

On Auction Models of Conflict with Network Applications¹

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Abstract

We consider several models of complex systems with active elements and show that the auction mechanism appears very natural in attaining proper equilibrium states, even in comparison with game theory ones. In particular, network equilibria are treated as implementation of the auction principle. An additional example of resource allocation in wireless communication networks is also described.

Key words: Complex systems; active elements; auction mechanism; network equilibria; wireless communication networks.

1 Introduction

Investigation of complex systems with active elements having their own interests and control parameters is usually based on a suitable equilibrium concept. Such a concept should equilibrate different interests and opportunities of the elements (agents, participants). For instance, the classical perfectly (Walrasian) and imperfectly (Cournot - Bertrand) competitive models are basic in economics; see e.g. [Nikaido, 1968], [Okuguchi and Szidarovszky, 1990] and references therein. Both these classes of models describe ways of equilibration of conflict interests of agents, however, they lead to different mathematical problems. The former is usually formulated as a complementarity problem, whereas the latter is based on the Nash equilibrium concept in non-cooperative game theory. This difference can be explained mainly by the information side of the models. In fact, within a perfect competition model an agent does not utilize (or finds it useless) the information about the behavior of other agents, because actions of any separate agent can not impact the state of the whole system. Hence, the agent utilizes values of a certain integral system parameter (say, prices). In imperfectly competitive models, agents however must utilize information about actions (interests, opportunities) of others, because their utility functions depend on these actions explicitly.

However, the recent development of information and telecommunication technologies together with great changes in several fields such as energy and electronic commerce

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yield new challenges in creation of new adequate mathematical models and derivation of efficient decisions; see e.g. [Ilic et al, 1998], [Courcoubetis and Weber, 2003], [Stańczak et al, 2006]. We suppose that auction market principles may represent one of possible alternative ways in attaining these goals. Traditionally, investigation of auction models was usually restricted by indivisible good ones and based on game theory techniques which evaluate strategies of players for capturing a desired lot; see e.g. [Weber, 1985], [Milgrom, 2004] and references therein. At the same time, auction models with infinitely divisible goods appeared very useful in describing new energy markets and resource allocation in telecommunication networks; see e.g. [Anderson and Philpott, 2002], [Beraldi et al, 2004], [Iosifidis and Koutsopoulos, 2010].

A new approach to modeling auction markets with infinitely divisible goods was proposed in [Konnov, 2006], [Konnov, 2007a], and further extended in [Konnov, 2007b], [Allevi et al, 2012], [Konnov, 2013], where variational inequality models of separate auctions with price functions of participants were suggested, i.e. they allowed for rather complex behavior of traders and buyers. Besides, unlike the other known models, auction models require minimal information about behavior of agents, which corresponds to many contemporary market conditions.

In this paper, we show that the auction mechanism appears very natural in describing proper equilibrium states of complex systems with active elements, even in comparison with game theory ones. In particular, we show that network equilibria are treated as implementation of the auction principle. An additional example of resource allocation in wireless communication networks is also described.

2 An auction of a homogeneous commodity

We first describe a simple auction market of a homogeneous divisible commodity. Denote by I and J the index sets of traders and buyers at this auction. For each $i \in I$, the i -th trader chooses some offer value x_i in his/her capacity segment $[\alpha'_i, \alpha''_i]$ and announces a price g_i . Similarly, for each $j \in J$, the j -th buyer chooses some bid value y_j in his/her capacity segment $[\beta'_j, \beta''_j]$ and also announces a price h_j . Denote by u the value of external excess demand for this market. That is, u reflects the excess demand of external economic agents who do not participate explicitly in this auction, but agree beforehand with its price. If $u = 0$, the market is closed. Then we can define the feasible set of offer/bid values

$$D = \left\{ (x, y) \left| \begin{array}{l} \sum_{i \in I} x_i - \sum_{j \in J} y_j - u = 0; \\ x_i \in [\alpha'_i, \alpha''_i], i \in I, y_j \in [\beta'_j, \beta''_j], j \in J \end{array} \right. \right\},$$

where $x = (x_i)_{i \in I}, y = (y_j)_{j \in J}$. The solution of the auction problem consists in finding a feasible volume vector $(\bar{x}, \bar{y}) \in D$ and a price \bar{p} such that

$$g_i \begin{cases} \geq \bar{p} & \text{if } \bar{x}_i = \alpha'_i, \\ = \bar{p} & \text{if } \bar{x}_i \in (\alpha'_i, \alpha''_i), \\ \leq \bar{p} & \text{if } \bar{x}_i = \alpha''_i, \end{cases} \quad i \in I, \quad (1)$$

and

$$h_j \begin{cases} \leq \bar{p} & \text{if } \bar{y}_j = \beta'_j, \\ = \bar{p} & \text{if } \bar{y}_j \in (\beta'_j, \beta''_j), \\ \geq \bar{p} & \text{if } \bar{y}_j = \beta''_j, \end{cases} \quad j \in J. \quad (2)$$

