

# Robust Stable Payoff Distribution in Stochastic Cooperative Games

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## Abstract

Cooperative games with transferable utilities belong to a branch of game theory where groups of players can enter into binding agreements and form coalitions in order to jointly achieve some objectives. In a cooperative setting, one of the most important questions to address is how to establish a payoff distribution among the players in such a way to ensure the stability of the game. Classical solution concepts such as the core and the least core are only defined in games with deterministic characteristic functions. However, the payoff function might not be exact due to estimation/approximation errors, and classical solution concepts are no longer applicable. We redefine the concept of stability in a stochastic setting and introduce new concepts for robust payoff distribution. We demonstrate these concepts with a number of games including the stochastic newsvendor games. Properties and numerical schemes for finding the robust solutions are presented.

## 1 Introduction

The world of game theory is often divided into two branches: cooperative game theory and non-cooperative game theory. This classification is based largely on whether or not players can enter into binding agreements before the game. In cooperative games, players can form coalitions in order to jointly achieve some outcomes. A special – and arguably the most popular – class of these games are cooperative games with transferable utilities (TU) in which the payoff of each coalition is represented by a single scalar number in the form of monetary gain/payment or some generic utilities, and can be transferred freely among the players. These games are then completely defined by characteristic

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functions that map each subset of players into a real number. A majority of research on classical cooperative game theory assumes that all players collaborate to form the grand coalition, and hence the natural follow-up question is on how to share the resulting total payoff among the players so that the cooperation can be sustained<sup>1</sup>.

Solution concepts for payoff distribution in cooperative games are designed to achieve two main criteria: (a) the fairness among the players, and (b) the stability of the game. The first group of solution concepts that aim for fairness includes the Banzhaf index and the Shapley value while those solution concepts in the second group include the imputation, the core, the least core and the nucleolus. The focus of our paper is on the core and the least core solutions; those that aim for the stability. The core was first formally defined by Gillies [11] as a way to distribute payoff such that no group of players has the incentive to leave the grand coalition. Unfortunately, the core might not exist in some games, i.e., no matter how the payoff is distributed, there is always at least one coalition that is not satisfied with its share. The least core is defined naturally for these cases where the greatest level of dissatisfaction among all the coalitions is minimized. We provide formal definitions of the core and the least core in Section 2.1.

Cooperative game theory has many applications in supply chain cooperation [7, 20], facility location [12], and logistics [1, 24]. In the fields of economics and business, cooperative games have been used to set insurance premiums [18] and interchange fees for ATM bank networks [13]. There are other applications of cooperative game theory such as voting power computation [17] and coalition structure formation in multi-agent systems [5] among many others. Traditionally, cooperative games with transferable utilities have been defined in deterministic settings where the characteristic functions are assumed to be given. However, in many applications, the payoff values are often estimations or approximations of reality. For example, in the facility location game, customer demand, supply and transportation costs are often random. Similarly, we can find many examples in which the payoff values are stochastic. In fact, we demonstrate this through the newsvendor game in Section 3 where the demand distributions are uncertain. The focus of our work is on stochastic cooperative games with transferable utilities. Under a stochastic setting, the value of the grand coalition, i.e., the total profit (or cost) is not fixed. Therefore,

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<sup>1</sup>Another important research question in cooperative game theory, particularly in large application domains such as multi-agent systems, is on the optimal coalition structure problem, i.e., how to divide the players into subgroups such that total payoff is maximized. Under the super-additivity assumption on the characteristic function, the grand coalition is also the optimal coalition structure. In more generic games, the optimal coalition structure involves solving a set-packing problem and is very challenging to solve. However, once an optimal coalition structure has been found, stable payoff distributions in such games are defined and computed in a very similar way to that of a super-additive game.

any distribution of some deterministic payoff among the players will not be efficient in general, i.e., payoffs do not sum up to the actual total profit (or cost). In addition, even if the payoff of the grand coalition is always fixed, a predefined payoff division suggested by a realization of the characteristic function (e.g., using the expected value) might not be stable under an actual realization of the payoff function. *Ideally, we would like to have a payoff distribution scheme that is stable given the uncertainty in the characteristic function.* This motivates us to study cooperative games with stochastic payoffs.

Decision making under uncertainty has been traditionally handled by using stochastic programming and more recently by robust optimization. In the context of stochastic cooperative games, some studies [6, 14] consider settings where the payoff to each coalition is a random quantity. The authors then transform the stochastic excess level of each coalition, i.e., the difference between the shares distributed to the coalition and its realized payoff, to a scalar number using chance constraints. The prior nucleolus of the corresponding stochastic game is defined in the same way as the deterministic nucleolus except that the deterministic excess levels are replaced by the transformed stochastic excess levels. For the computation of the prior nucleolus, the authors follow Kopelowitz [16] who models the nucleolus as successive linear programs. The problem of finding the prior nucleolus is then modelled as sequential nonlinear programs, each of which involves an exponentially large number of nonlinear constraints which are formed by the chance constraints. The authors then show some attractive theoretical properties of the prior nucleolus, such as its non-emptiness.

Cooperative games with stochastic payoffs depending on the actions of players and external stochastic terms are studied in [22, 23]. Each allocation rule for a coalition is represented by a pair of weights from which the final (stochastic) payoff to each player is scaled up by the realized coalition payoff. Each player is associated with a preference relation over the set of all allocation rules based on the corresponding stochastic payoff that the player will be allocated. The cores of these games are then defined as allocation rules that are not dominated; in the sense that there exists no coalition and no other allocations rule that is preferable to all player in that coalition. The authors then extend the Bondareva-Shapley theorem to provide conditions for the existence of the core in these games.

The literature on cooperative games with incomplete information has recently attracted attention from the research community in multi-agent systems where the application domain often involves a large number of players acting in selfish behavior. Here, the authors consider uncertainties from another angle in which players do not have complete information of the games (e.g., the players might not know the capabilities of others, as in [3, 4], or might not know the state of the world, as in [15]). This means the payoff to each coalition is uncertain, even to its members (such as in studies [3, 4]), and hence this

complicates the decisions of players to join coalitions, and the stability of games. Nevertheless, once the authors associate the uncertainties with some predefined probability distributions and the stability criteria are defined w.r.t. the corresponding expected payoff function, the stochastic game becomes a deterministic game and hence the related concepts such as the core and the least core are defined correspondingly.

Our research is closer to the direction taken by a number of studies [6, 14, 22, 23]; where we also consider stochastic games in an abstract setting and where the uncertainty is captured and represented through the stochastic payoff functions. Our method, however, departs from these existing methods in the way we model the uncertainty and the corresponding definition of the stochastic games as well as the solution concepts. Specifically, we use robust optimization approaches and model the uncertainty of the characteristic function through an uncertainty set. This uncertainty set could be as simple as an interval for the payoff value of a coalition or as complicated as the set of probability distribution of some random factors, as we show in the application of the stochastic newsvendor games. Our stochastic framework resolves some of the modelling challenges and computational challenges of these existing methods. Particularly, on the modelling side, our method does not require complete information on the distributions of payoff values for all coalitions as required in above-mentioned studies [6, 14]. Our method does not require the preference relation of the stochastic payoffs used in [22, 23] either. On the computational side, the computation of the prior nucleolus in other works [6, 14] is extremely difficult due to the nonlinearity, the exponentially large size of the formulations and the tedious task of handling optimal solutions so that subsequent LPs can be formulated<sup>2</sup>. Existing work in [3, 4, 15, 22, 23] focus more on providing conceptual frameworks and the authors do not discuss about the computation of the payoff distribution. In this paper, we introduce the concepts of robust imputation and robust core (and least core) and show that the computation of a robust core (and least core) is equivalent to computing the core of a deterministic game. This result makes it possible to compute robust payoff distribution in many classes of stochastic games, particularly when the number of players is of a reasonable size, i.e., less than 20.<sup>3</sup> For generic games, we study the existence of the robust solutions.

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<sup>2</sup>It is acknowledged that, even in a deterministic game, computing the nucleolus is much more difficult than computing the core (or least core). Therefore, it is reasonable to expect that the computation of the stochastic nucleolus will be more difficult than that of a stochastic core (or least core).

<sup>3</sup>Notice that, even in a generic deterministic game, computing a solution in the core (or least core) requires solving a linear program with  $(2^N + 1)$  constraints where  $N$  is the number of players. This is generally difficult to do for  $N \geq 20$  unless the characteristic function has some compact forms that we could exploit.

## Contributions and paper outline

In this paper, we introduce new solution concepts for stable payoff distribution under stochastic settings. We analyze properties of these solution concepts and demonstrate them with newsvendor games. We also develop numerical schemes to handle these solution concepts numerically. Specifically, our contributions and structure of the paper are as follows:

- (1) Section 2 provides definitions of solution concepts of cooperative games in stochastic settings. We study the existence of robust solutions in Section 2.2 for a general class of stochastic games and demonstrate these through stochastic weighted voting games and network cooperative games.
- (2) Section 3 studies the stochastic newsvendor game where complete information on the demand distributions is not available. We provide properties on the optimal order quantities and study the existence of the robust core of this stochastic newsvendor game.
- (3) Numerical schemes for finding the robust core in the stochastic newsvendor game are described in Section 4. We present a constraint generation method for speeding up the computation of the robust payoff distribution in larger games with more than 30 retailers. Demonstration of the properties of the robust solutions and computational results are provided in Section 5.

## 2 Stochastic Cooperative Games

### 2.1 Definitions

In this section, we propose the concept of *stochastic* cooperative games. Consider a set of  $N$  players,  $\mathcal{N} = \{1, \dots, N\}$ , and a function  $v : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ , which is called the *characteristic function*, such that  $v(\emptyset) = 0$ . We call  $\mathcal{G} = (\mathcal{N}, v)$  a *cooperative game*. Utility (payoff) is assumed to be transferable, i.e., for any coalition  $\mathcal{S} \subseteq \mathcal{N}$ , its total payoff is completely defined as  $v(\mathcal{S})$ , which can be transferred freely among its members. Given a cooperative game  $(\mathcal{N}, v)$ , we are interested in finding an allocation  $\mathbf{x} \in \mathbb{R}^N$  to allocate the total payoff  $v(\mathcal{N})$  among individual players. An allocation  $\mathbf{x}$  is called *efficient* if

$$\sum_{i=1}^N x_i = v(\mathcal{N}). \quad (1)$$

An important question regarding cooperative games is whether players are willing to join the grand coalition  $\mathcal{N}$ . Necessarily, there should be an allocation  $\mathbf{x} \in \mathbb{R}^N$  with which each individual player is

better off as compared to his/her standalone payoff. An allocation  $\mathbf{x}$  is called *individually rational* if

$$x_i \geq v(\{i\}), \quad \forall i = 1, \dots, N. \quad (2)$$

**Definition 1.** An imputation is an allocation that is both efficient and individually rational. The set of imputations of a cooperative game  $(\mathcal{N}, v)$  is written as follows:

$$\text{impu}(\mathcal{N}, v) := \left\{ \mathbf{x} \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = v(\mathcal{N}), x_i \geq v(\{i\}), \forall i \in \mathcal{N} \right\}. \quad (3)$$

If the characteristic function  $v$  is *super-additive*, i.e., for any two disjoint coalitions  $\mathcal{S}_1$  and  $\mathcal{S}_2$ ,  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ ,  $v(\mathcal{S}_1) + v(\mathcal{S}_2) \leq v(\mathcal{S}_1 \cup \mathcal{S}_2)$ , it is clear that there always exists at least one imputation or  $\text{impu}(\mathcal{N}, v) \neq \emptyset$ .

Individual rationality is not sufficient to guarantee that some players would prefer the grand coalition  $\mathcal{N}$  to a smaller coalition  $\mathcal{S} \subsetneq \mathcal{N}$ . An allocation is called *stable* with respect to a coalition  $\mathcal{S}$  if  $\sum_{i \in \mathcal{S}} x_i \geq v(\mathcal{S})$ .

**Definition 2.** The set of efficient allocations that are stable with respect to all coalitions  $\mathcal{S} \subseteq \mathcal{N}$  is called the *core*,

$$\text{core}(\mathcal{N}, v) := \left\{ \mathbf{x} \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = v(\mathcal{N}), \sum_{i \in \mathcal{S}} x_i \geq v(\mathcal{S}), \forall \mathcal{S} \subsetneq \mathcal{N} \right\}. \quad (4)$$

It is obvious that  $\text{core}(\mathcal{N}, v) \subseteq \text{impu}(\mathcal{N}, v)$ . However, the core might not exist. There are different solution concepts that take this issue into account. Given a parameter  $\epsilon \geq 0$ , the  $\epsilon$ -core is defined as follows:

$$\epsilon\text{-core}(\mathcal{N}, v) := \left\{ \mathbf{x} \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = v(\mathcal{N}), \sum_{i \in \mathcal{S}} x_i \geq v(\mathcal{S}) - \epsilon, \forall \mathcal{S} \subsetneq \mathcal{N} \right\}. \quad (5)$$

It is clear that  $\epsilon$ -core is non-empty for a given large enough  $\epsilon$ . The *least core* is the non-empty  $\epsilon$ -core with the smallest value of  $\epsilon$ , which is called the *least core value*. The least core value  $\epsilon(\mathcal{N}, v)$  is the optimal value of the following linear optimization problem:

$$\begin{aligned} \epsilon(\mathcal{N}, v) = \min_{\mathbf{x}, \epsilon} \quad & \epsilon \\ \text{s.t.} \quad & \sum_{i=1}^N x_i = v(\mathcal{N}), \\ & \sum_{i \in \mathcal{S}} x_i \geq v(\mathcal{S}) - \epsilon, \quad \forall \mathcal{S} \subseteq \mathcal{N}. \end{aligned} \quad (6)$$

If we are able to solve (6) and obtain  $\epsilon(\mathcal{N}, v) = 0$ , then the core is non-empty and coincides with the least core. Schulz and Uhan [21] consider the problem (6) in a slightly different way:

$$\begin{aligned} s(\mathcal{N}, v) = \min_{\mathbf{x}, \epsilon} \quad & \epsilon \\ \text{s.t.} \quad & \sum_{i=1}^N x_i = v(\mathcal{N}), \\ & \sum_{i \in \mathcal{S}} x_i \geq v(\mathcal{S}) - \epsilon, \quad \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset. \end{aligned} \quad (7)$$

In general, we have  $\epsilon(\mathcal{N}, v) = \max\{s(\mathcal{N}, v), 0\}$ , i.e.,  $s(\mathcal{N}, v)$  coincides with the least core value if it is non-negative. If  $s(\mathcal{N}, v) < 0$ , the absolute value  $|s(\mathcal{N}, v)|$  can be interpreted as the maximum possible increase in any coalition payoff  $v(\mathcal{S})$ ,  $\mathcal{S} \subsetneq \mathcal{N}$ , under the condition that at least one efficient and stable allocation still exists. We define  $\sigma(\mathcal{N}, v) = \max\{-s(\mathcal{N}, v), 0\}$  to be the *stability value* of the game  $(\mathcal{N}, v)$ . Intuitively, the larger the stability value is, the more stable the grand coalition is under changes in the payoffs of smaller coalitions. The stability value can be computed given a fixed characteristic function and proves to be a useful concept when we discuss cooperative games with uncertain characteristic functions. In several applications, it is indeed difficult to compute and/or represent the payoff  $v(\mathcal{S})$  for  $\mathcal{S} \subseteq \mathcal{N}$  by using a fixed value if there are uncertain factors. Let us consider an *uncertain* characteristic function  $\tilde{v}$  with uncertain values  $\tilde{v}(\mathcal{S})$ , i.e.,  $\tilde{v} \in \mathcal{V}$ , where  $\mathcal{V}$  is a given set of functions  $v : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ . Given a characteristic function in  $\mathcal{V}$ , all solution concepts regarding allocations as discussed above can be investigated. However, these allocations could be different and varied for different characteristic functions in  $\mathcal{V}$ . Given  $\mathcal{N}$  and a set  $\mathcal{V}$  of uncertain characteristic function  $\tilde{v}$ , we call  $\tilde{\mathcal{G}} = (\mathcal{N}, \mathcal{V})$  a *stochastic* cooperative game.

