

Finding the largest low-rank clusters with Ky Fan 2 - k -norm and ℓ_1 -norm*

Xuan Vinh Doan[†] Stephen Vavasis[‡]

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Abstract

We propose a convex optimization formulation with the Ky Fan 2 - k -norm and ℓ_1 -norm to find k largest approximately rank-one submatrix blocks of a given nonnegative matrix that has low-rank block diagonal structure with noise. We analyze low-rank and sparsity structures of the optimal solutions using properties of these two matrix norms. We show that, under certain hypotheses, with high probability, the approach can recover rank-one submatrix blocks even when they are corrupted with random noise and inserted into a much larger matrix with other random noise blocks.

1 Introduction

Given a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ that has low-rank block diagonal structure with noise, we would like to find that low-rank block structure of \mathbf{A} . Doan and Vavasis [5] have proposed a convex optimization formulation to find a large approximately rank-one submatrix of \mathbf{A} with the nuclear norm and ℓ_1 -norm. The proposed LAROS problem (for “large approximately rank-one submatrix”) in [5] can be used to sequentially extract features in data. For example, given a corpus of documents in some language, it can be used to co-cluster (or bicluster) both terms and documents, i.e., to identify simultaneously

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[†]DIMAP and ORMS Group, Warwick Business School, University of Warwick, Coventry, CV4 7AL, United Kingdom, xuan.doan@wbs.ac.uk. The research was partly done when the author was at the Department of Combinatorics and Optimization, University of Waterloo, Canada.

[‡]Department of Combinatorics and Optimization, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1, Canada, vavasis@uwaterloo.ca.

both subsets of terms and subsets of documents strongly related to each other from the *term-document matrix* $\mathbf{A} \in \mathbb{R}^{m \times n}$ of the underlying corpus of n documents with m defined terms (see, for example, Dhillon [4]). Here, “term” means a word in the language, excluding common words such as articles and prepositions. The (i, j) entry of \mathbf{A} is the number of occurrences of term i in document j , perhaps normalized. Another example is the biclustering of gene expression data to discover expression patterns of gene clusters with respect to different sets of experimental conditions (see the survey by Madeira and Oliveira [12] for more details). Gene expression data can be represented by a matrix \mathbf{A} whose rows are in correspondence with different genes and columns are in correspondence with different experimental conditions. The value a_{ij} is the measurement of the expression level of gene i under the experimental condition j .

If the selected terms in a bicluster occur proportionally in the selected documents, we can intuitively assign a topic to that particular term-document bicluster. Similarly, if the expression levels of selected genes are proportional in all selected experiments of a bicluster in the second example, we can identify a expression pattern for the given gene-experimental condition bicluster. Mathematically, for each bicluster i , we obtain a subset $\mathcal{I}_i \subset \{1, \dots, m\}$ and $\mathcal{J}_i \subset \{1, \dots, n\}$ and the submatrix block $\mathbf{A}(\mathcal{I}_i, \mathcal{J}_i)$ is approximately rank-one, i.e., $\mathbf{A}(\mathcal{I}_i, \mathcal{J}_i) \approx \mathbf{w}_i \mathbf{h}_i^T$. Assuming there are k biclusters and $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ and $\mathcal{J}_i \cap \mathcal{J}_j = \emptyset$ for all $i \neq j$, we then have the following approximation:

$$\mathbf{A} \approx [\bar{\mathbf{w}}_1, \dots, \bar{\mathbf{w}}_k][\bar{\mathbf{h}}_1, \dots, \bar{\mathbf{h}}_k]^T, \quad (1.1)$$

where $\bar{\mathbf{w}}_i$ and $\bar{\mathbf{h}}_i$ are the zero-padded extensions of \mathbf{w}_i and \mathbf{h}_i to vectors of length m and n respectively. If the matrix \mathbf{A} is nonnegative and consists of these k (row- and column-exclusive) biclusters, we may assume that $\mathbf{w}_i, \mathbf{h}_i \geq \mathbf{0}$ for all i (a consequence of Eckart-Young theorem, see, for example, Golub and Van Loan [7] for more details). Thus $\mathbf{A} \approx \mathbf{W}\mathbf{H}^T$, where $\mathbf{W}, \mathbf{H} \geq \mathbf{0}$, which is an approximate *nonnegative matrix factorization* (NMF) of the matrix \mathbf{A} . In this paper, we shall follow the NMF representation to find row- and column-exclusive biclusters. Note that there are different frameworks for biclustering problems such as the graph partitioning model used in Dhillon [4] and Ames [1].

