) = \text{conv}(S^\geq) \cap \text{conv}(S^\leq)$.*

Proof. Take $f(x)$ to be $x_3 - x_1 x_2^{b_2}$ and $X = [l, u]^3$ in Proposition 2.1. $\square$

In the ensuing sections, we make use several times of the following result, which help reduce the study of $\text{conv}(S^\geq)$ and $\text{conv}(S^\leq)$ down to a few canonical cases.

Lemma 2.3. *Let $T \subseteq [l, u] \subseteq \mathbb{R}^n$. For $j \in N := \{1, \ldots, n\}$ and $\theta \in [l_j, u_j]$, assume further that $A = \{x \in T \,|\, x_j \leq \theta\} = [l, u']$ where $u'_i = u_i$ for $i \in N \backslash \{j\}$ and $u'_j = \theta$. Then $\text{conv}(T) = A \cup \text{conv}(B)$ where $B = \{x \in T \,|\, x_j \geq \theta\}$.*

Proof. It is clear that $T = A \cup B$. Therefore $\text{conv}(T) \supseteq A \cup \text{conv}(B)$. We next argue that the reverse inclusion also holds. Assume by contradiction that there exists $x' \in \text{conv}(T) \subseteq [l, u]$ such that $x' \notin A \cup \text{conv}(B)$. If $x'_j \leq \theta$, then $x' \in A$, a contradiction. We may therefore assume that $x'_j > \theta$. It follows from the definition of $\text{conv}(T)$, the fact that A is convex, and the fact that $x' \notin \text{conv}(B)$ that $x' \in [\dot{x}, \ddot{x}]$ where $\dot{x} \in A$ and $\ddot{x} \in \text{conv}(B)$. Segment $[\dot{x}, \ddot{x}]$ must contain a point $\hat{x}$ such that $\hat{x}_j = \theta$ as $\dot{x}_j \leq \theta$ and $\ddot{x}_j > \theta$. Because A and $\text{conv}(B)$ are both subsets of $[l, u]$, then $\hat{x} \in [l, u]$. It now remains to observe that $\hat{x} \in A \cap B \subseteq B \subseteq \text{conv}(B)$ and that $x' \in [\hat{x}, \ddot{x}]$ to conclude that $x' \in \text{conv}(B)$, a contradiction. $\square$

Intuitively, Lemma 2.3 argues that if T contains a "slab," this slab can be removed from the set before convexification, and can be added back to the convexified object. Therefore, Lemma 2.3 states that the main difficulty in studying $\text{conv}(T)$ resides in the construction of $\text{conv}(B)$. The following result follows using the same proof (after transforming x_j to $-x_j$).

Lemma 2.4. *Let $T \subseteq [l, u] \subseteq \mathbb{R}^n$. For $j \in N := \{1, \ldots, n\}$ and $\theta \in [l_j, u_j]$, assume further that $A = \{x \in T \,|\, x_j \geq \theta\} = [l', u]$ where $l'_i = l_i$ for $i \in N \backslash \{j\}$ and $l'_j = \theta$. Then $\text{conv}(T) = A \cup \text{conv}(B)$ where $B = \{x \in T \,|\, x_j \leq \theta\}$.* $\square$

3 Convex hull of $S^\geq$

In this section, we study the convex hull of

$$S^\geq = \left\{ x \in [l, u]^3 \;\middle|\; x_1^{b_1} x_2^{b_2} \geq x_3 \right\},$$

where $b_1 \geq 1$, $b_2 \geq 1$, and where $l_i \in \mathbb{R}_\geq$, $u_i \in \mathbb{R}_\geq$ and $l_i \leq u_i$ for $i = 1, 2, 3$. To streamline notation, we define $a_1 := b_1^{-1}$ and $a_2 := b_2^{-1}$. In studying this set, we make the following assumptions:

(A1) $l_1 = 1$ and $l_2 = 1$,

(A2) $u_1 > 1$, $u_2 > 1$, and $u_3 > l_3$,

(A3) $1 < u_3$,

(A4) $u_3 \leq u_1^{b_1} u_2^{b_2}$,

(A5) $l_3 \geq 1$,

(A6) $l_3 \leq \min\{u_1^{b_1}, u_2^{b_2}\}$,

(A7) $u_3 \geq \max\{u_1^{b_1}, u_2^{b_2}\}$,

(A8) $u_1 \geq u_2$.

Assumption (A1) is without loss of generality (wlog) since the variables x_1 and x_2 can be rescaled. If, in addition, Assumption (A2) is not satisfied, the set S is not full-dimensional. In particular, when $u_1 = l_1 = 1$, then $\operatorname{conv}(S^{\geq})$ is polyhedral and is straightforward to derive. The case where $u_2 = l_2 = 1$ is symmetric. When $u_3 = l_3$, then $S^{\geq}$ is convex since its defining inequality can be rewritten as $x_1^{\frac{b_1}{b_1+b_2}} x_2^{\frac{b_2}{b_1+b_2}} \geq l_3^{\frac{1}{b_1+b_2}}$ where the left-hand-side is a concave function. When Assumption (A3) is not satisfied, inequality $x_1^{b_1} x_2^{b_2} \geq x_3$ is redundant in the description of $S^{\geq}$. In this case $\operatorname{conv}(S^{\geq}) = [l, u]$. Assumption (A4) is also wlog. In fact, when $u_3 > u_1^{b_1} u_2^{b_2}$, no point x with $x_3 = u_3$ satisfies inequality $x_1^{b_1} x_2^{b_2} \geq x_3$. Therefore, the bound u_3 on x_3 can be tightened to $u_1^{b_1} u_2^{b_2}$ without changing $S^{\geq}$. When Assumption (A5) is not satisfied, Lemma 2.3 can be applied with $j = 3$ and $\theta = 1$. It is therefore sufficient to assume that $l_3 \geq 1$. Assumption (A6) is also wlog. Assume $l_3 > u_2^{b_2}$. We observe that, for any feasible point, $x_1^{b_1} u_2^{b_2} \geq l_3$ and $u_1^{b_1} x_2^{b_2} \geq l_3$. Therefore, $x_1 \geq l_3^{a_1} u_2^{-a_1 b_2}$ and $x_2 \geq \max\{1, l_3^{a_2} u_1^{-b_1 a_2}\}$. Now, define $\tilde{x}_1 = x_1 u_2^{a_1 b_2} l_3^{-a_1}$, $\tilde{x}_2 = x_2 \min\{1, u_1^{b_1 a_2} l_3^{-a_2}\}$, and $\tilde{x}_3 = x_3 u_2^{b_2} l_3^{-2} \min\{u_1^{b_1}, l_3\}$. Then, $x_1^{b_1} x_2^{b_2} \geq x_3$ reduces to $\tilde{x}_1^{b_1} \tilde{x}_2^{b_2} \geq \tilde{x}_3$ and the bound inequalities reduce to $1 \leq \tilde{x}_1 \leq u_1 u_2^{a_1 b_2} l_3^{-a_1}$, $1 \leq \tilde{x}_2 \leq u_2 l_3^{-a_2} \min\{u_1^{b_1 a_2}, l_3^{a_2}\}$, and $u_2^{b_2} l_3^{-1} \min\{u_1^{b_1}, l_3\} \leq \tilde{x}_3 \leq u_3 u_2^{b_2} l_3^{-2} \min\{u_1^{b_1}, l_3\}$. It is easy to check that the new system satisfies the above assumption. Further, the new system continues to satisfy Assumptions (A3) and (A4). A similar discussion shows that assuming $l_3 \leq u_1^{b_1}$ is wlog. Suppose that Assumption (A7) is not satisfied. Then Lemma 2.4 can be applied with $j = 1$ and $\theta = u_3^{a_1}$ or with $j = 2$ and $\theta = u_3^{a_2}$. Therefore, we may reduce the upper bound of x_1 to $u_3^{a_1}$and/or reduce the upper bound of x_2 to $u_3^{a_2}$. Finally, Assumption (A8) is wlog since variables x_1 and x_2 can be interchanged.

3.1 Linear description of $\operatorname{conv}(S^{\geq})$

It is well-known that a full-dimensional closed convex set can be described as the intersection of all its tangent halfspaces; see [13, Theorem 18.8]. We will use this basic result to construct $\operatorname{conv}(S^{\geq})$ as it is clear that $\operatorname{conv}(S^{\geq})$ is compact since $S^{\geq}$ is.

In particular, we derive all nondominated linear valid inequalities for $\operatorname{conv}(S^{\geq})$ that are lifted from valid linear constraints in the space of variables (x_1, x_2). We thereby provide a semi-infinite description of $\operatorname{conv}(S^{\geq})$. We turn this semi-infinite description into one that contains only a finite number of linear and nonlinear inequalities in Section 3.2.

The derivation of the desired inequalities requires the solution of a certain optimization problem, called *lifting problem.* The solution of this problem involves the function $\phi_{\alpha_1,\alpha_2}(.) : \mathbb{R} \mapsto \mathbb{R}$ that we define as

$$\phi_{\alpha_1,\alpha_2}(t) := \min_{x_1,x_2} \left\{ \alpha_1 x_1 + \alpha_2 x_2 \;\middle|\; x_1^{b_1} x_2^{b_2} \geq t, 1 \leq x_1 \leq u_1, 1 \leq x_2 \leq u_2 \right\}, \tag{1}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, and $t \in \mathbb{R}$. If (1) is infeasible for some $t \in \mathbb{R}$, then we write that $\phi_{\alpha_1,\alpha_2}(t) = \infty$. For $t \in \mathbb{R}$, we denote $S^{\geq} \cap \{x \in \mathbb{R}^3 \mid x_3 = t\}$ by $S^{\geq}_t$.

Given $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, we wish to determine suitable values of (α_3, δ) for which

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \geq \delta \tag{2}$$

is valid for $S^{\geq}$. First, observe that if for a given (α_1, α_2), (x_1^a, x_2^a, x_3^a) and (x_1^b, x_2^b, x_3^b) are tight on (2) and are such that $x_3^a \neq x_3^b$, then α_3 and δ are uniquely determined. In general (2) is valid for $S^{\geq}$ if and only if $(\alpha_3, \delta) \in V_{\alpha_1,\alpha_2}$ where

$$\begin{aligned} V_{\alpha_1,\alpha_2} &= \big\{(\alpha_3,\delta) \mid \min_{(x_1,x_2,t)\in S} \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 t \geq \delta\big\} \\ &= \Big\{(\alpha_3,\delta) \,\Big|\, \min_{t\in[l_3,u_3]} \big\{\alpha_3 t + \min\{\alpha_1 x_1 + \alpha_2 x_2 \mid (x_1,x_2) \in S_t\}\big\} \geq \delta\Big\} \\ &= \big\{(\alpha_3,\delta) \mid \min_{t\in[l_3,u_3]} \big\{\alpha_3 t + \phi_{\alpha_1,\alpha_2}(t)\big\} \geq \delta\big\} \\ &= \big\{(\alpha_3,\delta) \mid \delta - \alpha_3 t \leq \phi_{\alpha_1,\alpha_2}(t),\ \forall t \in [l_3,u_3]\big\}. \end{aligned}$$

The previous derivation shows that V_{α_1,α_2} corresponds to the set of linear underestimators of the epigraph of $\phi_{\alpha_1,\alpha_2}(t)$ over $t \in [l_3, u_3]$. Therefore, the only linear inequalities that are non-dominated are those that support the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$, together with the inequalities that define the domain of t. If a pair (α_3, δ) describes an inequality that supports the epigraph of $\phi_{\alpha_1,\alpha_2}(t)$ at two distinct points, $(t_1, \phi_{\alpha_1,\alpha_2}(t_1))$ and $(t_2, \phi_{\alpha_1,\alpha_2}(t_2))$ then

$$(\alpha_3, \delta) = \left(-\frac{\phi_{\alpha_1,\alpha_2}(t_2) - \phi_{\alpha_1,\alpha_2}(t_1)}{t_2 - t_1}, \frac{t_2\phi_{\alpha_1,\alpha_2}(t_1) - t_1\phi_{\alpha_1,\alpha_2}(t_2)}{t_2 - t_1}\right). \tag{3}$$

In particular, when $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[l_3, u_3]$, its convex envelope over $[l_3, u_3]$ is affine, and supports its epigraph at the points $t = l_3$ and $t = u_3$. In this case, (2) can be rewritten, after scaling, as

$$d_3\alpha_1 x_1 + d_3\alpha_2 x_2 - \phi_{\alpha_1,\alpha_2}(l_3)(u_3 - x_3) - \phi_{\alpha_1,\alpha_2}(u_3)(x_3 - l_3) \geq 0 \tag{4}$$

where $d_3 = u_3 - l_3$.

We next derive in Section 3.1.1 linear inequalities that describe the part of $\mathrm{conv}(S^{\geq})$ whose geometry is simple. We refer to these inequalities as *trivial*. In Section 3.1.2, we derive inequalities that belong to the part of $\mathrm{conv}(S^{\geq})$ that may not be polyhedral. We refer to these inequalities as *nontrivial*.

3.1.1 Trivial inequalities for $\mathrm{conv}(S^{\geq})$

In this section, we derive all nondominated valid inequalities (2) for $\mathrm{conv}(S^{\geq})$ where $\alpha_1 \leq 0$ or $\alpha_2 \leq 0$. Proposition 3.1 provides a closed-form expression for $\phi_{\alpha_1,\alpha_2}(t)$ for these values of α_1 and α_2.

Proposition 3.1. *Assume that* $\alpha_1 \leq 0$ *or* $\alpha_2 \leq 0$. *For* $t \in [1, u_1^{b_1} u_2^{b_2}]$,

(i) $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 u_1 + \alpha_2 u_2$ *if* $\alpha_1 \leq 0$ *and* $\alpha_2 \leq 0$.

(ii) $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 u_1 + \alpha_2 \max\{t^{a_2} u_1^{-b_1 a_2}, 1\}$ *if* $\alpha_1 \leq 0$ *and* $\alpha_2 > 0$.

(iii) $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 \max\{t^{a_1} u_2^{-a_1 b_2}, 1\} + \alpha_2 u_2$ *if* $\alpha_1 > 0$ *and* $\alpha_2 \leq 0$.

Proof. Assume that $t \in [1, u_1^{b_1} u_2^{b_2}]$. In this case, (1) has (u_1, u_2) for feasible solution. It then follows from Weierstraß' theorem that (1) has an optimal solution. Let (x_1^*, x_2^*) be such an optimal solution. First assume that $\alpha_1 \leq 0$. It is clear that we can choose $x_1^* = u_1$. In fact, if $x_1^* < u_1$, increasing x_1^* by $\epsilon > 0$ maintains feasibility of the solution and does not deteriorate its objective value. Similarly, if $\alpha_2 \leq 0$, we can choose $x_2^* = u_2$. Using these observations, we consider three cases. If $\alpha_1 \leq 0$ and $\alpha_2 > 0$, then $x_1^* = u_1$. Since the objective function of (1) is increasing in x_2, it is optimal to let x_2 take its lowest admissible value, *i.e.*, $x_2^* = \max\{t^{a_2} u_1^{-b_1 a_2}, 1\}$, yielding (ii). The case where $\alpha_1 > 0$ and $\alpha_2 \leq 0$ is symmetric, yielding (iii). Finally, if $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$, then (u_1, u_2) is optimal for (1), yielding (i). $\square$

We next derive lifted inequalities for all values of (α_1, α_2) studied in Proposition 3.1. We obtain the inequalities

$$\begin{aligned}
x_1 &\geq 1, \text{ (if } l_3 < u_2^{b_2}), && (5)\\
-x_1 &\geq -u_1, && (6)\\
x_2 &\geq 1, \text{ (if } l_3 < u_1^{b_1}), && (7)\\
-x_2 &\geq -u_2, && (8)\\
x_3 &\geq l_3, && (9)\\
-x_3 &\geq -u_3, && (10)\\
(u_3 - u_1^{b_1})x_2 + (1 - u_3^{a_2} u_1^{-b_1 a_2})x_3 &\geq u_3 - u_1^{b_1 - b_1 a_2} u_3^{a_2}, \text{ (if } u_3 > u_1^{b_1}), && (11)\\
(u_3 - u_2^{b_2})x_1 + (1 - u_3^{a_1} u_2^{-a_1 b_2})x_3 &\geq u_3 - u_2^{b_2 - a_1 b_2} u_3^{a_1}, \text{ (if } u_3 > u_2^{b_2}), && (12)
\end{aligned}$$

as shown in the following proposition.

Proposition 3.2. *The only linear inequalities* (2) *with coefficients* $\alpha_1 \leq 0$ *or* $\alpha_2 \leq 0$ *necessary in the description of* $\operatorname{conv}(S^{\geq})$ *are among (5)-(12).*

Proof. We have established earlier that nondominated lifted inequalities (2) are either of the form (9) or (10), or can be derived from the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$. There are three cases:

(i) Assume that $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$. In this case, the function $\phi_{\alpha_1,\alpha_2}(t)$ is constant, and therefore convex over $[l_3, u_3]$. By (3), the corresponding lifting coefficients (α_3, δ) are $(0, \alpha_1 u_1 + \alpha_2 u_2)$. Since these coefficients are linear in (α_1, α_2), it is sufficient to consider the cases where (α_1, α_2) equals $(-1, 0)$, $(0, -1)$ and $(0, 0)$, which correspond to the extreme points and rays of the region $\{(\alpha_1, \alpha_2) \in \mathbb{R}^2 \mid \alpha_1 \leq 0, \alpha_2 \leq 0\}$. Ray $(-1, 0)$ yields (6) and ray $(0, -1)$ yields (8).

(ii) Assume now that $\alpha_1 \leq 0$ and $\alpha_2 > 0$. In this case, $\phi_{\alpha_1,\alpha_2}(t)$ is a piecewise function that is constant over $[l_3, u_1^{b_1}]$ and increasing concave over $[u_1^{b_1}, u_3]$. We conclude that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[1, u_1^{b_1} u_2^{b_2}]$ is constant over $[l_3, u_1^{b_1}]$ and affine over $[u_1^{b_1}, u_3]$. We can therefore develop two inequalities for each (α_1, α_2) when the corresponding intervals do not reduce to a single point. By (3), the corresponding lifting coefficients are $(\alpha_3, \delta) = (0, \alpha_1 u_1 + \alpha_2)$ if $l_3 < u_1^{b_1}$ and $(\alpha_3, \delta) = \left(-\alpha_2 \frac{u_3^{a_2} u_1^{-b_1 a_2} - 1}{u_3 - u_1^{b_1}}, \alpha_1 u_1 + \alpha_2 \frac{u_3 - u_3^{a_2} u_1^{b_1 - b_1 a_2}}{u_3 - u_1^{b_1}}\right)$ if $u_3 > u_1^{b_1}$. Since the coefficients (α_3, δ) are linear in (α_1, α_2), it is sufficient to consider the ray $(\alpha_1, \alpha_2) = (0, 1)$. This ray yields (7) and (11), respectively.

(iii) Assume finally that $\alpha_1 > 0$ and $\alpha_2 \leq 0$. Because this case is symmetric to the one discussed in (ii), we obtain (5) and (12).

$\square$

3.1.2 Nontrivial inequalities for $\text{conv}(S^{\geq})$

In this section, we derive all nondominated valid inequalities (2) for $\text{conv}(S^{\geq})$ where $\alpha_1 > 0$ and $\alpha_2 > 0$. To streamline notation, we let $B := b_1 + b_2$, $A := B^{-1} = (b_1 + b_2)^{-1}$, $\beta_1 := \alpha_1^{-1}$ and $\beta_2 := \alpha_2^{-1}$. Further, we define

$$c := \left(\tfrac{b_2}{b_1}\right)^{b_1/(b_1+b_2)} + \left(\tfrac{b_1}{b_2}\right)^{b_2/(b_1+b_2)} = (a_1 b_2)^{Ab_1} + (b_1 a_2)^{Ab_2},$$
$$\bar{c} := c(\alpha_1^{b_1}\alpha_2^{b_2})^{1/(b_1+b_2)} = c\alpha_1^{Ab_1}\alpha_2^{Ab_2}.$$

We first derive in Proposition 3.3 a closed-form expression for the lifting function $\phi_{\alpha_1,\alpha_2}(t)$ that we use to construct these inequalities.