We now propose a *consistent* solution concept for cooperative games under the setting of uncertain characteristic functions as follows. We define a *consistent allocation policy* as a vector  $\mathbf{y} \in \mathbb{R}^N$  such that for any realized characteristic function  $\tilde{v} \in \mathcal{V}$ , the vector  $\tilde{v}(\mathcal{N}) \cdot \mathbf{y}$  is chosen as the allocation for the (deterministic) cooperative game  $(\mathcal{N}, \tilde{v})$ . The proposed allocation policy is consistent in a sense that the allocations of individual players are proportionally the same in all cases. To enforce this concept of proportional allocation, we make the following assumption:

**Assumption 1.** *The payoff of the grand coalition  $\mathcal{N}$  is positive,  $\tilde{v}(\mathcal{N}) > 0$  for all  $\tilde{v} \in \mathcal{V}$ .*

We can now define similar concepts for stochastic cooperative games. An allocation policy  $\mathbf{y}$  is *efficient* if

$$\sum_{i=1}^N y_i = 1. \quad (8)$$

An allocation policy  $\mathbf{y}$  is *individually rational* if the resulting allocations in all cases are individually rational; or equivalently,

$$\forall \tilde{v} \in \mathcal{V} : y_i \geq \tilde{v}(\{i\})/\tilde{v}(\mathcal{N}), \forall i \in \mathcal{N}. \quad (9)$$

**Definition 3.** A robust imputation of a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$  is an allocation policy that is both efficient and individually rational. The set of robust imputations of the stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$  is written as follows:

$$\text{rimpu}(\mathcal{N}, \mathcal{V}) := \left\{ \mathbf{y} \in \mathbb{R}^N \mid \sum_{i=1}^N y_i = 1, \forall \tilde{v} \in \mathcal{V} : y_i \geq \tilde{v}(\{i\})/\tilde{v}(\mathcal{N}), \forall i \in \mathcal{N} \right\}. \quad (10)$$

Given a robust imputation  $\mathbf{y}$  of the stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$ , we can construct an imputation for *any* (deterministic) cooperative game  $(\mathcal{N}, \tilde{v})$ ,  $\tilde{v} \in \mathcal{V}$ , by scaling up  $\mathbf{y}$  with  $\tilde{v}(\mathcal{N})$ , which explains why we choose the term “robust imputation” for this solution concept.

We define a *stable* allocation policy as an allocation policy  $\mathbf{y}$  whose resulting allocations in all cases are stable, i.e., for all  $\tilde{v} \in \mathcal{V}$ ,  $\sum_{i \in \mathcal{S}} y_i \geq \tilde{v}(\mathcal{S})/\tilde{v}(\mathcal{N})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ .

**Definition 4.** Given a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$ , the set of efficient allocation policies that are stable is called the robust core:

$$\text{rcore}(\mathcal{N}, \mathcal{V}) := \left\{ \mathbf{y} \in \mathbb{R}^N \mid \sum_{i=1}^N y_i = 1, \forall \tilde{v} \in \mathcal{V} : \sum_{i \in \mathcal{S}} y_i \geq \tilde{v}(\mathcal{S})/\tilde{v}(\mathcal{N}), \forall \mathcal{S} \subseteq \mathcal{N} \right\}. \quad (11)$$

Similar to the robust imputation, given an allocation policy from the robust core of a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$ , we can construct an allocation in the core of *any* (deterministic) cooperative game  $(\mathcal{N}, \tilde{v})$ ,  $\tilde{v} \in \mathcal{V}$ .

**Example 1.** To demonstrate these robust solution concepts, let us first consider a nominal cooperative game  $\mathcal{G} = (\mathcal{N}, \bar{v})$  with  $\mathcal{N} = \{1, 2, 3\}$  and with a characteristic function

$$\bar{v}(\{1\}) = \bar{v}(\{2\}) = \bar{v}(\{3\}) = 1; \bar{v}(\{1, 2\}) = \bar{v}(\{1, 3\}) = 2; \bar{v}(\{2, 3\}) = 3; \bar{v}(\{1, 2, 3\}) = 5.$$

Given a  $\Delta > 0$ , consider the stochastic game  $\tilde{\mathcal{G}} = (\mathcal{N}, \mathcal{V}(\Delta))$  where  $\mathcal{V}(\Delta)$  is the set of all uncertain characteristic function  $\tilde{v}$  that are defined as

$$\tilde{v}(S) = \bar{v}(S) + \sum_{i \in S} z_i,$$

where  $z_i \in [-\Delta, \Delta]$ .



The set of all imputations of the nominal game is  $\mathcal{I} = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 5, x_1 \geq 1, x_2 \geq 1, x_3 \geq 1\}$ , and the core of that game is

$$\mathcal{C} = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 5, x_1 \geq 1, x_2 \geq 1, x_3 \geq 1, x_1 + x_2 \geq 2, x_1 + x_3 \geq 2, x_2 + x_3 \geq 3\}.$$

For each realization of  $\tilde{v}$ , we have a corresponding deterministic characteristic function game and a corresponding set of imputations and core (or least core) solutions. The four large triangles in Figure 1(a) represent the sets of imputations of four different games constructed with  $\mathbf{z}^{(0)} = (0, 0, 0)$ ,  $\mathbf{z}^{(a)} = (0.25, 0, 0)$ ,  $\mathbf{z}^{(b)} = (0, 0.25, 0)$ , and  $\mathbf{z}^{(c)} = (0, 0, 0.25)$  under the normalization  $x_1 + x_2 + x_3 = 1$ . Figure 1(b) shows the normalized cores for those four games as four large trapezoids. For other games, their cores can also be represented by different trapezoids.

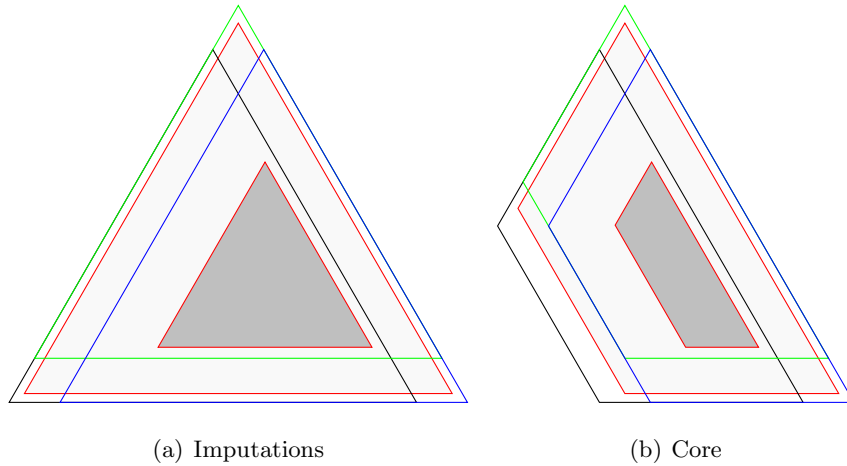


Figure 1: An example to demonstrate robust solution concepts in cooperative games.

Under the normalization, the intersection of all trapezoidal cores, which is the shaded area in Figure 1(b), is actually the robust core of the robust game  $(\mathcal{N}, \mathcal{V}(\Delta))$ , i.e., the set of all efficient and stable allocation policies for all games with a characteristic function  $\tilde{v} \in \mathcal{V}(\Delta)$ . Similarly, the shaded triangle in Figure 1(a) is the set of robust imputations of the robust game  $(\mathcal{N}, \mathcal{V}(\Delta))$ .

Given a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$ , we are concerned with the existence of robust imputations and the non-emptiness of its robust core. The following theorem shows the equivalence between these solution concepts of stochastic cooperative games and those of some related deterministic cooperative games.

**Theorem 1.** *Given a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$ , let  $v_{\mathcal{V}}^{\max} : 2^{\mathcal{N}} \rightarrow \mathbb{R}$  be the deterministic*

characteristic function with

$$v_{\mathcal{V}}^{\max}(\mathcal{S}) = \max_{\tilde{v} \in \mathcal{V}} \frac{\tilde{v}(\mathcal{S})}{\tilde{v}(\mathcal{N})}, \quad \forall \mathcal{S} \subseteq \mathcal{N}. \quad (12)$$

Then  $\text{rimpu}(\mathcal{N}, \mathcal{V}) = \text{impu}(\mathcal{N}, v_{\mathcal{V}}^{\max})$  and  $\text{rcore}(\mathcal{N}, \mathcal{V}) = \text{core}(\mathcal{N}, v_{\mathcal{V}}^{\max})$ .

**Proof.** The stable condition of allocation policy  $\mathbf{y}$  can be written as follows:

$$\min_{\tilde{v} \in \mathcal{V}} \left( \min_{\mathcal{S} \subseteq \mathcal{N}} \left\{ \sum_{i \in \mathcal{S}} y_i - \tilde{v}(\mathcal{S})/\tilde{v}(\mathcal{N}) \right\} \right) \geq 0.$$

It is easy to show that we can change the order of minimization, i.e., the equivalent stable condition is

$$\min_{\mathcal{S} \subseteq \mathcal{N}} \left\{ \min_{\tilde{v} \in \mathcal{V}} \left( \sum_{i \in \mathcal{S}} y_i - \tilde{v}(\mathcal{S})/\tilde{v}(\mathcal{N}) \right) \right\} \geq 0.$$

We can rewrite this condition using the following constraints:

$$\min_{\tilde{v} \in \mathcal{V}} \left( \sum_{i \in \mathcal{S}} y_i - \tilde{v}(\mathcal{S})/\tilde{v}(\mathcal{N}) \right) \geq 0, \quad \forall \mathcal{S} \subseteq \mathcal{N} \Leftrightarrow \sum_{i \in \mathcal{S}} y_i \geq \max_{\tilde{v} \in \mathcal{V}} \frac{\tilde{v}(\mathcal{S})}{\tilde{v}(\mathcal{N})} = v_{\mathcal{V}}^{\max}(\mathcal{S}), \quad \forall \mathcal{S} \subseteq \mathcal{N}.$$

These constraints are indeed the exact constraints for  $\mathbf{y}$  to be a stable allocation (with respect to all coalitions  $\mathcal{S} \subseteq \mathcal{N}$ ) of the deterministic cooperative game  $(\mathcal{N}, v_{\mathcal{V}}^{\max})$ . Thus we have  $\text{rcore}(\mathcal{N}, \mathcal{V}) = \text{core}(\mathcal{N}, v_{\mathcal{V}}^{\max})$ . Using similar arguments, we can also show that  $\text{rimpu}(\mathcal{N}, \mathcal{V}) = \text{impu}(\mathcal{N}, v_{\mathcal{V}}^{\max})$ .  $\square$

Based on Theorem 1, we can define the *robust  $\epsilon$ -core* of a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$ ,  $\epsilon$ - $\text{rcore}(\mathcal{N}, \mathcal{V})$ , as the  $\epsilon$ -core of the corresponding deterministic cooperative game  $(\mathcal{N}, v_{\mathcal{V}}^{\max})$ . Similarly, the *robust least core* of a stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$  is defined as the least core of the deterministic cooperative game  $(\mathcal{N}, v_{\mathcal{V}}^{\max})$  with the *robust least core value*  $\epsilon(\mathcal{N}, \mathcal{V}) = \epsilon(\mathcal{N}, v_{\mathcal{V}}^{\max})$ . A similar definition can be introduced for the notion of *robust stability value*.

## 2.2 Existence of Robust Imputations and Non-emptiness of Robust Cores

According to Theorem 1, the existence of robust imputations and the non-emptiness of the robust core of a stochastic cooperative game can be checked by investigating its corresponding deterministic cooperative game. The characteristic function  $v_{\mathcal{V}}^{\max}$  depends on the uncertainty set  $\mathcal{V}$ . To start with, we consider the following weighted voting game.

## Stochastic Weighted Voting Games

Consider a weighted voting game in which each player  $i$ ,  $i \in \mathcal{N}$  has a voting weight  $w_i$ . A coalition  $\mathcal{S}$  has the total voting weight of  $w(\mathcal{S}) = \sum_{i \in \mathcal{S}} w_i$ . The characteristic function of the game is defined as

$$v(\mathcal{S}) = \begin{cases} 1, & \text{if } w(\mathcal{S}) \geq \kappa, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\kappa$  is a threshold that the total voting weight of a coalition needs to exceed in order to win the game. The stochastic setting arises when the weight vector  $\mathbf{w}$  contains some uncertainty. Such cases occur when  $\mathbf{w}$  needs to be estimated; for example, through surveys. If we consider the simple box uncertainty for each weight  $w_i$ ,  $w_i \in [\bar{w}_i - \delta_i; \bar{w}_i + \delta_i]$ ,  $i \in \mathcal{N}$ , where  $\bar{\mathbf{w}}$  is the nominal weight vector and  $\boldsymbol{\delta} \geq \mathbf{0}$  is given, then the uncertainty set  $\mathcal{V}_v(\boldsymbol{\delta})$  can be defined as follows:

$$\mathcal{V}_v(\boldsymbol{\delta}) = \left\{ \tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists z_i \in [-\delta_i; \delta_i], \forall i \in \mathcal{N} : \tilde{v}(\mathcal{S}) = \mathbb{I} \left( \bar{w}(\mathcal{S}) + \sum_{i \in \mathcal{S}} z_i \geq \kappa \right), \forall \mathcal{S} \subseteq \mathcal{N} \right\}.$$

Under Assumption 1, we are required to have  $\sum_{i \in \mathcal{N}} \bar{w}_i - \sum_{i \in \mathcal{N}} \delta_i \geq \kappa$  so that  $\tilde{v}(\mathcal{N}) = 1 > 0$  for all  $\tilde{v} \in \mathcal{V}_v(\boldsymbol{\delta})$ . Since weighted voting games are *simple* games, i.e., games with characteristic function  $v : 2^{\mathcal{N}} \rightarrow \{0, 1\}$ , the concept of imputation is less important. Subsequently, we only focus on the core of the weighted voting games here. To use Theorem 1, we need to construct the characteristic function  $v_{\mathcal{V}_v(\boldsymbol{\delta})}^{\max}$ . We have

$$v_{\mathcal{V}_v(\boldsymbol{\delta})}^{\max}(\mathcal{S}) = \max_{\tilde{v} \in \mathcal{V}_v(\boldsymbol{\delta})} \frac{\tilde{v}(\mathcal{S})}{\tilde{v}(\mathcal{N})} = \max_{\tilde{v} \in \mathcal{V}_v(\boldsymbol{\delta})} \tilde{v}(\mathcal{S}) = \max_{z_i \in [-\delta_i; \delta_i]} \mathbb{I} \left( \bar{w}(\mathcal{S}) + \sum_{i \in \mathcal{S}} z_i \geq \kappa \right) = \mathbb{I} \left( \bar{w}(\mathcal{S}) + \sum_{i \in \mathcal{S}} \delta_i \geq \kappa \right).$$

