Noname manuscript No.
(will be inserted by the editor)

About the Convexity of a Special Function on Hadamard Manifolds

Cruz Neto J. X. · **Melo I. D.** · **Sousa P. A.** · **Silva J. P.**

the date of receipt and acceptance should be inserted later

Abstract In this article we provide an erratum to Proposition 3.4 of E.A. Papa Quiroz and P.R. Oliveira. Proximal Point Methods for Quasiconvex and Convex Functions with Bregman Distances on Hadamard Manifolds, Journal of Convex Analysis 16 (2009), 49-69. More specifically, we prove that the function defined by the product of a fixed vector in the space tangent by inverse of the exponential application at a given pointis is not convex on Hadamard manifolds with negative sectional curvature and present an alternative proof for the result in the case of null sectional curvature.

Cruz Neto J. X. (**Corresponding Author**)

CCN, DM, Universidade Federal do Piauí, Teresina, PI 64049-550, BR

E-mail: jxavier@ufpi.edu.br

Melo I. D.

CCN, DM, Universidade Federal do Piauí, Teresina, PI 64049-550, BR

E-mail: italodowell@ufpi.edu.br

Sousa P. A.

CCN, DM, Universidade Federal do Piauí, Teresina, PI 64049-550, BR

E-mail: paulosousa@ufpi.edu.br

Silva J. P.

CCN, DM, Universidade Federal do Piauí, Teresina, PI 64049-550, BR

E-mail: jsilva@ufpi.edu.br

Keywords Proximal algorithms; Hadamard manifold · Convex Functions.

AMS subject classification: 65K05 · 47J25 · 90C33 · 91E10.

1 Introduction

In the paper [QO], Proposition 3.4, Quiroz and Oliveira claim that the function of the Theorem 1 below, is an affine linear function, where M is a Hadamard manifold. Furthermore, the authors use this fact to prove the good definition of the Proximal Point Algorithm with Bregman Distances. In 2012, Colao et all [CLMM] generalized the result of Quiroz and Oliveria (see Proposition 2.9) and established existence results for the problem Mixed variational inequalities (Theorem 3.5) and Fixed points of set-valued mappings (Theorem 3.10). Moreover, in the Theorem 4.9 the authors give sufficient conditions for the resolvent to have full domain.

More recently, in [ZH] Zhou-Huang made use of the same fact to establish the Existence of weak minimum (Theorem 3.2).

In this work, we present a detailed proof that the the function g is an affine linear function if the manifold has null sectional curvature and prove that the result is not true for Hadamard manifold with negative sectional curvature.

2 Hadamard Manifold with Null Sectional Curvature

Theorem 1 *Let M be a Hadamard manifold with null sectional curvature, $y \in M$ and $u \in T_yM$ nonzero. Then the function $g : M \to \mathbb{R}$, defined by $g(x) = \langle \exp_y^{-1} x, u\rangle_y$, is an affine linear function.*

Proof Let $v \in T_yM$ such that $\exp_y(v) = x$. Now consider $\gamma : \mathbb{R} \to M$ the geodesic such that $\gamma(0) = y$ and $\gamma'(0) = v$, then $\gamma(1) = x$. Given a unit vector $w \in T_yM$, consider the curve $\alpha : \mathbb{R} \to T_yM$ satisfying $\alpha(0) = v$ and $\alpha'(0) = w$. By Corollary 2.5 of [dC], we have that $J(t) = (d\exp_y)_{t\gamma'(0)}(tJ'(0))$ is the Jacobi field along γ satisfying $J(0) = 0$.

On the other hand, since the sectional curvature of M is zero it follows that $J(t) = t \cdot w(t)$ where $w(t)$ is a parallel field along γ. Then, $J'(0) = w(0)$ and $J(1) = w(1)$. In what follows we will choose $w(t)$ as the parallel transport of w along of γ, soon $J'(0) = w$. As $\exp_y : T_yM \to M$ is a diffeomorphism, we get $g(\exp_y(v)) = \langle u, v\rangle$ and $g(\exp_y(\alpha(s))) = \langle u, \alpha(s)\rangle$. Deriving the equality obtained and making $s = 0$, we get

$$\langle \operatorname{grad} g(\exp_y(\alpha(0))), (d\exp_y)_{\alpha(0)}(\alpha'(0))\rangle = \langle u, \alpha'(0)\rangle. \tag{1}$$