Proposition 3.3. *Assume that* $\alpha_1 > 0$ *and* $\alpha_2 > 0$. *For* $t \in [1, u_1^{b_1}u_2^{b_2}]$,

$$\phi_{\alpha_1,\alpha_2}(t) = \begin{cases} \left.\begin{array}{lll} \phi^a_{\alpha_1,\alpha_2}(t) & \textit{if} & t \leq u_2^{b_2} \\ \phi^b_{\alpha_1,\alpha_2}(t) & \textit{if} & t > u_2^{b_2} \end{array}\right\} & \textit{when} \quad \alpha_1\beta_2 \geq \lambda(t), \\ \phi^c_{\alpha_1,\alpha_2}(t) & \textit{when} \quad \mu(t) < \alpha_1\beta_2 < \lambda(t), \\ \left.\begin{array}{lll} \phi^d_{\alpha_1,\alpha_2}(t) & \textit{if} & t \leq u_1^{b_1} \\ \phi^e_{\alpha_1,\alpha_2}(t) & \textit{if} & t > u_1^{b_1} \end{array}\right\} & \textit{when} \quad \alpha_1\beta_2 \leq \mu(t), \end{cases}$$

where

- $\phi^a_{\alpha_1,\alpha_2}(t) = \alpha_1 + \alpha_2 t^{a_2}$,
- $\phi^b_{\alpha_1,\alpha_2}(t) = \alpha_1 u_2^{-a_1 b_2} t^{a_1} + \alpha_2 u_2$,
- $\phi^c_{\alpha_1,\alpha_2}(t) = \bar{c} t^{A}$,
- $\phi^d_{\alpha_1,\alpha_2}(t) = \alpha_1 t^{a_1} + \alpha_2$,
- $\phi^e_{\alpha_1,\alpha_2}(t) = \alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} t^{a_2}$,

and

- $\lambda(t) = b_1 a_2 \min\{t^{a_2}, u_2^{Ba_1} t^{-a_1}\}$,
- $\mu(t) = b_1 a_2 \max\{t^{-a_1}, u_1^{-Ba_2} t^{a_2}\}$.

Proof. Assume that $t \in [1, u_1^{b_1}u_2^{b_2}]$. Weierstraß' theorem implies that (1) has an optimal solution since (u_1, u_2) belongs to its feasible region. We claim that there is an optimal solution $(\tilde{x}_1, \tilde{x}_2)$ with $\tilde{x}_1^{b_1}\tilde{x}_2^{b_2} = t$. Assume by contradiction that $\tilde{x}_1^{b_1}\tilde{x}_2^{b_2} > t$. There are two cases. Assume first that $\tilde{x}_1 > 1$. For ϵ positive but sufficiently small, the solution $(\bar{x}_1, \bar{x}_2) = (\tilde{x}_1 - \epsilon, \tilde{x}_2)$ is feasible and has a better objective value than $(\tilde{x}_1, \tilde{x}_2)$. Assume second that $\tilde{x}_1 = 1$. It follows that $\tilde{x}_2 > t \geq 1$. For ϵ positive but sufficiently small, the solution $(\bar{x}_1, \bar{x}_2) = (\tilde{x}_1, \tilde{x}_2 - \epsilon)$ is feasible and has a better objective value than $(\tilde{x}_1, \tilde{x}_2)$.

By eliminating variable x_2 using the relation $x_2 = t^{a_2} x_1^{-b_1 a_2}$, we can reformulate (1) as

$$\phi_{\alpha_1,\alpha_2}(t) := \min\{f_t(x_1) \,|\, L(t) \leq x_1 \leq U(t)\}, \tag{13}$$

where $f_t(x) = \alpha_1 x + \alpha_2 x^{-b_1 a_2} t^{a_2}$, $L(t) := \max\{1, t^{a_1} u_2^{-a_1 b_2}\} > 0$ and $U(t) := \min\{t^{a_1}, u_1\}$. It is easily verified that $L(t) \leq U(t)$ when $t \in [1, u_1^{b_1}u_2^{b_2}]$. Since $f_t(\cdot)$ is convex over $\mathbb{R}_{\geq}$, an optimal solution to (13) is (i) $x_1^* = L(t)$ if $f_t'(L(t)) \geq 0$, *i.e.*, $\alpha_1\beta_2 \geq b_1 a_2 L(t)^{-Ba_2} t^{a_2} = \lambda(t)$, and (ii)

$x_1^* = U(t)$ if $f_t'(U(t)) \le 0$, *i.e.*, $\alpha_1\beta_2 \le b_1a_2U(t)^{-Ba_2}t^{a_2} = \mu(t)$. When $\mu(t) < \alpha_1\beta_2 < \lambda(t)$, the intermediate value theorem implies that $f_t'(.)$ takes value zero for some $x_1 \in (L(t), U(t))$. It is then simple to verify that the only point that makes $f_t'(.)$ equal to zero is $x_1^* = (b_1a_2\beta_1\alpha_2)^{Ab_2}\, t^A$. Since (13) is a convex program, x_1^* must be an optimal solution. This yields the desired result. In particular, when $\alpha_1\beta_2 \ge \lambda(t)$ and $t \le u_2^{b_2}$, then $x_1^* = L(t) = 1$, yielding $\phi_{\alpha_1,\alpha_2}(t) = f_t(1) = \phi^a_{\alpha_1,\alpha_2}(t)$. The other cases are similar. □

We now make a few observations that can be verified through direct computation.

(O1) For $t \in [1, u_1^{b_1}u_2^{b_2}]$, $\mu(t) \le \lambda(t)$.

(O2) For $t \in \{1, u_1^{b_1}u_2^{b_2}\}$, $\mu(t) = \lambda(t)$.

(O3) For $t \in [1, u_1^{b_1}]$, *(i)* $\mu(t) = \mu_1(t) := b_1a_2t^{-a_1}$, *(ii)* $\mu(t)$ is decreasing, and *(iii)* $\mu([1, u_1^{b_1}]) = [b_1a_2u_1^{-1}, b_1a_2]$.

(O4) For $t \in [u_1^{b_1}, u_1^{b_1}u_2^{b_2}]$, *(i)* $\mu(t) = \mu_2(t) := b_1a_2u_1^{-Ba_2}t^{a_2}$, *(ii)* $\mu(t)$ is increasing, and *(iii)* $\mu([u_1^{b_1}, u_1^{b_1}u_2^{b_2}]) = [b_1a_2u_1^{-1}, b_1a_2u_1^{-1}u_2]$.

(O5) For $t \in [1, u_2^{b_2}]$, *(i)* $\lambda(t) = \lambda_1(t) := b_1a_2t^{a_2}$, *(ii)* $\lambda(t)$ is increasing, and *(iii)* $\lambda([1, u_2^{b_2}]) = [b_1a_2, b_1a_2u_2]$.

(O6) For $t \in [u_2^{b_2}, u_1^{b_1}u_2^{b_2}]$, *(i)* $\lambda(t) = \lambda_2(t) := b_1a_2u_2^{Ba_1}t^{-a_1}$, *(ii)* $\lambda(t)$ is decreasing, and *(iii)* $\lambda([u_2^{b_2}, u_1^{b_1}u_2^{b_2}]) = [b_1a_2u_1^{-1}u_2, b_1a_2u_2]$.

(O7) For $t \in [1, u_1^{b_1}u_2^{b_2}]$, functions $\phi^a_{\alpha_1,\alpha_2}(t)$, $\phi^b_{\alpha_1,\alpha_2}(t)$, $\phi^c_{\alpha_1,\alpha_2}(t)$, $\phi^d_{\alpha_1,\alpha_2}(t)$ and $\phi^e_{\alpha_1,\alpha_2}(t)$ are concave.

Further by combining Observations (O3) and (O4), and by combining Observations (O5) and (O6), respectively, we obtain using Assumption (A8) that

(O8) For $t \in [1, u_1^{b_1}u_2^{b_2}]$, $b_1a_2u_1^{-1} \le \mu(t) \le b_1a_2$.

(O9) For $t \in [1, u_1^{b_1}u_2^{b_2}]$, $b_1a_2u_1^{-1}u_2 \le \lambda(t) \le b_1a_2u_2$.

Our next goal is to determine properties of the function $\phi_{\alpha_1,\alpha_2}(t)$ that help in deriving its convex envelope over $[l_3, u_3]$.

The following ancillary result establishes relations between the slopes of the functions $\phi^a_{\alpha_1,\alpha_2}(t)$, …, $\phi^e_{\alpha_1,\alpha_2}(t)$.

Lemma 3.4. *For $t > 0$,*

(i) $\frac{d}{dt}\phi^d_{\alpha_1,\alpha_2}(t) \ge \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t)$ *if and only if* $t \ge t^{dc} := (b_1a_2\beta_1\alpha_2)^{b_1}$.

(ii) $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t) \ge \frac{d}{dt}\phi^e_{\alpha_1,\alpha_2}(t)$ *if and only if* $t \le t^{ce} := (a_1b_2\alpha_1\beta_2)^{b_2}u_1^B$.

(iii) $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t) \ge \frac{d}{dt}\phi^b_{\alpha_1,\alpha_2}(t)$ *if and only if* $t \le t^{cb} := (b_1a_2\beta_1\alpha_2)^{b_1}u_2^B$.

(iv) $\frac{d}{dt}\phi^a_{\alpha_1,\alpha_2}(t) \ge \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t)$ *if and only if* $t \ge t^{ac} := (a_1b_2\alpha_1\beta_2)^{b_2}$.

Proof. First, it is simple to compute that $\frac{d}{dt}\phi^a_{\alpha_1,\alpha_2}(t) = \alpha_2a_2t^{a_2-1}$, $\frac{d}{dt}\phi^b_{\alpha_1,\alpha_2}(t) = \alpha_1a_1u_2^{-a_1b_2}t^{a_1-1}$, $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t) = \bar{c}At^{A-1}$, $\frac{d}{dt}\phi^d_{\alpha_1,\alpha_2}(t) = \alpha_1a_1t^{a_1-1}$, and $\frac{d}{dt}\phi^e_{\alpha_1,\alpha_2}(t) = \alpha_2a_2u_1^{-b_1a_2}t^{a_2-1}$. We now derive the desired inequalities. In these derivations, we make use of the fact that $c = (a_1b_2)^{Ab_1} + (b_1a_2)^{Ab_2} = (b_1a_2)^{Ab_2}Ba_1$ or equivalently that $c = (a_1b_2)^{Ab_1}Ba_2$. Therefore

$$(i) \quad \bar{c} = \alpha_1^{Ab_1}(b_1a_2\alpha_2)^{Ab_2}Ba_1, \quad \text{and} \quad (ii) \quad \bar{c} = \alpha_2^{Ab_2}(a_1b_2\alpha_1)^{Ab_1}Ba_2. \tag{14}$$

It follows from the above expressions that

(i) $\frac{d}{dt}\phi^d_{\alpha_1,\alpha_2}(t) \geq \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t)$ iff $t \geq (\bar{c}Ab_1\beta_1)^{Bb_1a_2} = t^{dc}$. The equality holds because of (14i).

(ii) $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t) \geq \frac{d}{dt}\phi^e_{\alpha_1,\alpha_2}(t)$ iff $t \leq (\bar{c}Ab_2u_1^{b_1a_2}\beta_2)^{Ba_1b_2} = t^{ce}$. The equality holds because of (14ii).

(iii) $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t) \geq \frac{d}{dt}\phi^b_{\alpha_1,\alpha_2}(t)$ iff $t \leq (\bar{c}Ab_1u_2^{b_2a_1}\beta_1)^{Ba_2b_1} = t^{cb}$. The equality holds because of (14i).

(iv) $\frac{d}{dt}\phi^a_{\alpha_1,\alpha_2}(t) \geq \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t)$ iff $t \geq (\bar{c}Ab_2\beta_2)^{Ba_1b_2} = t^{ac}$. The equality holds because of (14ii).

□

We next show that $\phi_{\alpha_1,\alpha_2}(t)$ is concave for most values of $\alpha_1 > 0$ and $\alpha_2 > 0$ using Lemma 3.4.

Lemma 3.5. *For $\alpha_1 > 0$ and $\alpha_2 > 0$,*

(i) When $\alpha_1\beta_2 \leq b_1a_2u_1^{-1}$, then $\phi_{\alpha_1,\alpha_2}(t) = \phi^d_{\alpha_1,\alpha_2}(t)$ for $t \in [1, u_1^{b_1}]$ and $\phi_{\alpha_1,\alpha_2}(t) = \phi^e_{\alpha_1,\alpha_2}(t)$ for $t \in [u_1^{b_1}, u_1^{b_1}u_2^{b_2}]$.

(ii) When $\alpha_1\beta_2 \in [b_1a_2u_1^{-1}, b_1a_2u_2]$, then $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[1, u_1^{b_1}u_2^{b_2}]$.

(iii) When $\alpha_1\beta_2 \geq b_1a_2u_2$, then $\phi_{\alpha_1,\alpha_2}(t) = \phi^a_{\alpha_1,\alpha_2}(t)$ for $t \in [1, u_2^{b_2}]$ and $\phi_{\alpha_1,\alpha_2}(t) = \phi^b_{\alpha_1,\alpha_2}(t)$ for $t \in [u_2^{b_2}, u_1^{b_1}u_2^{b_2}]$.

Proof. We make use of the notation introduced in the statement of Lemma 3.4.

(i) From Observation (O8), we know that $\mu(t) \geq b_1a_2u_1^{-1}$ for $t \in [1, u_1^{b_1}u_2^{b_2}]$. The result then follows directly from Proposition 3.3.

(ii) There are three subcases. Assume first that $\alpha_1\beta_2 \in [b_1a_2u_1^{-1}, b_1a_2u_1^{-1}u_2]$. It follows from Observations (O3), (O4) and (O9) that $\phi_{\alpha_1,\alpha_2}(t) = \phi^d_{\alpha_1,\alpha_2}(t)$ for $t \in [1, \tau^{dc}]$, $\phi_{\alpha_1,\alpha_2}(t) = \phi^c_{\alpha_1,\alpha_2}(t)$ for $t \in [\tau^{dc}, \tau^{ce}]$, and $\phi_{\alpha_1,\alpha_2}(t) = \phi^e_{\alpha_1,\alpha_2}(t)$ for $t \in [\tau^{ce}, u_1^{b_1}u_2^{b_2}]$, where $\mu_1(\tau^{dc}) = \mu_2(\tau^{ce}) = \alpha_1\beta_2$. Direct computations show that $\tau^{dc} = t^{dc}$ and $\tau^{ce} = t^{ce}$. Lemma 3.4 then shows that $\frac{d}{dt}\phi^d_{\alpha_1,\alpha_2}(t^{dc}) = \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t^{dc})$, and $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t^{ce}) = \frac{d}{dt}\phi^e_{\alpha_1,\alpha_2}(t^{ce})$. Combined with Observation (O7), this establishes that $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[1, u_1^{b_1}u_2^{b_2}]$.

Assume next that $\alpha_1\beta_2 \in [b_1a_2u_1^{-1}u_2, b_1a_2]$. It follows from Observations (O3), (O4), (O5), (O6) and (O9) that $\phi_{\alpha_1,\alpha_2}(t) = \phi^d_{\alpha_1,\alpha_2}(t)$ for $t \in [1, \tau^{dc}]$, $\phi_{\alpha_1,\alpha_2}(t) = \phi^c_{\alpha_1,\alpha_2}(t)$ for $t \in [\tau^{dc}, \tau^{cb}]$, and $\phi_{\alpha_1,\alpha_2}(t) = \phi^b_{\alpha_1,\alpha_2}(t)$ for $t \in [\tau^{cb}, u_1^{b_1}u_2^{b_2}]$, where $\mu_1(\tau^{dc}) = \lambda_2(\tau^{cb}) = \alpha_1\beta_2$. Direct computations show that $\tau^{dc} = t^{dc}$ and $\tau^{cb} = t^{cb}$. Lemma 3.4 then shows that $\frac{d}{dt}\phi^d_{\alpha_1,\alpha_2}(t^{dc}) = \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t^{dc})$, and $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t^{cb}) = \frac{d}{dt}\phi^b_{\alpha_1,\alpha_2}(t^{cb})$. Combined with Observation (O7), this establishes that $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[1, u_1^{b_1}u_2^{b_2}]$.

Assume finally that $\alpha_1\beta_2 \in [b_1a_2, b_1a_2u_2]$. It follows from Observations (O5), (O6) and (O8) that $\phi_{\alpha_1,\alpha_2}(t) = \phi^a_{\alpha_1,\alpha_2}(t)$ for $t \in [1, \tau^{ac}]$, $\phi_{\alpha_1,\alpha_2}(t) = \phi^c_{\alpha_1,\alpha_2}(t)$ for $t \in [\tau^{ac}, \tau^{cb}]$, and $\phi_{\alpha_1,\alpha_2}(t) = \phi^b_{\alpha_1,\alpha_2}(t)$ for $t \in [\tau^{cb}, u_1^{b_1}u_2^{b_2}]$, where $\lambda_1(\tau^{ac}) = \lambda_2(\tau^{cb}) = \alpha_1\beta_2$. Direct computations show that $\tau^{ac} = t^{ac}$ and $\tau^{cb} = t^{cb}$. Lemma 3.4 then shows that $\frac{d}{dt}\phi^a_{\alpha_1,\alpha_2}(t^{ac}) = \frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t^{ac})$, and $\frac{d}{dt}\phi^c_{\alpha_1,\alpha_2}(t^{cb}) = \frac{d}{dt}\phi^b_{\alpha_1,\alpha_2}(t^{cb})$. Combined with Observation (O7), this establishes that $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[1, u_1^{b_1}u_2^{b_2}]$.

(iii) From Observation (O9), we know that $\lambda(t) \leq b_1a_2u_2$ for $t \in [1, u_1^{b_1}u_2^{b_2}]$. The result then follows directly from Proposition 3.3.

□

It follows from Lemma 3.5 that $\phi_{\alpha_1,\alpha_2}(t)$ is either concave or piecewise concave on two intervals. It is therefore clear that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ is piecewise affine. When $\phi_{\alpha_1,\alpha_2}(t)$ has two concave pieces, we still need to determine whether its envelope has one or two affine pieces in order to derive lifting coefficients. To streamline the discussions associated with computing the slope of convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$, we introduce the notation $\Delta(t_1,t_2) = \frac{\phi_{\alpha_1,\alpha_2}(t_2)-\phi_{\alpha_1,\alpha_2}(t_1)}{t_2-t_1}$ for $t_2 \neq t_1$. We also define

$$K_1 = \left(\frac{u_1^{-b_1a_2}u_3^{a_2}-1}{u_3-u_1^{b_1}}\right)\left(\frac{u_1^{b_1}-l_3}{u_1-l_3^{a_1}}\right) = u_1^{-1}\left(\frac{u_1^{-b_1a_2}u_3^{a_2}-1}{u_1^{-b_1}u_3-1}\right)\left(\frac{1-u_1^{-b_1}l_3}{1-u_1^{-1}l_3^{a_1}}\right),$$

where $l_3 < u_1^{b_1} < u_3$, and

$$K_2 = \left(\frac{u_3-u_2^{b_2}}{u_2^{-a_1b_2}u_3^{a_1}-1}\right)\left(\frac{u_2-l_3^{a_2}}{u_2^{b_2}-l_3}\right) = u_2\left(\frac{u_2^{-b_2}u_3-1}{u_2^{-a_1b_2}u_3^{a_1}-1}\right)\left(\frac{1-u_2^{-1}l_3^{a_2}}{1-u_2^{-b_2}l_3}\right)$$

where $l_3 < u_2^{b_2} < u_3$.

Lemma 3.6. *The following inequalities hold true*

$$K_1 \leq b_1a_2u_1^{-1}, \qquad \text{when } l_3 < u_1^{b_1} < u_3, \text{and} \tag{15}$$

$$K_2 \geq b_1a_2u_2, \qquad \text{when } l_3 < u_2^{b_2} < u_3. \tag{16}$$

Proof. Let $I \subseteq \mathbb{R}$ be an open interval and let $f(x) : I \mapsto \mathbb{R}$ be a differentiable concave function over I. Given $x_0 \in I$, we know that $f(x) \leq f(x_0) + f'(x_0)(x-x_0)$ for $x \in I$. For $y \in I$ with $y > x_0$ and for $z \in I$ with $z < x_0$, we write that

$$(i) \quad f'(x_0) \geq \frac{f(y)-f(x_0)}{y-x_0} \qquad \text{and} \qquad (ii) \quad f'(x_0) \leq \frac{f(x_0)-f(z)}{x_0-z}. \tag{17}$$

To prove (15), we apply (17)(i) with $I = \mathbb{R}^+$, $f(x) = x^{a_2}$ where $a_2 \in (0,1]$, $x_0 = 1$, and $y = u_1^{-b_1}u_3$ (observe that $y > x_0$ because of the assumption that $u_1^{b_1} < u_3$.) We obtain $\frac{u_1^{-b_1a_2}u_3^{a_2}-1}{u_1^{-b_1}u_3-1} \leq a_2$. We then apply (17)(ii) with $I = \mathbb{R}^+$, $f(x) = x^{a_1}$ where $a_1 \in (0,1]$, $x_0 = 1$, and $z = u_1^{-b_1}l_3$ (observe that $z < x_0$ because of the assumption that $l_3 < u_1^{b_1}$.) We obtain $a_1 \leq \frac{1-u_1^{-1}l_3^{a_1}}{1-u_1^{-b_1}l_3}$. Multiplying the two inequalities derived above, we obtain (15) after scaling throughout by u_1^{-1}.