However, we are interested in investigation of more complicated behavior of participants, when the announced prices may depend on all the offer/bid values. That is, now the i -th trader has a price (inverse supply) function $g_i(x, y)$, and the j -th buyer has a price (inverse demand) function $h_j(x, y)$. Moreover, this dependence of the price functions may appear regardless of participants intentions; see Section 5. Then, relations (1)–(2) are replaced with the following:

$$g_i(\bar{x}, \bar{y}) \begin{cases} \geq \bar{p} & \text{if } \bar{x}_i = \alpha'_i, \\ = \bar{p} & \text{if } \bar{x}_i \in (\alpha'_i, \alpha''_i), \\ \leq \bar{p} & \text{if } \bar{x}_i = \alpha''_i, \end{cases} \quad i \in I, \quad (3)$$

and

$$h_j(\bar{x}, \bar{y}) \begin{cases} \leq \bar{p} & \text{if } \bar{y}_j = \beta'_j, \\ = \bar{p} & \text{if } \bar{y}_j \in (\beta'_j, \beta''_j), \\ \geq \bar{p} & \text{if } \bar{y}_j = \beta''_j, \end{cases} \quad j \in J. \quad (4)$$

In this model, participants report their price functions and capacity bounds to an auction manager (regulator). The latter has to solve problem (3)–(4) and to report the auction clearing price, which equilibrates the market and yields also the offer/bid values. The auction procedure should be accomplished within a limited time period. Note that the participants may not know price functions of the others.

In [Konnov, 2006] (see also [Konnov, 2007a], [Konnov, 2007b]), the following basic relation between the auction market problem (3)–(4) and a variational inequality (VI, for short) was established.

Proposition 2.1 (a) *If $(\bar{x}, \bar{y}, \bar{p})$ satisfies (3)–(4) and $(\bar{x}, \bar{y}) \in D$, then (\bar{x}, \bar{y}) solves VI: Find $(\bar{x}, \bar{y}) \in D$ such that*

$$\sum_{i \in I} g_i(\bar{x}, \bar{y})(x_i - \bar{x}_i) - \sum_{j \in J} h_j(\bar{x}, \bar{y})(y_j - \bar{y}_j) \geq 0 \quad \forall (x, y) \in D. \quad (5)$$

(b) *If a pair (\bar{x}, \bar{y}) solves VI (5), then there exists \bar{p} such that $(\bar{x}, \bar{y}, \bar{p})$ satisfies (3)–(4).*

Therefore, we can apply various results from the theory of VIs and efficient methods (see, e.g., [Facchinei and Pang, 2003], [Konnov, 2007a]) for investigation and solution of the auction market problem. Nevertheless, we observe that VI (5) becomes equivalent to an optimization problem only under additional conditions. In fact, if all the mappings $g_i, i \in I$ and $h_j, j \in J$ are integrable, VI (5) gives an optimality condition for an optimization problem. The integrability condition holds if $g_i = g_i(x_i)$ for all $i \in I$ and $h_j = h_j(y_j)$ for all $j \in J$, i.e., when each agent does not take into account interests of the others. Next, if all the functions $g_i, i \in I$ and $-h_j, j \in J$ are continuous, we can define differentiable functions $\mu_i : [0, \alpha_i] \rightarrow \mathbb{R}, i \in I$ and $\eta_j : [0, \beta_j] \rightarrow \mathbb{R}, j \in J$ such that

$$\mu'_i(x_i) = g_i(x_i), \quad \eta'_j(y_j) = h_j(y_j),$$

and the optimization problem:

$$\begin{aligned} \min \rightarrow & \sum_{i \in I} \mu_i(x_i) - \sum_{j \in J} \eta_j(y_j) \\ \text{subject to} & (x, y) \in D; \end{aligned} \tag{6}$$

which maximizes the profit of the auction manager subject to the balance and participants' capacity constraints. VI (5) then gives an optimality condition for (6). Moreover, it is equivalent to the convex optimization problem (6) if the mappings $g_i, i \in I$ and $-h_j, j \in J$ are monotone.

Remark 2.1 *The absence of a common cost function (system goal) for the auction market model in the general case indicates one its difference from the centralized planning mechanisms. The other difference is that agents report only their desired information to the manager, rather than peculiarities of their technology and financial abilities. They can even indicate the desired dependence of their commodity shares (volumes) from the other ones, which gives the information scheme of this allocation mechanism rather different from the perfect competition. At the same time, unlike the game theoretic setting, this information can be very limited and not be reported to other participants. Indeed, players must have information (even incomplete) about the interests (utility functions) and abilities (strategy sets) of others in order to elaborate a proper and rational strategy decision. Under the full absence of such information, player's behavior can be described by the known Chinese sentence (Confucius): "The hardest thing of all is to find a black cat in a dark room, especially if there is no cat". Therefore, any attempt to place such conflict models without necessary information in the game theoretic setting seems artificial despite the existing tendency to treat any conflict as a game problem.*