Therefore, $\langle \operatorname{grad} g(\exp_y(v)), (d\exp_y)_v(w)\rangle = \langle u, w\rangle$ implying $\langle \operatorname{grad} g(x), J(1)\rangle = \langle u, w\rangle$. Further,

$$\langle \operatorname{grad} g(x), w(1)\rangle = \langle u, w\rangle. \tag{2}$$

As the parallel transport is an isometry and the vector w is arbitrary, it follows that $\operatorname{grad} g(x)$ is the parallel transport of u along of γ. We now show that the function g is affine. For this we use the fact that it is sufficient to show the result to normalized geodesic segments. Then be $\beta : (-\varepsilon, \varepsilon) \to M$ that $\beta(0) = x$ and $\beta'(0) = v$ a normalized geodesic. If for some $t^* \in (-\varepsilon, 0)$ we have $\beta(t^*) = y$ then

$$(g \circ \beta)'(0) = \langle \operatorname{grad} g(x), \beta'(0)\rangle_x = \langle T_{\beta,y,x}(u), v\rangle_x, \tag{3}$$

where $T_{\beta,y,x}$ denotes o Parallel Transport along of β from y to x.

Hence, Parallel Transport is a parallel fields ($\nabla_t T = 0$) e β is a geodesic segment conclude that $(g \circ \beta)''(0) = 0$. If t^* does existe then choise any $\beta(\bar{\varepsilon})$, $\bar{\varepsilon} \in (-\varepsilon, 0)$, and consider a geodesic segment normalized $\xi : [\iota, \bar{\varepsilon}] \to M$ such that $\xi(\iota) = y$ and $\xi(\bar{\varepsilon}) = \beta(\bar{\varepsilon})$. So then just repeat the above argument for the broken geodesic segment $\xi \cup \beta$, because a sectional curvature is null than a Parallel Transport independent of the chosen curve. So the result is proved. □

3 Hadamard Manifold with Negative Sectional Curvature

Theorem 2 *Let M be a Hadamard manifold with negative sectional curvature, $y \in M$ and $u \in T_yM$ nonzero. Then the function $g : M \to \mathbb{R}$, defined by $g(x) = \langle \exp_y^{-1} x, u\rangle_y$, is not convex.*

Proof Initially we will consider $q = \exp_y(-u)$, which is on the geodesic $\gamma(t) = \exp_y(tu)$. We can assume without loss that generalizade that $u \in T_yM$ is unitary. Now consider a normalized geodesic $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = q$ and $\langle \alpha'(0), \gamma'(-1) \rangle_q = 0$, i.e., $\alpha(s)$ is orthogonal to $\gamma(t)$ in q.

Assuming by contradiction that the function $g(x) = \langle \exp_y^{-1} x, u \rangle_y$ is convex, then for all $s \in (-\varepsilon, \varepsilon)$ we get

$$-1 = g(\alpha(0)) = g\left(\alpha\left(\frac{1}{2}\cdot s + \frac{1}{2}(-s)\right)\right) \le \frac{1}{2}g(\alpha(s)) + \frac{1}{2}g(\alpha(-s)). \tag{4}$$

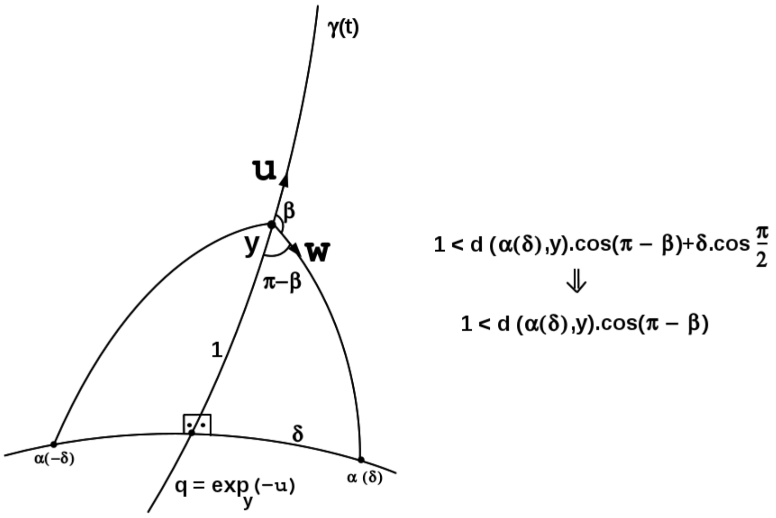
On the other hand, $g(\alpha(s)) = |\exp_y^{-1}(\alpha(s))| \cos(\beta) = \mathrm{dist}(y, \alpha(s)) \cos\beta$ where β is the angle between the vectors u e $\exp_y^{-1}(\alpha(s))$ and $g(\alpha(-s)) = \mathrm{dist}(y, \alpha(-s)) \cos\theta$ where θ is the angle between the vectors u e $\exp_y^{-1}(\alpha(-s))$. Moreover, we can choose s small enough, such that the angles β and θ are larger than $\frac{\pi}{2}$. Substituting the equalities obtained in (4) have