For (16), we apply (17)(ii) with $I = \mathbb{R}^+$, $f(x) = x^{a_2}$ where $a_2 \in (0,1]$, $x_0 = 1$, and $z = u_2^{-b_2}l_3$ (observe that $z < x_0$ because of the assumption that $l_3 > u_2^{b_2}$.) We obtain $\frac{1-u_2^{-1}l_3^{a_2}}{1-u_2^{-b_2}l_3} \geq a_2$. We then apply (17)(i) with $I = \mathbb{R}^+$, $f(x) = x^{a_1}$ where $a_1 \in (0,1]$, $x_0 = 1$, and $y = u_2^{-b_2}u_3$ (observe that $y > x_0$ because of the assumption that $u_3 > u_2^{b_2}$.) We obtain $a_1 \geq \frac{u_2^{-a_1b_2}u_3^{a_1}-1}{u_2^{-b_2}u_3-1}$. Multiplying the two inequalities derived above, we obtain (16) after scaling throughout by u_2. □

Observe that $\phi_{\alpha_1,\alpha_2}(t)$ can be overestimated using $\phi^b_{\alpha_1,\alpha_2}(t)$ (resp. $\phi^e_{\alpha_1,\alpha_2}(t)$) when $t > u_2^{b_2}$ (resp. $t > u_1^{b_1}$) irrespective of the values of α_1 and α_2. Combining this observation with the geometric insights about the solutions that attain $\phi_{\alpha_1,\alpha_2}(t)$, it is possible to argue that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is piecewise affine over various regions as detailed in Lemma 3.7. We provide a direct algebraic proof instead.

Lemma 3.7. *For $\alpha_1 > 0$ and $\alpha_2 > 0$,* *(i) If* $\alpha_1\beta_2 < K_1$*, the convex envelope of* $\phi_{\alpha_1,\alpha_2}(t)$ *is piecewise affine over* $[l_3, u_1^{b_1}]$ *and* $[u_1^{b_1}, u_3]$ *with* $\operatorname{conv}_{[l_3,u_3]} \phi_{\alpha_1,\alpha_2}(t) = \phi_{\alpha_1,\alpha_2}(t)$ *for* $t \in \{l_3, u_1^{b_1}, u_3\}$*.*

(ii) If $K_1 \le \alpha_1\beta_2 \le K_2$*, the convex envelope of* $\phi_{\alpha_1,\alpha_2}(t)$ *is affine over* $[l_3, u_3]$ *and for* $t \in \{l_3, u_3\}$*,* $\operatorname{conv}_{[l_3,u_3]} \phi_{\alpha_1,\alpha_2}(t) = \phi_{\alpha_1,\alpha_2}(t)$*.*

(iii) If $\alpha_1\beta_2 > K_2$*, the convex envelope of* $\phi_{\alpha_1,\alpha_2}(t)$ *is piecewise affine over* $[l_3, u_2^{b_2}]$ *and* $[u_2^{b_2}, u_3]$ *with* $\operatorname{conv}_{[l_3,u_3]} \phi_{\alpha_1,\alpha_2}(t) = \phi_{\alpha_1,\alpha_2}(t)$ *for* $t \in \{l_3, u_2^{b_2}, u_3\}$*.*

Proof. (i) We have that $\alpha_1\beta_2 < K_1 \le b_1a_2u_1^{-1}$, where the inequality holds because of Lemma 3.6. It follows from Lemma 3.5 that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_1^{b_1}]$ and $[u_1^{b_1}, u_3]$. If $l_3 = u_1^{b_1}$ or $u_3 = u_1^{b_1}$, the result is clear. Therefore, we assume that $l_3 < u_1^{b_1} < u_3$. We have that

$$\Delta(l_3, u_1^{b_1}) = \alpha_1 \frac{u_1 - l_3^{a_1}}{u_1^{b_1} - l_3} < \alpha_2 \frac{u_1^{-b_1a_2}u_3^{a_2} - 1}{u_3 - u_1^{b_1}} = \Delta(u_1^{b_1}, u_3),$$

where the inequality holds because of the definition of K_1 and the fact that $\alpha_1 < \alpha_2 K_1$. It follows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ has two affine pieces.

(ii) When $K_1 \le \alpha_1\beta_2 \le b_1a_2u_1^{-1}$, the expressions for $\Delta(l_3, u_1^{b_1})$ and $\Delta(u_1^{b_1}, u_3)$ are identical to those computed in (i). If $l_3 = u_1^{b_1}$ or $u_3 = u_1^{b_1}$, the result is clear. Therefore we assume that $l_3 < u_1^{b_1} < u_3$. It can be readily verified that $\Delta(l_3, u_1^{b_1}) \ge \Delta(u_1^{b_1}, u_3)$ as $\alpha_2 K_1 \le \alpha_1$. It follows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ is affine. When $\alpha_1\beta_2 \in [b_1a_2u_1^{-1}, b_1a_2u_2]$, Lemma 3.5 shows that $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[l_3, u_3]$. It follows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_3]$. When $b_1a_2u_2 \le \alpha_1\beta_2 \le K_2$, Lemma 3.5 shows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_2^{b_2}]$ and $[u_2^{b_2}, u_3]$. If $l_3 = u_2^{b_2}$ or $u_3 = u_2^{b_2}$, the result is clear. Therefore, we assume that $l_3 < u_2^{b_2} < u_3$. We compute that

$$\Delta(l_3, u_2^{b_2}) = \alpha_2 \frac{u_2 - l_3^{a_2}}{u_2^{b_2} - l_3} \ge \alpha_1 \frac{u_2^{-a_1b_2}u_3^{a_1} - 1}{u_3 - u_2^{b_2}} = \Delta(u_2^{b_2}, u_3),$$

where the inequality holds because of the definition of K_2 and the fact that $\alpha_2 \ge \alpha_1 K_2^{-1}$. We conclude that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ is affine.

(iii) We have that $b_1a_2u_2 \le K_2 < \alpha_1\beta_2$, where the inequality holds because of Lemma 3.6. It follows from Lemma 3.5 that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_2^{b_2}]$ and $[u_2^{b_2}, u_3]$. If $l_3 = u_2^{b_2}$ or $u_3 = u_2^{b_2}$, the result is clear. Therefore, we assume that $l_3 < u_2^{b_2} < u_3$. Observe that the expressions for $\Delta(l_3, u_2^{b_2})$ and $\Delta(u_2^{b_2}, u_3)$ are identical to those computed in (ii). It is simple to verify that $\Delta(l_3, u_2^{b_2}) < \Delta(u_2^{b_2}, u_3)$ as $\alpha_2 < \alpha_1 K_2^{-1}$. It follows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ has two affine pieces.

□

As the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ is polyhedral, the values of $\phi_{\alpha_1,\alpha_2}(t)$ at $t = l_3$ and $t = u_3$ are of particular interest. These values can be obtained directly from Proposition 3.3.

Lemma 3.8. *We have that*

$$\phi_{\alpha_1,\alpha_2}(l_3) = \begin{cases} \alpha_1 + \alpha_2 {l_3}^{a_2} & \text{when} \quad \alpha_1\beta_2 \ge b_1a_2l_3^{a_2} \\ \bar{c}{l_3}^{A} & \text{when} \quad b_1a_2l_3^{-a_1} \le \alpha_1\beta_2 \le b_1a_2l_3^{a_2} \\ \alpha_1 {l_3}^{a_1} + \alpha_2 & \text{when} \quad \alpha_1\beta_2 \le b_1a_2l_3^{-a_1}, \end{cases}$$

and

$$\phi_{\alpha_1,\alpha_2}(u_3) = \begin{cases} \alpha_1 u_2^{-a_1 b_2} u_3{}^{a_1} + \alpha_2 u_2 & \textit{when} \quad \alpha_1\beta_2 \geq b_1 a_2 u_2^{Ba_1} u_3^{-a_1} \\ \bar{c} u_3{}^{A} & \textit{when} \quad b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \leq \alpha_1\beta_2 \leq b_1 a_2 u_2^{Ba_1} u_3^{-a_1} \\ \alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} u_3{}^{a_2} & \textit{when} \quad \alpha_1\beta_2 \leq b_1 a_2 u_1^{-Ba_2} u_3^{a_2}. \end{cases}$$

□

It is simple to verify that

$$b_1 a_2 u_1^{-1} \leq b_1 a_2 l_3^{-a_1} \leq b_1 a_2 l_3^{a_2} \leq b_1 a_2 u_2 \tag{18}$$

because of Assumptions (A6), (A5), and (A6), respectively, and that

$$b_1 a_2 u_1^{-1} \leq b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \leq b_1 a_2 u_2^{Ba_1} u_3^{-a_1} \leq b_1 a_2 u_2 \tag{19}$$

because of Assumptions (A7), (A4), and (A7), respectively. Further, $b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \leq b_1 a_2 l_3^{a_2}$ because of Assumptions (A4), (A5) and (A8). The exact relations between $b_1 a_2 l_3^{-a_1}$ and $b_1 a_2 u_1^{-Ba_2} u_3^{a_2}$, between $b_1 a_2 l_3^{-a_1}$ and $b_1 a_2 u_2^{Ba_1} u_3^{-a_1}$, and between $b_1 a_2 l_3^{a_2}$ and $b_1 a_2 u_2^{Ba_1} u_3^{-a_1}$ depend on the particular instance studied.

Next, we derive lifted valid inequalities for each (α_1, α_2) when $\alpha_1 > 0$ and $\alpha_2 > 0$. During this process, we obtain the isolated inequalities

$$x_1 + K_1^{-1} x_2 + \frac{l_3^{a_1} - u_1}{u_1^{b_1} - l_3} x_3 \geq \frac{u_1^{b_1} l_3^{a_1} - u_1 l_3}{u_1^{b_1} - l_3} + K_1^{-1}, \ (\text{if } l_3 < u_1^{b_1}) \tag{20}$$

$$K_2 x_1 + x_2 + \frac{l_3^{a_2} - u_2}{u_2^{b_2} - l_3} x_3 \geq \frac{u_2^{b_2} l_3^{a_2} - u_2 l_3}{u_2^{b_2} - l_3} + K_2, \ (\text{if } l_3 < u_2^{b_2}) \tag{21}$$

together with the families of inequalities

$$\begin{array}{l} d_3\alpha_1 x_1 + d_3\alpha_2 x_2 - (\alpha_1 l_3{}^{a_1} + \alpha_2)(u_3 - x_3) - (\bar{c} u_3{}^{A})(x_3 - l_3) \geq 0, \\ \text{if } b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \leq \alpha_1\beta_2 \leq \min\{b_1 a_2 l_3^{-a_1}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\} \end{array} \tag{22}$$

$$\begin{array}{l} d_3\alpha_1 x_1 + d_3\alpha_2 x_2 - (\bar{c} l_3{}^{A})(u_3 - x_3) - (\alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} u_3{}^{a_2})(x_3 - l_3) \geq 0, \\ \text{if } b_1 a_2 l_3^{-a_1} \leq \alpha_1\beta_2 \leq b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \end{array} \tag{23}$$

$$\begin{array}{l} d_3\alpha_1 x_1 + d_3\alpha_2 x_2 - (\bar{c} l_3{}^{A})(u_3 - x_3) - (\bar{c} u_3{}^{A})(x_3 - l_3) \geq 0, \\ \text{if } \max\{b_1 a_2 l_3^{-a_1}, b_1 a_2 u_1^{-Ba_2} u_3^{a_2}\} < \alpha_1\beta_2 < \min\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\} \end{array} \tag{24}$$

$$\begin{array}{l} d_3\alpha_1 x_1 + d_3\alpha_2 x_2 - (\alpha_1 + \alpha_2 l_3{}^{a_2})(u_3 - x_3) - (\bar{c} u_3{}^{A})(x_3 - l_3) \geq 0, \\ \text{if } b_1 a_2 l_3^{a_2} \leq \alpha_1\beta_2 \leq b_1 a_2 u_2^{Ba_1} u_3^{-a_1} \end{array} \tag{25}$$

$$\begin{array}{l} d_3\alpha_1 x_1 + d_3\alpha_2 x_2 - (\bar{c} l_3{}^{A})(u_3 - x_3) - (\alpha_1 u_2^{-a_1 b_2} u_3{}^{a_1} + \alpha_2 u_2)(x_3 - l_3) \geq 0, \\ \text{if } \max\{b_1 a_2 l_3^{-a_1}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\} \leq \alpha_1\beta_2 \leq b_1 a_2 l_3^{a_2}. \end{array} \tag{26}$$

Proposition 3.9. *The only linear inequalities with coefficients $\alpha_1 > 0$ and $\alpha_2 > 0$ necessary in the description of* $\text{conv}(S^{\geq})$ *are among (20)-(26).*

Proof. In light of (18), (19) and Lemma 3.5, we differentiate several cases based on the value that $\alpha_1\beta_2$ takes.

Case 1: $0 < \alpha_1\beta_2 < K_1$**.** Assume first that $l_3 < u_1^{b_1} < u_3$, an assumption that implies that the two intervals $[l_3, u_1^{b_1}]$ and $[u_1^{b_1}, u_3]$ do not reduce to a point. It follows from Lemma 3.7 that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ has two pieces. Because $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 t^{a_1} + \alpha_2$ for $t \in [l_3, u_1^{b_1}]$ and $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} t^{a_2}$ for $t \in [u_1^{b_1}, u_3]$, it follows from (3) that the corresponding lifting coefficients (α_3, δ) are linear in (α_1, α_2). Since the interval of interest for $\alpha_1\beta_2$ is open, these inequalities are not needed in the description of $\operatorname{conv}(S^{\geq})$. When $l_3 = u_1^{b_1}$ (resp. $u_3 = u_1^{b_1}$), $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[l_3, u_3]$. It follows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_3]$. Since $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 {l_3}^{a_1} + \alpha_2$ and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} {u_3}^{a_2}$, we obtain from (3) that lifting coefficients (α_3, δ) are linear in (α_1, α_2). Therefore, no such lifted inequality is needed in the description of $\operatorname{conv}(S^{\geq})$.

Case 2: $K_1 \leq \alpha_1\beta_2 \leq K_2$**.** It follows from Lemma 3.7 that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_3]$, we therefore only need to compute the values of $\phi_{\alpha_1,\alpha_2}(t)$ at $t = l_3$ and $t = u_3$. There are three subcases.

Case 2.1: $K_1 \leq \alpha_1\beta_2 < \min\{b_1 a_2 l_3^{-a_1}, b_1 a_2 u_1^{-Ba_2} u_3^{a_2}\}$**.** Since $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 {l_3}^{a_1} + \alpha_2$ and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} {u_3}^{a_2}$, we obtain from (3) that lifting coefficients (α_3, δ) are linear in (α_1, α_2). Therefore, since the interval of interest for $\alpha_1\beta_2$ is semi-open, the only such lifted inequality that is needed in the description of $\operatorname{conv}(S^{\geq})$ is that for which $\alpha_1\beta_2 = K_1$. Using (4), we obtain (20).

Case 2.2: $\min\{b_1 a_2 l_3^{-a_1}, b_1 a_2 u_1^{-Ba_2} u_3^{a_2}\} \leq \alpha_1\beta_2 \leq \max\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\}$**.** We consider two subcases.

Case 2.2.1: $b_1 a_2 l_3^{-a_1} \leq b_1 a_2 u_1^{-Ba_2} u_3^{a_2}$, *i.e.*, $u_1^{Ba_2} \leq l_3^{a_1} u_3^{a_2}$. Whenever $b_1 a_2 l_3^{-a_1} \leq \alpha_1\beta_2 \leq b_1 a_2 u_1^{-Ba_2} u_3^{a_2}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \bar{c} {l_3}^A$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_1 + \alpha_2 u_1^{-b_1 a_2} {u_3}^{a_2}$, yielding (23). When $b_1 a_2 u_1^{-Ba_2} u_3^{a_2} < \alpha_1\beta_2 < \min\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \bar{c} {l_3}^A$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \bar{c} {u_3}^A$, yielding (24). When $\min\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\} \leq \alpha_1\beta_2 \leq \max\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\}$, we have two possibilities. If $l_3^{a_2} u_3^{a_1} \leq u_2^{Ba_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 + \alpha_2 {l_3}^{a_2}$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \bar{c} {u_3}^A$, yielding (25). If $l_3^{a_2} u_3^{a_1} > u_2^{Ba_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \bar{c} {l_3}^A$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_2^{-a_1 b_2} {u_3}^{a_1} + \alpha_2 u_2$, yielding (26).

Case 2.2.2: $b_1 a_2 l_3^{-a_1} > b_1 a_2 u_1^{-Ba_2} u_3^{a_2}$, *i.e.*, $u_1^{Ba_2} > l_3^{a_1} u_3^{a_2}$. We assume first that $b_1 a_2 l_3^{-a_1} \leq b_1 a_2 u_2^{Ba_1} u_3^{-a_1}$, *i.e.*, $l_3^{-a_1} u_3^{a_1} \leq u_2^{Ba_1}$. When $b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \leq \alpha_1\beta_2 \leq b_1 a_2 l_3^{-a_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 {l_3}^{a_1} + \alpha_2$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \bar{c} {u_3}^A$, yielding (22). When $b_1 a_2 l_3^{-a_1} < \alpha_1\beta_2 < \min\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \bar{c} {l_3}^A$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \bar{c} {u_3}^A$, yielding (24). When $\min\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\} \leq \alpha_1\beta_2 \leq \max\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\}$ then similar to Case 2.2.1, we have two possibilities. If $l_3^{a_2} u_3^{a_1} \leq u_2^{Ba_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 + \alpha_2 {l_3}^{a_2}$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \bar{c} {u_3}^A$, yielding (25). If $l_3^{a_2} u_3^{a_1} > u_2^{Ba_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \bar{c} {l_3}^A$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_2^{-a_1 b_2} {u_3}^{a_1} + \alpha_2 u_2$, yielding (26). We next assume that $b_1 a_2 l_3^{-a_1} > b_1 a_2 u_2^{Ba_1} u_3^{-a_1}$, *i.e.*, $l_3^{-a_1} u_3^{a_1} > u_2^{Ba_1}$. When $b_1 a_2 u_1^{-Ba_2} u_3^{a_2} \leq \alpha_1\beta_2 \leq b_1 a_2 u_2^{Ba_1} u_3^{-a_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 {l_3}^{a_1} + \alpha_2$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \bar{c} {u_3}^A$, yielding (22). When $b_1 a_2 u_2^{Ba_1} u_3^{-a_1} < \alpha_1\beta_2 < b_1 a_2 l_3^{-a_1}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 {l_3}^{a_1} + \alpha_2$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_2^{-a_1 b_2} {u_3}^{a_1} + \alpha_2 u_2$. It follows from (3) that the corresponding lifting coefficients (α_3, δ) are linear in (α_1, α_2). Therefore, none of these inequalities are required in the description of $\operatorname{conv}(S^{\geq})$ as the interval of interest for $\alpha_1\beta_2$ is open. When $b_1 a_2 l_3^{-a_1} \leq \alpha_1\beta_2 \leq b_1 a_2 l_3^{a_2}$ then $\phi_{\alpha_1,\alpha_2}(l_3) = \bar{c} {l_3}^A$, and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_2^{-a_1 b_2} {u_3}^{a_1} + \alpha_2 u_2$, yielding (26).

Case 2.3: $\max\{b_1 a_2 l_3^{a_2}, b_1 a_2 u_2^{Ba_1} u_3^{-a_1}\} < \alpha_1\beta_2 < K_2$**.** This case is symmetric to Case 2.1. As $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 + \alpha_2 {l_3}^{a_2}$ and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_2^{-a_1 b_2} {u_3}^{a_1} + \alpha_2 u_2$, we obtain from (3) that lifting coefficients (α_3, δ) are linear in (α_1, α_2). Therefore, the only such lifted inequalities needed in the description of $\text{conv}(S^{\geq})$ is that for which $\alpha_1\beta_2 = K_2$. We obtain (21).

Case 3: $K_2 < \alpha_1\beta_2$**.** Assume first that $l_3 < u_2^{b_2} < u_3$, an assumption that implies that these two intervals $[l_3, u_2^{b_2}]$ and $[u_2^{b_2}, u_3]$ do not reduce to a point. It follows from Lemma 3.7 that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ over $[l_3, u_3]$ has two pieces. Because Lemma 3.5 shows that $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 + \alpha_2 t^{a_2}$ for $t \in [l_3, u_2^{b_2}]$ and $\phi_{\alpha_1,\alpha_2}(t) = \alpha_1 u_2^{-a_1 b_2} t^{a_1} + \alpha_2 u_2$ for $t \in [u_2^{b_2}, u_3]$, it follows from (3) that the corresponding lifting coefficients (α_3, δ) are linear in (α_1, α_2). Since the interval of interest for $\alpha_1\beta_2$ is open, these inequalities are not needed in the description of $\text{conv}(S^{\geq})$. When $l_3 = u_2^{b_2}$ (resp. $u_3 = u_2^{b_2}$), $\phi_{\alpha_1,\alpha_2}(t)$ is concave over $[l_3, u_3]$. It follows that the convex envelope of $\phi_{\alpha_1,\alpha_2}(t)$ is affine over $[l_3, u_3]$. Since $\phi_{\alpha_1,\alpha_2}(l_3) = \alpha_1 + \alpha_2 {l_3}^{a_2}$ and $\phi_{\alpha_1,\alpha_2}(u_3) = \alpha_1 u_2^{-a_1 b_2} {u_3}^{a_1} + \alpha_2 u_2$, we obtain from (3) that lifting coefficients (α_3, δ) are linear in (α_1, α_2). Therefore, the corresponding lifted inequalities are not needed in the description of $\text{conv}(S^{\geq})$.