Approximate and exact NMF problems are difficult to solve. The LAROS problem proposed by Doan and Vavasis [5] can be used as a subroutine for a greedy algorithm with which columns of \mathbf{W} and \mathbf{H} are constructed sequentially. Each pair of columns corresponds to a feature (or pattern) in the original data matrix \mathbf{A} . Given the properties of LAROS problem, the most significant feature (in size and magnitude) will be constructed first with the appropriate parameter.

The iterated use of the LAROS algorithm of [5] to extract blocks one at a time, however, will not

succeed in the case that there are two or more hidden blocks of roughly the same magnitude. In order to avoid this issue, we propose a new convex formulation that allows us to extract several (non-overlapping) features simultaneously. In Section 2, we study the proposed convex relaxation and the properties of its optimal solutions. In Section 3, we provide conditions to recover low-rank block structure of the block diagonal data matrix \mathbf{A} in the presence of random noise.

Notation. $\langle \mathbf{A}, \mathbf{X} \rangle = \text{trace}(\mathbf{A}^T \mathbf{X})$ is used to denote the inner product of two matrices \mathbf{A} and \mathbf{X} in $\mathbb{R}^{m \times n}$. $\|\mathbf{X}\|_1$ means the sum of the absolute values of all entries of \mathbf{X} , i.e., the ℓ_1 -norm of $\text{vec}(\mathbf{X})$, the long vector constructed by the concatenation of all columns of \mathbf{X} . Similarly, $\|\mathbf{X}\|_\infty$ is the maximum absolute value of entries of \mathbf{X} , i.e, the ℓ_∞ -norm of $\text{vec}(\mathbf{X})$.

2 Matrix norm minimization

We start with the following general norm minimization problem, which has been considered in [5].

$$\begin{aligned} \min \quad & \|\mathbf{X}\| \\ \text{s.t.} \quad & \langle \mathbf{A}, \mathbf{X} \rangle \geq 1, \end{aligned} \tag{2.1}$$

where $\|\cdot\|$ is an arbitrary norm function on $\mathbb{R}^{m \times n}$. The associated dual norm $\|\cdot\|^\star$ is defined as

$$\begin{aligned} \|\mathbf{A}\|^\star = \max \quad & \langle \mathbf{A}, \mathbf{Y} \rangle \\ \text{s.t.} \quad & \|\mathbf{Y}\| \leq 1. \end{aligned} \tag{2.2}$$

Main properties of optimal solutions of this problem are captured in the following theorem.

Theorem 1 (Doan and Vavasis [5]). *The following statements are true:*

- (i) *The set of optimal solutions of Problem (2.1) is $(\|\mathbf{A}\|^\star)^{-1} \partial \|\mathbf{A}\|^\star$, where $\partial \|\cdot\|^\star$ is the subgradient of the dual norm function $\|\cdot\|^\star$.*
- (ii) *Problem (2.1) has a unique optimal solution if and only if the dual norm function $\|\cdot\|^\star$ is differentiable at \mathbf{A} .*

The LAROS problem in [5] belongs to a special class of (2.1) with parametric matrix norms of the form $\|\mathbf{X}\|_\theta = \|\mathbf{X}\| + \theta \|\mathbf{X}\|_1$ where $\|\cdot\|$ is a *unitarily invariant norm* and θ is a nonnegative parameter, $\theta \geq 0$:

$$\begin{aligned} \min \quad & \|\mathbf{X}\| + \theta \|\mathbf{X}\|_1 \\ \text{s.t.} \quad & \langle \mathbf{A}, \mathbf{X} \rangle \geq 1. \end{aligned} \tag{2.3}$$