Thus the corresponding cooperative game  $(\mathcal{N}, v_{\mathcal{V}_v(\boldsymbol{\delta})}^{\max})$  is a weighted voting game with the weight  $\bar{w}_i + \delta_i$  for all  $i \in \mathcal{N}$ . The non-emptiness of the robust core of the stochastic weighted voting game  $(\mathcal{N}, \mathcal{V}_v(\boldsymbol{\delta}))$  is then characterized by the existence of an *veto* player for the weighted voting game  $(\mathcal{N}, v_{\mathcal{V}_v(\boldsymbol{\delta})}^{\max})$ , i.e., there exists a player  $i$ ,  $i \in \mathcal{N}$ , such that  $v_{\mathcal{V}_v(\boldsymbol{\delta})}^{\max}(\mathcal{N} \setminus \{i\}) = 0$  (see [10] for details). The condition can be written as follows:

$$\sum_{i \in \mathcal{N}} \bar{w}_i + \sum_{i \in \mathcal{N}} \delta_i - \max_{i \in \mathcal{N}} \{\bar{w}_i + \delta_i\} < \kappa.$$

Assuming that the nominal weighted voting game  $(\mathcal{N}, \bar{v})$ , where  $\bar{v}(\mathcal{S}) = \mathbb{I}(\bar{w}(\mathcal{S}) \geq \kappa)$ , has a non-empty core, i.e.,  $\sum_{i \in \mathcal{N}} \bar{w}_i - \max_{i \in \mathcal{N}} \bar{w}_i < \kappa$ , we can then derive the following sufficient condition on  $\boldsymbol{\delta}$  such that the robust core of the stochastic weighted voting game  $(\mathcal{N}, v_{\mathcal{V}_v(\boldsymbol{\delta})}^{\max})$  is non-empty based on the fact that  $\boldsymbol{\delta} \geq \mathbf{0}$ :

$$\|\boldsymbol{\delta}\|_1 \leq \min \left\{ \sum_{i \in \mathcal{N}} \bar{w}_i - \kappa, \kappa + \max_{i \in \mathcal{N}} \bar{w}_i - \sum_{i \in \mathcal{N}} \bar{w}_i \right\}. \quad (13)$$

A more restrictive condition on individual elements of  $\delta$  can also be derived:

$$\|\delta\|_\infty \leq \frac{1}{N} \min \left\{ \sum_{i \in \mathcal{N}} \bar{w}_i - \kappa, \kappa + \max_{i \in \mathcal{N}} \bar{w}_i - \sum_{i \in \mathcal{N}} \bar{w}_i \right\}. \quad (14)$$

For this simple game, the quantity  $\left\{ \kappa + \max_{i \in \mathcal{N}} \bar{w}_i - \sum_{i \in \mathcal{N}} \bar{w}_i \right\}$ , if positive, acts like the stability value of the game; that is, it provides an indication of how stable the grand coalition is with respect to changes in the weights. It is shown in (13) and (14) that this quantity affects the allowable degree of uncertainty while still guaranteeing the existence of a robust core solution.

For general cooperative games, we now study a similar additive uncertainty model for the characteristic function and show that the stability value has a strong influence on the allowable degree of uncertainty of data. We define the following parametric uncertainty set  $\mathcal{V}(\delta)$ :

$$\mathcal{V}(\delta) = \left\{ \tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists z_i \in [-\delta_i, \delta_i], \forall i \in \mathcal{N} : \tilde{v}(\mathcal{S}) = \bar{v}(\mathcal{S}) + \sum_{i \in \mathcal{S}} z_i, \forall \mathcal{S} \subseteq \mathcal{N} \right\}, \quad (15)$$

where  $\delta \geq \mathbf{0}$  and  $\bar{v}$  is a given nominal characteristic function. Assuming the nominal cooperative game  $(\mathcal{N}, \bar{v})$  has imputations and its core is non-empty, we seek to identify conditions in which the stochastic cooperative game  $(\mathcal{N}, \mathcal{V}(\delta))$  has robust imputations and its robust core is non-empty. The following theorem provides some sufficient conditions.

**Theorem 2.** *Consider the stochastic cooperative game  $(\mathcal{N}, \mathcal{V}(\delta))$  with the uncertainty set  $\mathcal{V}(\delta)$  defined as in (15) with a given nominal characteristic function  $\bar{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ .*

- i) *Assuming the nominal game  $(\mathcal{N}, \bar{v})$  has a non-empty core and has a positive stability value,  $\sigma(\mathcal{N}, \bar{v}) > 0$ , then the stochastic game  $(\mathcal{N}, \mathcal{V}(\delta))$  has a non-empty robust core if*

$$\|\delta\|_\infty < \min \left\{ \frac{\bar{v}(\mathcal{N})}{N}, \frac{\sigma(\mathcal{N}, \bar{v})}{\max_{\mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset} \{ |\mathcal{S}| \cdot |1 - y_\sigma(\mathcal{S})| + (N - |\mathcal{S}|) \cdot |y_\sigma(\mathcal{S})| \}} \right\}, \quad (16)$$

where  $y_\sigma(\mathcal{S}) = \frac{\bar{v}(\mathcal{S}) + \sigma(\mathcal{N}, \bar{v})}{\bar{v}(\mathcal{N})}$  for all  $\mathcal{S} \subsetneq \mathcal{N}$  and  $y_\sigma(\mathcal{N}) = 1$ .

- ii) *Assuming the nominal game  $(\mathcal{N}, \bar{v})$  has at least one imputation and*

$$\iota(\mathcal{N}, \bar{v}) = \frac{1}{N} \left( \bar{v}(\mathcal{N}) - \sum_{i \in \mathcal{N}} \bar{v}(\{i\}) \right) > 0,$$

*then the stochastic game  $(\mathcal{N}, \mathcal{V}(\delta))$  has at least one robust imputation if*

$$\|\delta\|_\infty < \min \left\{ \frac{\bar{v}(\mathcal{N})}{N}, \frac{\iota(\mathcal{N}, \bar{v})}{\max_{i \in \mathcal{N}} |1 - y_i^t| + (N - 1) |y_i^t|} \right\}, \quad (17)$$

where  $y_i^t = \frac{\bar{v}(\{i\}) + \iota(\mathcal{N}, \bar{v})}{\bar{v}(\mathcal{N})}$  for all  $i \in \mathcal{N}$ .

**Proof.** We consider the corresponding deterministic cooperative game  $(\mathcal{N}, v_{\mathcal{V}(\delta)}^{\max})$ . We have

$$v_{\mathcal{V}(\delta)}^{\max}(\mathcal{S}) = \max_{z_i \in [-\delta_i; \delta_i]} \frac{\bar{v}(\mathcal{S}) + \sum_{i \in \mathcal{S}} z_i}{\bar{v}(\mathcal{N}) + \sum_{i \in \mathcal{N}} z_i} = \max_{z_i \in [-\delta_i; \delta_i]} \frac{\bar{v}(\mathcal{S}) + \sum_{i \in \mathcal{S}} z_i}{\bar{v}(\mathcal{N}) + \sum_{i \in \mathcal{S}} z_i - \sum_{i \notin \mathcal{S}} \delta_i}.$$

This is due to the fact that  $\bar{v}(\mathcal{N}) > \sum_{i \in \mathcal{N}} \delta_i$  based on Assumption 1 and  $z_i \in [-\delta_i; \delta_i]$  for all  $i \in \mathcal{N}$ . Note that Assumption 1 is satisfied under the condition  $\|\delta\|_{\infty} \leq \bar{v}(\mathcal{N})/N$ . Using the monotonicity of linear fractional functions, we can then compute the values of the characteristic function  $v_{\mathcal{V}(\delta)}^{\max}$  as follows:

$$v_{\mathcal{V}(\delta)}^{\max}(\mathcal{S}) = \max \left\{ \frac{\bar{v}(\mathcal{S}) + \sum_{i \in \mathcal{S}} \delta_i}{\bar{v}(\mathcal{N}) + \sum_{i \in \mathcal{S}} \delta_i - \sum_{i \notin \mathcal{S}} \delta_i}, \frac{\bar{v}(\mathcal{S}) - \sum_{i \in \mathcal{S}} \delta_i}{\bar{v}(\mathcal{N}) - \sum_{i \in \mathcal{N}} \delta_i} \right\}. \quad (18)$$

i) We now claim that  $v_{\mathcal{V}(\delta)}^{\max}(\mathcal{S}) \leq y_{\sigma}(\mathcal{S})$  for all  $\mathcal{S} \subsetneq \mathcal{N}$ . Indeed, we have

$$\frac{\bar{v}(\mathcal{S}) + \sum_{i \in \mathcal{S}} \delta_i}{\bar{v}(\mathcal{N}) + \sum_{i \in \mathcal{S}} \delta_i - \sum_{i \notin \mathcal{S}} \delta_i} \leq y_{\sigma}(\mathcal{S}) \Leftrightarrow (1 - y_{\sigma}(\mathcal{S})) \cdot \sum_{i \in \mathcal{S}} \delta_i + y_{\sigma}(\mathcal{S}) \cdot \sum_{i \notin \mathcal{S}} \delta_i \leq \sigma(\mathcal{N}, \bar{v}).$$

Similarly, we have

$$\frac{\bar{v}(\mathcal{S}) - \sum_{i \in \mathcal{S}} \delta_i}{\bar{v}(\mathcal{N}) - \sum_{i \in \mathcal{N}} \delta_i} \leq y_{\sigma}(\mathcal{S}) \Leftrightarrow (y_{\sigma}(\mathcal{S}) - 1) \cdot \sum_{i \in \mathcal{S}} \delta_i + y_{\sigma}(\mathcal{S}) \cdot \sum_{i \notin \mathcal{S}} \delta_i \leq \sigma(\mathcal{N}, \bar{v}).$$

Thus we have

$$v_{\mathcal{V}(\delta)}^{\max}(\mathcal{S}) \leq y_{\sigma}(\mathcal{S}) \Leftrightarrow |1 - y_{\sigma}(\mathcal{S})| \cdot \sum_{i \in \mathcal{S}} \delta_i + y_{\sigma}(\mathcal{S}) \cdot \sum_{i \notin \mathcal{S}} \delta_i \leq \sigma(\mathcal{N}, \bar{v}).$$

Since  $\sigma(\mathcal{N}, \bar{v}) > 0$ , it is clear that (16) implies the above inequality. Thus we have  $v_{\mathcal{V}(\delta)}^{\max}(\mathcal{S}) \leq y_{\sigma}(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ . We have:  $\sigma(\mathcal{N}, \bar{v}) > 0$  is the stability value of the game  $(\mathcal{N}, \bar{v})$ , which means the core of  $(\mathcal{N}, \bar{v}(\mathcal{N}) \cdot y_{\sigma})$  is non-empty and so is that of the game  $(\mathcal{N}, v_{\mathcal{V}(\delta)}^{\max})$ . According to Theorem 1, the robust core of the stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$  is non-empty.

ii) Similar arguments can be used to prove this part. We can show that  $v_{\mathcal{V}(\delta)}^{\max}(\{i\}) \leq y_i^t$  for all  $i \in \mathcal{N}$ .

This implies the existence of imputations of the game  $(\mathcal{N}, v_{\mathcal{V}(\delta)}^{\max})$  given the fact that  $\bar{v}(\mathcal{N}) \cdot \mathbf{y}^t$  is an imputation of the nominal game  $(\mathcal{N}, \bar{v})$ . Using results from Theorem 1, we finally prove that the stochastic cooperative game  $(\mathcal{N}, \mathcal{V})$  has at least one robust imputation.

□

We now provide another example of stochastic cooperative games whose characteristic functions can be naturally modelled with the additive model of uncertainty similar to  $\mathcal{V}(\delta)$ .

## Stochastic Network Cooperative Games

Consider an undirected graph  $(\mathcal{N}, \mathcal{E})$  with a weight  $v_{ij}$  for each edge  $(i, j) \in \mathcal{E}$ . For each coalition  $\mathcal{S} \subseteq \mathcal{N}$ , we define  $v(\mathcal{S}) = \sum_{(i,j) \in \mathcal{E}: i,j \in \mathcal{S}} v_{ij}$ . Again, the concept of imputation is trivial with these network games and, instead, we focus on their cores. The cooperative game  $(\mathcal{N}, v)$  has a non-empty core if and only if there is no negative cut, i.e.,  $c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S}) = \sum_{(i,j) \in \mathcal{E}: i \in \mathcal{S}, j \in \mathcal{N} \setminus \mathcal{S}} v_{ij} \geq 0$  for all  $\mathcal{S} \subsetneq \mathcal{N}$  (see Deng and Papadimitriou [8] for details). If  $(\mathcal{N}, v)$  has a non-empty core, one of the core elements is the allocation  $\mathbf{a}$ , where  $a_i = \sum_{j: (i,j) \in \mathcal{E}} v_{ij}/2$ . We show how to compute the stability value of these games in the following lemma.

**Lemma 1.** *The stability value of the above network game is  $\sigma(\mathcal{N}, v) = \min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$  if its core is non-empty.*

**Proof.** Let us consider the linear optimization problem (7) and the solution  $\mathbf{x} = \mathbf{a}$  and  $\epsilon = -\min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$ . We have

$$a(\mathcal{S}) - v(\mathcal{S}) = c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2 \geq \min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2, \quad \forall \mathcal{S} \subsetneq \mathcal{N}.$$

Thus, the above solution is feasible, which means  $s(\mathcal{N}, v) \leq -\min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$ . Suppose that  $s(\mathcal{N}, v) < -\min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$ ; let us consider an optimal solution  $(\mathbf{x}^*, s(\mathcal{N}, v))$  of the problem (7). We have

$$x^*(\mathcal{S}) + x^*(\mathcal{N} \setminus \mathcal{S}) = x^*(\mathcal{N}) = a(\mathcal{N}) = a(\mathcal{S}) + a(\mathcal{N} \setminus \mathcal{S}).$$

Thus we have

$$-2s(\mathcal{N}, v) \leq x^*(\mathcal{S}) - v(\mathcal{S}) + x^*(\mathcal{N} \setminus \mathcal{S}) - v(\mathcal{N} \setminus \mathcal{S}) = a(\mathcal{S}) - v(\mathcal{S}) + a(\mathcal{N} \setminus \mathcal{S}) - a(\mathcal{N} \setminus \mathcal{S}) = c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S}),$$

which means  $s(\mathcal{N}, v) \geq -c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$  for all  $\mathcal{S} \subseteq \mathcal{N}$ , which is a contradiction. Thus we have:  $s(\mathcal{N}, v) = -\min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$ . Given the fact that the game has a non-empty core, we can conclude that  $\sigma(\mathcal{N}, v) = -s(\mathcal{N}, v) = \min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S})/2$ .  $\square$

We now assume that the weights are uncertain and  $v_{ij} \in [\bar{v}_{ij} - \delta_{ij}, \bar{v}_{ij} + \delta_{ij}]$  for all  $(i, j) \in \mathcal{E}$ . The uncertainty set is

$$\mathcal{V}_n(\boldsymbol{\delta}) = \{\tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists z_{ij} \in [-\delta_{ij}, \delta_{ij}], \forall (i, j) \in \mathcal{E} : \tilde{v}(\mathcal{S}) = \bar{v}(\mathcal{S}) + z(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N}\},$$

where  $z(\mathcal{S}) = \sum_{(i,j) \in \mathcal{E}: i,j \in \mathcal{S}} z_{ij}$ . Applying similar arguments used in Theorem 2, we achieve the following sufficient condition for the non-emptiness of the robust core of the stochastic cooperative game

$(\mathcal{N}, \mathcal{V}_n(\delta))$ :

$$\max_{(i,j) \in \mathcal{E}} \delta_{ij} < \min \left\{ \frac{1}{|\mathcal{E}|} \cdot \sum_{(i,j) \in \mathcal{E}} v_{ij}, \frac{\sigma(\mathcal{N}, \bar{v})}{\max_{\mathcal{S} \subsetneq \mathcal{N}} \{|E(\mathcal{S})| \cdot |1 - y_{\sigma}(\mathcal{S})| + (|\mathcal{E}| - |E(\mathcal{S})|) \cdot |y_{\sigma}(\mathcal{S})|\}} \right\}, \quad (19)$$

where  $E(\mathcal{S}) = \{(i, j) \in \mathcal{E} : i, j \in \mathcal{S}\}$  and  $y_{\sigma}(\mathcal{S}) = \frac{\bar{v}(\mathcal{S}) + \sigma(\mathcal{N}, \bar{v})}{\bar{v}(\mathcal{N})}$  for all  $\mathcal{S} \subsetneq \mathcal{N}$ , assuming that  $\sigma(\mathcal{N}, \bar{v}) > 0$  or  $\min_{\mathcal{S} \subsetneq \mathcal{N}} c(\mathcal{S}, \mathcal{N} \setminus \mathcal{S}) > 0$  according to Lemma 1.