□

Observe that the families of inequalities (22)-(26) are applicable for all values of $\alpha_1\beta_2$ in a certain (possibly empty) interval. Because the intervals on $\alpha_1\beta_2$ associated with inequalities (22)-(23) cannot be nonempty simultaneously (unless they reduce to a single point), and because the intervals on $\alpha_1\beta_2$ associated with inequalities (25)-(26) cannot be nonempty simultaneously (unless they reduce to a single point), it can be verified that at most three of the five identified families of linear inequalities are needed in any instance of $\text{conv}(S^{\geq})$.

Also we observe that, when $b_1 = b_2 = 1$, $K_1 = u_1^{-1}$ and $K_2 = u_2$. It follows that, in this case, inequalities (20) and (21) simplify to $x_1 + u_1 x_2 - u_1 \geq x_3$ and $u_2 x_1 + x_2 - u_2 \geq x_3$, which can be easily derived from McCormick envelopes of the function $x_1 x_2$ over $[1, u_1] \times [1, u_2]$.

3.2 Nonlinear description of $\text{conv}(S^{\geq})$

In this section, we turn the infinite families of linear inequalities (22)-(26) obtained in Proposition 3.9 into nonlinear inequalities. Since $\alpha_2 > 0$, We may assume that $\alpha_2 = 1$ through scaling. It is then simple to verify that (22)-(26) can be written as

$$\alpha_1 p + q - (\alpha_1)^{A b_1} s \geq 0, \qquad \text{for} \qquad g \leq \alpha_1 \leq h \tag{27}$$

where p, q, and s are affine functions of (x_1, x_2, x_3) and (g, h) are nonnegative parameters. It is easy to see that including the boundaries of the interval (g, h) for (24) is admissible. We next describe these functions and parameters for (22)-(26), in this order:

$$\begin{aligned}
p_1 &= d_3 x_1 - l_3^{a_1}(u_3 - x_3)\\
q_1 &= d_3 x_2 - (u_3 - x_3)\\
s_1 &= c u_3^{A}(x_3 - l_3)\\
g_1 &= b_1 a_2 u_1^{-B a_2} u_3^{a_2}\\
h_1 &= \min\{b_1 a_2 l_3^{-a_1}, b_1 a_2 u_2^{B a_1} u_3^{-a_1}\},
\end{aligned} \tag{28}$$

$$\begin{aligned}
p_2 &= d_3 x_1 - u_1(x_3 - l_3)\\
q_2 &= d_3 x_2 - u_1^{-b_1 a_2} u_3^{a_2}(x_3 - l_3)\\
s_2 &= c l_3^{A}(u_3 - x_3)\\
g_2 &= b_1 a_2 l_3^{-a_1}\\
h_2 &= b_1 a_2 u_1^{-B a_2} u_3^{a_2},
\end{aligned} \tag{29}$$

$$\begin{aligned}
p_3 &= d_3x_1\\
q_3 &= d_3x_2\\
s_3 &= c(l_3^A(u_3-x_3)+u_3^A(x_3-l_3))\\
g_3 &= \max\{b_1a_2l_3^{-a_1}, b_1a_2u_1^{-Ba_2}u_3^{a_2}\}\\
h_3 &= \min\{b_1a_2l_3^{a_2}, b_1a_2u_2^{Ba_1}u_3^{-a_1}\},
\end{aligned} \tag{30}$$

$$\begin{aligned}
p_4 &= d_3x_1-(u_3-x_3)\\
q_4 &= d_3x_2-l_3^{a_2}(u_3-x_3)\\
s_4 &= cu_3^A(x_3-l_3)\\
g_4 &= b_1a_2l_3^{a_2}\\
h_4 &= b_1a_2u_2^{Ba_1}u_3^{-a_1},
\end{aligned} \tag{31}$$

$$\begin{aligned}
p_5 &= d_3x_1-u_2^{-a_1b_2}u_3^{a_1}(x_3-l_3)\\
q_5 &= d_3x_2-u_2(x_3-l_3)\\
s_5 &= cl_3^A(u_3-x_3)\\
g_5 &= \max\{b_1a_2l_3^{-a_1}, b_1a_2u_2^{Ba_1}u_3^{-a_1}\}\\
h_5 &= b_1a_2l_3^{a_2}.
\end{aligned} \tag{32}$$

Proposition 3.10. *For the values of* (p,q,s,g,h) *presented in* (28)-(32)*, the families of inequalities* (22)-(26) *for which* $g \le h$ *can be expressed as*

$$\begin{aligned}
gp+q-g^{Ab_1}s \ge 0\\
\textit{when } p \ge Ab_1g^{-Ab_2}s,
\end{aligned} \tag{33}$$

$$\begin{aligned}
B\,(a_1p)^{Ab_1}\,(a_2q)^{Ab_2} \ge s\\
\textit{when } p \ge 0, Ab_1h^{-Ab_2}s < p < Ab_1g^{-Ab_2}s,
\end{aligned} \tag{34}$$

$$\begin{aligned}
hp+q-h^{Ab_1}s \ge 0\\
\textit{when } p \le Ab_1h^{-Ab_2}s.
\end{aligned} \tag{35}$$

Proof. It can be readily verified that, for each family (in (28)-(32)), $s \ge 0$ whenever $x \in [l,u]^3$. For (x_1,x_2,x_3), the value of α_1 that yields the tighter valid inequality in the family (27) is obtained by solving

$$\min\{\alpha_1p+q-(\alpha_1)^{Ab_1}s \mid g \le \alpha_1 \le h\}, \tag{36}$$

which is a feasible convex program since its objective function is strictly convex as $0 < Ab_1 < 1$, $s \ge 0$ and $g \le h$. We claim that an optimal solution to (36) is given by

$$\alpha_1^* = \begin{cases} g & p \ge Ab_1g^{-Ab_2}s,\\ \left(\frac{Ab_1s}{p}\right)^{Ba_2} & Ab_1h^{-Ab_2}s < p < Ab_1g^{-Ab_2}s,\\ h & p \le Ab_1h^{-Ab_2}s. \end{cases}$$

We next give a proof of this claim. If $p \le 0$, an optimal solution is obtained by setting $\alpha_1^* = h$ since the objective function is nonincreasing. We can therefore assume that $p > 0$. If $s = 0$, then an optimal solution is obtained by setting $\alpha_1^* = g$, since the objective function is increasing. We therefore assume that $s > 0$. In this case, the derivative of the objective function of (36) is zero at $\gamma = \left(\frac{Ab_1s}{p}\right)^{Ba_2}$. It is then clear that $\alpha_1^* = \min\{\max\{g,\gamma\},h\}$, yielding the desired result. In fact, $\alpha_1^* = g$ when $\gamma \leq g$, *i.e.*, $Ab_1 g^{-Ab_2} s \leq p$, producing (33). Similarly $\alpha_1^* = h$ when $\gamma \geq h$, *i.e.*, $Ab_1 h^{-Ab_2} s \geq p$, producing (35). Otherwise, $\alpha_1^* = \gamma$. Plugging the expression for γ in (27) yields the inequality $q \geq (Ab_2)(\frac{Ab_1 s}{p})^{b_1 a_2} s$, which implies that $q \geq 0$. This nonlinear inequality is therefore valid when $(p, q, s) \in F$ where

$$F = \{(p, q, s) \,|\, p > 0, q \geq 0, Ab_1 h^{-Ab_2} s < p < Ab_1 g^{-Ab_2} s\}.$$

Over $F \cap [l, u]$, it can be rewritten as (34). ☐

We mention that each of the above inequalities (33)-(35) is associated with a subset of values of (x_1, x_2, x_3) over which the inequality is valid and strong. In the case of (33) and (35), these inequalities are valid outside of their prescribed subsets, since they were obtained for a specific value of α that is clearly feasible, even if not optimal. The same conclusion does not hold in general for the nonlinear inequality (34), which may become invalid outside of its prescribed range.

We summarize the description of $\mathrm{conv}(S^{\geq})$ in the following theorem.

Theorem 3.11. *Under Assumptions (A1)-(A8), a description of* $\mathrm{conv}(S^{\geq})$ *is obtained by combining inequalities* (5)-(12), (20)-(21), *and* (33)-(35) *for all* (p, q, s, g, h) *defined in* (28)-(32) *for which* $g \leq h$. ☐

Among the nonlinear inequalities developed above, inequality (34) with $(p_3, q_3, s_3, g_3, h_3)$ is special in that it does not involve bounds on variables x_1 and x_2, and admits a straightforward derivation. First, after filling in the functional forms for p_3, q_3 and s_3, we see that (34) can be written as

$$x_1^{\frac{b_1}{b_1+b_2}} x_2^{\frac{b_2}{b_1+b_2}} \geq l_3^{\frac{1}{b_1+b_2}} \left(\frac{u_3 - x_3}{u_3 - l_3}\right) + u_3^{\frac{1}{b_1+b_2}} \left(\frac{x_3 - l_3}{u_3 - l_3}\right). \tag{37}$$

The validity of (37) can be argued as follows. First, we observe that the inequality defining $S^{\geq}$ can be equivalently written in the form $x_1^{\frac{b_1}{b_1+b_2}} x_2^{\frac{b_2}{b_1+b_2}} \geq x_3^{\frac{1}{b_1+b_2}}$. Since the function $x_3^{\frac{1}{b_1+b_2}}$ is concave over $[l_3, u_3]$, it can be lower-estimated by the linear function $l_3^{\frac{1}{b_1+b_2}} \left(\frac{u_3-x_3}{u_3-l_3}\right) + u_3^{\frac{1}{b_1+b_2}} \left(\frac{x_3-l_3}{u_3-l_3}\right)$, which takes values $l_3^{\frac{1}{b_1+b_2}}$ at $x_3 = l_3$ and $u_3^{\frac{1}{b_1+b_2}}$ at $x_3 = u_3$, yielding the desired result. In particular, the above derivation shows that (37) is globally valid for $\mathrm{conv}(S^{\geq})$. Theorem 3.11 shows that this inequality is in fact, an important component of the convex hull of $S^{\geq}$.

We conclude this section by illustrating the result of Theorem 3.11 on an example.

Example 3.12. *Consider an instance of* $S^{\geq}$ *where* $l_1 = l_2 = 1$, $l_3 = 16$, $u_1 = 36$, $u_2 = 5$, $u_3 = 54$, $b_1 = 1$ *and* $b_2 = 2$. *The trivial inequalities* (5)-(12) *can be written as*

$$\begin{aligned} 1 \leq x_1 &\leq 36, \\ 1 \leq x_2 &\leq 5, \\ 16 \leq x_3 &\leq 54, \\ 18x_2 + \left(1 - \frac{\sqrt{6}}{2}\right) x_3 &\geq 54 - 18\sqrt{6}, \\ 25x_1 - x_3 &\geq 0. \end{aligned}$$

Further, inequalities (20)-(21) *take the form*

$$\begin{aligned} x_1 + \frac{36}{\sqrt{6}-2} x_2 - x_3 &\geq \frac{36}{\sqrt{6}-2}, \\ \left(\frac{25}{9}\right) x_1 + x_2 - \left(\frac{1}{9}\right) x_3 &\geq \frac{20}{9}, \end{aligned}$$

since $K_1 = \frac{\sqrt{6}-2}{36}$ *and* $K_2 = \frac{25}{9}$.

Next, we compute that $g_1 = \frac{\sqrt{6}}{144} < h_1 = \frac{1}{32}$, $g_2 = \frac{1}{32} > h_2 = \frac{\sqrt{6}}{144}$, $g_3 = \frac{1}{32} < h_3 = \frac{125}{108}$, $g_4 = 2 > h_4 = \frac{125}{108}$, *and* $g_5 = \frac{125}{108} < h_5 = 2$. *This shows that families 1, 3, 5 are applicable while the other two families are not. For the first family, we obtain*

$$\left(\frac{\sqrt{6}}{144}\right)(38x_1 + 16x_3 - 864) + (38x_2 + x_3 - 54) - \left(\frac{\sqrt{6}}{144}\right)^{\frac{1}{3}} \frac{9}{2}\sqrt[3]{4}(x_3 - 16) \geq 0,$$

$$3\,(38x_1 + 16x_3 - 864)^{\frac{1}{3}} \left(\frac{38x_2 + x_3 - 54}{2}\right)^{\frac{2}{3}} \geq \frac{9}{2}\sqrt[3]{4}(x_3 - 16),$$

$$\left(\frac{1}{32}\right)(38x_1 + 16x_3 - 864) + (38x_2 + x_3 - 54) - \left(\frac{1}{32}\right)^{\frac{1}{3}} \frac{9}{2}\sqrt[3]{4}(x_3 - 16) \geq 0,$$

where the nonlinear inequality is applicable when $38x_1 + 16x_3 - 864 \geq 0$ *and* $8x_3 + 480 < 38x_1 < 20x_3 + 288$. *For the third family, we obtain*

$$\left(\frac{1}{32}\right)(38x_1) + (38x_2) - \left(\frac{1}{32}\right)^{\frac{1}{3}} \frac{3}{2}\sqrt[3]{4}x_3 + 90\sqrt[3]{4} \geq 0,$$

$$3\,(38x_1)^{\frac{1}{3}} \left(\frac{38x_2}{2}\right)^{\frac{2}{3}} \geq \frac{3}{2}\sqrt[3]{4}x_3 + 90\sqrt[3]{4},$$

$$\left(\frac{125}{108}\right)(38x_1) + (38x_2) - \left(\frac{125}{108}\right)^{\frac{1}{3}} \frac{3}{2}\sqrt[3]{4}x_3 + 90\sqrt[3]{4} \geq 0,$$

where the nonlinear inequality is applicable when $38x_1 \geq 0$ *and* $\frac{18}{25}x_3 + \frac{216}{5} < 38x_1 < 8x_3 + 480$. *For the fifth family, we obtain*

$$\left(\frac{125}{108}\right)\left(38x_1 - \frac{54}{25}x_3 + \frac{864}{25}\right) + (38x_2 - 5x_3 + 80) - \left(\frac{125}{108}\right)^{\frac{1}{3}} 3\sqrt[3]{4}(54 - x_3) \geq 0,$$

$$3\left(38x_1 - \frac{54}{25}x_3 + \frac{864}{25}\right)^{\frac{1}{3}} \left(\frac{38x_2 - 5x_3 + 80}{2}\right)^{\frac{2}{3}} \geq 3\sqrt[3]{4}(54 - x_3),$$

$$2\left(38x_1 - \frac{54}{25}x_3 + \frac{864}{25}\right) + (38x_2 - 5x_3 + 80) - (2)^{\frac{1}{3}}\, 3\sqrt[3]{4}(54 - x_3) \geq 0,$$

where the nonlinear inequality is applicable when $38x_1 - \frac{54}{25}x_3 + \frac{864}{25} \geq 0$ *and* $\frac{29}{25}x_3 + \frac{486}{25} < 38x_1 < \frac{18}{25}x_3 + \frac{216}{5}$.

In Figure 1, we give a representation of the region defined by the nontrivial inequalities. In particular, we clearly observe that the three families of nonlinear inequalities have a preponderant role in this description.

4 Convex hull of $S^{\leq}$

In this section, we study the convex hull of

$$S^{\leq} = \{x \in [l, u]^3 \mid x_1^{b_1} x_2^{b_2} \leq x_3\},$$

where $b_1 \geq 1$, $b_2 \geq 1$, and where $l_i \in \mathbb{R}_{\geq}$ and $u_i \in \mathbb{R}_{\geq}$ with $l_i \leq u_i$ for $i = 1, 2, 3$. In the remainder of this section, we also impose the following restrictive assumption

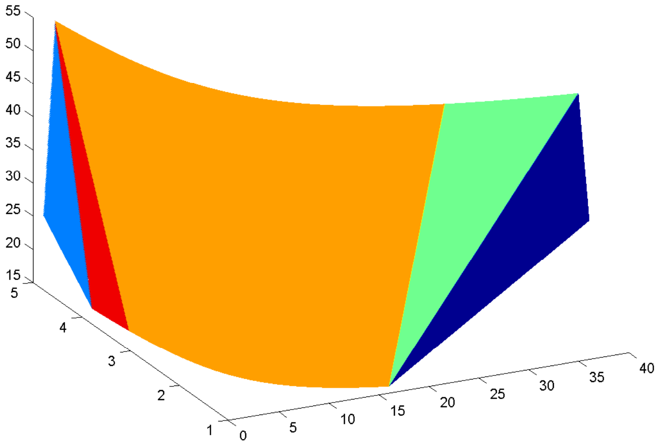

Figure 1: Convex hull description of Example 3.12

(B0) $b_1 = 1$.

When Assumption (B0) is not satisfied, the geometry of the problem is more complicated and, as a result, the derivation of lifted inequalities is harder. In studying $S^{\le}$, we pose the following additional assumptions

(B1) $l_2 = 1$, and $l_3 = 1$,

(B2) $l_1 \le \min\{1, u_2^{-b_2} u_3\}$,

(B3) $u_1 > l_1$, $u_2 > 1$, and $u_3 > 1$,

(B4) $l_1 \ge u_2^{-b_2}$,

(B5) $u_1 \ge \max\{1, u_2^{-b_2} u_3\}$,

(B6) $u_1 \le u_3$.

We next argue that Assumptions (B1)-(B6) are made without loss of generality. Assumption (B1) can be achieved by scaling variables x_2 and x_3. For Assumption (B2), we observe that there is no feasible solution with $x_3 = 1$ when $l_1 > 1$ as the relation $x_3 \ge x_1 x_2^{b_2} \ge l_1$ must be satisfied by all feasible solutions of $S^{\le}$. The lower bound on x_3 can therefore be increased to l_1. After rescaling variables x_1 and x_3 so that the lower bound on x_3 equals 1, we obtain that $l_1 \le 1$. Similarly, we observe that the defining inequality of $S^{\le}$ implies that $x_2^{b_2} \le \frac{x_3}{x_1} \le \frac{u_3}{l_1}$ in all feasible solutions. If $u_2^{b_2} > l_1^{-1} u_3$, we can tighten the upper bound on variable x_2 to be $l_1^{-a_2} u_3^{a_2}$. After this tightening, $l_1 u_2^{b_2} \le u_3$. For Assumption (B3), we note that $S^{\le}$ is convex when $u_i = l_i$ for $i \in \{1, 2\}$, and therefore does not warrant further study. When $u_3 = l_3$, $S^{\le} = \{(x_1, x_2) \in [l, u]^2 \,|\, x_1 \le l_3 x_2^{-b_2}\} \times \{l_3\}$. It is readily verified that $\operatorname{conv}(S^{\le})$ is a polyhedron with a simple linear description. For Assumption (B4), we note that, when $l_1 < u_2^{-b_2}$, $S^{\le} = A \cup B$ where $A = [l, u] \cap \{x_1 \le u_2^{-b_2}\}$ and $B = S^{\le} \cap \{x_1 \ge u_2^{-b_2}\}$. Applying Lemma 2.3 with $j = 1$ and $\theta = u_2^{-b_2}$, we obtain that $\operatorname{conv}(S^{\le}) = A \cup \operatorname{conv}(B)$. Therefore $\operatorname{conv}(S^{\le})$ can be obtained by focusing on the case where $l_1 \ge u_2^{-b_2}$. We proceed similarly for Assumption (B5). When $u_1 < 1$, we see that $S^{\le} = A \cup B$

where $A = [l, u] \cap \{x_2 \le u_1^{-a_2}\}$ and $B = S^{\le} \cap \{x_2 \ge u_1^{-a_2}\}$. Applying Lemma 2.3 with $j = 2$ and $\theta = u_1^{-a_2}$, we obtain that $\operatorname{conv}(S^{\le}) = A \cup \operatorname{conv}(B)$. When $u_1 u_2^{b_2} < u_3$, we see that $S^{\le} = A \cup B$ where $A = [l, u] \cap \{x_3 \ge u_1 u_2^{b_2}\}$ and $B = S^{\le} \cap \{x_3 \le u_1 u_2^{b_2}\}$. It then follows from Lemma 2.4 with $j = 3$ and $\theta = u_1 u_2^{b_2}$ that $\operatorname{conv}(S^{\le}) = A \cup \operatorname{conv}(B)$. For Assumption (B6), we observe that, when $u_1 > u_3$, there is no solution in $S^{\le}$ with $x_1 = u_1$. In this case, we can tighten the upper bound on variable x_1 to $\min\{u_1, u_3\}$.

4.1 Linear description of $\operatorname{conv}(S^{\le})$

In this section, we derive all nondominated linear valid inequalities for $\operatorname{conv}(S^{\le})$ that are lifted from linear constraints in the space of variables (x_2, x_3). We thereby provide a semi-infinite description of $\operatorname{conv}(S^{\le})$. We turn this semi-infinite description into one that contains only a finite number of linear and nonlinear inequalities in Section 4.2.

To obtain the desired linear inequalities, we introduce the function, $\psi_{\alpha_2,\alpha_3}(\cdot) : \mathbb{R} \mapsto \mathbb{R}$, defined as

$$\psi_{\alpha_2,\alpha_3}(t) := \min\{-\alpha_2 x_2 + \alpha_3 x_3 \mid t x_2^{b_2} \le x_3, 1 \le x_2 \le u_2, 1 \le x_3 \le u_3\}, \tag{38}$$

where $\alpha_2, \alpha_3 \in \mathbb{R}$, and $t \in \mathbb{R}$. If (38) is infeasible for some $t \in \mathbb{R}$, then we write that $\psi_{\alpha_2,\alpha_3}(t) = \infty$.