A norm $\|\cdot\|$ is unitarily invariant if $\|\mathbf{U}\mathbf{X}\mathbf{V}\| = \|\mathbf{X}\|$ for all pairs of unitary matrices \mathbf{U} and \mathbf{V} (see, for example, Lewis [11] for more details). For the LAROS problem, $\|\mathbf{X}\|$ is the nuclear norm, $\|\mathbf{X}\| = \|\mathbf{X}\|_*$, which is the sum of singular values of \mathbf{X} . In order to characterize the optimal solutions of (2.3), we need to compute the dual norm $\|\cdot\|_\theta^*$:

$$\begin{aligned} \|\mathbf{A}\|_\theta^* &= \max \langle \mathbf{A}, \mathbf{Y} \rangle \\ \text{s.t. } &\|\mathbf{Y}\| + \theta \|\mathbf{Y}\|_1 \leq 1. \end{aligned} \quad (2.4)$$

The following proposition, which is a straightforward generalization of Proposition 7 in [5], provides a dual formulation to compute $\|\cdot\|_\theta^*$.

Proposition 1. *The dual norm $\|\mathbf{A}\|_\theta^*$ with $\theta > 0$ is the optimal value of the following optimization problem:*

$$\begin{aligned} \|\mathbf{A}\|_\theta^* &= \min \max \{ \|\mathbf{Y}\|_\theta^*, \theta^{-1} \|\mathbf{Z}\|_\infty \} \\ \text{s.t. } &\mathbf{Y} + \mathbf{Z} = \mathbf{A}. \end{aligned} \quad (2.5)$$

The optimality conditions of (2.3) are described in the following proposition, which is again a generalization of Proposition 9 in [5].

Proposition 2. *Consider a feasible solution \mathbf{X} of Problem (2.3). If there exists (\mathbf{Y}, \mathbf{Z}) that satisfies the conditions below,*

- (i) $\mathbf{Y} + \mathbf{Z} = \mathbf{A}$ and $\|\mathbf{Y}\|_\theta^* = \theta^{-1} \|\mathbf{Z}\|_\infty$,
- (ii) $\mathbf{X} \in \alpha \partial \|\mathbf{Y}\|_\theta^*$, $\alpha \geq 0$,
- (iii) $\mathbf{X} \in \beta \partial \|\mathbf{Z}\|_\infty$, $\beta \geq 0$,
- (iv) $\alpha + \theta\beta = (\|\mathbf{Y}\|_\theta^*)^{-1}$,

then \mathbf{X} is an optimal solution of Problem (2.3). In addition, if

- (v) $\|\cdot\|_\theta^*$ is differentiable at \mathbf{Y} or $\|\cdot\|_\infty$ is differentiable at \mathbf{Z} ,

then \mathbf{X} is the unique optimal solution.

The low-rank structure of solutions obtained from the LAROS problem comes from the fact that the dual norm of the nuclear norm is the spectral norm (or 2-norm), $\|\mathbf{X}\| = \sigma_1(\mathbf{X})$, the largest singular value of \mathbf{X} . More exactly, it is due to the structure of the subgradient $\partial \|\cdot\|$. According to Ziętak [15],

if $\mathbf{Y} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ is a singular value decomposition of \mathbf{Y} and s is the multiplicity of the largest singular value of \mathbf{Y} , the subgradient $\partial \|\mathbf{Y}\|$ is written as follows:

$$\partial \|\mathbf{Y}\| = \left\{ \mathbf{U} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{V}^T : \mathbf{S} \in \mathcal{S}_+^s, \|\mathbf{S}\|_* = 1 \right\},$$