$$-2 \le \mathrm{dist}(y, \alpha(s)) \cos\beta + \mathrm{dist}(y, \alpha(-s)) \cos\theta.$$

Therefore,

$$\mathrm{dist}(y, \alpha(s)) \cos(\pi - \beta) + \mathrm{dist}(y, \alpha(-s)) \cos(\pi - \theta) \le 2. \tag{5}$$

Now consider the geodesic triangle whose vertices are y, q e $\alpha(s)$. The unique geodesic connecting the points y and $\alpha(s)$ is $\exp_y(t(\exp_y^{-1}(\alpha(s))))$. Since the angle of the geodesic triangle, in the vertice y, equal to the angle between the vectors $-u$ and w its measure is $\pi - \beta$. By construction, the angle in the vertice q measure $\frac{\pi}{2}$. See figure below.



Now, applying twice the law of cosines (see reference [PP] page 167) to the triangle $\Delta(y,q,\alpha(s))$ we obtain

$$\operatorname{dist}^2(y,\alpha(s)) > \operatorname{dist}^2(q,\alpha(s)) + \operatorname{dist}^2(y,q) - 2\operatorname{dist}(q,\alpha(s))\operatorname{dist}(y,q)\cos\frac{\pi}{2};$$

$$\operatorname{dist}^2(q,\alpha(s)) > \operatorname{dist}^2(y,\alpha(s)) + \operatorname{dist}^2(y,q) - 2\operatorname{dist}(y,\alpha(s))\operatorname{dist}(y,q)\cos(\pi-\beta).$$

Since $\operatorname{dist}(y,q) = 1$, the obtained inequalities imply that

$$1 < \operatorname{dist}(y,\alpha(s))\cos(\pi-\beta).$$

Similarly, using the triangle $\Delta(y,q,\alpha(-s))$ obtain

$$1 < \operatorname{dist}(y,\alpha(-s))\cos(\pi-\theta).$$

Therefore, $2 < \operatorname{dist}(y,\alpha(s))\cos(\pi-\beta) + \operatorname{dist}(y,\alpha(-s))\cos(\pi-\theta)$ contradicting (5). With this we show that g can not be a convex function. □

4 The Particular case $M = \mathbb{H}^2$

Let $\mathbb{L}^3$ denote the 3-dimensional Lorentz space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle p, q\rangle = p_1q_1 + p_2q_2 - p_3q_3.$$

The 2-dimensional hyperbolic space

$$\mathbb{H}^2 = \{p \in \mathbb{L}^3; \langle p, p\rangle = -1,\, p_3 \geq 1\}$$

as is well known, is a spacelike hypersurface in $\mathbb{L}^3$, that is, the induced metric via the inclusion $\iota : \mathbb{H}^2 \to \mathbb{L}^3$ is a Riemannian metric on $\mathbb{H}^2$. Furthermore, the hyperbolic space has Gaussian curvature -1. A parameterization for the hyperbolic space is

$$\Psi(u, v) = (\sinh u \cos v, \sinh u \sin v, \cosh u),$$

where $(u, v) \in [0, +\infty) \times [0, 2\pi)$. Note that Ψ is bijective when restricted $(0, +\infty) \times [0, 2\pi)$.