The above auction market is two-side since it involves traders and buyers. Suppose that one side (say, buyers) is absent, i.e., $J = \emptyset$. Then, $g_i = g_i(x)$ and we have the feasible set

$$X = \left\{ x \mid \sum_{i \in I} x_i = u; x_i \in [\alpha'_i, \alpha''_i], i \in I \right\};$$

where u is treated as the fixed demand. The auction relations (3) are replaced with the following:

$$g_i(\bar{x}) \begin{cases} \geq \bar{p} & \text{if } \bar{x}_i = \alpha'_i, \\ = \bar{p} & \text{if } \bar{x}_i \in (\alpha'_i, \alpha''_i), \\ \leq \bar{p} & \text{if } \bar{x}_i = \alpha''_i, \end{cases} \quad i \in I. \quad (7)$$

Similarly to Proposition 2.1, (\bar{x}, \bar{p}) satisfies (7) and $\bar{x} \in X$, iff \bar{x} solves VI: Find $\bar{x} \in X$ such that

$$\sum_{i \in I} g_i(\bar{x})(x_i - \bar{x}_i) \geq 0 \quad \forall x \in X.$$

Furthermore, if $\alpha'_i = 0$ and $\alpha''_i = +\infty$ for each $i \in I$, i.e. traders admit any non-negative capacity, X is replaced with

$$\tilde{X} = \left\{ x \mid \sum_{i \in I} x_i = u; x_i \geq 0, i \in I \right\};$$

relations (7) are replaced with the following:

$$g_i(\bar{x}) \begin{cases} \geq \bar{p} & \text{if } \bar{x}_i = 0, \\ = \bar{p} & \text{if } \bar{x}_i > 0; \end{cases} \quad i \in I. \quad (8)$$

Clearly, (\bar{x}, \bar{p}) satisfies (8) and $\bar{x} \in \tilde{X}$, iff \bar{x} solves VI: Find $\bar{x} \in \tilde{X}$ such that

$$\sum_{i \in I} g_i(\bar{x})(x_i - \bar{x}_i) \geq 0 \quad \forall x \in \tilde{X}.$$

Similar capacity modifications can be made in the general two-side model (3)–(4).

3 Network equilibria as auction models: fixed demands

We now recall the well known network equilibrium problems within the so-called path flow formulation. These models are determined on an oriented graph, each its arc being associated with some flow (for instance, traffic) and some expense (dis-utility, for instance, time of delay, or cost etc), which depends on the values of arc flows. Usually, arc dis-utilities are monotone increasing functions of arc flows.

Let us given a graph with a finite set of nodes \mathcal{N} and a set of oriented arcs \mathcal{A} which join the nodes so that any arc $a = (i, j)$ has the origin i and the destination j . Next, among all the pairs of nodes of the graph we extract a subset of origin-destination (O/D) pairs \mathcal{W} of the form $w = (i \rightarrow j)$. Besides, each pair $w \in \mathcal{W}$ is associated with a non-negative flow demand d_w and with the set of paths \mathcal{P}_w which connect the origin and destination for this pair. Also, denote by x_p the path flow for the path p . First we

consider the fixed demand case, where all the values d_w are fixed. Then one can define the feasible set of flows:

$$X = \left\{ x \mid \sum_{p \in \mathcal{P}_w} x_p = d_w, x_p \geq 0, p \in \mathcal{P}_w, w \in \mathcal{W} \right\}, \quad (9)$$

with $x = (x_p)_{p \in \mathcal{P}_w, w \in \mathcal{W}}$. Given a flow vector x , one can determine the value of the arc flow

$$f_a = \sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} \alpha_{pa} x_p \quad (10)$$

for each arc $a \in \mathcal{A}$, where

$$\alpha_{pa} = \begin{cases} 1 & \text{if arc } a \text{ belongs to path } p, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

If the vector $f = (f_a)_{a \in \mathcal{A}}$ of arc flows is known, one can determine the dis-utility value $c_a(f)$ for each arc with the help of some mapping c that is defined in the space of flows. Then one can compute the dis-utility value for each path p :

$$g_p(x) = \sum_{a \in \mathcal{A}} \alpha_{pa} c_a(f). \quad (12)$$

A feasible flows vector $x^* \in X$ is said to be an *equilibrium point* if it satisfies the following conditions:

$$\forall w \in \mathcal{W}, \forall p \in \mathcal{P}_w, \quad x_p^* > 0 \implies g_p(x^*) = \min_{q \in \mathcal{P}_w} g_q(x^*). \quad (13)$$

In other words, positive values of flow for any (O/D) pair correspond to paths with minimal dis-utility at the current flows distribution and all the dis-utilities along used routes are equal; see [Wardrop, 1952], [Smith, 1979]. It is known that conditions (9)–(13) can be equivalently rewritten in the form of the variational inequality: Find a vector $x^* \in X$ such that

$$\sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} g_p(x^*) (x_p - x_p^*) \geq 0 \quad \forall x \in X. \quad (14)$$

Proposition 3.1 (see [Dafermos, 1980]) *A flow vector $x^* \in X$ solves problem (14) if and only if it satisfies conditions (9)–(13).*

At the same time, the equilibrium conditions in (13) represent implementation of the auction mechanism. In fact, they can be equivalently rewritten as

$$\forall w \in \mathcal{W}, \exists \lambda_w \text{ such that } g_p(x^*) \begin{cases} \geq \lambda_w & \text{if } x_p^* = 0, \\ = \lambda_w & \text{if } x_p^* > 0; \end{cases} \quad \forall p \in \mathcal{P}_w. \quad (15)$$