where \mathcal{S}_+^s is the set of positive semidefinite matrices of size s . The description of the subgradient shows that the maximum possible rank of $\mathbf{X} \in \alpha \partial \|\mathbf{Y}\|$ is the multiplicity of the largest singular value of \mathbf{Y} and if $s = 1$, we achieve rank-one solutions. This structural property of the subgradient $\partial \|\cdot\|$ motivates the norm optimization formulation for the LAROS problem, which aims to find a *single* approximately rank-one submatrix of the data matrix \mathbf{A} . We now propose a new pair of norms that would allow us to handle several approximately rank-one submatrices simultaneously instead of individual ones. Let consider the following norm, which we call Ky Fan 2- k -norm given its similar formulation to that of the classical Ky Fan k -norm:

$$\|\mathbf{A}\|_{k,2} = \left(\sum_{i=1}^k \sigma_i^2(\mathbf{A}) \right)^{\frac{1}{2}}, \quad (2.6)$$

where $\sigma_1 \geq \dots \geq \sigma_k \geq 0$ are the first k largest singular values of \mathbf{A} , $k \leq k_0 = \text{rank}(\mathbf{A})$. The dual norm of the Ky Fan 2- k -norm is denoted by $\|\cdot\|_{k,2}^*$. According to Bhatia [3], Ky Fan 2- k -norm is a Q -norm, which is unitarily invariant (Definition IV.2.9 [3]). Since Ky Fan 2- k -norm is unitarily invariant, we can define its corresponding symmetric gauge function, $\|\cdot\|_{k,2} : \mathbb{R}^n \rightarrow \mathbb{R}$, as follows:

$$\|\mathbf{x}\|_{k,2} = \left(\sum_{i=1}^k |x|_{(i)}^2 \right)^{\frac{1}{2}}, \quad (2.7)$$

where $|x|_{(i)}$ is the $(n - i + 1)$ -st order statistic of $|\mathbf{x}|$. The dual norm of this gauge function (or more exactly, its square), has been used in Argyriou et al. [2] as a regularizer in sparse prediction problems. More recently, its matrix counterpart is considered in McDonald et al. [13] as a special case of the matrix cluster norm defined in [9], whose square is used for multi-task learning regularization. On the other hand, the square Ky Fan 2- k -norm is considered as a penalty in low-rank regression analysis in Giraud [6]. In this paper, we are going to use dual Ky Fan 2- k -norm, not its square, in our formulation given its structural properties, which will be explained later.

When $k = 1$, the Ky Fan 2- k -norm becomes the spectral norm, whose subgradient has been used to characterize the low-rank structure of the optimal solutions of the LAROS problem. We now propose the following optimization problem, of which the LAROS problem is a special instance with $k = 1$:

$$\begin{aligned} \min \quad & \|\mathbf{X}\|_{k,2}^* + \theta \|\mathbf{X}\|_1 \\ \text{s.t.} \quad & \langle \mathbf{A}, \mathbf{X} \rangle \geq 1, \end{aligned} \quad (2.8)$$

where θ is a nonnegative parameter, $\theta \geq 0$. The proposed formulation is an instance of the parametric problem (2.3) and we can use results obtained in Proposition 1 and 2 to characterize its optimal solutions. Before doing so, we first provide an equivalent semidefinite optimization formulation for (2.8) in the following proposition.

Proposition 3. *Assuming $m \geq n$, the optimization problem (2.8) is then equivalent to the following semidefinite optimization problem:*

$$\begin{aligned}
\min_{p, \mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{X}} \quad & p + \text{trace}(\mathbf{R}) + \theta \langle \mathbf{E}, \mathbf{Q} \rangle \\
\text{s.t.} \quad & kp - \text{trace}(\mathbf{P}) = 0, \\
& p\mathbf{I} - \mathbf{P} \succeq 0, \\
& \begin{pmatrix} \mathbf{P} & -\frac{1}{2}\mathbf{X}^T \\ -\frac{1}{2}\mathbf{X} & \mathbf{R} \end{pmatrix} \succeq 0, \\
& \mathbf{Q} \succeq \mathbf{X}, \mathbf{Q} \succeq -\mathbf{X}, \\
& \langle \mathbf{A}, \mathbf{X} \rangle \geq 1,
\end{aligned} \tag{2.9}$$

where \mathbf{E} is the matrix of all ones.