Similar to Section 3, the problem of finding, for each (α_2, α_3), coefficients (α_1, δ) that make

$$\alpha_1 x_1 - \alpha_2 x_2 + \alpha_3 x_3 \ge \delta \tag{39}$$

valid and undominated for $S^{\le}$ corresponds to finding the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$. If a pair (α_1, δ) describes a hyperplane that supports the epigraph of $\psi_{\alpha_2,\alpha_3}(t)$ at two distinct points $(t_1, \psi_{\alpha_2,\alpha_3}(t_1))$ and $(t_2, \psi_{\alpha_2,\alpha_3}(t_2))$, then

$$(\alpha_1, \delta) = \left(-\frac{\psi_{\alpha_2,\alpha_3}(t_2) - \psi_{\alpha_2,\alpha_3}(t_1)}{t_2 - t_1}, \frac{t_2 \psi_{\alpha_2,\alpha_3}(t_1) - t_1 \psi_{\alpha_2,\alpha_3}(t_2)}{t_2 - t_1}\right). \tag{40}$$

In particular, when the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ supports the epigraph of $\psi_{\alpha_2,\alpha_3}(t)$ at $t_1 = l_1$ and $t_2 = u_1$, (39) can be written, after scaling, as

$$\psi_{\alpha_2,\alpha_3}(l_1)(u_1 - x_1) + \psi_{\alpha_2,\alpha_3}(u_1)(x_1 - l_1) + \alpha_2 d_1 x_2 - \alpha_3 d_1 x_3 \le 0, \tag{41}$$

where $d_1 = u_1 - l_1$.

We next derive in Section 4.1.1 linear inequalities that describe the part of $\operatorname{conv}(S^{\le})$ whose geometry is simple. We refer to these inequalities as *trivial*. In Section 4.1.2, we derive inequalities that belong to the part of $\operatorname{conv}(S^{\le})$ that may not be polyhedral. We refer to these inequalities as *nontrivial*.

4.1.1 Trivial inequalities for $\operatorname{conv}(S^{\le})$

In this section, we derive all nondominated valid inequalities (39) for $\operatorname{conv}(S^{\le})$ where $\alpha_2 \le 0$ or $\alpha_3 \le 0$. Proposition 4.1 provides a closed-form expression for $\psi_{\alpha_2,\alpha_3}(t)$ for these values of α_2 and α_3.

Proposition 4.1. *Assume that $\alpha_2 \le 0$ or $\alpha_3 \le 0$. For $t \in [u_2^{-b_2}, u_3]$, then*

(i) $\psi_{\alpha_2,\alpha_3}(t) = -\alpha_2 + \alpha_3 u_3$, *if $\alpha_2 \le 0$ and $\alpha_3 \le 0$,*

(ii) $\psi_{\alpha_2,\alpha_3}(t) = -\alpha_2 + \alpha_3 \max\{t, 1\}$, *if $\alpha_2 \le 0$ and $\alpha_3 > 0$,*

(iii) $\psi_{\alpha_2,\alpha_3}(t) = -\alpha_2 \min\{u_2, u_3^{a_2} t^{-a_2}\} + \alpha_3 u_3$*, if* $\alpha_2 > 0$ *and* $\alpha_3 \leq 0$.

Proof. Let $t \in [u_2^{-b_2}, u_3]$. In this case, (38) has $(1, u_3)$ for feasible solution. It then follows from Weierstraß' theorem that (38) has an optimal solution. Let (x_2^*, x_3^*) be such an optimal solution. First assume that $\alpha_3 \leq 0$. It is clear that we can choose $x_3^* = u_3$. In fact if $x_3^* < u_3$, increasing x_3^* by $\epsilon > 0$ maintains feasibility of the solution, and does not deteriorate its objective value. There are two possible situations. If $\alpha_2 > 0$, increasing x_2^* does not deteriorate the solution value. It is therefore optimal to let x_2 take its highest admissible value, *i.e.*, $x_2^* = \min\{u_2, u_3^{a_2} t^{-a_2}\}$. It is clear that $x_2^* \geq 1$ since $t \leq u_3$. We obtain that $\psi_{\alpha_2,\alpha_3}(t) = -\alpha_2 \min\{u_2, u_3^{a_2} t^{-a_2}\} + \alpha_3 u_3$, yielding (iii). If $\alpha_2 \leq 0$, we can similarly choose $x_2^* = 1$. We obtain that $\psi_{\alpha_2,\alpha_3}(t) = -\alpha_2 + \alpha_3 u_3$, yielding (i). Second assume that $\alpha_3 > 0$. Because in this case $\alpha_2 \leq 0$, we can choose as before $x_2^* = 1$ and $x_3^* = \max\{t, 1\}$. It is clear that $x_3^* \leq u_3$ since $t \leq u_3$. We obtain that $\psi_{\alpha_2,\alpha_3}(t) = -\alpha_2 + \alpha_3 \max\{t, 1\}$, yielding (ii). $\square$

We next derive lifted inequalities for all values of (α_2, α_3) studied in Proposition 4.1. We obtain the inequalities

$$
\begin{align}
-x_1 &\leq -l_1 \tag{42}\\
x_1 &\leq u_1 \tag{43}\\
-x_2 &\leq -1 \tag{44}\\
x_2 &\leq u_2 \quad (\text{if } l_1 < u_2^{-b_2} u_3) \tag{45}\\
-x_3 &\leq -1 \quad (\text{if } l_1 < 1) \tag{46}\\
x_3 &\leq u_3 \tag{47}\\
x_1 &\leq x_3 \quad (\text{if } u_1 > 1) \tag{48}\\
\left(u_2 - u_1^{-a_2} u_3^{a_2}\right) x_1 + (u_1 - u_2^{-b_2} u_3) x_2 &\leq u_1 u_2 - u_1^{-a_2} u_2^{-b_2} u_3^{1+a_2} \qquad (\text{if } u_1 > u_2^{-b_2} u_3) \tag{49}
\end{align}
$$

as shown in the following proposition.

Proposition 4.2. *The only linear inequalities* (39) *with coefficients* $\alpha_2 \leq 0$ *or* $\alpha_3 \leq 0$ *necessary in the description of* $\operatorname{conv}(S^{\leq})$ *are among* (42)-(49).

Proof. It follows from our discussion in Section 3.1 that nondominated lifted inequalities (39) are either of the form (42)-(43), or can be derived from the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$. There are three cases.

(i) Assume that $\alpha_2 \leq 0$ and $\alpha_3 \leq 0$. In this case, function $\psi_{\alpha_2,\alpha_3}(t)$ is constant, and therefore convex, over $[l_1, u_1]$. By (40), the corresponding lifting coefficients (α_1, δ) are $(0, -\alpha_2 + \alpha_3 u_3)$. Since these coefficients are linear in (α_2, α_3), it is sufficient to consider the cases where (α_2, α_3) equals $(-1, 0)$, $(0, -1)$ and $(0, 0)$, which correspond to the extreme rays and point of the region $\{(\alpha_2, \alpha_3) \in \mathbb{R}^2 \,|\, \alpha_2 \leq 0, \alpha_3 \leq 0\}$. Ray $(-1, 0)$ yields (44) and ray $(0, -1)$ yields (47). Extreme point $(0, 0)$ does not yield a useful inequality.

(ii) Assume now that $\alpha_2 \leq 0$ and $\alpha_3 > 0$. In this case, $\psi_{\alpha_2,\alpha_3}(t)$ is a piecewise affine function over $[l_1, 1]$ and $[1, u_1]$. Further, it is convex as the maximum of two convex functions. We can therefore develop two inequalities for each (α_2, α_3), when the corresponding intervals do not reduce to a single point. By (40), the corresponding lifting coefficients are $(\alpha_1, \delta) = (0, -\alpha_2 + \alpha_3)$ if $l_1 < 1$, and $(\alpha_1, \delta) = (-\alpha_3, -\alpha_2)$ if $u_1 > 1$. Since these coefficients are linear in (α_2, α_3), it is sufficient to consider the ray $(\alpha_2, \alpha_3) = (0, 1)$. This ray yields (46) and (48), respectively.

(iii) Assume finally that $\alpha_2 > 0$ and $\alpha_3 \leq 0$. In this case, $\psi_{\alpha_2,\alpha_3}(t)$ is a piecewise concave function. Since $\psi_{\alpha_2,\alpha_3}(t)$ is constant over $[l_1, u_2^{-b_2}u_3]$ and nondecreasing concave over $[u_2^{-b_2}u_3, u_1]$, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is affine over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$. We can therefore develop two inequalities for each (α_2, α_3) when the corresponding intervals do not reduce to a single point. By (40), the corresponding lifting coefficients are $(\alpha_1, \delta) = (0, -\alpha_2 u_2 + \alpha_3 u_3)$ if $l_1 < u_2^{-b_2}u_3$, and $(\alpha_1, \delta) = \left(-\frac{u_2 - u_1^{-a_2}u_3^{a_2}}{u_1 - u_2^{-b_2}u_3}\alpha_2, \frac{u_1^{-a_2}u_2^{-b_2}u_3^{1+a_2} - u_1 u_2}{u_1 - u_2^{-b_2}u_3}\alpha_2 + \alpha_3 u_3\right)$ if $u_1 > u_2^{-b_2}u_3$. Since these coefficients are linear in (α_2, α_3), it is sufficient to consider the ray $(\alpha_2, \alpha_3) = (1, 0)$. This ray yields (45) and (49), respectively.

□

Note that (45), (46), (48), and (49) are valid for $\text{conv}(S^{\leq})$, even when the inequality in the associated condition on parameters is not strict. In this case, however, these linear constraints are not necessary in the description of $\text{conv}(S^{\leq})$.

4.1.2 Nontrivial inequalities for $\text{conv}(S^{\leq})$

In this section, we derive all nondominated valid inequalities (39) for $\text{conv}(S^{\leq})$ where $\alpha_2 > 0$ and $\alpha_3 > 0$. When $b_2 > 1$, we pose $c_2 := (b_2 - 1)^{-1}$. We first derive in Proposition 4.3 a closed-form expression for $\psi_{\alpha_2,\alpha_3}(t)$, which we use to construct these inequalities.

Proposition 4.3. *Assume that $\alpha_2 > 0$ and $\alpha_3 > 0$. For $t \in [u_2^{-b_2}, u_3]$,*

$$\psi_{\alpha_2,\alpha_3}(t) = \begin{cases} \psi^a_{\alpha_2,\alpha_3}(t) & \text{if} \quad t \leq u_2^{-b_2}u_3 \\ \psi^b_{\alpha_2,\alpha_3}(t) & \text{if} \quad t > u_2^{-b_2}u_3 \end{cases} \quad \text{when} \quad \alpha_2\beta_3 \geq \mu(t)$$
$$\psi_{\alpha_2,\alpha_3}(t) = \psi^c_{\alpha_2,\alpha_3}(t) \quad \text{when} \quad \lambda(t) < \alpha_2\beta_3 < \mu(t)$$
$$\psi_{\alpha_2,\alpha_3}(t) = \begin{cases} \psi^d_{\alpha_2,\alpha_3}(t) & \text{if} \quad t \leq 1 \\ \psi^e_{\alpha_2,\alpha_3}(t) & \text{if} \quad t > 1 \end{cases} \quad \text{when} \quad \alpha_2\beta_3 \leq \lambda(t)$$

where

- $\psi^a_{\alpha_2,\alpha_3}(t) = -\alpha_2 u_2 + \alpha_3 u_2^{b_2} t$,
- $\psi^b_{\alpha_2,\alpha_3}(t) = -\alpha_2 u_3^{a_2} t^{-a_2} + \alpha_3 u_3$,
- $\psi^c_{\alpha_2,\alpha_3}(t) = \alpha_2(a_2 - 1)\left(a_2\alpha_2\beta_3\right)^{c_2} t^{-c_2}$,
- $\psi^d_{\alpha_2,\alpha_3}(t) = -\alpha_2 t^{-a_2} + \alpha_3$,
- $\psi^e_{\alpha_2,\alpha_3}(t) = -\alpha_2 + \alpha_3 t$,

and

- $\mu(t) = b_2 \min\{u_2^{b_2-1}t, u_3^{1-a_2}t^{a_2}\}$,
- $\lambda(t) = b_2 \max\{t, t^{a_2}\}$.

Proof. Assume that $t \in [u_2^{-b_2}, u_3]$. Weierstraß' theorem implies that (38) has an optimal solution since $(1, u_3)$ belongs to its feasible region. We claim that all optimal solutions $(\tilde{x}_2, \tilde{x}_3)$ satisfy $t\tilde{x}_2^{b_2} = \tilde{x}_3$. Assume by contradiction that $(\tilde{x}_2, \tilde{x}_3)$ is an optimal solution that satisfies $t\tilde{x}_2^{b_2} < \tilde{x}_3$. There are two cases. Assume first that $\tilde{x}_3 > 1$. For ϵ positive but sufficiently small, the solution

$(\bar{x}_2, \bar{x}_3) = (\tilde{x}_2, \tilde{x}_3 - \epsilon)$ is feasible and has a better objective value than $(\tilde{x}_2, \tilde{x}_3)$, a contradiction. Assume second that $\tilde{x}_3 = 1$. It follows that $x_2 < t^{-a_2} \leq u_2$. For ϵ positive but sufficiently small, the solution $(\bar{x}_2, \bar{x}_3) = (\tilde{x}_2 + \epsilon, \tilde{x}_3)$ is feasible and has a better objective value than $(\tilde{x}_2, \tilde{x}_3)$.

By eliminating variable x_3 using the relation $x_3 = tx_2^{b_2}$, we can reformulate (38) as

$$\psi_{\alpha_2,\alpha_3}(t) := \min\{f_t(x_2) \,|\, L(t) \leq x_2 \leq U(t)\}, \tag{50}$$

where $f_t(x_2) = -\alpha_2 x_2 + \alpha_3 t x_2^{b_2}$, $L(t) := \max\{1, t^{-a_2}\} > 0$ and $U(t) := \min\{u_2, u_3^{a_2} t^{-a_2}\}$. It is easily verified that $L(t) \leq U(t)$ when $t \in [u_2^{-b_2}, u_3]$. Since $f_t(.)$ is convex over $\mathbb{R}_{\geq}$, an optimal solution to (50) is (i) $x_2^* = L(t)$ if $f_t'(L(t)) \geq 0$, *i.e.*, $\alpha_2\beta_3 \leq b_2 t L(t)^{b_2-1} = \lambda(t)$, and (ii) $x_2^* = U(t)$ if $f_t'(U(t)) \leq 0$, *i.e.*, $\alpha_2\beta_3 \geq b_2 t U(t)^{b_2-1} = \mu(t)$. When $\lambda(t) < \alpha_2\beta_3 < \mu(t)$ (which also implies that $b_2 > 1$), the intermediate value theorem implies that $f_t'(.)$ takes value zero for some $x_2 \in (L(t), U(t))$. It is then simple to verify that the only point that makes $f_t'(.)$ equal to zero is $x_2^* = (a_2\alpha_2\beta_3 t^{-1})^{c_2}$. Since (50) is a convex program, x_2^* must be an optimal solution. This yields the desired result. In particular, when $\alpha_2\beta_3 \geq \mu(t)$ and $t \leq u_2^{-b_2} u_3$, the optimal solution found above is $x_2^* = u_2$, yielding $\psi_{\alpha_2,\alpha_3}(t) = f_t(u_2) = -\alpha_2 u_2 + \alpha_3 u_2^{b_2} t = \psi^a_{\alpha_2,\alpha_3}(t)$. The derivation of the other functions is similar. □

Observe that, in the above description, function $\psi^c_{\alpha_2,\alpha_3}(\cdot)$ is not well-defined when $b_2 = 1$. However, in this case $\lambda(t) = \mu(t)$ for $t \in [u_2^{-b_2}, u_3]$ and therefore, the interval of values of $\alpha_2\beta_3$ over which it would be applied is empty. Also observe that in this case, we define the function $\psi_{\alpha_2,\alpha_3}(t)$ twice for values of t with $\alpha_2\beta_3 = \mu(t) = \lambda(t)$. It is simple to verify, however, that the two values we assign to $\psi_{\alpha_2,\alpha_3}(t)$ are identical in such cases.

We now make a few observations that can be verified through direct computation.

(P1) For $t \in [u_2^{-b_2}, u_3]$, $\lambda(t) \leq \mu(t)$.

(P2) For $t \in [u_2^{-b_2}, u_3]$, $\lambda(t)$ and $\mu(t)$ are continuous increasing functions.

(P3) For $t \in [u_2^{-b_2}, u_3]$, $\psi^a_{\alpha_2,\alpha_3}(t)$, $\psi^b_{\alpha_2,\alpha_3}(t)$, $\psi^c_{\alpha_2,\alpha_3}(t)$, $\psi^d_{\alpha_2,\alpha_3}(t)$ and $\psi^e_{\alpha_2,\alpha_3}(t)$ are concave functions.

Observation (P2) directly implies:

(P4) For $t \in [l_1, u_1]$, $\lambda(t) \geq \lambda(l_1) = b_2 l_1^{a_2}$.

(P5) For $t \in [l_1, u_1]$, $\mu(t) \leq \mu(u_1) = b_2 u_1^{a_2} u_3^{1-a_2}$.

Our next goal is to derive, in Lemma 4.5, properties of the function $\psi_{\alpha_2,\alpha_3}(t)$ that help in constructing its convex envelope over $[l_1, u_1]$ in Lemma 4.7. To facilitate the proof of these results, we first derive the following ancillary results.

Lemma 4.4. *The following inequalities hold true:*

$$b_2 u_2^{b_2-1} \geq \frac{u_2^{b_2} - 1}{u_2 - 1}, \tag{51}$$

$$b_2 \leq \frac{u_2^{b_2} - 1}{u_2 - 1}, \tag{52}$$

$$b_2 \geq u_2 \frac{1 - u_2^{-b_2}}{u_2 - 1}. \tag{53}$$

The following inequalities also hold

$$a_2 \le \frac{1 - l_1^{a_2}}{1 - l_1}, \tag{54}$$

$$b_2 \le \frac{1 - u_1 u_2^{b_2} u_3^{-1}}{1 - u_1^{a_2} u_2 u_3^{-a_2}}, \tag{55}$$

if $l_1 < 1$ *and* $u_1 > u_2^{-b_2} u_3$*, respectively.*

Proof. Let $I \subseteq \mathbb{R}$ be an open interval and let $f(x) : I \mapsto \mathbb{R}$ be a differentiable convex function. Given $x_0 \in I$, we know that $f(x) \ge f(x_0) + f'(x_0)(x - x_0)$ for $x \in I$. For $y \in I$ with $y > x_0$, and for $z \in I$ with $z < x_0$, we write that

$$(i) \quad f'(x_0) \le \frac{f(y) - f(x_0)}{y - x_0} \qquad \text{and} \qquad (ii) \quad f'(x_0) \ge \frac{f(x_0) - f(z)}{x_0 - z}. \tag{56}$$

To prove (51), we apply (56)(ii) with $I = \mathbb{R}^+$, $f(x) = x^{b_2}$ where $b_2 \in [1, \infty)$, $x_0 = u_2$, and $z = 1$ (observe that $z < x_0$ because of Assumption (B3).) For (52), we apply (56)(i) with $I = \mathbb{R}_{\ge}$, $f(x) = x^{b_2}$ where $b_2 \in [1, \infty)$, $x_0 = 1$, and $y = u_2$ (observe that $y > x_0$ because of Assumption (B3).) To prove (53), we apply (56)(ii) with $I = \mathbb{R}_{\ge}$, $f(x) = x^{b_2}$ where $b_2 \in [1, \infty)$, $x_0 = 1$, and $z = \frac{1}{u_2}$ (observe that $z < x_0$ because of Assumption (B3).) To prove (54), we apply (56)(ii) with $I = \mathbb{R}_{\ge}$, $f(x) = -x^{a_2}$ where $a_2 \in (0, 1]$, $z = l_1 < 1$ and $x_0 = 1$. Finally for (55), we apply (56)(i) with $I = \mathbb{R}_{\ge}$, $f(x) = x^{b_2}$ where $b_2 \in [1, \infty)$, $y = u_1^{a_2} u_2 u_3^{-a_2} > 1$, and $x_0 = 1$.

□

The requirements that $l_1 < 1$ and $u_1 > u_2^{-b_2} u_3$ in the statements of (54) and (55) are satisfied by most of our problem instances since Assumption (B2) requires that $l_1 \le 1$ and Assumption (B5) requires that $u_1 u_2^{b_2} \ge u_3$. Further, the quantities $\frac{1-l_1}{l_1^{-a_2}-1}$ and $\frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$ play an important role in the ensuing derivations. When $l_1 = 1$ and $u_1 u_2^{b_2} = u_3$, however, these expressions are not well-defined. In these cases, we will take them to be equal to b_2 and $b_2 u_1^{a_2} u_3^{1-a_2}$, respectively (which can be shown to be their limit as $l_1 \to 1$ and $(\frac{u_1}{u_3})^{a_2} u_2 \to 1$.)