Let us temporarily consider the case when there exists only one (O/D) pair \bar{w} , i.e. $\mathcal{W} = \{\bar{w}\}$. Then (15) clearly coincides with (8). This means that the network equilibrium problem now falls into the single-commodity one-side (trader) auction model, where $d_{\bar{w}}$ gives the fixed demand volume, paths $p \in \mathcal{P}_{\bar{w}}$ correspond to carriers (traders) with price functions $g_p(x)$ depending on volumes. Within this auction, an allocation of shares of the unique (O/D) pair demand volume is accomplished in conformity with the given carrier cost functions. In the general case, (15) corresponds to a collection of single-commodity one-side auctions, where each commodity is associated with an (O/D) pair $w \in \mathcal{W}$ with a unique buyer having the fixed demand volume d_w and a set of carriers (traders) with price functions $g_p(x)$ for $p \in \mathcal{P}_w$. At the same time, these auctions are closely related to each other, because carriers of different (O/D) pairs utilize the same links (arcs), hence each price function $g_p(x)$ depend on offer values of other carriers, including those of other (O/D) pairs.

Assertion 3.1 *The network equilibrium problem (9)–(13) is a multi-commodity one-side auction, represented by a linked system of single-commodity one-side auctions.*

It should be observed that there exist a great number of works where the above network equilibrium problem is treated as an application of the known Nash equilibrium principle from the non-cooperative game theory; see, e.g., [Nisan et al, 2007], [Kliemann and Srivastav, 2009] and references therein. Indeed, condition (13) also means that any single user (vehicle) can not receive profit from his/her unilateral action by taking an unused route, which seems similar to the Nash equilibrium concept.

We recall that the latter is defined in an *m-person noncooperative game*, where the *i*-th player has a cost function $f_i : U \rightarrow \mathbb{R}$ and a strategy set U_i so that

$$U = U_1 \times \dots \times U_m.$$

A strategy profile $u^* = (u_1^*, \dots, u_m^*)^\top \in U$ is said to be a Nash equilibrium point for this game [Nash, 1951], if

$$f_i(u_{-i}^*, u_i) \geq f_i(u^*) \quad \forall u_i \in U_i, \quad i = 1, \dots, m. \quad (16)$$

We see the fundamental difference between both the equilibrium concepts. In (13), each participant utilizes only information about some integral parameters of the whole system, such as total flows distribution, instead of opportunities and actions of others, as in (16), where an action of any separate participant (player) changes values of cost functions of others. Therefore, equilibrium points in (13) or (14) differ from the Nash equilibrium points. Otherwise, solutions of any variational inequality should be treated as game equilibria. We observe also that the difference between the perfectly and imperfectly competitive market models is similar; see Section 1. Moreover, perfect market equilibria can be obtained as limits of imperfect ones; see [Mas-Colell, 1983], [Novshek and Sonnenschein, 1983], however, network equilibria can

be also obtained as limits of Nash equilibrium points under certain additional conditions; see [Haurie and Marcotte, 1985], [Altman et al, 2011].

Application of the noncooperative game equilibrium concepts to network problems requires a more detailed evaluation. Let us first consider an opportunity of such an application to traffic network problems. Then, clearly, each single participant (vehicle) can not affect the total state of the system since

- (i) there are a great number of other vehicles in the system;
- (ii) topology of the transportation network involving all the routes is too complex for evaluation.

Due to the same reasons, it seems too difficult or useless for a single participant to collect information about actions and abilities of other separate vehicles, and he/she will prefer to utilize parameters (delays or traffic distribution) of the whole system. Therefore, the direct application of noncooperative game equilibrium concepts seems rather artificial, although there exists a conflict situation here. One can try to apply the so-called non-atomic game approach, where players are associated with groups of vehicles having the same (O/D) pair of nodes. The number of vehicles is supposed to be real and reflects the traffic flow. However, it is too difficult to assume that such an aggregated “player” possesses rational behavior features, because vehicle users do not coordinate their road selection actions, unlike cooperative non-atomic games, where this assumption seems very natural; see [Aumann and Shapley, 1974].

Let us consider an application of noncooperative game equilibrium concepts to communication networks. Here, each player is attributed to a single (O/D) pair and chooses his/her traffic allocation shares among joining routes. Again, this approach meets difficulties related to certain lack of information since

- (i) the number of users may be too large and they may not have any information on each other in general;
- (ii) topology of the communication network may be too complex for evaluation.

Such information drawbacks of games have been discussed in Remark 2.1. Therefore, the noncooperative game approach is applicable here only if the topology of the communication network is relatively simple and users are able to analyze activity of each other during a rather long time period. At the same time, the auction approach can be used in all the above network systems due to Assertion 3.1. Let us consider an example illustrating relationships between both these approaches.