### 3 Stochastic Newsvendor Games

#### 3.1 Deterministic Newsvendor Games

In this section, we investigate newsvendor games under a stochastic setting. Consider the set  $\mathcal{N}$  of  $N$  retailers and let  $\tilde{d}_i \in \mathbb{Z}_+$  be the random demand for retailer  $i$ ,  $i \in \mathcal{N}$ . In the setting of newsvendor games, we assume that the unit ordering cost  $c$  and the unit selling price  $p$  are the same for all retailers,  $0 < c < p$ . Individual retailers need to decide the optimal ordering quantity  $\bar{y}_i^*$  to maximize their expected profit,

$$\bar{y}_i^* \in \arg \max_{y \geq 0} \mathbb{E}_{P_i} [p \min\{\tilde{d}_i, y\} - cy],$$

which is the  $(p - c)/p$ -quantile of  $P_i$ , the distribution function of  $\tilde{d}_i$  for all  $i \in \mathcal{N}$ . The optimal expected profit is  $\bar{v}_i = \mathbb{E}_{P_i} [p \min\{\tilde{d}_i, \bar{y}_i^*\} - cy]$ . In the newsvendor games, we are concerned about whether individual retailers should form a coalition to make orders together and share the inventories with each other. For a coalition  $\mathcal{S} \subseteq \mathcal{N}$ , the aggregate demand is  $\tilde{d}(\mathcal{S}) = \sum_{i \in \mathcal{S}} \tilde{d}_i$  and we assume that the joint distribution  $P(\mathcal{S})$  of  $\tilde{d}_i$ ,  $i \in \mathcal{S}$ , is known. The coalition  $\mathcal{S}$  of retailers then needs to decide the optimal order quantity  $\bar{y}^*(\mathcal{S})$  to maximize their total expected profit,

$$\bar{y}^*(\mathcal{S}) \in \arg \max_{y \geq 0} \mathbb{E}_{P(\mathcal{S})} [p \min\{\tilde{d}(\mathcal{S}), y\} - cy], \quad (20)$$

which again is the  $(p - c)/p$ -quantile of the distribution of  $\tilde{d}(\mathcal{S})$ . The optimal total expected profit of the coalition  $\mathcal{S}$  is  $\bar{v}(\mathcal{S}) = \mathbb{E}_{P(\mathcal{S})} [p \min\{\tilde{d}_i, \bar{y}^*(\mathcal{S})\} - cy]$ . The cooperative game  $(\mathcal{N}, \bar{v})$  is called the (simple) newsvendor game. The characteristic function  $\bar{v}$  is super-additive. Indeed, we have

$$\bar{v}(\mathcal{S}) = \mathbb{E}_{P(\mathcal{S})} [p \min\{\tilde{d}_i, \bar{y}^*(\mathcal{S})\} - cy] = (p - c)\bar{y}^*(\mathcal{S}) - p \mathbb{E}_{P(\mathcal{S})} \left[ \left( \bar{y}^*(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right],$$

where  $x^+ = \max\{x, 0\}$ . Consider two disjoint coalitions  $\mathcal{S}_1$  and  $\mathcal{S}_2$ ,  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ ; we have  $\tilde{d}(\mathcal{S}_1 \cup \mathcal{S}_2) = \tilde{d}(\mathcal{S}_1) + \tilde{d}(\mathcal{S}_2)$ . Using (20) and the fact that  $(x + y)^+ \leq x^+ + y^+$ , we have

$$\begin{aligned} \bar{v}(\mathcal{S}_1 \cup \mathcal{S}_2) &\geq (p - c)(\bar{y}^*(\mathcal{S}_1) + \bar{y}^*(\mathcal{S}_2)) - p \mathbb{E}_{P(\mathcal{S}_1 \cup \mathcal{S}_2)} \left[ \left( \bar{y}^*(\mathcal{S}_1) + \bar{y}^*(\mathcal{S}_2) - \tilde{d}(\mathcal{S}_1 \cup \mathcal{S}_2) \right)^+ \right] \\ &\geq (p - c)\bar{y}^*(\mathcal{S}_1) - p \mathbb{E}_{P(\mathcal{S}_1)} \left[ \left( \bar{y}^*(\mathcal{S}_1) - \tilde{d}(\mathcal{S}_1) \right)^+ \right] + \dots \\ &\quad (p - c)\bar{y}^*(\mathcal{S}_2) - p \mathbb{E}_{P(\mathcal{S}_2)} \left[ \left( \bar{y}^*(\mathcal{S}_2) - \tilde{d}(\mathcal{S}_2) \right)^+ \right] \\ &= \bar{v}(\mathcal{S}_1) + \bar{v}(\mathcal{S}_2). \end{aligned}$$

This shows that newsvendor games always have imputations. In addition, Müller et al. [20] prove that every newsvendor game has a non-empty core and Montrucchio and Scarsini [19] show how to construct an allocation in the core of newsvendor games. The results are based on the assumption that the joint distribution of  $\tilde{d}_i$ ,  $i \in \mathcal{N}$  is known. In the next section, we relax this assumption and develop a stochastic newsvendor game with ambiguity in distributions.

### 3.2 Stochastic Newsvendor Games with Ambiguity in Distributions

Individual retailers usually collect historical demands independently before they join any coalition and the current assumption of a known joint distribution can be considered quite strong. In order to make the problem is more realistic, we only assume that some (multivariate) marginal distributions are known. For example, it is more reasonable to assume the knowledge of the joint demand distribution of a subset of retailers that are located close to each other and hence likely serve customers from a same area. More concretely, consider a partition of  $\mathcal{N}$  with  $R$  subsets  $\mathcal{N}_1, \dots, \mathcal{N}_R$  such that

$$\mathcal{N} = \bigcup_{r=1}^R \mathcal{N}_r \quad \text{and} \quad \mathcal{N}_r \cap \mathcal{N}_s = \emptyset \quad \text{for all } r \neq s.$$

Given a vector  $\mathbf{d} \in \mathbb{R}^n$ , let  $\mathbf{d}_r \in \mathbb{R}^{N_r}$  denote the sub-vector formed with the elements in the  $r$ th subset  $\mathcal{N}_r$  where  $N_r = |\mathcal{N}_r|$  is the size of the subset. We assume that probability measures  $P_r$  of random vectors  $\tilde{\mathbf{d}}_r$  are known for all  $r = 1, \dots, R$ . Let  $\mathcal{P}(P_1, \dots, P_R)$  denote the set of joint probability measures of the random vector  $\tilde{\mathbf{d}}$  consistent with the prescribed probability measures of the random vectors  $\tilde{\mathbf{d}}_r$  for all  $r = 1, \dots, R$ . We now develop a stochastic newsvendor game under this distributional ambiguity setting.

Given a subset  $\mathcal{S} \subseteq \mathcal{N}$ , we define  $\mathcal{S}_r = \mathcal{S} \cap \mathcal{N}_r$  for all  $r = 1, \dots, R$ . Clearly, if all retailers  $i$ ,  $i \in \mathcal{S}$ , join together, we know the non-overlapping marginal distributions of the joint demand vector with respect to the partition  $(\mathcal{S}_1, \dots, \mathcal{S}_R)$  of  $\mathcal{S}$ . If  $\mathcal{S} \subseteq \mathcal{N}_r$  for some  $r$ , it is straightforward to define  $v(\mathcal{S}) = \bar{v}(\mathcal{S})$  since



the joint distribution of  $\tilde{d}_i$ ,  $i \in \mathcal{S}$ , is completely known. The optimal quantity order is again  $\bar{y}^*(\mathcal{S})$  as defined in (20). For arbitrary  $\mathcal{S}$ , the joint distribution of  $\tilde{d}_i$ ,  $i \in \mathcal{S}$ , is not completely known in general.

We consider two possible cases in the collaborative newsvendor game:

**Case 1:** The retailers collaborate only at the demand/supply redistribution stage. In this case, the retailers make their ordering decision individually, e.g., based on their best knowledge of their demand distribution. After that, a coalition, if formed, will allow its players (retailers) to pool the demand and supply and act as a single retailer. The optimal ordering quantity for retailer  $i$  is

$$y_i^* \in \arg \max_{y \geq 0} \left\{ (p - c)y - p \mathbb{E}_{P_i} \left[ \left( y - \tilde{d}_i \right)^+ \right] \right\}. \quad (21)$$

The aggregated order quantity for a coalition  $\mathcal{S}$  is then  $\bar{y}(\mathcal{S}) = \sum_{i \in \mathcal{S}} y_i^*$ .

**Case 2:** The retailers collaborate and make their joint decision on both the ordering stage and the demand/supply redistribution stage. In this case, we assume that the retailers follow the worst-case (or robust optimization) principle to make ordering decisions. The optimal ordering quantity that a coalition  $\mathcal{S}$  made is computed as follows:

$$y^*(\mathcal{S}) \in \arg \max_{y \geq 0} \left\{ (p - c)y - p \max_{P \in \mathcal{P}(P_1, \dots, P_R)} \mathbb{E}_P \left[ \left( y - \tilde{d}(\mathcal{S}) \right)^+ \right] \right\}. \quad (22)$$

The first method of collaboration is much easier to implement in practice as the ordering quantity can be decided independently on how the coalitions are formed. The second method, on the other hand, is ‘optimal’ for each coalition as it uses the optimal ordering quantity. We will show an interesting result in Corollary 1 that the two ordering policies are equivalent in the case where only the demand distributions of individual retailers are known. We now focus on the second approach for the general case when multivariate marginal distributions are known.

Given an optimal ordering quantity  $y^*(\mathcal{S})$  and a joint distribution  $P \in \mathcal{P}(P_1, \dots, P_R)$ , the total expected profit of the coalition  $\mathcal{S}$  is

$$v(\mathcal{S}) = v_P(\mathcal{S}) = (p - c)y^*(\mathcal{S}) - p \mathbb{E}_P \left[ \left( y^*(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right].$$

The uncertainty set of the characteristic function  $\tilde{v}$  is written as follows:

$$\mathcal{V}(P_1, \dots, P_R) = \left\{ \tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists P \in \mathcal{P}(P_1, \dots, P_R) : \tilde{v}(\mathcal{S}) = v_P(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N} \right\}. \quad (23)$$

The stochastic cooperative game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, P_R))$  is the stochastic newsvendor game that we intend to consider. We start with Assumption 1. We have

$$\begin{aligned} \tilde{v}(\mathcal{N}) > 0, \quad \forall \tilde{v} \in \mathcal{V}(P_1, \dots, P_R) &\Leftrightarrow \min_{P \in \mathcal{P}(P_1, \dots, P_R)} v_P(\mathcal{N}) > 0 \\ &\Leftrightarrow (p - c)y^*(\mathcal{N}) - p \max_{P \in \mathcal{P}(P_1, \dots, P_R)} \mathbb{E}_P \left[ \left( y^*(\mathcal{N}) - \tilde{d}(\mathcal{N}) \right)^+ \right] > 0 \\ &\Leftrightarrow \max_{y \geq 0} \left\{ (p - c)y - p \max_{P \in \mathcal{P}(P_1, \dots, P_R)} \mathbb{E}_P \left[ \left( y - \tilde{d}(\mathcal{N}) \right)^+ \right] \right\} > 0. \end{aligned}$$

For  $0 \leq y \leq d_{\min}(\mathcal{N}) = \min\{\tilde{d}(\mathcal{N})\}$ , we have  $(p - c)y - p \mathbb{E}_P \left[ \left( y - \tilde{d}(\mathcal{N}) \right)^+ \right] = (p - c)y \geq 0$  for any distribution function  $P$ . Since  $p > c$ , we have  $y^*(\mathcal{N}) \geq d_{\min}(\mathcal{N})$  and it shows that if  $d_{\min}(\mathcal{N}) > 0$ , Assumption 1 is satisfied. Given the fact that marginal distributions  $P_1, \dots, P_R$  are known, we make the following assumption to ensure  $d_{\min}(\mathcal{N}) > 0$ :

**Assumption 2.** *There exists  $r = 1, \dots, R$  such that  $d_{\min}(\mathcal{S}_r) = \min\{\tilde{d}(\mathcal{S}_r)\} > 0$ .*

We now investigate properties of the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, P_R))$ . For the deterministic stochastic games, optimal ordering quantities are  $(p - c)/p$ -quantiles of some distributions. For the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, P_R))$ , the following lemma shows how to calculate optimal ordering quantities  $y^*(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ .

**Lemma 2.** *Consider the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}((P_1, \dots, P_R)))$ . Then for all  $\mathcal{S} \subseteq \mathcal{N}$ , the optimal ordering quantity  $y^*(\mathcal{S})$  defined in (22) can be calculated as follows:*

$$y^*(\mathcal{S}) = \sum_{r=1}^R \bar{y}^*(\mathcal{S}_r), \quad (24)$$

where  $\mathcal{S}_r = \mathcal{S} \cap \mathcal{N}_r$  for all  $r = 1, \dots, R$ , and  $\bar{y}^*(\emptyset) = 0$ .