In light of the previous discussion, it follows from (54) that $\frac{1-l_1}{l_1^{-a_2}-1} \le b_2 l_1^{a_2}$, and from (55) that $b_2 u_1^{a_2} u_3^{1-a_2} \le \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$ for the problems we study. We can then establish that

$$\frac{1 - l_1}{l_1^{-a_2} - 1} \le b_2 l_1^{a_2} \le b_2 \le b_2 u_1 \le b_2 u_1^{a_2} u_3^{1-a_2} \le \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}} \tag{57}$$

using Assumptions (B2), (B5), and (B6), respectively. Similarly, we can show that

$$\frac{1 - l_1}{l_1^{-a_2} - 1} \le b_2 l_1^{a_2} \le b_2 l_1 u_2^{b_2 - 1} \le b_2 \frac{u_3}{u_2} \le b_2 u_1^{a_2} u_3^{1-a_2} \le \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}} \tag{58}$$

using Assumptions (B4), (B2) and (B5), respectively. Note that the exact ordering of the points b_2 and $b_2 u_1$ with respect to $b_2 l_1 u_2^{b_2-1}$ and $b_2 \frac{u_3}{u_2}$ depends on the problem instance.

Next, for given α_2 and β_3, we define

$$\begin{aligned}
I^{ab}(\alpha_2, \beta_3) &= \{t \in [l_1, u_1] \,|\, \alpha_2 \beta_3 \ge \mu(t)\}, \\
I^{c}(\alpha_2, \beta_3) &= \{t \in [l_1, u_1] \,|\, \lambda(t) < \alpha_2 \beta_3 < \mu(t)\}, \\
I^{de}(\alpha_2, \beta_3) &= \{t \in [l_1, u_1] \,|\, \alpha_2 \beta_3 \le \lambda(t)\}.
\end{aligned}$$

In the sequel, we do not explicitly write the dependence of the sets I^{ab}, I^c, and I^{de} on α_2 and β_3 to streamline notation. It is clear from its definition that I^{ab} describes the subset of $[l_1, u_1]$ where $\psi_{\alpha_2,\alpha_3}(t)$ equals $\psi^a_{\alpha_2,\alpha_3}(t)$ or $\psi^b_{\alpha_2,\alpha_3}(t)$. I^c and I^{de} have similar interpretations. Because of Observations (P1) and (P2), it is easily verified that

(P6) I^{ab}, I^c and I^{de} are (possibly empty) intervals.

(P7) For $t^{ab} \in I^{ab}$, $t^c \in I^c$ and $t^{de} \in I^{de}$, then $t^{ab} \le t^c$, $t^{ab} \le t^{de}$ and $t^c \le t^{de}$.

(P8) $[l_1, u_1] = I^{ab} \cup I^c \cup I^{de}$.

Next, we use Proposition 4.3 to describe intervals I^{ab}, I^c and I^{de} more precisely. We obtain

(P9) When $\alpha_2\beta_3 \in (b_2 l_1^{a_2}, b_2 u_1^{a_2} u_3^{1-a_2})$, then

1. $I^{ab} = [l_1, \max\{a_2 u_2^{1-b_2}(\alpha_2\beta_3), a_2^{b_2} u_3^{1-b_2}(\alpha_2\beta_3)^{b_2}\}]$
2. $\mathrm{int}(I^c) = (\max\{a_2 u_2^{1-b_2}(\alpha_2\beta_3), a_2^{b_2} u_3^{1-b_2}(\alpha_2\beta_3)^{b_2}\}, \min\{a_2^{b_2}(\alpha_2\beta_3)^{b_2}, a_2(\alpha_2\beta_3)\}) \cap (l_1, u_1)$, and
3. $I^{de} = [\min\{a_2^{b_2}(\alpha_2\beta_3)^{b_2}, a_2(\alpha_2\beta_3)\}, u_1]$.

(P10) When $\alpha_2\beta_3 \in (b_2 l_1^{a_2}, b_2 u_1^{a_2} u_3^{1-a_2})$ and $b_2 > 1$, then $\mathrm{int}(I^c) \neq \emptyset$.

We can then define $I^a = I^{ab} \cap [l_1, u_2^{-b_2}u_3]$, $I^b = I^{ab} \cap (u_2^{-b_2}u_3, u_1]$, $I^d = I^{de} \cap [l_1, 1]$ and $I^e = I^{de} \cap (1, u_1]$. With these definitions, it is clear that $\psi_{\alpha_2,\alpha_3}(t) = \psi^\sigma_{\alpha_2,\alpha_3}(t)$ for $t \in I^\sigma$ for all $\sigma \in \{a, b, c, d, e\}$, *i.e.*, $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over the nonempty intervals among I^a, I^b, I^c, I^d and I^e. Given a function $\psi(t)$ that is concave over nonempty intervals $[t_1, t_2]$ and $[t_2, t_3]$, we say that $\psi(t)$ is *concave at* t_2 if $\psi(t)$ is concave over $[t_1, t_3]$.

Lemma 4.5. *(i) When $\alpha_2\beta_3 \le b_2 l_1^{a_2}$, $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, 1]$ and $[1, u_1]$.*

(ii) When $b_2 l_1^{a_2} < \alpha_2\beta_3 < b_2 u_1^{a_2} u_3^{1-a_2}$, $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, \min\{u_2^{-b_2}u_3, 1\}]$, $[\min\{u_2^{-b_2}u_3, 1\}, \max\{u_2^{-b_2}u_3, 1\}]$ and $[\max\{u_2^{-b_2}u_3, 1\}, u_1]$. Further, $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $[l_1, 1]$ and $[1, u_1]$ if $\alpha_2\beta_3 \le b_2 \frac{u_3}{u_2}$ and $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$ if $\alpha_2\beta_3 \ge b_2$.

(iii) When $\alpha_2\beta_3 \ge b_2 u_1^{a_2} u_3^{1-a_2}$, $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$.

Proof. (i) In this case, we have that $\alpha_2\beta_3 \le b_2 l_1^{a_2} \le \lambda(t)$ for $t \in [l_1, u_1]$, because of Observation (P4). We conclude that $\psi_{\alpha_2,\alpha_3}(t) = \psi^d_{\alpha_2,\alpha_3}(t)$ for $t \in [l_1, 1]$, and $\psi_{\alpha_2,\alpha_3}(t) = \psi^e_{\alpha_2,\alpha_3}(t)$ for $t \in [1, u_1]$. The result follows because of Observation (P3).

(ii) We first show that the only points at which $\psi_{\alpha_2,\alpha_3}(t)$ might not be concave are $t^{ab} = u_2^{-b_2}u_3$ and $t^{de} = 1$. There are two cases. First assume that $b_2 > 1$. Since $b_2 l_1^{a_2} < \alpha_2\beta_3 < b_2 u_1^{a_2} u_3^{1-a_2}$, we conclude from Observation (P10) that $\mathrm{int}(I^c) \neq \emptyset$. We verify next that $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $I^{ab} \cup I^c$. Define $t^{ac} = a_2 u_2^{1-b_2}(\alpha_2\beta_3)$ and $t^{bc} = a_2^{b_2} u_3^{1-b_2}(\alpha_2\beta_3)^{b_2}$. It follows from Observations (P9-1) and (P9-2) that $I^{ab} \cap \mathrm{cl}(I^c) = \emptyset$ if $\max\{t^{ac}, t^{bc}\} < l_1$, *i.e.*, $\alpha_2\beta_3 < b_2 u_2^{b_2-1} l_1$, and $I^{ab} \cap \mathrm{cl}(I^c) = \{\max\{t^{ac}, t^{bc}\}\}$ otherwise. In the former case, I^{ab} is empty and the claim is trivial. In the latter case, direct computation shows that $\frac{d}{dt}\psi^a_{\alpha_2,\alpha_3}(t^{ac}) = \alpha_3 u_2^{b_2} = \frac{d}{dt}\psi^c_{\alpha_2,\alpha_3}(t^{ac})$, and $\frac{d}{dt}\psi^b_{\alpha_2,\alpha_3}(t^{bc}) = \alpha_3 b_2^{b_2} u_3^{b_2}(\alpha_2\beta_3)^{-b_2} = \frac{d}{dt}\psi^c_{\alpha_2,\alpha_3}(t^{bc})$. We verify next that $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $I^c \cup I^{de}$. Define $t^{cd} = a_2^{b_2}(\alpha_2\beta_3)^{b_2}$ and $t^{ce} = a_2(\alpha_2\beta_3)$. It follows from Observations (P9-2) and (P9-3) that $\mathrm{cl}(I^c) \cap I^{de} = \emptyset$ if $\min\{t^{cd}, t^{ce}\} > u_1$, *i.e.*,

$\alpha_2\beta_3 > b_2u_1$, and $\mathrm{cl}(I^c) \cap I^{de} = \{\min\{t^{cd}, t^{ce}\}\}$ otherwise. In the former case, I^{de} is empty and the claim is trivial. In the latter case, direct computation shows that $\frac{d}{dt}\psi^c_{\alpha_2,\alpha_3}(t^{cd}) = \alpha_3 b_2^{b_2}(\alpha_2\beta_3)^{-b_2} = \frac{d}{dt}\psi^d_{\alpha_2,\alpha_3}(t^{cd})$, and $\frac{d}{dt}\psi^c_{\alpha_2,\alpha_3}(t^{ce}) = \alpha_3 = \frac{d}{dt}\psi^e_{\alpha_2,\alpha_3}(t^{ce})$. We conclude from Observation (P7) and the fact that $\mathrm{int}(I^c) \neq \emptyset$ that $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $[l_1, \min\{u_2^{-b_2}u_3, 1\}]$, $[\min\{u_2^{-b_2}u_3, 1\}, \max\{u_2^{-b_2}u_3, 1\}]$ and $[\max\{u_2^{-b_2}u_3, 1\}, u_1]$. Second, assume that $b_2 = 1$. Since $\lambda(t) = \mu(t)$ for $t \in [u_2^{-b_2}, u_3]$, we conclude that $I^c = \emptyset$. It follows from Observations (P9-1) and (P9-3) with $b_2 = 1$ that $I^{ab} \cap I^{de} = \{\alpha_2\beta_3\}$. Direct computation shows that $\frac{d}{dt}\psi^a_{\alpha_2,\alpha_3}(\alpha_2\beta_3) = \alpha_3u_2 \geq \frac{\alpha_3}{l_1} > \frac{\alpha_3^2}{\alpha_2} = \frac{d}{dt}\psi^d_{\alpha_2,\alpha_3}(\alpha_2\beta_3)$ since $l_1u_2 \geq 1$ by Assumption (B4) and $\alpha_2\beta_3 > l_1$, $\frac{d}{dt}\psi^a_{\alpha_2,\alpha_3}(\alpha_2\beta_3) = \alpha_3u_2 \geq \alpha_3 = \frac{d}{dt}\psi^e_{\alpha_2,\alpha_3}(\alpha_2\beta_3)$ since $u_2 \geq 1$ by Assumption (B5), $\frac{d}{dt}\psi^b_{\alpha_2,\alpha_3}(\alpha_2\beta_3) = \frac{\alpha_3^2}{\alpha_2}u_3 \geq \frac{\alpha_3^2}{\alpha_2} = \frac{d}{dt}\psi^d_{\alpha_2,\alpha_3}(\alpha_2\beta_3)$ since $u_3 > 1$ by Assumption (B3), and $\frac{d}{dt}\psi^b_{\alpha_2,\alpha_3}(\alpha_2\beta_3) = \frac{\alpha_3^2}{\alpha_2}u_3 \geq \frac{u_3}{u_1}\alpha_3 \geq \alpha_3 = \frac{d}{dt}\psi^e_{\alpha_2,\alpha_3}(\alpha_2\beta_3)$ since $\alpha_2\beta_3 \leq u_1$ and since $u_3 \geq u_1$ by Assumption (B6). It follows that $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $[l_1, \min\{u_2^{-b_2}u_3, 1\}]$, $[\min\{u_2^{-b_2}u_3, 1\}, \max\{u_2^{-b_2}u_3, 1\}]$ and $[\max\{u_2^{-b_2}u_3, 1\}, u_1]$.

Next we give conditions for $\psi_{\alpha_2,\alpha_3}(t)$ to be concave at t^{ab} and t^{de}. We verify that $\frac{d}{dt}\psi^a_{\alpha_2,\alpha_3}(t^{ab}) = \alpha_3u_2^{b_2} \geq \alpha_2\frac{u_2}{b_2u_3}u_2^{b_2} = \frac{d}{dt}\psi^b_{\alpha_2,\alpha_3}(t^{ab})$ if and only if $\alpha_2\beta_3 \leq \frac{b_2u_3}{u_2}$. This shows that $\psi_{\alpha_2,\alpha_3}(t)$ is concave at t^{ab} when $\alpha_2\beta_3 \leq \frac{b_2u_3}{u_2}$. Finally, we compute that $\frac{d}{dt}\psi^d_{\alpha_2,\alpha_3}(t^{de}) = \alpha_2a_2 \geq \alpha_3 = \frac{d}{dt}\psi^e_{\alpha_2,\alpha_3}(t^{de})$ if and only if $\alpha_2\beta_3 \geq b_2$. This shows that $\psi_{\alpha_2,\alpha_3}(t)$ is concave at t^{de} when $\alpha_2\beta_3 \geq b_2$.

(iii) In this case, we have that $\alpha_2\beta_3 \geq b_2u_1^{a_2}u_3^{1-a_2} \geq \mu(t)$ for $t \in [l_1, u_1]$, because of Observation (P5). We conclude that $\psi_{\alpha_2,\alpha_3}(t) = \psi^a_{\alpha_2,\alpha_3}(t)$ for $t \in [l_1, u_2^{-b_2}u_3]$, and $\psi_{\alpha_2,\alpha_3}(t) = \psi^b_{\alpha_2,\alpha_3}(t)$ for $t \in [u_2^{-b_2}u_3, u_1]$. The result follows because of Observation (P3).

□

It follows from Lemma 4.5 that, for each value of $\alpha_2\beta_3$, there exists $t_0 < t_1 < \ldots < t_l$ where $t_0 = l_1$ and $t_l = u_1$ such that $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $[t_j, t_{j+1}]$ for each $j = 0, \ldots, l-1$. Computing the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is therefore equivalent to determining the convex envelope of the piecewise function $\tilde{\psi}_{\alpha_2,\alpha_3}(t)$ that, for each $j = 0, \ldots, l-1$ is affine in $[t_j, t_{j+1}]$ and takes the value $\psi_{\alpha_2,\alpha_3}(t_j)$ at t_j, and $\psi_{\alpha_2,\alpha_3}(t_{j+1})$ at t_{j+1}. In turn, this implies that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is polyhedral. To streamline the discussions associated with computing the slope of function $\tilde{\psi}_{\alpha_2,\alpha_3}(t)$, we introduce the notation $\Delta(t_1, t_2) = \frac{\psi_{\alpha_2,\alpha_3}(t_2) - \psi_{\alpha_2,\alpha_3}(t_1)}{t_2 - t_1}$ for $t_2 \neq t_1$. We will shown next in Lemma 4.7 that the function $\tilde{\psi}_{\alpha_2,\alpha_3}(t)$ is often concave. It follows that, when deriving the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$, the values of $\psi_{\alpha_2,\alpha_3}(t)$ at $t = l_1$ and $t = u_1$ are of particular interest.

Lemma 4.6. *We have that*

$$\psi_{\alpha_2,\alpha_3}(l_1) = \begin{cases} \psi^a_{\alpha_2,\alpha_3}(l_1) = -\alpha_2u_2 + \alpha_3l_1u_2^{b_2} & \text{when} \quad b_2l_1u_2^{b_2-1} \leq \alpha_2\beta_3 \\ \psi^c_{\alpha_2,\alpha_3}(l_1) = \alpha_2(a_2-1){l_1}^{-c_2}(a_2\alpha_2\beta_3)^{c_2} & \text{when} \quad b_2l_1^{a_2} < \alpha_2\beta_3 < b_2l_1u_2^{b_2-1} \\ \psi^d_{\alpha_2,\alpha_3}(l_1) = -\alpha_2{l_1}^{-a_2} + \alpha_3 & \text{when} \quad \alpha_2\beta_3 \leq b_2l_1^{a_2}, \end{cases}$$

and

$$\psi_{\alpha_2,\alpha_3}(u_1) = \begin{cases} \psi^b_{\alpha_2,\alpha_3}(u_1) = -\alpha_2{u_1}^{-a_2}u_3^{a_2} + \alpha_3u_3 & \text{when} \quad b_2u_1^{a_2}u_3^{1-a_2} \leq \alpha_2\beta_3 \\ \psi^c_{\alpha_2,\alpha_3}(u_1) = \alpha_2(a_2-1){u_1}^{-c_2}(a_2\alpha_2\beta_3)^{c_2} & \text{when} \quad b_2u_1 < \alpha_2\beta_3 < b_2u_1^{a_2}u_3^{1-a_2} \\ \psi^e_{\alpha_2,\alpha_3}(u_1) = -\alpha_2 + \alpha_3u_1 & \text{when} \quad \alpha_2\beta_3 \leq b_2u_1. \end{cases}$$

Proof. The proof follows directly from Proposition 4.3 since $l_1 \le u_2^{-b_2} u_3$ by Assumption (B2), and $u_1 > 1$ by Assumption (B3). $\square$

The shape of the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is obtained next. Observe that $\psi_{\alpha_2,\alpha_3}(t)$ can be overestimated using $\psi^b_{\alpha_2,\alpha_3}(t)$ (resp. $\psi^e_{\alpha_2,\alpha_3}(t)$) when $t > u_2^{-b_2} u_3$ (resp. $t > 1$) irrespective of the values of α_2 and α_3. Combining this observation with the geometric insights about the solutions that attain $\psi_{\alpha_2,\alpha_3}(t)$, it is possible to argue that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise affine over various regions as detailed in Lemma 4.7. We provide a direct algebraic proof instead.

Lemma 4.7. *For $\alpha_2 > 0$ and $\alpha_3 > 0$,*

(i) If $\alpha_2\beta_3 \le \frac{1-l_1}{l_1^{-a_2}-1}$, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise affine over $[l_1, 1]$ and $[1, u_1]$ with $\operatorname{conv}_{[l_1,u_1]} \psi_{\alpha_2,\alpha_3}(t) = \psi_{\alpha_2,\alpha_3}(t)$ for $t \in \{l_1, 1, u_1\}$.

(ii) If $\frac{1-l_1}{l_1^{-a_2}-1} < \alpha_2\beta_3 < \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is affine over $[l_1, u_1]$ with $\operatorname{conv}_{[l_1,u_1]} \psi_{\alpha_2,\alpha_3}(t) = \psi_{\alpha_2,\alpha_3}(t)$ for $t \in \{l_1, u_1\}$.

(iii) If $\alpha_2\beta_3 \ge \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise affine over $[l_1, u_2^{-b_2} u_3]$ and $[u_2^{-b_2} u_3, u_1]$ with $\operatorname{conv}_{[l_1,u_1]} \psi_{\alpha_2,\alpha_3}(t) = \psi_{\alpha_2,\alpha_3}(t)$ for $t \in \{l_1, u_2^{-b_2} u_3, u_1\}$.

Proof. (i) From Lemma 4.5(i), we know that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, 1]$ and $[1, u_1]$. Because the result is clear when $l_1 = 1$ or $u_1 = 1$, we assume that $l_1 < 1 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^d_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(1) = \psi^d_{\alpha_2,\alpha_3}(1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, 1) = \alpha_2 \frac{l_1^{-a_2}-1}{1-l_1}$ and $\Delta(1, u_1) = \alpha_3$. Since the relations $\Delta(l_1, 1) \le \Delta(1, u_1)$ and $\alpha_2\beta_3 \le \frac{1-l_1}{l_1^{-a_2}-1}$ are equivalent, we conclude that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is the linear interpolation of $\psi_{\alpha_2,\alpha_3}(t)$ at the points $\{l_1, 1, u_1\}$.

(ii) There are 5 cases.

Case 1: $\frac{1-l_1}{l_1^{-a_2}-1} < \alpha_2\beta_3 < b_2 l_1^{a_2}$**:** From Lemma 4.5(i), we know that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, 1]$ and $[1, u_1]$. Because the result is clear when $l_1 = 1$ or $u_1 = 1$, we assume that $l_1 < 1 < u_1$. As in (i), we compute that $\Delta(l_1, 1) = \alpha_2 \frac{l_1^{-a_2}-1}{1-l_1}$ and $\Delta(1, u_1) = \alpha_3$. Since the relations $\Delta(l_1, 1) > \Delta(1, u_1)$ and $\alpha_2\beta_3 > \frac{1-l_1}{l_1^{-a_2}-1}$ are equivalent, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine.