Example 3.1 The simple graph represented at Figure 1 contains two nodes and two two-directional edges. User 1 is associated with (O/D) pair $(1 \rightarrow 2)$, user 2 is associated with (O/D) pair $(2 \rightarrow 1)$. They have the same unit flow demands and the feasible flow profiles $X = \{x \in \mathbb{R}_+^2 \mid x_1 + x_2 = 1\}$ and $Y = \{y \in \mathbb{R}_+^2 \mid y_1 + y_2 = 1\}$ with flow costs per unit $c_1(x, y) = 2(x_1 + y_1)$ and $c_2(x, y) = 3(x_2 + y_2)$, respectively. Hence, the costs depend on flows in both directions.

On account of (15), the network equilibrium (\bar{x}, \bar{y}) can be calculated from the relations

$$c_1(\bar{x}, \bar{y}) = c_2(\bar{x}, \bar{y}) = \lambda,$$

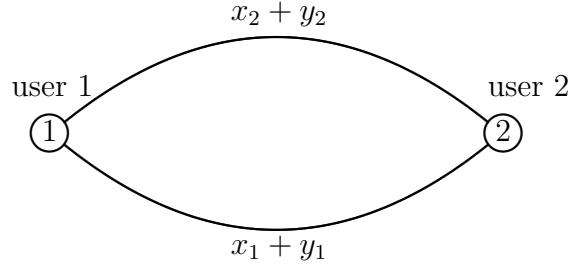


Figure 1: A simple network example

which gives $\bar{x}_1 + \bar{y}_1 = 1.2$ and $\lambda = 2.4$. This means that there exists a set of equilibrium flows, e.g.,

$$x' = (0.6, 0.4)^\top, y' = (0.6, 0.4)^\top \text{ and } x'' = (0.4, 0.6)^\top, y'' = (0.8, 0.2)^\top.$$

Let us turn to Nash equilibrium points. In the non-cooperative game, players have the cost functions

$$h_1(x, y) = x_1 c_1(x, y) + x_2 c_2(x, y)$$

and

$$h_2(x, y) = y_1 c_1(x, y) + y_2 c_2(x, y).$$

Take first the point (x', y') above, then, setting $t = x_1$, we have

$$h_1(x, y') = 2t(t + 0.6) + 3(t - 1)(t - 1.4).$$

This strongly convex quadratic function attains its minimum at the unique point $x'_1 = 0.6$. Note that $h_1(x, y') = h_2(x', y)$, and the best response of the second player is the same. Hence, (x', y') is a Nash equilibrium point of this game.

Take now the point (x'', y'') above, then, setting $\tilde{x} = (0.5, 0.5)^\top$, we have

$$h_1(\tilde{x}, y'') = 2.35 < h_1(x'', y'') = 2.4.$$

Similarly, setting $\tilde{x} = (0.7, 0.3)^\top$, we have

$$h_2(x'', \tilde{y}) = 2.35 < h_2(x'', y'') = 2.4.$$

This means that this network equilibrium point (x'', y'') is not a Nash equilibrium point of this game.

4 Network equilibria as auction models: elastic demands

Let us turn to the more general elastic (inverse) demand case, where all the flow demands d_w are variables. Then, the feasible set of flows/demands Y can be defined as follows:

$$Y = \left\{ y = (x, d) \mid \sum_{p \in \mathcal{P}_w} x_p = d_w, x_p \geq 0, p \in \mathcal{P}_w, w \in \mathcal{W} \right\}, \quad (17)$$

where $d = (d_w)_{w \in \mathcal{W}}$.

Denote by τ_w the minimal path dis-utility for each pair $w \in \mathcal{W}$ and suppose that it can be in principle dependent of the flow demand d , i.e.

$$\tau_w = \tau_w(d). \quad (18)$$

A feasible flows/demands pair $(x^*, d^*) \in Y$ is said to be an *equilibrium point* if it satisfies the following conditions:

$$g_p(x^*) - \tau_w(d^*) \begin{cases} = 0 & \text{if } x_p^* > 0, \\ \geq 0 & \text{if } x_p^* = 0; \end{cases} \quad \forall p \in \mathcal{P}_w, w \in \mathcal{W}. \quad (19)$$

Again, positive values of flow for any (O/D) pair must correspond to paths with minimal dis-utility. Besides, it is known that conditions (17)–(19) can be equivalently rewritten in the form of the variational inequality: Find a pair $(x^*, d^*) \in Y$ such that

$$\sum_{w \in \mathcal{W}} \sum_{p \in \mathcal{P}_w} g_p(x^*)(x_p - x_p^*) - \sum_{w \in \mathcal{W}} \tau_w(d^*)(d_w - d_w^*) \geq 0 \quad \forall (x, d) \in Y. \quad (20)$$

Proposition 4.1 (see [Dafermos, 1982]) *A pair $(x^*, d^*) \in Y$ solves problem (20) if and only if it satisfies conditions (10)–(12), (17)–(19).*