**Proof.** Consider the optimization problem in (22). We have for  $y \leq d_{\min}(\mathcal{S}) = \min\{\tilde{d}(\mathcal{S})\}$ , we can write  $(p - c)y - p \mathbb{E}_P \left[ \left( y - \tilde{d}(\mathcal{S}) \right)^+ \right] = (p - c)y$  for any distribution  $P$ . Since  $p - c > 0$ , it is an increasing function in  $y$  in  $(-\infty; d_{\min}(\mathcal{S})]$ . Since  $d_{\min}(\mathcal{S}) \geq 0$ , we can then remove the non-negative constraint  $y \geq 0$  from (22) when calculating  $y^*(\mathcal{S})$ . Now, consider the inner optimization problem of (22). This is an instance of the distributionally robust optimization problem studied in Doan and Natarajan [9]. Without loss of generality, we can assume that  $\mathcal{S}_r \neq \emptyset$  for all  $r = 1, \dots, R$  knowing that  $\bar{y}^*(\emptyset) = 0$ . Applying Proposition 1(ii) from [9], we obtain the following reformulation:

$$\begin{aligned} \max_{P \in \mathcal{P}(P_1, \dots, P_R)} \mathbb{E}_P \left[ \left( y - \tilde{d}(\mathcal{S}) \right)^+ \right] &= \min_{\mathbf{x}} \sum_{r=1}^R \mathbb{E}_{P_r} \left[ \left( x_r - \tilde{d}(\mathcal{S}_r) \right)^+ \right] \\ \text{s.t.} \quad &\sum_{r=1}^R x_r = y. \end{aligned}$$

Thus, in order to find  $y^*(\mathcal{S})$ , we can solve the following optimization problem

$$\begin{aligned} \max_{y, \mathbf{x}} \quad & (p - c)y - p \sum_{r=1}^R \mathbb{E}_{P_r} \left[ \left( x_r - \tilde{d}(\mathcal{S}_r) \right)^+ \right] \\ \text{s.t.} \quad & \sum_{r=1}^R x_r = y. \end{aligned}$$

The optimal ordering quantity  $y^*(\mathcal{S})$  can be calculated as  $y^*(\mathcal{S}) = \sum_{r=1}^R x_r^*$ , where  $\mathbf{x}^*$  is the optimal solution of the following separable optimization problem:

$$\max_{\mathbf{x}} \sum_{r=1}^R \left( (p - c)x_r - p \sum_{r=1}^R \mathbb{E}_{P_r} \left[ \left( x_r - \tilde{d}(\mathcal{S}_r) \right)^+ \right] \right),$$

or equivalently,

$$\sum_{r=1}^R \max_{x_r} \left( (p - c)x_r - p \mathbb{E}_{P_r} \left[ \left( x_r - \tilde{d}(\mathcal{S}_r) \right)^+ \right] \right).$$

For each sub-problem,  $x_r^*$  is the  $(p - c)/p$ -quantile of the distribution of  $\tilde{d}(\mathcal{S}_r)$ , which means  $x_r^* = \bar{y}^*(\mathcal{S}_r)$  for all  $r = 1, \dots, R$ . Thus we have  $y^*(\mathcal{S}) = \sum_{r=1}^R \bar{y}^*(\mathcal{S}_r)$ .  $\square$

**Corollary 1.** *Consider the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}((P_1, \dots, P_N)))$  where only individual demand distributions are known. Then, for all coalition  $\mathcal{S} \subseteq \mathcal{N}$ , the optimal ordering quantity  $y^*(\mathcal{S})$  defined in (22) is the sum of its players' optimal ordering quantities; that is,*

$$y^*(\mathcal{S}) = \bar{y}(\mathcal{S}) = \sum_{i \in \mathcal{S}} y_i^*, \quad (25)$$

where  $y_i^* = \arg \max_{y \geq 0} \left\{ (p - c)y - p \mathbb{E}_{P_i} \left[ \left( y - \tilde{d}_i \right)^+ \right] \right\}$  is the optimal ordering quantity for retailer  $i$  for all  $i = 1, \dots, N$ .

Based on Theorem 1, we now consider the equivalent deterministic cooperative game  $(\mathcal{N}, v_{\mathcal{V}(\mathbf{P})}^{\max})$  of the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$ , where

$$v_{\mathcal{V}(\mathbf{P})}^{\max}(\mathcal{S}) = \max_{P \in \mathcal{P}(P_1, \dots, P_R)} \frac{v_P(\mathcal{S})}{v_P(\mathcal{N})}, \quad \forall \mathcal{S} \subseteq \mathcal{N}. \quad (26)$$

The deterministic newsvendor games always have imputations since the corresponding characteristic function is super-additive. The following theorem claims the existence of robust imputations of the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$ .

**Theorem 3.** *The stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  always has robust imputations.*

**Proof.** Applying Lemma 2 for  $\mathcal{S} = \mathcal{N}$ , we have

$$y^*(\mathcal{N}) = \sum_{i=1}^R \bar{y}^*(\mathcal{N}_r).$$

Using the fact that  $(x + y)^+ \leq x^+ + y^+$ , we have

$$\begin{aligned} v_P(\mathcal{N}) &= (p - c)y^*(\mathcal{N}) - p \mathbb{E}_P \left[ \left( y^*(\mathcal{N}) - \tilde{d}(\mathcal{N}) \right)^+ \right] \\ &\geq (p - c) \sum_{i=1}^R \bar{y}^*(\mathcal{N}_r) - p \sum_{r=1}^R \mathbb{E}_{P_r} \left[ \left( \bar{y}^*(\mathcal{N}_r) - \tilde{d}(\mathcal{N}_r) \right)^+ \right] \\ &= \sum_{r=1}^R \left\{ (p - c) \bar{y}^*(\mathcal{N}_r) - p \mathbb{E}_{P_r} \left[ \left( \bar{y}^*(\mathcal{N}_r) - \tilde{d}(\mathcal{N}_r) \right)^+ \right] \right\} \\ &= \sum_{r=1}^R \bar{v}(\mathcal{N}_r), \end{aligned}$$

where  $\bar{v}(\mathcal{N}_r)$  is computed using (20) for all  $r = 1, \dots, R$  since  $P_1, \dots, P_R$  are completely known. We have  $\bar{v}(\mathcal{N}_r) \geq 0$  for all  $r = 1, \dots, R$  and under Assumption 2, there exists  $r$  such that  $\bar{v}(\mathcal{N}_r) > 0$ . Thus, we have

$$v_P(\mathcal{N}) \geq \sum_{r=1}^R \bar{v}(\mathcal{N}_r) > 0.$$

Now consider the deterministic newsvendor game for the coalition  $\mathcal{N}_r$  with the complete knowledge of the joint distribution  $P_r$ . There exists at least one imputation for this cooperative game; thus, we have

$$\sum_{i \in \mathcal{N}_r} \bar{v}(\{i\}) \leq \bar{v}(\mathcal{N}_r), \quad \forall r = 1, \dots, R.$$

In addition,  $v_P(\{i\}) = \bar{v}(\{i\})$  for all  $i \in \mathcal{N}$  since marginal distributions  $P_1, \dots, P_R$  are known. We then have

$$v_{\mathcal{V}(\mathbf{P})}^{\max}(\{i\}) = \max_{P \in \mathcal{P}(P_1, \dots, P_R)} \frac{v_P(\{i\})}{v_P(\mathcal{N})} \leq \bar{v}(\{i\}) \left( \sum_{r=1}^R \bar{v}(\mathcal{N}_r) \right)^{-1}, \quad \forall i \in \mathcal{N}.$$

Using the fact that  $\mathcal{N} = \bigcup_{r=1}^R \mathcal{N}_r$  and  $\mathcal{N}_r \cap \mathcal{N}_s = \emptyset$  for all  $r \neq s$ , we have

$$\sum_{i \in \mathcal{N}} v_{\mathcal{V}(\mathbf{P})}^{\max}(\{i\}) \leq \left( \sum_{i \in \mathcal{N}} \bar{v}(\{i\}) \right) \left( \sum_{r=1}^R \bar{v}(\mathcal{N}_r) \right)^{-1} \leq \left( \sum_{r=1}^R \bar{v}(\mathcal{N}_r) \right) \left( \sum_{r=1}^R \bar{v}(\mathcal{N}_r) \right)^{-1} = 1.$$

Thus, there exists at least one imputation for the cooperative game  $(\mathcal{N}, v_{\mathcal{V}(\mathbf{P})}^{\max})$ . According to Theorem 1, the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  has at least one robust imputation.  $\square$

The stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  always has robust imputations but its core is not always non-empty in general. The following theorem shows the non-emptiness of the core of  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  in some special settings in which the characteristic function  $v_{\mathcal{V}(\mathbf{P})}^{\max}$  can be computed analytically.

**Theorem 4.** *There exist  $\underline{\kappa} > 0$  and  $\bar{\kappa} < 1$  such that the robust core of the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  is non-empty if the cost-to-price ratio  $c/p \leq \underline{\kappa}$  or  $c/p \geq \bar{\kappa}$ .*

**Proof.** We have  $v_P(\mathcal{S}) = (p - c)y^*(\mathcal{S}) - p \mathbb{E}_P \left[ \left( y^*(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right]$ . If  $y^*(\mathcal{S}) \leq d_{\min}(\mathcal{S})$ , we can easily compute  $v_P(\mathcal{S}) = (p - c)y^*(\mathcal{S})$ . According to Lemma 2, we have

$$y^*(\mathcal{S}) = \sum_{r=1}^R \bar{y}^*(\mathcal{S}_r),$$

where  $\bar{y}^*(\mathcal{S}_r)$  is the  $(p - c)/p$ -quantile of the distribution of  $\tilde{d}(\mathcal{S}_r)$ . We define

$$\bar{\kappa} = \max_{r=1, \dots, R, \mathcal{S}_r \subseteq \mathcal{N}_r} \left\{ 1 - \mathbb{P} \left( \tilde{d}(\mathcal{S}_r) = d_{\min}(\mathcal{S}_r) \right) \right\} < 1.$$

If  $c/p \geq \bar{\kappa}$ , then  $\bar{y}^*(\mathcal{S}_r) = d_{\min}(\mathcal{S}_r)$  for all  $\mathcal{S}_r \subseteq \mathcal{N}_r$ ,  $r = 1, \dots, R$ . In addition,

$$d_{\min}(\mathcal{S}) \geq \sum_{r=1}^R d_{\min}(\mathcal{S}_r), \quad \forall \mathcal{S} \subseteq \mathcal{N}.$$

Therefore,  $v_P(\mathcal{S}) = (p - c)y^*(\mathcal{S}) = (p - c) \sum_{r=1}^R d_{\min}(\mathcal{S}_r)$  for all  $\mathcal{S} \subseteq \mathcal{N}$ , which does not depend on  $P$ , if  $c/p \geq \bar{\kappa}$ . This implies that the game  $(\mathcal{N}, v_{\mathcal{V}(\mathbf{P})}^{\max})$  is equivalent to a deterministic newsvendor game with an arbitrary joint probability measure  $P \in \mathcal{P}(P_1, \dots, P_R)$  (up to a scaling factor) with a non-empty core. Subsequently, the robust core of the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  is non-empty if  $c/p \geq \bar{\kappa}$ .

Using similar arguments, we can define

$$\underline{\kappa} = \min_{r=1, \dots, R, \mathcal{S}_r \subseteq \mathcal{N}_r} \mathbb{P} \left( \tilde{d}(\mathcal{S}_r) = d_{\max}(\mathcal{S}_r) \right) > 0,$$

where  $d_{\max}(\mathcal{S}) = \max\{\tilde{d}(\mathcal{S})\}$  for all  $\mathcal{S} \subseteq \mathcal{N}$ . If  $c/p \leq \underline{\kappa}$ , we have:  $\bar{y}^*(\mathcal{S}_r) = d_{\max}(\mathcal{S}_r)$  for all  $\mathcal{S}_r \subseteq \mathcal{N}_r$ ,  $r = 1, \dots, R$ . We also have

$$d_{\max}(\mathcal{S}) \leq \sum_{r=1}^R d_{\max}(\mathcal{S}_r), \quad \forall \mathcal{S} \subseteq \mathcal{N}.$$

Therefore,  $y^*(\mathcal{S}) \geq d_{\max}(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ , which means  $v_P(\mathcal{S}) = \sum_{r=1}^R p \mathbb{E}_{P_r} [\tilde{d}(\mathcal{S}_r)] - c d_{\max}(\mathcal{S}_r)$ , which again does not depend on  $P$ , for all  $\mathcal{S} \subseteq \mathcal{N}$ . Subsequently, the game  $(\mathcal{N}, v_{\mathcal{V}(\mathbf{P})}^{\max})$  in this case is also equivalent to a deterministic newsvendor game with an arbitrary joint probability measure  $P \in \mathcal{P}(P_1, \dots, P_R)$  (up to a scaling factor) with a non-empty core. This shows that the robust core of the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, V_R))$  is non-empty if  $c/p \leq \underline{\kappa}$ .  $\square$

To complete this section, we briefly discuss the case when individual retailers make their own ordering decisions. Given the ordering quantity  $\bar{y}(\mathcal{S})$ , the expected profit of the coalition  $\mathcal{S}$  is

$$\bar{v}(\mathcal{S}) = \bar{v}_P(\mathcal{S}) = (p - c)\bar{y}(\mathcal{S}) - p \mathbb{E}_P \left[ \left( \bar{y}(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right]$$

for the joint distribution  $P \in \mathcal{P}(P_1, \dots, P_R)$ . The uncertainty set  $\bar{\mathcal{V}}(P_1, \dots, P_R)$  of characteristic functions can then be defined in a similar way to (23):

$$\bar{\mathcal{V}}(P_1, \dots, P_R) = \left\{ \tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists P \in \mathcal{P}(P_1, \dots, P_R) : \tilde{v}(\mathcal{S}) = \bar{v}_P(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N} \right\}. \quad (27)$$

Consider the stochastic game  $(\mathcal{N}, \bar{\mathcal{V}}(P_1, \dots, P_R))$ , we have

$$\bar{v}_P(\mathcal{S}) = (p - c)\bar{y}(\mathcal{S}) - p \mathbb{E}_P \left[ \left( \bar{y}(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right] \geq \sum_{i \in \mathcal{S}} (p - c)y_i^* - p \mathbb{E}_{Q_i} \left[ \left( y_i^* - \tilde{d}_i \right)^+ \right],$$

where  $Q_i$ ,  $i = 1, \dots, N$ , are the univariate marginal distributions of  $P_r$ ,  $r = 1, \dots, R$ . Using an analysis similar to that above, Assumption 1 is satisfied under the following condition:

**Assumption 3.** *There exists  $i = 1, \dots, N$  such that  $\min\{\tilde{d}_i\} > 0$ .*

We now consider the game  $(\mathcal{N}, \mathcal{V}(Q_1, \dots, Q_N))$ . Assumption 3 for the game  $(\mathcal{N}, \bar{\mathcal{V}}(P_1, \dots, P_R))$  is actually the same as Assumption 2 applied for this game. In addition, according to Corollary 1, the ordering policies of these two games are the same. The only difference between  $(\mathcal{N}, \bar{\mathcal{V}}(P_1, \dots, P_R))$  and  $(\mathcal{N}, \mathcal{V}(Q_1, \dots, Q_N))$  is the uncertainty set of characteristic functions

$$\bar{\mathcal{V}}(P_1, \dots, P_R) \subseteq \mathcal{V}(Q_1, \dots, Q_N).$$

This follows directly from the fact that  $\mathcal{P}((P_1, \dots, P_R) \subseteq \mathcal{P}(Q_1, \dots, Q_N)$  and from the definitions of uncertainty sets in (23) and (27). In order to further analyze some relationships between these two games, we first prove the following simple but general result:

**Proposition 1.** *Consider two stochastic newsvendor games  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_1))$  and  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_2))$  with ordering policies resulting in the same ordering quantities; that is,  $y^1(\mathcal{S}) = y^2(\mathcal{S}) = y(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ , and uncertainty sets of characteristic functions,  $\mathcal{V}(\mathcal{P}_k)$ ,  $k = 1, 2$ , defined as follows:*

$$\mathcal{V}(\mathcal{P}_k) = \left\{ \tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists P \in \mathcal{P}_k : \tilde{v}(\mathcal{S}) = (p - c)y(\mathcal{S}) - p \mathbb{E}_P \left[ \left( y(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right], \forall \mathcal{S} \subseteq \mathcal{N} \right\}.$$

*The following statements are true if  $\mathcal{P}_2 \subseteq \mathcal{P}_1$ :*

- i) The existence of robust imputations of  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_1))$  implies the existence of robust imputations of  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_2))$ .*

ii) If the robust core of  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_1))$  is non-empty, then so is the robust core of  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_2))$ .