Case 2: $b_2 l_1^{a_2} \le \alpha_2\beta_3 < \min\{b_2, b_2 l_1 u_2^{b_2-1}\}$**:** This case only occurs when $b_2 > 1$. We know from Lemma 4.5(ii) that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, 1]$ and $[1, u_1]$. Because the result is clear when $l_1 = 1$ or $u_1 = 1$, we assume that $l_1 < 1 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^c_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(1) = \psi^d_{\alpha_2,\alpha_3}(1) = \psi^e_{\alpha_2,\alpha_3}(1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, 1) = \frac{-\alpha_2 + \alpha_3 - \alpha_2(a_2-1)(l_1 b_2)^{-c_2}(\alpha_2\beta_3)^{c_2}}{1-l_1}$ and $\Delta(1, u_1) = \alpha_3$. We claim that $\Delta(l_1, 1) \ge \Delta(1, u_1)$, which is equivalent to $\xi(\alpha_2\beta_3) \ge -l_1$ where $\xi(t) = -t - (a_2 - 1)(l_1 b_2)^{-c_2} t^{c_2+1} \ge -l_1$ for $t \in \mathbb{R}_{\ge}$. Function $\xi(t)$ is strictly convex over $\mathbb{R}_{\ge}$ as $c_2 + 1 > 0$ and $(a_2 - 1) \le 0$. Setting its derivative to zero, we can verify that its minimum is $t^* = l_1 b_2$ as $(c_2 + 1)(1 - a_2) = 1$. Direct computations then show that $\xi(t^*) = -l_1$ as $a_2 b_2 = 1$. It follows that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is affine over $[l_1, u_1]$.

Case 3: $\min\{b_2, b_2 l_1 u_2^{b_2-1}\} \le \alpha_2\beta_3 \le \max\{b_2 u_1, b_2 \frac{u_3}{u_2}\}$**:** There are two subcases.

Case 3.1: $b_2 < b_2\frac{u_3}{u_2}$ Consider $\alpha_2\beta_3 \in [\min\{b_2, b_2 l_1 u_2^{b_2-1}\}, b_2)$. When this case exist, we have that $b_2 > b_2 l_1 u_2^{b_2-1}$. Lemma 4.5(ii) shows that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, 1]$ and $[1, u_1]$. Because the result is clear when $l_1 = 1$ or $u_1 = 1$, we assume that $l_1 < 1 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(1) = \psi^d_{\alpha_2,\alpha_3}(1) = \psi^e_{\alpha_2,\alpha_3}(1)$, $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, 1) = \frac{\alpha_2(u_2-1)+\alpha_3(1-l_1u_2^{b_2})}{1-l_1}$ and $\Delta(1, u_1) = \alpha_3$. Using (51), we write that $\alpha_2\beta_3 \geq b_2 l_1 u_2^{b_2-1} \geq l_1\frac{u_2^{b_2}-1}{u_2-1}$. This relation shows that $\Delta(l_1, 1) \geq \Delta(1, u_1)$, thereby proving that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine. Consider next $\alpha_2\beta_3 \in [b_2, b_2\frac{u_3}{u_2}]$. Lemma 4.5(ii) shows that $\psi_{\alpha_2,\alpha_3}(t)$ is concave over $[l_1, u_1]$. This implies that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine. Finally consider $\alpha_2\beta_3 \in (b_2\frac{u_3}{u_2}, \max\{b_2u_1, b_2\frac{u_3}{u_2}\}]$. When this case exist, we have that $b_2\frac{u_3}{u_2} < b_2u_1$. Lemma 4.5(ii) shows that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$. Because the result is clear when $l_1 = u_2^{-b_2}u_3$ or $u_1 = u_2^{-b_2}u_3$, we assume that $l_1 < u_2^{-b_2}u_3 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3) = \psi^a_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3)$, $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, u_2^{-b_2}u_3) = \alpha_3u_2^{b_2}$ and $\Delta(u_2^{-b_2}u_3, u_1) = \frac{\alpha_2(u_2-1)+\alpha_3(u_1-u_3)}{u_1-u_2^{-b_2}u_3}$. Using the relation $\alpha_2\beta_3 \leq b_2u_1 \leq \frac{u_2^{b_2}-1}{u_2-1}u_1$, which holds because of (52), it is readily verified that $\Delta(l_1, u_2^{-b_2}u_3) \geq \Delta(u_2^{-b_2}u_3, u_1)$, showing that the envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine.

Case 3.2: $b_2 \geq b_2\frac{u_3}{u_2}$ When $\alpha_2\beta_3 \in [b_2 l_1 u_2^{b_2-1}, b_2\frac{u_3}{u_2}]$, then Lemma 4.5(ii) shows that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, 1]$ and $[1, u_1]$. Because the result is clear when $l_1 = 1$ or $u_1 = 1$, we assume that $l_1 < 1 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(1) = \psi^d_{\alpha_2,\alpha_3}(1) = \psi^e_{\alpha_2,\alpha_3}(1)$, and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, 1) = \frac{\alpha_2(u_2-1)+\alpha_3(1-l_1u_2^{b_2})}{1-l_1}$ and $\Delta(1, u_1) = \alpha_3$. Using (51), we write that $\alpha_2\beta_3 \geq b_2u_2^{b_2-1}l_1 \geq \frac{u_2^{b_2}-1}{u_2-1}l_1$. This relation shows that $\Delta(l_1, 1) \geq \Delta(1, u_1)$, thereby proving that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine. When $\alpha_2\beta_3 \in (b_2\frac{u_3}{u_2}, b_2)$, then Lemma 4.5(ii) shows that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$, $[u_2^{-b_2}u_3, 1]$ and $[1, u_1]$. We assume first that these intervals do not reduce to a single point. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3) = \psi^a_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3)$, $\psi_{\alpha_2,\alpha_3}(1) = \psi^d_{\alpha_2,\alpha_3}(1) = \psi^e_{\alpha_2,\alpha_3}(1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, u_2^{-b_2}u_3) = \alpha_3u_2^{b_2}$, $\Delta(u_2^{-b_2}u_3, 1) = \frac{\alpha_2(u_2-1)+\alpha_3(1-u_3)}{1-u_2^{-b_2}u_3}$, $\Delta(1, u_1) = \alpha_3$. Using (52), we write that $\alpha_2\beta_3 \leq b_2 \leq \frac{u_2^{b_2}-1}{u_2-1}$. This relation shows that $\Delta(l_1, u_2^{-b_2}u_3) \geq \Delta(u_2^{-b_2}u_3, 1)$. Using (53), we write that $\alpha_2\beta_3 \geq u_3\frac{b_2}{u_2} \geq \frac{1-u_2^{-b_2}}{u_2-1}$. This relation shows that $\Delta(u_2^{-b_2}u_3, 1) \geq \Delta(1, u_1)$. It is also easily verified that $\Delta(l_1, u_2^{-b_2}u_3) \geq \Delta(1, u_1)$ since $u_2^{b_2} \geq 1$ by Assumption (B3). We conclude that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine. When exactly one of the above intervals reduces to a single point, repeating the above derivation for the other two intervals shows that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine. When two of the above intervals reduce to a single point, $\psi_{\alpha_2,\alpha_3}(t)$ is concave, and therefore, its convex envelope over $[l_1, u_1]$ is affine. When $\alpha_2\beta_3 \in [b_2, b_2u_1]$, then Lemma 4.5(ii) shows that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$. Because the result is clear when $l_1 = u_2^{-b_2}u_3$ or $u_1 = u_2^{-b_2}u_3$, we assume that $l_1 < u_2^{-b_2}u_3 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3) = \psi^a_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3)$, and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, u_2^{-b_2}u_3) = \alpha_3u_2^{b_2}$ and $\Delta(u_2^{-b_2}u_3, u_1) = \frac{\alpha_2(u_2-1)+\alpha_3(u_1-u_3)}{u_1-u_2^{-b_2}u_3}$. Using (52), we write that $\alpha_2\beta_3 \leq b_2u_1 \leq \frac{u_2^{b_2}-1}{u_2-1}u_1$.

This relation shows that $\Delta(l_1, u_2^{-b_2}u_3) \geq \Delta(u_2^{-b_2}u_3, u_1)$, thereby proving that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ is affine.

Case 4: $\max\{b_2u_1, b_2\frac{u_3}{u_2}\} < \alpha_2\beta_3 \leq b_2u_1^{a_2}u_3^{1-a_2}$**:** This case only occurs when $b_2 > 1$. We know from Lemma 4.5(ii) that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$. Because the result is clear when $l_1 = u_2^{-b_2}u_3$ or $u_1 = u_2^{-b_2}u_3$, we assume that $l_1 < u_2^{-b_2}u_3 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3) = \psi^a_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^c_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, u_2^{-b_2}u_3) = \alpha_3u_2^{b_2}$ and $\Delta(u_2^{-b_2}u_3, u_1) = \frac{\alpha_2(a_2-1)(\alpha_2\beta_3)^{c_2}(u_1b_2)^{-c_2}+\alpha_2u_2-\alpha_3u_3}{u_1-u_2^{-b_2}u_3}$. We claim that $\Delta(l_1, u_2^{-b_2}u_3) \geq \Delta(u_2^{-b_2}u_3, u_1)$, which is equivalent to $\xi(\alpha_2\beta_3) \leq u_1u_2^{b_2}$, where $\xi(t) = (a_2 - 1)(u_1b_2)^{-c_2}t^{c_2+1} + tu_2 \leq u_1u_2^{b_2}$ for $t \in \mathbb{R}_{\geq}$. Function $\xi(t)$ is strictly concave over $\mathbb{R}_{\geq}$ as $c_2 + 1 > 0$ and $(a_2 - 1) \leq 0$. Setting its derivative to zero, we can verify that its maximum is $t^* = (u_1b_2)u_2^{b_2-1}$ as $(c_2+1)(a_2-1) = -1$. Direct computations then show that $\xi(t^*) = u_1u_2^{b_2}$. It follows that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is affine over $[l_1, u_1]$.

Case 5: $b_2u_1^{a_2}u_3^{1-a_2} < \alpha_2\beta_3 < \frac{u_1u_2^{b_2}-u_3}{u_2-u_1^{-a_2}u_3^{a_2}}$**:** From Lemma 4.5(iii), we know that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$ and $[u_2^{-b_2}u_3, u_1]$. Because the result is clear when $l_1 = u_2^{-b_2}u_3$ or $u_1 = u_2^{-b_2}u_3$, we assume that $l_1 < u_2^{-b_2}u_3 < u_1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3) = \psi^a_{\alpha_2,\alpha_3}(u_2^{-b_2}u_3)$, and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^b_{\alpha_2,\alpha_3}(u_1)$, we compute that $\Delta(l_1, u_2^{-b_2}u_3) = \alpha_3u_2^{b_2}$ and $\Delta(u_2^{-b_2}u_3, u_1) = \alpha_2\frac{u_2-u_1^{-a_2}u_3^{a_2}}{u_1-u_2^{-b_2}u_3}$. As the relations $\Delta(l_1, u_2^{-b_2}u_3) \geq \Delta(u_2^{-b_2}u_3, u_1)$ and $\alpha_2\beta_3 \leq \frac{u_1u_2^{b_2}-u_3}{u_2-u_1^{-a_2}u_3^{a_2}}$ are equivalent, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is affine over $[l_1, u_1]$.

(iii) It follows from Lemma 4.5(iii) that $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise concave over $[l_1, u_2^{-b_2}u_3]$ and over $[u_2^{-b_2}u_3, u_1]$. Because the result is clear when $l_1 = u_2^{-b_2}u_3$ or $u_1 = u_2^{-b_2}u_3$, we assume that $l_1 < u_2^{-b_2}u_3 < u_1$. As in (ii)(5), the relations $\Delta(l_1, u_2^{-b_2}u_3) < \Delta(u_2^{-b_2}u_3, u_1)$ and $\alpha_2\beta_3 > \frac{u_1u_2^{b_2}-u_3}{u_2-u_1^{-a_2}u_3^{a_2}}$ are equivalent. We conclude that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is the linear interpolation of $\psi_{\alpha_2,\alpha_3}(t)$ at the points $\{l_1, u_2^{-b_2}u_3, u_1\}$.

$\square$

Next, we derive lifted valid inequalities for all pairs (α_2, α_3) where $\alpha_2 > 0$ and $\alpha_3 > 0$. During this process, we obtain the isolated inequalities

$$x_1 + \frac{1-l_1}{l_1^{-a_2}-1}(x_2 - 1) - x_3 \leq 0 \quad \text{(if } l_1 < 1\text{)}, \tag{59}$$

$$(1 - b_2)l_1u_2^{b_2}(u_1 - x_1) + (u_1 - b_2l_1u_2^{b_2-1})(x_1 - l_1) + d_1b_2l_1u_2^{b_2-1}x_2 - d_1x_3 \leq 0 \quad \text{(if } l_1u_2^{b_2-1} \leq \min\{1, u_2^{-1}u_3\}\text{)}, \tag{60}$$

$$(l_1u_2^{b_2} - b_2u_1u_2)(u_1 - x_1) + (1 - b_2)u_1(x_1 - l_1) + d_1b_2u_1x_2 - d_1x_3 \leq 0 \quad \text{(if } \max\{1, l_1u_2^{b_2-1}, u_2^{-1}u_3\} \leq u_1\text{)}, \tag{61}$$

$$u_2^{b_2}x_1 + \frac{u_1u_2^{b_2} - u_3}{u_2 - u_1^{-a_2}u_3^{a_2}}(x_2 - u_2) - x_3 \le 0 \qquad \text{(if } l_1 < u_2^{-b_2}u_3 < u_1\text{)}, \tag{62}$$

together with the families of inequalities

$$(a_2 - 1)(l_1b_2)^{-c_2}(\alpha_2\beta_3)^{c_2+1}(u_1 - x_1) + (u_1 - \alpha_2\beta_3)(x_1 - l_1) + \alpha_2\beta_3d_1x_2 - d_1x_3 \le 0 \qquad \text{(if } b_2l_1^{a_2} \le \alpha_2\beta_3 \le \min\{b_2u_1, b_2l_1u_2^{b_2-1}\}, b_2 > 1\text{)}, \tag{63}$$

$$(a_2 - 1)b_2^{-c_2}(\alpha_2\beta_3)^{c_2+1}(l_1^{-c_2}(u_1 - x_1) + u_1^{-c_2}(x_1 - l_1)) + \alpha_2\beta_3d_1x_2 - d_1x_3 \le 0 \qquad \text{(if } b_2u_1 < \alpha_2\beta_3 < b_2l_1u_2^{b_2-1}\text{)}, \tag{64}$$

$$(l_1u_2^{b_2} - \alpha_2\beta_3u_2)(u_1 - x_1) + (a_2 - 1)(u_1b_2)^{-c_2}(\alpha_2\beta_3)^{c_2+1}(x_1 - l_1) + \alpha_2\beta_3d_1x_2 - d_1x_3 \le 0 \qquad \text{(if } \max\{b_2u_1, b_2l_1u_2^{b_2-1}\} \le \alpha_2\beta_3 \le b_2u_1^{a_2}u_3^{1-a_2}, b_2 > 1\text{)}. \tag{65}$$

Proposition 4.8. *The only linear inequalities* (39) *with coefficients $\alpha_2 > 0$ and $\alpha_3 > 0$ necessary in the description of* $\operatorname{conv}(S^{\le})$ *are among* (59)-(65).

Proof. **Case 1: $0 < \alpha_2\beta_3 \le \frac{1-l_1}{l_1^{-a_2}-1}$:** It follows from Lemma 4.7(i) that the envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise affine over $[l_1, 1]$ and $[1, u_1]$ (if any of these segments reduces to a point, it can be ignored in the foregoing discussion.) We can therefore develop at most two inequalities for each such (α_2, α_3). Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^d_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(1) = \psi^d_{\alpha_2,\alpha_3}(1) = \psi^e_{\alpha_2,\alpha_3}(1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, it follows from (40) that (α_1, δ) are linear functions of (α_2, α_3). (For the values of $\psi_{\alpha_2,\alpha_3}(\cdot)$ attained at l_1 or u_1 see Lemma 4.6.) As a result, it is sufficient to consider the situation where $\alpha_2\beta_3 = \frac{1-l_1}{l_1^{-a_2}-1}$. In this case, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ reduces to an affine function over $[l_1, u_1]$. We obtain (59).

Case 2: $\frac{1-l_1}{l_1^{-a_2}-1} < \alpha_2\beta_3 < \frac{u_1u_2^{b_2}-u_3}{u_2-u_1^{-a_2}u_3^{a_2}}$: It follows from Lemma 4.7(ii) that the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is affine over $[l_1, u_1]$. It is therefore sufficient to consider the values of $\psi_{\alpha_2,\alpha_3}(t)$ at $t = l_1$ and $t = u_1$.

Case 2.1: $\frac{1-l_1}{l_1^{-a_2}-1} < \alpha_2\beta_3 < b_2l_1^{a_2}$: As in Case 1, it follows from (40) that lifting coefficients (α_1, δ) are linear in α_2 and α_3. Since the interval of interest for $\alpha_2\beta_3$ is open, we conclude that these inequalities are not needed in the description of $\operatorname{conv}(S^{\le})$.

Case 2.2: $b_2l_1^{a_2} \le \alpha_2\beta_3 < \min\{b_2, b_2l_1u_2^{b_2-1}\}$: Here, $b_2 > 1$. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^c_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$, (41) reduces to (63).

Case 2.3: $\min\{b_2, b_2l_1u_2^{b_2-1}\} \le \alpha_2\beta_3 \le \max\{b_2u_1, b_2\frac{u_3}{u_2}\}$: There are two subcases.

Case 2.3.1: $b_2 < b_2l_1u_2^{b_2-1}$: In order for this case to occur, we must have that $b_2 > 1$. It follows that $\psi^c_{\alpha_2,\alpha_3}(t)$ is well-defined. Assume first that $b_2u_1 < b_2l_1u_2^{b_2-1}$. When $\alpha_2\beta_3 \in [b_2, b_2u_1]$, we obtain (63) similarly to Case 2.2. When $\alpha_2\beta_3 \in (b_2u_1, b_2l_1u_2^{b_2-1})$, then $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^c_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^c_{\alpha_2,\alpha_3}(u_1)$. Therefore, (41) reduces to (64). When $\alpha_2\beta_3 \in [b_2l_1u_2^{b_2-1}, b_2\frac{u_3}{u_2}]$, then $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^c_{\alpha_2,\alpha_3}(u_1)$. Applying (41), we obtain (65). Assume second that $b_2l_1u_2^{b_2-1} \le b_2u_1$. When $\alpha_2\beta_3 \in [b_2, b_2l_1u_2^{b_2-1}]$, we obtain (63). When $\alpha_2\beta_3 \in (b_2l_1u_2^{b_2-1}, \max\{b_2\frac{u_3}{u_2}, b_2u_1\}]$, there are two cases. If $b_2\frac{u_3}{u_2} \le b_2u_1$, it follows from (40) that lifting coefficients (α_1, δ) are linear in

α_2 and α_3 when $\alpha_2\beta_3 \in (b_2 l_1 u_2^{b_2-1}, b_2 u_1]$ since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$. Since the interval of interest for $\alpha_2\beta_3$ is semi-open, we conclude that the only inequality that is required in the description of $\operatorname{conv}(S^{\leq})$ occurs when $\alpha_2\beta_3 = b_2 u_1$. We obtain (61). If $b_2\frac{u_3}{u_2} > b_2 u_1$, it follows similarly that lifting coefficients (α_1, δ) are linear in α_2 and α_3 when $\alpha_2\beta_3 \in (b_2 l_1 u_2^{b_2-1}, b_2 u_1)$. Since the interval of interest for $\alpha_2\beta_3$ is open, we conclude that none of these inequalities are required in the description of $\operatorname{conv}(S^{\leq})$. When $\alpha_2\beta_3 \in [b_2 u_1, b_2\frac{u_3}{u_2}]$, we obtain (65) since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^c_{\alpha_2,\alpha_3}(u_1)$.

Case 2.3.2: $b_2 \geq b_2 l_1 u_2^{b_2-1}$**:** Assume first that $b_2 u_1 \geq b_2\frac{u_3}{u_2}$. When $\alpha_2\beta_3 \in [b_2 l_1 u_2^{b_2-1}, b_2 u_1]$, it is easily verified from (40) that lifting coefficients (α_1, δ) are linear in α_2 and α_3 since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^e_{\alpha_2,\alpha_3}(u_1)$. It is therefore sufficient to consider the case where $\alpha_2\beta_3 = b_2 l_1 u_2^{b_2-1}$ and $\alpha_2\beta_3 = b_2 u_1$. We obtain (60) and (61) respectively. Assume second that $b_2 u_1 < b_2\frac{u_3}{u_2}$. For $\alpha_2\beta_3 \in [b_2 l_1 u_2^{b_2-1}, b_2 u_1)$, the derivation in the previous line shows that it is sufficient to consider the case where $\alpha_2\beta_3 = b_2 l_1 u_2^{b_2-1}$. We obtain (60). For $\alpha_2\beta_3 \in [b_2 u_1, b_2\frac{u_3}{u_2}]$, $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^c_{\alpha_2,\alpha_3}(u_1)$. We obtain (65).

Case 2.4: $\max\{b_2 u_1, b_2\frac{u_3}{u_2}\} < \alpha_2\beta_3 \leq b_2 u_1^{a_2} u_3^{1-a_2}$**:** This case only occurs when $b_2 > 1$ and is similar to Case 2.2. Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^c_{\alpha_2,\alpha_3}(u_1)$, (41) reduces to (65).

Case 2.5: $b_2 u_1^{a_2} u_3^{1-a_2} < \alpha_2\beta_3 < \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$**:** It follows from (40) that (α_1, δ) are linear in α_2 and α_3 since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^b_{\alpha_2,\alpha_3}(u_1)$. Since the interval of interest for $\alpha_2\beta_3$ is open, we conclude that these inequalities are not needed in the description of $\operatorname{conv}(S^{\leq})$.