We now intend to show that the equilibrium conditions in (19) also represent implementation of the auction mechanism. In fact, they can be equivalently rewritten as

$$\forall w \in \mathcal{W}, \exists \lambda_w \text{ such that } g_p(x^*) \begin{cases} \geq \lambda_w & \text{if } x_p^* = 0, \\ = \lambda_w & \text{if } x_p^* > 0; \end{cases} \quad \forall p \in \mathcal{P}_w; \quad (21)$$

and

$$\tau_w(d^*) \begin{cases} \leq \lambda_w & \text{if } d_w^* = 0, \\ = \lambda_w & \text{if } d_w^* > 0; \end{cases} \quad \forall p \in \mathcal{P}_w; \quad (22)$$

cf. (3)–(4). Take an arbitrary pair $w \in \mathcal{W}$. Suppose a pair $(x^*, d^*) \in Y$ satisfies conditions (21)–(22). Then, for any $p \in \mathcal{P}_w$, the relation $x_p^* > 0$ implies $d_w^* > 0$, hence

we have $g_p(x^*) = \lambda_w = \tau_w(d^*)$. Next, the relation $x_p^* = 0$ implies $d_w^* \geq 0$, hence we have $g_p(x^*) \geq \lambda_w \geq \tau_w(d^*)$, and (19) holds true.

Conversely, suppose a pair $(x^*, d^*) \in Y$ satisfies conditions (19). Set $\lambda_w = \tau_w(d^*)$. If $d_w^* = 0$, then $x_p^* = 0$ for any $p \in \mathcal{P}_w$ and (19) implies $g_p(x^*) \geq \lambda_w$ for all $p \in \mathcal{P}_w$. If $d_w^* > 0$, take any $p \in \mathcal{P}_w$. If $x_p^* > 0$, (19) implies $g_p(x^*) = \lambda_w$, otherwise, for $x_p^* = 0$ (19) implies $g_p(x^*) \geq \lambda_w$, and (21)–(22) hold true. We now give the precise statement.

Proposition 4.2 *For any pair $(x^*, d^*) \in Y$, conditions (19) and (21)–(22) are equivalent.*

In order to explain the auction mechanism in (10)–(12), (17)–(19), we temporarily consider the case when there exists only one (O/D) pair \bar{w} , i.e. $\mathcal{W} = \{\bar{w}\}$. Then the network equilibrium problem falls into the single-commodity two-side auction model, where $d_{\bar{w}}$ gives the traffic demand volume, $\tau_{\bar{w}}(d)$ gives the price function of the unique user \bar{w} , paths $p \in \mathcal{P}_{\bar{w}}$ correspond to carriers (traders) with price functions $g_p(x)$ depending on their volumes. Within this auction, an allocation of shares of the unique (O/D) pair demand volume is accomplished in conformity with the given carrier cost functions and bid prices. In the general case, we thus obtain a collection of single-commodity two-side auctions, where each commodity is associated with an (O/D) pair $w \in \mathcal{W}$ with a unique buyer with price function $\tau_w(d)$ and a set of carriers (traders) with price functions $g_p(x)$ for $p \in \mathcal{P}_w$. At the same time, these auctions are closely related to each other, because carriers of different (O/D) pairs utilize the same links (arcs), hence each price function $g_p(x)$ depends on offer values of other carriers, including those of other (O/D) pairs. Besides, each (O/D) dis-utility function $\tau_w(d)$ may depend on bid values of other (O/D) pairs.

Assertion 4.1 *The network equilibrium problem (10)–(12), (17)–(19) is a multi-commodity two-side auction, represented by a linked system of single-commodity two-side auctions.*

5 An auction model of resource allocation in wireless networks

In contemporary wireless networks, increasing demand of services leads to serious congestion effects, whereas significant network resources (say, bandwidth and batteries capacity) are utilized inefficiently, especially in the case when fixed allocation mechanisms are implemented. This situation forces us to apply more flexible and dynamical allocation mechanisms based on market type models; see e.g. [Wyglinski et al, 2010], [Zhao and Sadler, 2009].

Due to the presence of conflict of interests, most papers on allocation mechanisms are devoted to game-theoretic models; see, e.g., [Leshem and Zehavi, 2009],

[Raouf and Al-Raweshidy, 2010]. Another way consists in application of auction market models due to its minimal requirements on involved users, which is very essential for wireless telecommunication networks; see, e.g., [Iosifidis and Koutsopoulos, 2010], [Raouf and Al-Raweshidy, 2010] and references therein.

In [Hayrapetyan et al, 2007] and [Zhang and Zhang, 2013], game-theoretic models for wired and wireless network resource distribution were presented. We now describe a modification of that approach and show that it leads to a two-side auction with complicated price functions.

Let us consider a problem of allocation of network resources of several competitive providers to a large number of users. For this reason, users are considered as a unique buyer with the price function $h(y)$ and the scalar bid volume y . Each i -th provider (trader) announce his/her price function $b_i(x_i)$ depending on offer volume x_i in his/her capacity segment $[0, \alpha_i]$ for $i = 1, \dots, m$. Besides, this joint utilization of resources of providers yields the dis-utility $l_i(x)$ where $x = (x_1, \dots, x_m)^\top$. For instance, common spectrum sharing in wireless communication results in decrease of signal quality, the corresponding example is given in [Zhang and Zhang, 2013]. Therefore, the price function of the i -th trader is $g_i(x) = b_i(x_i) + l_i(x)$. This conflict situation can be resolved by using the auction mechanism. We define the feasible set of offer/bid values

$$D = \left\{ (x, y) \in \mathbb{R}^{m+1} \left| \sum_{i=1}^m x_i = y; x_i \in [0, \alpha_i], i = 1, \dots, m, y \geq 0 \right. \right\}.$$