Given a point $p \in \mathbb{H}^2$, consider (u_0, v_0) such that $\Psi(u_0, v_0) = p$. Then, the curve $\gamma(u) = \Psi(u, v_0) : [0, +\infty) \to \mathbb{H}^2$ is the minimizing geodesic satisfying $\gamma(0) = e_3$ and $\gamma(u_0) = p$. Therefore,

$$\text{dist}(e_3, p) = \int_0^{u_0} |\gamma'(u)| du = \int_0^{u_0} \sqrt{\langle \gamma'(u), \gamma'(u)\rangle} du = u_0. \tag{6}$$

Now consider the exponential map $\exp_{e_3} : T_{e_3}\mathbb{H}^2 \to \mathbb{H}^2$. Since $\gamma'(0) = (\cos v_0, \sin v_0, 0)$ unitary, we have by (6) that

$$\exp_{e_3}^{-1}(\Psi(u_0, v_0)) = \exp_{e_3}^{-1}(p) = u_0 \cdot \gamma'(0) = u_0 \cdot \Psi_u(0, v_0). \tag{7}$$

It is easy to see that $\gamma(t) = \cosh t\, e_3 + \sinh t\, e_1$ is the geodesic such that $\gamma(0) = e_3$ and $\gamma'(0) = e_1 \in T_{e_3}\mathbb{H}^2$. Let $q = \gamma(-1) = \cosh 1\, e_3 - \sinh 1\, e_1$.

It is not difficult to see that $e_2 \in T_q\mathbb{H}^2$ and $\langle e_2, \gamma'(-1)\rangle = 0$. Now consider the geodesic $\beta : \mathbb{R} \to \mathbb{H}^2$, defined by $\beta(s) = \cosh s\, q + \sinh s\, e_2$.

We prove that the function $g : \mathbb{H}^2 \to \mathbb{R}$, defined as $g(p) = \langle \exp_{e_3}^{-1}(p), e_1 \rangle$, is not convex. For this, we analyze $g(\beta(s))$. If $\Psi(u_0(s), v_0(s)) = \beta(s)$, it follows that

$$\begin{aligned}(\sinh u_0(s)\cos v_0(s), \sinh u_0(s)\sin v_0(s), \cosh u_0(s)) &= \cosh s\, q + \sinh s\, e_2 \\ &= \cosh s\,(\cosh 1\, e_3 - \sinh 1\, e_1) + \sinh s\, e_2 \\ &= (-\sinh 1 \cosh s, \sinh s, \cosh 1 \cosh s).\end{aligned}$$

Therefore, using (7) we get

$$\begin{aligned} g(\beta(s)) &= \langle \exp_{e_3}^{-1}(\beta(s)), e_1 \rangle = \langle u_0(s) \cdot \Psi_u(0, v_0(s)), e_1 \rangle \\ &= -\frac{u_0(s) \cdot \sinh 1 \cdot \cosh s}{\sinh u_0(s)} = -\tanh 1 \frac{u_0(s) \cdot \cosh 1 \cdot \cosh s}{\sinh u_0(s)} \\ &= -\tanh 1 \frac{u_0(s) \cdot \cosh u_0(s)}{\sinh u_0(s)} = -\tanh 1 \cdot u_0(s) \cdot \tanh u_0(s). \end{aligned}$$

As a consequence of the equality $\cosh u_0(s) = \cosh 1 \cosh s$, it follows that $u_0(s)$ is an even function and

$$\lim_{s \to \pm\infty} u_0(s) = +\infty.$$

Thus, $g(\beta(s))$ is also an even function and

$$\lim_{s \to \pm\infty} g(\beta(s)) = -\infty. \tag{8}$$

Now assume by contradiction that g is convex, then for all s

$$-1 = g(\beta(0)) = g\left(\beta\left(\frac{1}{2}s + \frac{1}{2}(-s)\right)\right) \leq \frac{1}{2}g(\beta(s)) + \frac{1}{2}g(\beta(-s)) = g(\beta(s))$$

contradicting (8).

References

[QO] E.A. Papa Quiroz and P.R. Oliveira. *Proximal Point Methods for Quasiconvex and Convex Functions with Bregman Distances on Hadamard Manifolds*, Journal of Convex Analysis **16** (2009), 49–69.

[CLMM] V. Colao, G. López, G. Marino and V. Martín-Márquez. *Equilibrium problems in Hadamard manifolds*, J. Math. Analysis and Applications **388** (2012), 61–77.

[ZH] L. Zhou and N. Huang. *Existence of Solutions for Vector Optimization on Hadamard Manifolds*, J Optim. Theory Appl. **157** (2013), 44–53.

[dC] M.P. do Carmo. *Geometria Riemannaina*, Projeto Euclides - IMPA (1988).

[PP] P. Petersen. *Riemannaian Geometry - Second Edition*, Springer (2006).