Case 3: $\frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}} \leq \alpha_2\beta_3$**:** It follows from Lemma 4.7(iii) that the envelope of $\psi_{\alpha_2,\alpha_3}(t)$ is piecewise affine over $[l_1, u_2^{-b_2} u_3]$ and $[u_2^{-b_2} u_3, u_1]$ (if any of these segments reduces to a point, it can be ignored in the foregoing discussion.) We can therefore develop at most two inequalities for each such (α_2, α_3). Since $\psi_{\alpha_2,\alpha_3}(l_1) = \psi^a_{\alpha_2,\alpha_3}(l_1)$, $\psi_{\alpha_2,\alpha_3}(u_2^{-b_2} u_3) = \psi^a_{\alpha_2,\alpha_3}(u_2^{-b_2} u_3)$ and $\psi_{\alpha_2,\alpha_3}(u_1) = \psi^b_{\alpha_2,\alpha_3}(u_1)$, it follows from (40) that (α_1, δ) are linear in (α_2, α_3). It is therefore sufficient to consider the case where $\alpha_2\beta_3 = \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$. (Observe that the inequality obtained as $\alpha_2\beta_3 \to +\infty$ was already obtained when considering the case $\alpha_3 = 0$). In this case, the convex envelope of $\psi_{\alpha_2,\alpha_3}(t)$ over $[l_1, u_1]$ reduces to an affine function. We obtain (62).

□

Observe that the families of inequalities (63)-(65) are applicable for all values of $\alpha_2\beta_3$ in a certain interval (*conditions on slope*). These intervals are sometimes empty, and sometimes also reduce to a single point. An interesting special case is that where $b_2 = 1$. In that situation, it is easy to see that $\operatorname{conv}(S^{\leq})$ is polyhedral. The proof of Proposition 4.2 is simpler as Cases 2.1, 2.2, 2.3.1, 2.4 and 2.5 do not exist. The remaining cases yield only linear inequalities, as desired. We record this linear description of $\operatorname{conv}(S^{\leq})$ in Corollary 4.9.

Corollary 4.9. *When* $b_2 = 1$, $\operatorname{conv}(S^{\leq})$ *is described by the trivial inequalities* $l_1 \leq x_1 \leq u_1$, $1 \leq x_2 \leq u_2$, $1 \leq x_3 \leq u_3$, $x_1 \leq x_3$ *together with*

$$x_1 + l_1 x_2 \leq l_1 + x_3, \tag{66}$$

$$u_2 x_1 + u_1 x_2 \leq u_1 u_2 + x_3. \tag{67}$$

Proof. The proof follows by simplifying the inequalities obtained in Propositions 4.2 and 4.8 using $b_2 = 1$, by removing unnecessary inequalities, and by using the fact that, when $b_2 = 1$, inequalities (63)-(65) do not apply. In particular, inequalities (59) and (60) reduce to (66), while inequalities (61), and (62) reduce to (67) and inequality (49) is dominated by (67). $\square$

One possible way of deriving valid inequalities for $S^{\leq}$ is as follows

$$x_3 \geq x_1 x_2^{b_2} \geq x_1(1 + b_2(x_2 - 1)) = (1 - b_2)x_1 + b_2 x_1 x_2,$$

where the second inequality holds because $x_2^{b_2}$ is a convex function and $x_1 \geq 0$. Then we underestimate x_1x_2 by its convex envelope over $[l_1, u_1] \times [1, u_2]$, which is $\max\{l_1x_2 + x_1 - l_1, u_1x_2 + u_2x_1 - u_1u_2\}$ to obtain, after rearranging their terms, the two linear inequalities

$$x_3 \geq x_1 + b_2 l_1(x_2 - 1) \tag{68}$$

$$x_3 \geq (1 - b_2)u_1 \frac{x_1 - l_1}{d_1} + ((1 - b_2)l_1 - d_1 b_2 u_2)\frac{u_1 - x_1}{d_1} + b_2 u_1 x_2. \tag{69}$$

Similarly, we can write

$$x_3 \geq x_1 x_2^{b_2} \geq x_1(u_2^{b_2} + b_2 u_2^{b_2 - 1}(x_2 - u_2)) = (1 - b_2)u_2^{b_2} x_1 + b_2 u_2^{b_2 - 1} x_1 x_2.$$

and underestimate x_1x_2 as above to obtain the following two linear inequalities

$$x_3 \geq (1 - b_2) l_1 u_2^{b_2} \frac{u_1 - x_1}{d_1} + ((1 - b_2)u_2^{b_2} u_1 + d_1 b_2 u_2^{b_2 - 1})\frac{x_1 - l_1}{d_1} + b_2 u_2^{b_2 - 1} l_1 x_2 \tag{70}$$

$$x_3 \geq u_2^{b_2} x_1 + b_2 u_2^{b_2 - 1} u_1 (x_2 - u_2). \tag{71}$$

When $b_2 = 1$, the above derivations are equivalent to underestimating x_1x_2 by its McCormick envelopes over $[l_1, u_1] \times [1, u_2]$. It is easy to see that, in this case, inequalities (68) and (70) are identical, and similarly that inequalities (69) and (71) are identical. It is interesting to observe that inequalities (59), (60), (61) and (62) are in fact, strengthenings of inequalities (68), (70), (69), and (71), respectively, as $\frac{1 - l_1}{l_1^{-a_2} - 1} \geq b_2 l_1$, $(1 - b_2)u_2^{b_2} u_1 + d_1 b_2 u_2^{b_2 - 1} \leq u_1 - b_2 l_1 u_2^{b_2 - 1}$, $(1 - b_2)l_1 - d_1 b_2 u_2 \leq l_1 u_2^{b_2} - b_2 u_1 u_2$, and $b_2 u_2^{b_2 - 1} u_1 \geq \frac{u_1 u_2^{b_2} - u_3}{u_2 - u_1^{-a_2} u_3^{a_2}}$.

4.2 Nonlinear description of conv$(S^{\leq})$

We next turn the infinite families of linear inequalities (63)-(65) into nonlinear inequalities. Note that such an operation is meaningful only for those cases where the convex hull is not polyhedral. In particular, we assume from here on that $b_2 > 1$. Since $\beta_3 > 0$ for all inequalities (63)-(65), we may assume that $\beta_3 = 1$ through scaling. After this substitution is performed, (63)-(65) can be written in the form

$$\alpha_2 p + q - \alpha_2^{c_2 + 1} s \leq 0 \quad \text{for} \quad g \leq \alpha_2 \leq h \tag{72}$$

where p, q, and s are affine functions of (x_1, x_2, x_3) and (g, h) are nonnegative parameters. It is easy to see that including the boundaries of the interval (g, h) for (64) is admissible when $b_2 > 1$.

We next describe these functions and parameters for (63)-(65) respectively:

$$\begin{aligned}
&p_1 = -(x_1 - l_1) + d_1 x_2\\
&q_1 = u_1(x_1 - l_1) - d_1 x_3\\
&s_1 = (1 - a_2)(l_1 b_2)^{-c_2}(u_1 - x_1)\\
&g_1 = b_2 l_1^{a_2}\\
&h_1 = \min\{b_2 u_1, b_2 l_1 u_2^{b_2-1}\},
\end{aligned} \tag{73}$$

$$\begin{aligned}
&p_2 = d_1 x_2\\
&q_2 = -d_1 x_3\\
&s_2 = (1 - a_2) b_2^{-c_2}\left(l_1^{-c_2}(u_1 - x_1) + u_1^{-c_2}(x_1 - l_1)\right)\\
&g_2 = b_2 u_1\\
&h_2 = b_2 l_1 u_2^{b_2-1},
\end{aligned} \tag{74}$$

$$\begin{aligned}
&p_3 = -u_2(u_1 - x_1) + d_1 x_2\\
&q_3 = l_1 u_2^{b_2}(u_1 - x_1) - d_1 x_3\\
&s_3 = (1 - a_2)(u_1 b_2)^{-c_2}(x_1 - l_1)\\
&g_3 = \max\{b_2 u_1, b_2 l_1 u_2^{b_2-1}\}\\
&h_3 = b_2 u_1^{a_2} u_3^{1-a_2}.
\end{aligned} \tag{75}$$

Proposition 4.10. *For the values of* p, q, s, g, *and* h *presented in* (73)-(75), *the families of valid linear inequalities* (63)-(65) *for* $\operatorname{conv}(S^{\leq})$ *for which* $g \leq h$ *can be expressed as:*

$$\begin{gathered}
gp + q - g^{c_2+1}s \leq 0\\
\text{when } p \leq (c_2+1)g^{c_2}s,
\end{gathered} \tag{76}$$

$$\begin{gathered}
p \leq b_2^{a_2}(c_2+1)^{1-a_2}(-q)^{a_2}s^{1-a_2}\\
\text{when } q \leq 0, (c_2+1)g^{c_2}s < p < (c_2+1)h^{c_2}s,
\end{gathered} \tag{77}$$

$$\begin{gathered}
hp + q - h^{c_2+1}s \leq 0\\
\text{when } p \geq (c_2+1)h^{c_2}s.
\end{gathered} \tag{78}$$

Proof. It can be readily verified that $s \geq 0$ (in (73)-(75)) whenever $x \in [l, u]^3$. For (x_1, x_2, x_3), the coefficient α_2 that yields the tighter inequality in (72) is obtained by solving

$$\max\{\alpha_2 p + q - \alpha_2^{c_2+1}s \,|\, g \leq \alpha_2 \leq h\}, \tag{79}$$

which is a convex program since its objective function is strictly concave as $c_2 + 1 > 1$ and $s \geq 0$. We claim that an optimal solution to (79) is given by

$$\alpha_2^* = \begin{cases} g & p \leq (c_2+1)g^{c_2}s\\ (\frac{p}{(c_2+1)s})^{b_2-1} & (c_2+1)g^{c_2}s < p < (c_2+1)h^{c_2}s\\ h & p \geq (c_2+1)h^{c_2}s. \end{cases}$$

We next give a proof of this claim. If $p \leq 0$, an optimal solution is obtained by setting $\alpha_2^* = g$ since the objective function is nonincreasing. We can therefore assume that $p > 0$. If $s = 0$, then an optimal solution is obtained by setting $\alpha_2^* = h$ since the objective function is increasing. We can therefore assume that $s > 0$. In this case the derivative of the objective function of (79) is zero at $\gamma = (\frac{p}{(c_2+1)s})^{b_2-1}$. It is then clear that $\alpha_2^* = \min\{\max\{g, \gamma\}, h\}$, yielding the desired result.

In fact, $\alpha_2^* = g$ when $\gamma \leq g$, *i.e.*, $(c_2+1)g^{c_2}s \geq p$, producing (76). Similarly $\alpha_2^* = h$ when $\gamma \geq h$, *i.e.*, $(c_2+1)h^{c_2}s \leq p$ producing (78). Otherwise, $\alpha_2^* = \gamma$. Plugging the expression for γ in (72) yields inequality $q + a_2 \frac{p^{b_2}}{((c_2+1)s)^{b_2-1}} \leq 0$, which implies that $q \leq 0$. This nonlinear inequality is therefore valid when $(p,q,s) \in F^{\leq}$ where

$$F^{\leq} = \{(p,q,s) \,|\, p > 0, q \leq 0, (c_2+1)g^{c_2}s < p < (c_2+1)h^{c_2}s\}.$$

Over $F^{\leq} \cap [l,u]^3$, it can be rewritten as (77). □

Similar to the discussion given in Section 3, we mention that each of the above inequalities (76)-(78) is associated with a subset of values of (x_1, x_2, x_3) over which the inequality is valid and strong. Inequalities (76) and (78) are valid outside of their prescribed subsets, since these were derived for specific values of $\alpha_2\beta_3$ which are feasible though not necessarily optimal. The same conclusion does not hold in general for the nonlinear inequality (77), which may become invalid outside of its prescribed range.

We summarize the description of $\text{conv}(S^{\leq})$ in the following theorem.

Theorem 4.11. *Under Assumptions (B0)-(B6), a description of* $\text{conv}(S^{\leq})$ *for* $b_2 > 1$*, is obtained by combining inequalities* (42)-(49)*,* (59)-(62)*, and* (76)-(78) *for all* (p,q,s,g,h) *defined in* (73)-(75) *for which* $g \leq h$*.* □

Among the nonlinear inequalities developed above, inequality (77) with $(p_2, q_2, s_2, g_2, h_2)$ is special in that it does not involve bounds on variables x_1 and x_2, and admits a straightforward derivation. First, after filling in the functional forms for p_2, q_2 and s_2, we see that (77) can be written as

$$x_2^{\frac{b_2}{b_2-1}} x_3^{-\frac{1}{b_2-1}} \leq \left(l_1^{-\frac{1}{b_2-1}} \left(\frac{u_1 - x_1}{u_1 - l_1} \right) + u_1^{-\frac{1}{b_2-1}} \left(\frac{x_1 - l_1}{u_1 - l_1} \right) \right). \tag{80}$$

The validity of (80) can be argued as follows. First, we observe that the inequality defining $S^{\leq}$ can be equivalently written in the form $x_2^{b_2} x_3^{-1} \leq x_1^{-1}$ or $x_2^{\frac{b_2}{b_2-1}} x_3^{-\frac{1}{b_2-1}} \leq x_1^{-\frac{1}{b_2-1}}$. Since the function $x_1^{-\frac{1}{b_2-1}}$ is convex over $[l_1, u_1]$, it can be upper-estimated by the linear function $l_1^{-\frac{1}{b_2-1}} \left(\frac{u_1 - x_1}{u_1 - l_1} \right) + u_1^{-\frac{1}{b_2-1}} \left(\frac{x_1 - l_1}{u_1 - l_1} \right)$, which takes values $l_1^{-\frac{1}{b_2-1}}$ at $x_1 = l_1$ and $u_1^{-\frac{1}{b_2-1}}$ at $x_1 = u_1$, yielding the desired result. In particular, the above derivation shows that (80) is globally valid for $\text{conv}(S^{\leq})$. Theorem 4.11 shows that this inequality is in fact, an important component of the convex hull of $S^{\leq}$.

We conclude this section by illustrating the result of Theorem 4.11 on an example.

Example 4.12. *Consider an instance of* $S^{\leq}$ *where* $l_1 = \frac{1}{16}$, $l_2 = l_3 = 1$, $u_1 = \frac{9}{4}$, $u_2 = 5$, $u_3 = \frac{27}{8}$, $b_1 = 1$ *and* $b_2 = 2$*. We note that these parameters match those we use in Example 3.12 after scaling the variables* x_1 *and* x_3*. It follows from Corollary 2.2 that the inequalities we derive next can be combined with those of Example 3.12 (after suitable scaling) to obtain a convex hull description of* *the corresponding set S. The trivial inequalities (42)-(49) of $\operatorname{conv}(S^{\leq})$ can be written as*

$$
\begin{aligned}
\frac{1}{16} \leq x_1 \leq \frac{9}{4}&,\\
1 \leq x_2 \leq 5&,\\
1 \leq x_3 \leq \frac{27}{8}&,\\
x_1 \leq x_3&,\\
\left(5 - \frac{\sqrt{6}}{2}\right) x_1 + \frac{423}{200} x_2 \leq \frac{1125 - 54\sqrt{6}}{100}&.
\end{aligned}
$$

Further, inequalities (59)-(62) take the form

$$
\begin{aligned}
x_1 + \frac{5}{16}(x_2 - 1) - x_3 \leq 0&,\\
\frac{51}{16} x_1 + \frac{350}{256} x_2 - \frac{35}{16} x_3 \leq \frac{463}{128}&,\\
\frac{299}{16} x_1 + \frac{315}{32} x_2 - \frac{35}{16} x_3 \leq \frac{1503}{32}&,\\
25 x_1 + \frac{423}{4(10 - \sqrt{6})}(x_2 - 5) - x_3 \leq 0&.
\end{aligned}
$$

Next, we compute that $g_1 = \frac{1}{2} < h_1 = \frac{5}{8}$, $g_2 = \frac{9}{2} > h_2 = \frac{5}{8}$, and $g_3 = \frac{9}{2} < h_3 = \frac{9}{4}\sqrt{6}$. This shows that families 1 and 3 are applicable while the family 2 is not. For the first family, we obtain

$$
\begin{gathered}
\left(\frac{1}{2}\right)\left(\frac{1}{16}\left(-16x_1 + 35x_2 + 1\right)\right) + \left(\frac{1}{16}\left(36x_1 - 35x_3 - \frac{9}{4}\right)\right) - \left(\frac{1}{2}\right)^2 \left(-4x_1 + 9\right) \leq 0,\\
2\left(\frac{1}{16}\left(-36x_1 + 35x_3 + \frac{9}{4}\right)\right)^{\frac{1}{2}} \left(-4x_1 + 9\right)^{\frac{1}{2}} \leq \frac{1}{16}\left(-16x_1 + 35x_2 + 1\right),\\
\left(\frac{5}{8}\right)\left(\frac{1}{16}\left(-16x_1 + 35x_2 + 1\right)\right) + \left(\frac{1}{16}\left(36x_1 - 35x_3 - \frac{9}{4}\right)\right) - \left(\frac{5}{8}\right)^2 \left(-4x_1 + 9\right) \leq 0,
\end{gathered}
$$

where the nonlinear inequality applies when $\frac{1}{16}\left(36x_1 - 35x_3 - \frac{9}{4}\right) \leq 0$ and $-48x_1 + 143 < 35x_3 < -64x_1 + 179$. For the third family, we obtain

$$
\begin{gathered}
\left(\frac{9}{2}\right)\left(\frac{1}{16}\left(80x_1 + 35x_2 - 180\right)\right) + \left(\frac{1}{16}\left(-25x_1 - 35x_3 + \frac{225}{4}\right)\right)\\
\qquad - \left(\frac{9}{2}\right)^2 \left(\frac{1}{9}\left(x_1 - \frac{1}{16}\right)\right) \leq 0,\\
2\left(\frac{1}{16}\left(25x_1 + 35x_3 - \frac{225}{4}\right)\right)^{\frac{1}{2}} \left(\frac{1}{9}\left(x_1 - \frac{1}{16}\right)\right)^{\frac{1}{2}} \leq \frac{1}{16}\left(80x_1 + 35x_2 - 180\right),\\
\left(\frac{9}{4}\sqrt{6}\right)\left(\frac{1}{16}\left(80x_1 + 35x_2 - 180\right)\right) + \left(\frac{1}{16}\left(-25x_1 - 35x_3 + \frac{225}{4}\right)\right) -\\
\qquad \left(\frac{9}{4}\sqrt{6}\right)^2 \left(\frac{1}{9}\left(x_1 - \frac{1}{16}\right)\right) \leq 0,
\end{gathered}
$$

where the nonlinear inequality applies when $\frac{1}{16}\left(-25x_1 - 35x_3 + \frac{225}{4}\right) \leq 0$ and $-64x_1 + 179 < 35x_2 < (8\sqrt{6} - 80)x_1 + 180 - \frac{\sqrt{6}}{2}$.

In Figure 2, we give a representation of the region defined by nontrivial inequalities.

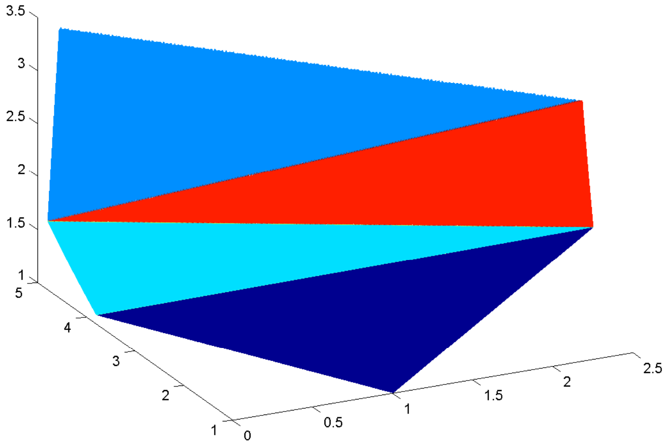

Figure 2: Convex hull description of Example 4.12

5 Conclusion

In this paper, we study a polynomial partitioning set that plays a central role in factorable programming. Although this set has been studied for many years, its convex hull was only known for specific parameters and specific situations where some of the variables are unbounded. We derive a convex hull description of this set in the space of original variables. To obtain this description, we study the packing and covering relaxations of the set independently. We then describe the convex hulls of these relaxations through infinite families of linear inequalities obtained through lifting. We then project this description into a collection of linear and nonlinear inequalities. We show that some of linear inequalities we derive dominate inequalities that would be obtained using McCormick envelopes. Among the nonlinear inequalities we obtain, we observe that some are globally valid for the set, while some others may only be valid for specific subsets of $\mathbb{R}^3$. We present constructive derivations for two of the families of nonlinear inequalities that are part of the description of $\text{conv}(S)$. We generalize this construction in [16].

The technique we use in this paper to obtain the convex hull of S is general and could be applied to study the convex hull of other sets. It, however, requires that the solution of both lifting and projection problems can be carried in closed-form. We believe that the results we obtain in this paper shed light on the functional forms that arise in the description of the convex hulls of polynomial partitioning sets, and that they provide ways of strengthening factorable relaxations that are currently used in global optimization software.

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