The solution of the two-side auction problem consists in finding a feasible volume vector $(\bar{x}, \bar{y}) \in D$ and a price \bar{p} such that

$$g_i(\bar{x}) \begin{cases} \geq \bar{p} & \text{if } \bar{x}_i = 0, \\ = \bar{p} & \text{if } \bar{x}_i \in (0, \alpha_i), \\ \leq \bar{p} & \text{if } \bar{x}_i = \alpha_i, \end{cases} \quad i = 1, \dots, m; \quad (23)$$

and

$$h(\bar{y}) \begin{cases} \leq \bar{p} & \text{if } \bar{y} = 0, \\ = \bar{p} & \text{if } \bar{y} > 0; \end{cases} \quad (24)$$

see (3)–(4). As in Proposition 2.1, conditions (23)–(24) can be replaced with the equivalent (VI): Find a pair $(\bar{x}, \bar{y}) \in D$ such that

$$\sum_{i=1}^m g_i(\bar{x})(x_i - \bar{x}_i) - h(\bar{y})(y - \bar{y}) \geq 0 \quad \forall (x, y) \in D. \quad (25)$$

After a simple substitution we can write this problem in x only: Find a pair $\bar{x} \in X$ such that

$$\langle F(\bar{x}), x - \bar{x} \rangle = \sum_{i=1}^m g_i(\bar{x})(x_i - \bar{x}_i) - h\left(\sum_{i=1}^m \bar{x}_i\right) \sum_{i=1}^m (x_i - \bar{x}_i) \geq 0 \quad \forall x \in X, \quad (26)$$

where

$$F_i(x) = g_i(x) - h \left(\sum_{i=1}^m x_i \right), \quad i = 1, \dots, m, \quad X = [0, \alpha_1] \times \dots \times [0, \alpha_m].$$

At the same time, if the mappings $l_i(x)$ are not integrable, this VI is not equivalent to an optimization problem; see Section 2.

Remark 5.1 *In [Hayrapetyan et al, 2007], the authors suggested an equilibrium model with fixed prices and link delay functions only as a starting point for formulation of a noncooperative equilibrium problem. This approach was modified and extended in [Zhang and Zhang, 2013], where the same equilibrium model with fixed prices and disutility functions was used for allocations of resources of several wireless access points. Further, it was also utilized for a two-level Stackelberg equilibrium model. In this work, we admit rather complex behavior of providers and users, give auction conditions (23)–(24), and show that the problem is equivalent to a usual variational inequality.*

There exist a number of efficient iterative methods to solve VI (25) or (26); see, e.g., [Facchinei and Pang, 2003], [Konnov, 2007a]. For instance, the simplest projection method for VI (26) is defined as follows:

$$x_i^{k+1} = \max\{0, x_i^k - \eta_k F_i(x^k)\}, \quad i = 1, \dots, m, \quad \text{and } \eta_k > 0.$$

Under a proper choice of stepsizes, the iteration sequence will converge to a solution.

6 Conclusions

We considered the known network equilibrium problems with fixed and elastic demands and showed they follow the auction equilibrium principle. Another auction model was suggested for resource allocation in wireless communication networks. In general, we can conclude that the auction mechanism seems promising for resolving conflict situations in the cases where participants do not have a sufficient information about each other, or are forced to act within a rather complicated system, or have certain time limitations in their decision making.

References

- [Nikaido, 1968] Nikaido, H., (1968), Convex Structures and Economic Theory. Academic Press, New York.
- [Okuguchi and Szidarovszky, 1990] Okuguchi, K., and Szidarovszky, F., (1990), The Theory of Oligopoly with Multi-product Firms. Springer-Verlag, Berlin.