**Proof.** The proof of this theorem is straightforward. We can calculate the corresponding deterministic characteristic functions as follows:

$$v_{\mathcal{V}(\mathcal{P}_k)}^{\max}(\mathcal{S}) = \max_{P \in \mathcal{P}_k} \frac{(p-c)y^*(\mathcal{S}) - p \mathbb{E}_P \left[ \left( y^*(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right]}{(p-c)y^*(\mathcal{N}) - p \mathbb{E}_P \left[ \left( y^*(\mathcal{N}) - \tilde{d}(\mathcal{N}) \right)^+ \right]}, \quad \forall \mathcal{S} \subseteq \mathcal{N}, k = 1, 2.$$

Since  $\mathcal{P}_2 \subseteq \mathcal{P}_1$ , we then have

$$v_{\mathcal{V}(\mathcal{P}_1)}^{\max}(\mathcal{S}) \geq v_{\mathcal{V}(\mathcal{P}_2)}^{\max}(\mathcal{S}), \quad \forall \mathcal{S} \subseteq \mathcal{N}.$$

If the game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_1))$  has a robust imputation  $\mathbf{y}$ , we have

$$y_i \geq v_{\mathcal{V}(\mathcal{P}_1)}^{\max}(\{i\}) \geq v_{\mathcal{V}(\mathcal{P}_2)}^{\max}(\{i\}), \quad i = 1, \dots, N,$$

which implies robust imputations exist for the game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_2))$ .

Similarly, suppose the core of the stochastic game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_1))$  is non-empty and let  $\mathbf{y}$  be a core member. Then we have

$$\sum_{i \in \mathcal{S}} y_i \geq v_{\mathcal{V}(\mathcal{P}_1)}^{\max}(\mathcal{S}) \geq v_{\mathcal{V}(\mathcal{P}_2)}^{\max}(\mathcal{S}), \quad \forall \mathcal{S} \subseteq \mathcal{N}.$$

Thus, the core of the stochastic game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}_2))$  is also non-empty.  $\square$

Applying the above result for the stochastic games  $(\mathcal{N}, \bar{\mathcal{V}}(P_1, \dots, P_R))$  and  $(\mathcal{N}, \mathcal{V}(Q_1, \dots, Q_N))$ , we can conclude that there are always robust imputations for the game  $(\mathcal{N}, \bar{\mathcal{V}}(P_1, \dots, P_R))$  given the result obtained from Theorem 3 for the game  $(\mathcal{N}, \mathcal{V}(Q_1, \dots, Q_N))$ . Similarly, if the cost-to-price ratio  $c/p \leq \underline{\kappa}$  or  $c/p \geq \bar{\kappa}$ , the robust core of the game  $(\mathcal{N}, \bar{\mathcal{V}}(P_1, \dots, P_R))$  is non-empty, where  $\underline{\kappa}$  and  $\bar{\kappa}$  can then be calculated based on results obtained from Theorem 4 for the game  $(\mathcal{N}, \mathcal{V}(Q_1, \dots, Q_N))$  as follows:

$$\underline{\kappa} = \min_{i=1, \dots, N} \mathbb{P} \left( \tilde{d}_i = \min\{\tilde{d}_i\} \right), \quad \bar{\kappa} = \max_{i=1, \dots, N} \left\{ 1 - \mathbb{P} \left( \tilde{d}_i = \max\{\tilde{d}_i\} \right) \right\}.$$

## 4 Computation of Stochastic Newsvendor Games

In this section, we consider the computational aspect of the general stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}))$  given the ordering quantities  $y(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ , and the uncertainty set of characteristic functions defined as follows:

$$\mathcal{V}(\mathcal{P}) = \left\{ \tilde{v} : 2^{\mathcal{N}} \rightarrow \mathbb{R} \mid \exists P \in \mathcal{P} : \tilde{v}(\mathcal{S}) = (p-c)y(\mathcal{S}) - p \mathbb{E}_P \left[ \left( y(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right], \forall \mathcal{S} \subseteq \mathcal{N} \right\}, \quad (28)$$

where  $\mathcal{P}$  is a given set of distributions. More concretely, we would like to consider the following optimization problem to find the robust least core value  $\epsilon(\mathcal{N}, \mathcal{V}(\mathcal{P}))$  as well as the robust stability value  $\sigma(\mathcal{N}, \mathcal{V}(\mathcal{P}))$ :

$$\begin{aligned} s(\mathcal{N}, \mathcal{V}(\mathcal{P})) = \min \quad & \epsilon \\ \text{s.t.} \quad & \sum_{i=1}^N x_i = 1, \\ & \sum_{i \in \mathcal{S}} x_i \geq v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) - \epsilon, \quad \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \end{aligned} \quad (29)$$

where the deterministic characteristic function  $v_{\mathcal{V}(\mathcal{P})}^{\max}$  is defined as follows:

$$v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) = \max_{P \in \mathcal{P}} \frac{(p-c)y(\mathcal{S}) - p \mathbb{E}_P \left[ \left( y(\mathcal{S}) - \tilde{d}(\mathcal{S}) \right)^+ \right]}{(p-c)y(\mathcal{N}) - p \mathbb{E}_P \left[ \left( y(\mathcal{N}) - \tilde{d}(\mathcal{N}) \right)^+ \right]}, \quad \forall \mathcal{S} \subsetneq \mathcal{N}. \quad (30)$$

#### 4.1 Scenario-based Stochastic Newsvendor Games

In order to completely define the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}))$ , we need to characterize the set  $\mathcal{P}$  of distributions and the ordering policy resulting in ordering quantities  $y(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ . With respect to  $\mathcal{P}$ , we focus on classes of scenario-based demand distributions, which can be constructed from historical sales or from market analysis. We assume there are  $K$  demand scenarios, each of which is represented by the demand vector  $\mathbf{d}_k \in \mathbb{R}_+^N$ ,  $k = 1, \dots, K$ . Let  $q_k$  be the probability of each scenario  $k$ ,  $k = 1, \dots, K$ , we have  $q_k \geq 0$  for all  $k = 1, \dots, K$ , and  $\sum_{k=1}^K q_k = 1$ .

For deterministic newsvendor games, the probability vector  $\mathbf{q}$  is assumed to be known. We consider the stochastic setting where  $\mathbf{q}$  belongs to an uncertainty set  $\mathcal{Q}$ , which can be characterized as a polyhedron:

$$\mathcal{Q} = \{ \mathbf{q} : \mathbf{A}\mathbf{q} = \mathbf{b}, \mathbf{q} \geq \mathbf{0} \}. \quad (31)$$

Note that the normalization constraint  $\sum_{k=1}^K q_k = 1$  should be included in the set of constraints defining  $\mathcal{Q}$ ,  $\mathbf{A}\mathbf{q} = \mathbf{b}$ . For scenario-based distributions, each uncertainty set  $\mathcal{Q}$  of the probability vector  $\mathbf{q}$  generates a set  $\mathcal{P}$  of distributions which can be used to define the stochastic game  $(\mathcal{N}, \mathcal{V}(\mathcal{P}))$ . The uncertainty set  $\mathcal{Q}$  defined in (31) can be used to impose additional information about the joint demand distribution represented by the probability vector  $\mathbf{q}$ . A general constraint for additional information, which can be written as a linear constraint in  $\mathbf{q}$ , is the following moment-based constraint:

$$\mathbb{E} [\psi(\tilde{\mathbf{d}})] = \mu \Leftrightarrow \sum_{k=1}^K q_k \psi(\mathbf{d}_k) = \mu. \quad (32)$$



There are several well-known functions that can be chosen for  $\psi$ . For example, we can choose  $\psi_j(\mathbf{d}) \equiv d_i$ ,  $i = 1, \dots, N$ , which will result in a constraint on the expected demand of retailer  $i$ . Similarly, we can choose appropriate function  $\psi$  to impose constraints on the correlations between demands of different retailers. The class  $\mathcal{P}(P_1, \dots, P_R)$  of distributions with known multivariate marginals studied in Section 3.2 can also be characterized by an uncertainty set  $\mathcal{Q}$  with this type of constraints. Consider the distribution  $P_r$  of  $\tilde{\mathbf{d}}_r$  represented by a scenario-based distribution with  $K_r$  values,  $\mathbf{d}_r^l$  of probability  $p_r^l$ , for  $l = 1, \dots, K_r$ ,  $r = 1, \dots, R$ . The total number of possible scenarios for the joint demand distribution is  $K = \prod_{r=1}^R K_r$ . If we define  $\psi_{rl}(\mathbf{d}) = \mathbb{I}\{\mathbf{d}_{k,r} = \mathbf{d}_r^l\}$  for all  $r = 1, \dots, R$  and  $l = 1, \dots, K_r$ , we can then represent  $\mathcal{P}(P_1, \dots, P_R)$  with the following constraints:

$$\mathbb{E}[\psi_{rl}(\tilde{\mathbf{d}})] = p_r^l \Leftrightarrow \sum_{k=1}^K \mathbb{I}\{\mathbf{d}_{k,r} = \mathbf{d}_r^l\} q_k = p_r^l, \quad r = 1, \dots, R, l = 1, \dots, K_r.$$

Finally, note that we can introduce inequalities for the moment-based constraints instead of equalities to impose bounds on the corresponding moments and the subsequent analysis will remain the same. For the clarity of the exposition, we only consider the uncertainty set  $\mathcal{Q}$  defined in (31). The following theorem shows that (29) is equivalent to a linear optimization problem.

**Proposition 2.**  $s(\mathcal{N}, \mathcal{V}(\mathcal{P}))$  is the optimal value of the following linear optimization problem:

$$\begin{aligned} \min_{\mathbf{x}, \lambda(\cdot), \gamma(\cdot)} \quad & \epsilon \\ \text{s.t.} \quad & \sum_{i=1}^N x_i = 1, \\ & \sum_{i \in \mathcal{S}} x_i + \epsilon - \lambda(\mathcal{S}) \geq 0, \quad \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \\ & (p - c)y(\mathcal{N})\lambda(\mathcal{S}) - \mathbf{b}^t \gamma(\mathcal{S}) \geq (p - c)y(\mathcal{S}), \quad \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \\ & p(y(\mathcal{N}) - \mathbf{d}(\mathcal{N}))^+ \lambda(\mathcal{S}) - \mathbf{A}^t \gamma(\mathcal{S}) \leq p(y(\mathcal{S}) - \mathbf{d}(\mathcal{S}))^+, \quad \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \end{aligned} \tag{33}$$

where  $\lambda(\mathcal{S})$  and  $\gamma(\mathcal{S})$  are (additional) decision variables for all  $\mathcal{S} \subsetneq \mathcal{N}$ .

**Proof.** The deterministic characteristic function  $v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S})$  can be computed as follows:

$$\begin{aligned} v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) = \max_{\mathbf{q}} \quad & \frac{(p - c)y(\mathcal{S}) - p \sum_{k=1}^K [(y(\mathcal{S}) - d_k(\mathcal{S}))^+] q_k}{(p - c)y(\mathcal{N}) - p \sum_{k=1}^K [(y(\mathcal{N}) - d_k(\mathcal{N}))^+] q_k} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{q} = \mathbf{b}, \\ & \mathbf{q} \geq \mathbf{0}. \end{aligned} \tag{34}$$

According to Cambini et al. [2], strong duality holds for the problem (34), which is a linear fractional programming problem. By introducing dual decision variables  $\gamma(\mathcal{S})$  for all constraints  $\mathbf{A}\mathbf{q} = \mathbf{b}$ , we can compute the characteristic function  $v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S})$  as follows:

$$\begin{aligned} v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) &= \min_{\lambda(\mathcal{S}), \gamma(\mathcal{S})} \lambda(\mathcal{S}) \\ \text{s.t. } & (p-c)y(\mathcal{N})\lambda(\mathcal{S}) - \mathbf{b}^t \gamma(\mathcal{S}) \geq (p-c)y(\mathcal{S}), \\ & p(y(\mathcal{N}) - \mathbf{d}(\mathcal{N}))^+ \lambda(\mathcal{S}) - \mathbf{A}^t \gamma(\mathcal{S}) \leq p(y(\mathcal{S}) - \mathbf{d}(\mathcal{S}))^+. \end{aligned} \quad (35)$$

Using this dual formulation for all  $\mathcal{S} \subsetneq \mathcal{N}$ , we can then obtain the final linear optimization formulation as in (36) to compute the robust least core value  $\epsilon(\mathcal{N}, \mathcal{V}(\mathcal{P}))$ .  $\square$

We can apply Proposition 2 directly for the stochastic newsvendor game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, P_R))$  with the appropriate uncertainty set  $\mathcal{Q}$  to obtain  $s(\mathcal{N}, \mathcal{V}(P_1, \dots, P_R))$  as the optimal value of the following optimization problem:

$$\begin{aligned} \min_{\mathbf{x}, \alpha(\cdot), \beta(\cdot), \lambda(\cdot)} \quad & \epsilon \\ \text{s.t. } \quad & \sum_{i=1}^N x_i = 1, \\ & \sum_{i \in \mathcal{S}} x_i \geq \lambda(\mathcal{S}) - \epsilon, \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \\ & (p-c)y^*(\mathcal{N})\lambda(\mathcal{S}) - \sum_{r=1}^R \sum_{l=1}^{K_r} p_r^l \alpha_r^l(\mathcal{S}) + \beta(\mathcal{S}) \geq (p-c)y^*(\mathcal{S}), \forall \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \\ & p(y^*(\mathcal{N}) - d_k(\mathcal{N}))^+ \lambda(\mathcal{S}) - \sum_{r=1}^R \sum_{l=1}^{K_r} \mathbb{I}\{\mathbf{d}_{k,r} = \mathbf{d}_r^l\} \alpha_r^l(\mathcal{S}) + \beta(\mathcal{S}) \leq p(y^*(\mathcal{S}) - d_k(\mathcal{S}))^+, \\ & \forall k = 1, \dots, K, \mathcal{S} \subsetneq \mathcal{N}, \mathcal{S} \neq \emptyset, \end{aligned} \quad (36)$$

where  $d_k(\mathcal{S}) = \sum_{i \in \mathcal{S}} d_{k,i}$  for  $k = 1, \dots, K$ , and  $\alpha_r^l(\mathcal{S})$  and  $\beta(\mathcal{S})$  are (additional) decision variables for all  $r = 1, \dots, R$ ,  $l = 1, \dots, K_r$ , and  $\mathcal{S} \subsetneq \mathcal{N}$ .

In order to find  $s(\mathcal{N}, \mathcal{V}(\mathcal{P}))$  using the linear optimization formulation in (36), we need to compute  $(2^N - 1)$  ordering quantities  $y(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$ . If we use the optimal ordering quantities, we need to solve the following optimization problem:

$$y^*(\mathcal{S}) \in \arg \max_{y \geq 0} (p-c)y - p \cdot \max_{\mathbf{q} \in \mathcal{Q}} \sum_{k=1}^K q_k (y - d_k(\mathcal{S}))^+.$$

We can dualize the inner optimization problem using  $\gamma$  as dual variables for the set of constraints  $\mathbf{A}\mathbf{q} = \mathbf{b}$ , and obtain the following equivalent optimization problem to calculate the optimal ordering

quantity  $y^*(\mathcal{S})$ :

$$\begin{aligned} y^*(\mathcal{S}) = \arg \max_{y, \gamma} \quad & (p - c)y + p\mathbf{b}^t\gamma, \\ \text{s.t.} \quad & \mathbf{A}^t\gamma + (y - \mathbf{d}(\mathcal{S}))^+ \leq 0 \\ & y \geq 0, \end{aligned} \tag{37}$$

where  $\mathbf{d}(\mathcal{S}) = \{d_k(\mathcal{S})\}_{k=1, \dots, K} \in \mathbb{R}^K$ . For the stochastic game  $(\mathcal{N}, \mathcal{V}(P_1, \dots, P_R))$ , this results in a closed-form formulation for  $y^*(\mathcal{S})$  for all  $\mathcal{S} \subseteq \mathcal{N}$  as shown in Lemma 2. Even in this case, we still need to compute a large number of ordering quantities in general, which is in the order of  $\sum_{r=1}^R 2^{N_r}$ , where  $N_r$  is the size of the subset  $\mathcal{N}_r$ ,  $r = 1, \dots, R$ . Under the special setting when all univariate marginals are known, i.e.,  $N_r = 1$  for all  $r = 1, \dots, N$ , we only need to compute  $N$  optimal ordering quantities  $y_i^*$ ,  $i = 1, \dots, N$ , and this would coincide with the ordering policy when retailers make their own ordering decisions,  $\bar{y}(\mathcal{S}) = \sum_{i \in \mathcal{S}} y_i^*$ .

## 4.2 Constraint Generation for Solving Robust Least Core

In order to find the robust least core  $\epsilon(\mathcal{N}, \mathcal{V}(\mathcal{P}))$ , we could obtain  $v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S})$  for each coalition  $\mathcal{S}$  by solving the corresponding linear program (LP) in (35); that is, to solve  $(2^N - 1)$  such linear programs, and then finally solve the linear program (7). This approach is appropriate for cases when  $N$  is reasonably small ( $\leq 10$ ) and when the constraint matrix  $\mathbf{A}$  is of moderate size so that each of the LPs in (35) can be solved in a reasonable time. For  $N \geq 20$ , solving more than one million LPs just to obtain the parameters before solving the final LP with more than one million constraints is not sufficient. Another option is to solve the linear program in (33). This would avoid the need to solve  $2^N$  LPs in (35) as all the parameters  $v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S})$  have been incorporated by using the analytical formulation of these parameters. However, this comes at the cost of having a larger final linear program with  $(2^{(N+1)} + N)$  decision variables and with  $((K + 1)2^N + 1)$  constraints. For problems with reasonable  $N$  and  $K$ , i.e.,  $N \leq 10$  and  $K \leq 100$ , this LP can be solved very efficiently. For larger  $N$  and  $K$ , finding the robust least core for stochastic games using (33) would prove to be quite challenging even for the current state-of-the-art LP solvers. We now propose a constraint generation framework to solve the original problem in (29) for large instances. Given the fact that the computation of optimal ordering quantities is time-consuming in general for large instances, we only focus on the ordering policy under which retailers make their own ordering decisions; that is,  $\bar{y}(\mathcal{S}) = \sum_{i \in \mathcal{S}} y_i^*$  for all  $\mathcal{S} \subseteq \mathcal{N}$ . We make the implicit assumption that information of all univariate marginal distributions can be extracted from the uncertainty set  $\mathcal{Q}$  so that  $y_i^*$  can be computed for  $i = 1, \dots, N$ . Under the constraint generation framework, we repeatedly solve a

relaxation problem  $\text{RLP}(\mathcal{C})$  of (29), where only constraints with respect to  $\mathcal{S} \in \mathcal{C}$ ,  $\mathcal{C} \subseteq 2^{\mathcal{N}}$ , are included:

$$\begin{aligned} \text{RLC}(\mathcal{C}) : \min \quad & \epsilon \\ \text{s.t.} \quad & \sum_{i=1}^N x_i = 1, \\ & \sum_{i \in \mathcal{S}} x_i \geq v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) - \epsilon, \quad \forall \mathcal{S} \in \mathcal{C}. \end{aligned} \quad (38)$$

The constraint generation algorithm CGA for (29) is described in detail as follows:

**The CGA.** Given an initial imputation  $\mathbf{x}^0$ , set  $k = 0$ ,  $\epsilon^0 = -\infty$ ,  $\tau^0 = +\infty$ , and  $\mathcal{C}^0 = \emptyset$ , and perform the following loop:

**Step 1.** Solve the constraint generation problem CG

$$\tau^{k+1} = \max_{\mathcal{S} \subseteq \mathcal{N}} \left\{ v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) - \sum_{i \in \mathcal{S}} x_i^k \right\}, \quad \mathcal{S}^{k+1} = \arg \max_{\mathcal{S} \subseteq \mathcal{N}} \left\{ v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) - \sum_{i \in \mathcal{S}} x_i^k \right\}. \quad (39)$$

**Step 2.** If  $\tau^{k+1} = \epsilon^k$ , stop and output  $(\mathbf{x}^k, \epsilon^k)$ ; otherwise, continue to Step 3.

**Step 3.** Update  $\mathcal{C}^{k+1} = \mathcal{C}^k \cup \{\mathcal{S}^{k+1}\}$ . Solve the relaxation problem  $\text{RLP}(\mathcal{C}^{k+1})$  to find an optimal solution  $(\mathbf{x}^{k+1}, \epsilon^{k+1})$ . Set  $k = k + 1$ .

The main difficulty of the CGA algorithm is how to solve the constraint generation problem CG. The characteristic function  $v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S})$  is the optimal value of the linear fractional optimization problem (34), which can be reformulated as the linear optimization problem (35). We now consider the dual formulation of (35) (or to be more precise, the direct linear optimization reformulation of (34)):

$$\begin{aligned} v_{\mathcal{V}(\mathcal{P})}^{\max}(\mathcal{S}) = \max_{\delta, \boldsymbol{\omega}} \quad & (p - c)\bar{y}(\mathcal{S})\delta - \sum_{k=1}^K p(\bar{y}(\mathcal{S}) - d_k(\mathcal{S}))^+ \omega_k \\ \text{s.t.} \quad & (p - c)\bar{y}(\mathcal{N})\delta - \sum_{k=1}^K p(\bar{y}(\mathcal{N}) - d_k(\mathcal{N}))^+ \omega_k = 1, \\ & \mathbf{A}\boldsymbol{\omega} = \delta \mathbf{b}, \\ & \delta \geq 0, \boldsymbol{\omega} \geq \mathbf{0}. \end{aligned} \quad (40)$$

Each coalition  $\mathcal{S}$  can be represented as an indicator vector  $\mathbf{z} \in \{0, 1\}^N$  where  $z_i = 1$  means that retailer  $i$  is in the coalition  $\mathcal{S}$  and  $z_i = 0$  means otherwise. The ordering quantity  $\bar{y}(\mathcal{S})$  can then be

written as  $\bar{y}(\mathcal{S}) = (\mathbf{y}^*)^t \mathbf{z}$ . Similarly,  $d_k(\mathcal{S}) = (\mathbf{d}_k)^t \mathbf{z}$ . The problem CG can be reformulated as follows:

$$\begin{aligned}
\max_{\mathbf{z}, \delta, \boldsymbol{\omega}} \quad & (p - c)(\mathbf{y}^*)^t \mathbf{z} \delta - p \sum_{k=1}^K ((\mathbf{y}^*)^t \mathbf{z} - (\mathbf{d}_k)^t \mathbf{z})^+ \omega_k - \mathbf{x}^t \mathbf{z} \\
\text{s.t.} \quad & (p - c)\bar{y}(\mathcal{N})\delta - \sum_{k=1}^K p(\bar{y}(\mathcal{N}) - d_k(\mathcal{N}))^+ \omega_k = 1, \\
& \mathbf{A}\boldsymbol{\omega} = \delta \mathbf{b}, \\
& \mathbf{z} \in \{0, 1\}^N, \delta \geq 0, \boldsymbol{\omega} \geq \mathbf{0},
\end{aligned} \tag{41}$$

or equivalently,

$$\begin{aligned}
\max_{\mathbf{z}, \delta, \boldsymbol{\omega}, \mathbf{u}} \quad & (p - c)(\mathbf{y}^*)^t \mathbf{z} \delta - p \sum_{k=1}^K u_k \omega_k - \mathbf{x}^t \mathbf{z} \\
\text{s.t.} \quad & (p - c)\bar{y}(\mathcal{N})\delta - \sum_{k=1}^K p(\bar{y}(\mathcal{N}) - d_k(\mathcal{N}))^+ \omega_k = 1, \\
& \mathbf{A}\boldsymbol{\omega} = \delta \mathbf{b}, \\
& u_k \geq (\mathbf{y}^* - \mathbf{d}_k)^t \mathbf{z}, \quad \forall k = 1, \dots, K, \\
& \mathbf{z} \in \{0, 1\}^N, \delta \geq 0, \boldsymbol{\omega}, \mathbf{u} \geq \mathbf{0}.
\end{aligned} \tag{42}$$

This is a mixed-integer nonlinear optimization problem (MINLP) that involves a vector of binary decision variable  $\mathbf{z} \in \{0, 1\}^n$  and  $(2K + 1)$  continuous decision variables  $(\delta, \boldsymbol{\omega}, \mathbf{u})$ . The objective function is piecewise bilinear on  $(\delta, \boldsymbol{\omega})$  and  $(\mathbf{z}, \mathbf{u})$ . The problem becomes a linear program once we fix  $\mathbf{z}$  (and  $\mathbf{u}$  accordingly) and a mixed-integer linear program (MILP) once we fix  $(\delta, \boldsymbol{\omega})$ . We propose a heuristic method to find the violating constraint in Step 2 of the CGA algorithm based on this MINLP problem. First, we iteratively fix  $\mathbf{z}$  (and  $\mathbf{u}$  accordingly) and solve for the optimal solution  $(\delta, \boldsymbol{\omega})$  and then fix the newly found  $(\delta, \boldsymbol{\omega})$  and solve the resulting MILP for the optimal solution  $(\mathbf{z}, \mathbf{u})$ . This process is repeated until we reach a local optimal solution. If this local optimal solution provides a violating constraint, we introduce it in Step 2 of the CGA algorithm. If not, we shall attempt to solve the MINLP problem (41) by linearizing the nonlinear terms with the big-M method. Details about this are presented in Appendix 7.

## 5 Numerical Results

In this section, we demonstrate some properties and computational results of robust payoff distributions. In Section 5.1, we start by showing how data uncertainty and stability values of nominal games affect the existence of robust solutions. For stochastic newsvendor games, we investigate the existence of robust cores given different cost-to-price ratios in Section 5.2. Finally, we present computational results for computing the robust core in large stochastic newsvendor games in Section 5.3.

### 5.1 Existence of Robust Cores and Stability Values of Nominal Games

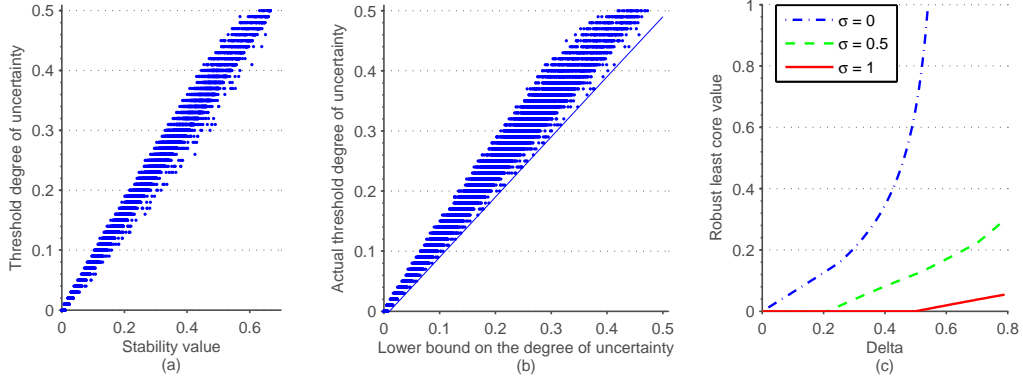


Figure 2: Existence of robust cores with respect to stability value and degree of data uncertainty.

Existence of the robust core depends on the stability value of the nominal game and the degree of data uncertainty in the stochastic game as shown in Theorem 2. We demonstrate these effects through the following simple simulation. We generate  $K = 10000$  nominal games, each with three players and with a characteristic function generated from uniform additive terms<sup>4</sup>. Consider the class of stochastic games  $\tilde{\mathcal{G}} = (\mathcal{N}, \mathcal{V}(\Delta))$  with nominal games generated as above, where  $\mathcal{V}(\Delta)$  is the set of all uncertain characteristic function  $\tilde{v}$  that are defined as  $\tilde{v}(S) = \bar{v}(S) + \sum_{i \in S} z_i$ , with  $z_i \in [-\Delta, \Delta]$  for all  $i \in \mathcal{N}$ . For each nominal game, there is a corresponding stability value  $\sigma$  and a corresponding threshold  $\Delta^*$  above which the robust core of the corresponding stochastic game no longer exists. Figure 2(a) shows the

<sup>4</sup>Each characteristic function is generated as follows:  $\bar{v}(\{i\}) = \epsilon_i, \forall i = 1, \dots, N$ ,  $\bar{v}(\{i, j\}) = \bar{v}(\{i\}) + \bar{v}(\{j\}) + \epsilon_{ij}, \forall i, j = 1, \dots, N, i \neq j$  and  $\bar{v}(\{1, 2, 3\}) = CoS(v) + \epsilon_{123}$  where  $\epsilon_i, \epsilon_{ij}$  and  $\epsilon_{123}$  are uniform random variables within  $[0, 1]$  generated in Matlab by using different seed generators (between 1 and  $K$ ), and  $CoS(v)$  is the minimum payoff value of the grand coalition such that the nominal game is stable.

pairs of  $(\sigma, \Delta^*)$  for these  $K$  nominal games. As we expect, there is a high degree of correlation between the stability value  $\sigma$  (shown in the horizontal axis) and the threshold degree of uncertainty (shown in the vertical axis). This makes sense as we expect that games with greater stability values are more stably-immune to changes in the characteristic functions.

Figure 2(b) shows the relationship between the lower bound on the threshold degree of uncertainty ( $DoU_{LB}$ ) found in Theorem 2, i.e., the value below which the robust core is guaranteed to exist, and the actual threshold degree of uncertainty ( $DoU$ ). As expected, the pairs of  $(DoU_{LB}, DoU)$  lie above the 45% line (shown by the solid line) by the definition of the lower bounds. In addition, we can observe that the pairs of  $(DoU_{LB}, DoU)$  do not lie too far above the 45% line which means the lower bounds provide a reasonably good estimate of the actual threshold degree of uncertainty.