- [Ilic et al, 1998] Ilic, M., Galiana, F., and Fink, L., (1998) Power Systems Restructuring: Engineering and Economics. Kluwer Academic Publishers, Boston.
- [Courcoubetis and Weber, 2003] Courcoubetis, C., and Weber, R., (2003) Pricing Communication Networks: Economics, Technology and Modelling. John Wiley & Sons, Chichester.
- [Stańczak et al, 2006] Stańczak, S., Wiczanowski, M., and Boche, H., (2006) Resource Allocation in Wireless Networks. Theory and Algorithms. Springer, Berlin.
- [Weber, 1985] Weber, R.J., (1985) Auctions and competitive bidding. In: Young, H.P.(ed.) Fair Allocation, Proceedings of Symposia in Applied Mathematics, vol.33. American Mathematical Society, Providence, pp.143–170.
- [Milgrom, 2004] Milgrom, P., (2004) Putting Auction Theory to Work. Cambridge University Press.
- [Anderson and Philpott, 2002] Anderson, E.J., and Philpott, A.B., (2002) Optimal offer construction in electricity markets. *Mathem. Oper. Res.*, vol.27, pp. 82–100.
- [Beraldi et al, 2004] Beraldi, P., Conforti, D., Triki, C., and Violi, A., (2004) Constrained auction clearing in the Italian electricity market. *4OR*, vol.2, pp.35–51.
- [Iosifidis and Koutsopoulos, 2010] Iosifidis, G., and Koutsopoulos, I., (2010) Double auction mechanisms for resource allocation in autonomous networks. *IEEE J. Sel. Areas in Commun.*, vol.28, nr.1, pp.95–102.
- [Konnov, 2006] Konnov, I.V., (2006) On modeling of auction type markets. *Issled. Inform.*, vol.10, pp.73–76 (in Russian).
- [Konnov, 2007a] Konnov, I.V., (2007a) Equilibrium Models and Variational Inequalities. Elsevier, Amsterdam.
- [Konnov, 2007b] Konnov, I.V., (2007b) On variational inequalities for auction market problems. *Optim. Lett.*, vol.1, pp.155–162.
- [Allevi et al, 2012] Allevi, E., Gnudi, A., and Konnov, I.V., (2012) Combined methods for dynamic spatial auction market models. *Optimization and Engineering*, vol.13, pp. 401–416.
- [Konnov, 2013] Konnov, I.V., (2013) Equilibrium models for multi-commodity auction market problems. *Advanced Modeling and Optimization*, vol.15, pp. 511–524.
- [Facchinei and Pang, 2003] Facchinei, F., and Pang, J.-S., (2003) Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer-Verlag, Berlin.

- [Wardrop, 1952] Wardrop, J.G., (1952) Some theoretical aspects of road traffic research. Proceedings of the Institute of Civil Engineering, Part II. pp.325–304.
- [Smith, 1979] Smith, M.J., (1979) Existence, uniqueness, and stability of traffic equilibria. Transp. Science, vol.13B, pp. 259–304.
- [Dafermos, 1980] Dafermos, S., (1980) Traffic equilibrium and variational inequalities. Transp. Science, vol.14, pp. 42–54.
- [Dafermos, 1982] Dafermos, S., (1982) The general multimodal network equilibrium problem. Networks, vol.12, pp. 231–240.
- [Nisan et al, 2007] Nisan, N., Roughgarden, T., Tardos, É., and Vazirani, V., (eds.) (2007) Algorithmic Game Theory. Cambridge University Press, Cambridge.
- [Kliemann and Srivastav, 2009] Kliemann, L., and Srivastav, A., (2009) Models of non-atomic congestion games: From unicast to multicast routing. In: Lerner, J., Wagner, D., and Zweig, K.A. (eds.) Algorithmics of Large and Complex Networks. Springer-Verlag, Berlin, pp.292-318.
- [Nash, 1951] Nash, J., (1951) Non-cooperative games. Ann. of Math., vol.54, pp.286–295.
- [Mas-Colell, 1983] Mas-Colell, A., (1983) Walrasian equilibria as limits of noncooperative equilibria. Part I: Mixed strategies. Journal of Economic Theory 30, pp.153–170.
- [Novshek and Sonnenschein, 1983] Novshek, W., and Sonnenschein, H., (1983) Walrasian equilibria as limits of noncooperative equilibria. Part II: Pure strategies. Journal of Economic Theory, vol. 30, pp.171–187.
- [Haurie and Marcotte, 1985] Haurie, A. and Marcotte, P., (1985) On the relationship between Nash-Cournot and Wardrop equilibria. Networks, vol.15, pp.295–308.
- [Altman et al, 2011] Altman, E., Combes, R., Altman, Z., and Sorin, S., (2011) Routing games in the many players regime. In: 4-th International ICST Workshop on Game Theory in Communication Networks, ACM, Paris, pp.525–527.
- [Aumann and Shapley, 1974] Aumann, R.J., and Shapley, L.S., (1974) Values of Non-Atomic Games. Princeton University Press, Princeton.
- [Wyglinski et al, 2010] Wyglinski, A.M., Nekovee, M., and Hou, Y.T., (eds.), (2010) Cognitive Radio Communications and Networks: Principles and Practice. Elsevier, Amsterdam.
- [Zhao and Sadler, 2009] Zhao, Q., and Sadler, B., (2009) A survey of dynamic spectrum access. IEEE Commun. Mag., vol. 24, pp.79–89.

- [Leshem and Zehavi, 2009] Leshem, A., and Zehavi, E., (2009) Game theory and the frequency selective interference channel: A practical and theoretic point of view. *IEEE Signal Process.*, vol.26, pp.28–40.
- [Raouf and Al-Raweshidy, 2010] Raouf, O., and Al-Raweshidy, H., (2010) Auction and game-based spectrum sharing in cognitive radio networks. In: Huang, Q., (ed.), *Game Theory*, Sciyo, Rijeka, ch. 2, pp.13–40.
- [Hayrapetyan et al, 2007] Hayrapetyan, A., Tardos, É., and Wexler, T., (2007) A network pricing game for selfish traffic. *Distributed Computing*, vol.19, pp.255-266.
- [Zhang and Zhang, 2013] Zhang, F., and Zhang, W., (2013) Competition between wireless service providers: Pricing, equilibrium and efficiency. In: 11th International Symposium and Workshops on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt 2013), IEEE, pp. 208–215.