In Figure 2(c), we choose three specific nominal games among  $K$  games generated with the stability values being the smallest ( $\sigma = 0$ ), the average ( $\sigma = 0.5$ ), and the highest ( $\sigma = 1$ ). We then consider stochastic games generated from these three nominal games and with  $\Delta$  varying between 0 and 1. For each pair of a nominal game  $\bar{v}(\cdot)$  and a choice of  $\Delta$ , we have a corresponding stochastic game. We solve each stochastic game and record its robust least core value. This robust least core value is equal to zero for stochastic games with non-empty robust core and is greater than zero otherwise. Figure 2(c) shows the change of the robust least core value (shown on the vertical axis) when we vary the degree of uncertainty  $\Delta$  (shown on the horizontal axis) for these three nominal games. We can see that the least core values increase when we increase  $\Delta$  in all three curves. For each nominal game, there is a threshold value for  $\Delta$  below which the least core value is equal to zero, i.e., the robust core exists. Above those thresholds, the robust core does not exist. In addition, we also observe the threshold degree of uncertainty increases with the increase of the stability values.

## 5.2 Robust Cores of Stochastic Newsvendor Games

In this section, we demonstrate how the cost-to-price ratio and the uncertainty of the demand distribution affect the existence of the robust core of a stochastic newsvendor game. Consider a simple case with  $N = 5$  retailers. From a set of  $N$  retailers, there is an exponentially large number of possible ways to partition these into subsets<sup>5</sup>. However, for the purpose of demonstration, we only consider four such partitions with subset sizes  $N_r$  which are (1,1,1,1,1), (2,1,2), (3,2) and (5). This corresponds to four multi-marginal distributions of the demands and are labelled as  $\mathcal{P}^a$ ,  $\mathcal{P}^b$ ,  $\mathcal{P}^c$ , and  $\mathcal{P}^d$  respectively. These

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<sup>5</sup>The total number of partitions is called the Bell number which is equal to 52 for the case of  $N = 5$

marginal distributions are reconstructed from the same multivariate demand distribution originally generated from a discrete uniform distribution. In this case,  $\mathcal{P}^a$  corresponds to the univariate case while  $\mathcal{P}^d$  corresponds to the multivariate case. We fix the price  $p = 2$  and vary the cost  $c$  between 0 and  $p$ , i.e., we vary the cost-to-price ratios  $c/p$  between 0 and 1. For each choice of  $c$  and for each distribution, we have a corresponding stochastic newsvendor game. In each game, we find the corresponding deterministic payoff distribution  $v_{\mathcal{V}(\cdot)}^{\max}(\cdot)$  and solve the least core problem shown in (6) to obtain the least core value.

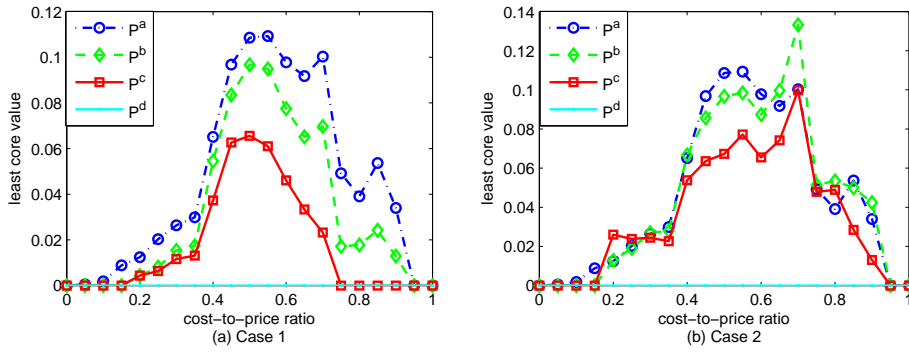


Figure 3: Robust cores of stochastic newsvendor games with different cost-to-price ratios and with different levels of uncertainty.

Figure 3(a) shows the least core values of the stochastic games generated when the ordering policy follows case 1, i.e., retailers make individual ordering decision before collaborating on the demand pooling. We can see two clear messages here. First of all, the least core values are equal to zero for the cases when the cost-to-price ratios are at the two extreme ends, which means the robust core exists for these cases. This result matches with the result from Theorem 4. In addition, we can observe that the least core for the multivariate distribution case is always zero and hence the core of the deterministic newsvendor game always exists. However, the least core values for other multi-marginal cases are non-zeros for the middle ranges cost-to-price ratios. Second of all, for each fixed cost-to-price ratio, the least core values for distribution  $\mathcal{P}^c$  are smaller than that of distribution  $\mathcal{P}^b$  which in turns are smaller than that of  $\mathcal{P}^a$ . This result matches with the result from Position 1 which states that stochastic games with less uncertainty have a higher chance of robust core existence. Figure 3(b) shows the least core values of the stochastic games generated when the ordering policy follows case 2, i.e., retailers make joint ordering decisions before collaborating on the demand pooling. We can see the same effect of the cost-to-price ratios on the robust core existence. Notice, however, that the least core value does not always decrease with less uncertainty in the demand distribution.



### 5.3 Computation of Large Stochastic Newsvendor Games

In the experiments for the newsvendor games, we let the number of retailers  $N$  range between 5 and 50. For each choice of  $N$ , we generate  $m = 10$  stochastic newsvendor games. In each game, we assume the decision maker has  $K = 100$  joint demand vectors  $\mathbf{d}_k$ ,  $k = 1, \dots, K$ , that could be collected either from historical data or from some other sources such as simulation or market analysis. Although these historical joint demand vectors are known to the decision maker, their joint probability distribution is not fully known. The decision maker, instead, has some partial information on this. In our experiments, we assume this partial information includes the univariate distributions of the demands for individual retailers. To generate the random historical data, we first generate a pool of  $K_i^N$  possible joint demand vectors where  $K_i = 10$  is the number of possible values that the demand of each retailer could realize. From this pool, we generate  $K$  samples using sampling with replacement. Next, we generate a valid probability distribution over the  $K$  observed joint demand vectors by assigning random probabilities for them with appropriate normalization. Note that this probability distribution is not known to retailers. Nevertheless, they know it belongs to the uncertainty set described in (31) with  $\mathbf{A}$  and  $\mathbf{b}$  constructed from the knowledge of the univariate distributions of the demands of the individual retailers. The selling price and the ordering price are fixed at  $p = 2$  and  $c = 1.5$ .

$N$	# iterations			Relaxed LPs			Constraint Generation			Total time (seconds)
	Ave.	Min	Max	Ave.	Min	Max	Ave.	Min	Max	
5	10.5	7	16	0.855	0.680	1.035	0.629	0.273	1.430	15.733
10	24.9	14	36	1.802	1.368	2.226	0.557	0.310	0.992	62.126
15	23.2	20	31	2.171	2.037	2.489	0.353	0.325	0.425	59.093
20	29.6	25	39	2.881	2.706	3.263	0.360	0.334	0.433	96.831
25	34.3	31	36	3.569	3.428	3.646	0.367	0.355	0.398	135.094
30	41.2	35	44	4.398	4.142	4.516	0.384	0.368	0.439	197.235
35	48.3	46	51	5.220	5.121	5.295	0.404	0.397	0.410	271.693
40	53.9	50	57	6.058	5.898	6.207	0.437	0.417	0.488	350.208
45	61.2	58	64	7.011	6.871	7.173	0.446	0.436	0.461	456.548
50	69.5	64	76	8.629	8.297	8.905	0.500	0.486	0.524	635.054

Table 1: Computational results for computing robust (least) cores of stochastic newsvendor games.

Table 1 shows the computational results for these games in detail. We report the performance of

the constraint generation in finding the robust (least) core. The first column shows the number of retailers. The next three columns show the numbers of iterations required by the constraint generation algorithm. Here, we report the average, the minimum, and the maximum numbers of iterations over  $m$  random instances generated. The next 6 columns report the same set of statistics for the total times to solve the LP relaxation problems and the constraint generation problems. The last column shows the average total time to find the robust payoff distributions. Overall, the number of iterations and the computational times for solving the sub-problems increase with the number of retailers. It takes about 11 minutes to solve for the robust payoff distribution of a stochastic newsvendor game with  $N = 50$  retailers.

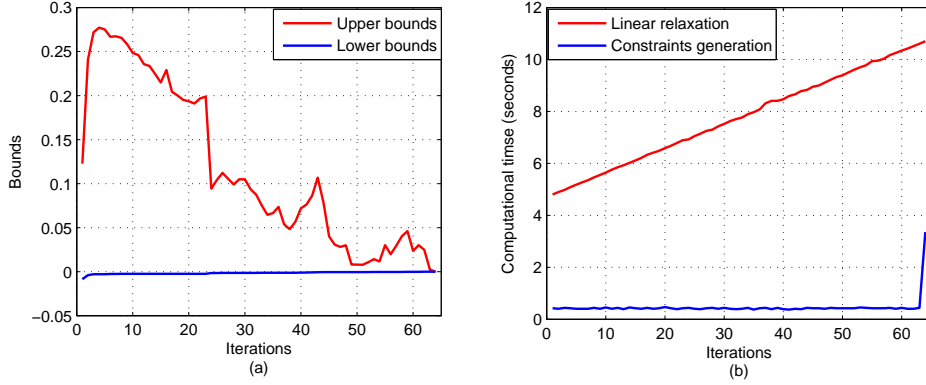


Figure 4: Performance of the constraint generation algorithm through iterations.

Figure 4 shows the convergence of the constraint generation algorithm in detail for a stochastic game with  $N = 50$  retailers. Figure 4(a) shows the lower and upper bounds generated by solving the LP relaxation and the constraint generation problems respectively. The LP bounds are always increasing but the upper bounds fluctuate with a decreasing trend. The algorithm converges after 67 iterations. Figure 4(b) shows the computational times for solving the LP relaxation problems and the constraint generation problems in each iteration. The times to solve the LP increase through iterations as we would expect since each subsequent LP is added with one additional violating constraint from the previous iteration. The times to solve the constraint generation problems are quite stable except for the last iterations because the local search involves solving sub-problems with similar sizes. The last iterations take a longer time because we need to solve a bigger MILP to confirm definitively that no more constraints are violated.

## 6 Conclusion

In this paper, we study stochastic cooperative games with uncertain characteristic functions. This setting appears in many practical situations in which the value functions need to be estimated or calculated with possible approximation errors. We introduce new solution concepts and show that the problem of finding a robust core – one that is defined as a consistent payoff distribution which is stable to all possible stochastic distribution functions within an uncertainty set – is equivalent to the problem of finding the core of a deterministic game. We study the existence and analyze properties of the robust imputation and the robust core. Specifically, we identify two important characteristics of a stochastic game – namely the stability value of the nominal game and the degree of uncertainty – that could affect the existence of the robust core. We demonstrate these concepts and properties through stochastic voting games and stochastic network games.

We also study the stochastic newsvendor games where the complete information on the demand distributions is not available. We provide properties of the optimal order quantities in the face of demand distribution uncertainty and study the existence of the robust core of the game. We develop numerical schemes to compute these solution concepts numerically and demonstrate these through reasonably large stochastic newsvendor games.

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## 7 Appendix

### Reformulation of the Constraint Generation Problem

We first introduce variables  $\psi_i = z_i \delta$  and  $\chi_{ik} = z_i \omega_k$ . We have

$$\begin{aligned}
 (\mathbf{y}^*)^t \mathbf{z} \delta &= \boldsymbol{\psi}^t \mathbf{y}^*, \\
 ((\mathbf{y}^*)^t \mathbf{z} - (\mathbf{d}_k)^t \mathbf{z}) \omega_k &= \boldsymbol{\chi}_k^t (\mathbf{y}^* - \mathbf{d}_k), \quad \forall k = 1, \dots, K,
 \end{aligned}$$

which are linear terms in a higher dimension space. In addition, we need to enforce  $\psi_i$  according to  $z_i$  as follows:

$$0 \leq \psi_i \leq \delta, \quad \text{and} \quad \psi_i \leq M z_i, \quad \forall i = 1, \dots, N,$$

where  $M$  is a sufficiently large constant and can be chosen as an upper bound for  $\delta$ . These two constraints ensure that if  $z_i = 0$ , then  $\psi_i$  must be equal to zero and if  $z_i = 1$ , then  $\psi_i$  should rather

be equal to  $\delta$  in order to drive the objective function to the maximum. For the choice of  $M$ , notice that in the derivation from the linear fractional function in (30) to the linear program in (35),  $\delta$  is a transformed variable that is equal to  $1/\tilde{v}(\mathcal{N})$ . Thus, we can choose an upper-bound  $M$  for  $\delta$  to be equal to  $1/\min_{q \in \mathcal{Q}} \tilde{v}(\mathcal{N})$  which can be calculated by solving a linear program that has a similar structure to (35), except that  $\mathcal{S}$  is replaced by  $\mathcal{N}$  and the min operator is replaced by a max operator.

Similarly, we need to enforce  $\chi_{ik}$  according to  $z_i$  as follows:

$$0 \leq \chi_{ik} \leq \omega_k, \quad \chi_{ik} \leq Mz_i, \quad \text{and} \quad \chi_{ik} + M(1 - z_i) \geq \omega_k, \quad \forall k = 1, \dots, K, \forall i = 1, \dots, N.$$

Since  $\sum_i \omega_i = \delta$ , we can choose the same upper-bound  $M$  for  $\omega_i$ . Putting these constraints altogether, we obtain the following mix-integer linear optimization problem:

$$\begin{aligned} \max_{\mathbf{z}, \delta, \boldsymbol{\omega}, \boldsymbol{\psi}, \mathbf{v}, \boldsymbol{\chi}} \quad & (p - c)\boldsymbol{\psi}^t \mathbf{y}^* - p \sum_{k=1}^K v_k - \mathbf{z}^t \mathbf{x} \\ \text{s.t.} \quad & (p - c)\bar{y}(\mathcal{N})\delta - \sum_{k=1}^K p(\bar{y}(\mathcal{N}) - d_k(\mathcal{N}))^+ \omega_k = 1, \\ & \mathbf{A}\boldsymbol{\omega} = \delta \mathbf{b}, \\ & v_k \geq \boldsymbol{\chi}_k^t (\mathbf{y}^* - \mathbf{d}_k), \quad \forall k = 1, \dots, K, \\ & 0 \leq \psi_i \leq \delta \text{ and } \psi_i \leq Mz_i, \quad \forall i = 1, \dots, N, \\ & 0 \leq \chi_{ik} \leq \omega_k, \text{ and } \chi_{ik} \leq Mz_i, \text{ and } \chi_{ik} + M(1 - z_i) \geq \omega_k, \quad \forall k = 1, \dots, K, i = 1, \dots, N, \\ & \mathbf{z} \in \{0, 1\}^N, \delta \geq 0, \boldsymbol{\omega}, \mathbf{v} \geq \mathbf{0}. \end{aligned} \tag{43}$$

This model has  $N$  binary variables,  $(NK + N + 2K + 1)$  continuous variables and  $(3NK + NK_i + 2N + K + 2)$  constraints. Although solving this MILP is still difficult for large  $(N, K)$ , we show in numerical examples that CPLEX can solve this problem for instances with up to 50 retailers.