Proof. We first consider the dual norm $\|\mathbf{X}\|_{k,2}^*$. We have:

$$\begin{aligned}
\|\mathbf{X}\|_{k,2}^* = \max \quad & \langle \mathbf{X}, \mathbf{Y} \rangle \\
\text{s.t.} \quad & \|\mathbf{Y}\|_{k,2} \leq 1.
\end{aligned} \tag{2.10}$$

Since $m \geq n$, we have: $(\|\mathbf{Y}\|_{k,2})^2 = \|\mathbf{Y}^T \mathbf{Y}\|_k$, where $\|\cdot\|_k$ is the Ky Fan k -norm, i.e., the sum of k largest singular values. Since $\mathbf{Y}^T \mathbf{Y}$ is symmetric, $\|\mathbf{Y}^T \mathbf{Y}\|_k$ is actually the sum of k largest eigenvalues of $\mathbf{Y}^T \mathbf{Y}$. Similar to $\|\mathbf{x}\|_k$, which is the sum of k largest elements of \mathbf{x} , we obtain the following (dual) optimization formulation for $\|\mathbf{Y}^T \mathbf{Y}\|_k$ (for example, see Laurent and Rendl [10]):

$$\begin{aligned}
\|\mathbf{Y}^T \mathbf{Y}\|_k = \min \quad & kz + \text{trace}(\mathbf{U}) \\
\text{s.t.} \quad & z\mathbf{I} + \mathbf{U} \succeq \mathbf{Y}^T \mathbf{Y}, \\
& \mathbf{U} \succeq 0.
\end{aligned}$$

Applying the Schur complement, we have:

$$\begin{aligned}
\|\mathbf{Y}^T \mathbf{Y}\|_k = \min \quad & kz + \text{trace}(\mathbf{U}) \\
\text{s.t.} \quad & \begin{pmatrix} z\mathbf{I} + \mathbf{U} & \mathbf{Y}^T \\ \mathbf{Y} & \mathbf{I} \end{pmatrix} \succeq 0, \\
& \mathbf{U} \succeq 0.
\end{aligned}$$

Thus, the dual norm $\|\cdot\|_{k,2}^*$ can be computed as follows:

$$\begin{aligned} \|\mathbf{X}\|_{k,2}^* &= \max \langle \mathbf{X}, \mathbf{Y} \rangle \\ \text{s.t. } &kz + \text{trace}(\mathbf{U}) \leq 1, \\ &\begin{pmatrix} z\mathbf{I} + \mathbf{U} & \mathbf{Y}^T \\ \mathbf{Y} & \mathbf{I} \end{pmatrix} \succeq 0, \\ &\mathbf{U} \succeq 0. \end{aligned}$$

Applying strong duality theory under Slater's condition, we have:

$$\begin{aligned} \|\mathbf{X}\|_{k,2}^* &= \min p + \text{trace}(\mathbf{R}) \\ \text{s.t. } &kp - \text{trace}(\mathbf{P}) = 0, \\ &p\mathbf{I} - \mathbf{P} \succeq 0, \\ &\begin{pmatrix} \mathbf{P} & -\frac{1}{2}\mathbf{X}^T \\ -\frac{1}{2}\mathbf{X} & \mathbf{R} \end{pmatrix} \succeq 0. \end{aligned} \tag{2.11}$$

The reformulation of $\|\mathbf{X}\|_1$ is straightforward with the new decision variable \mathbf{Q} and additional constraints $\mathbf{Q} \geq \mathbf{X}$ and $\mathbf{Q} \geq -\mathbf{X}$, given the fact that the main problem is a minimization problem.

□

Proposition 3 indicates that in general, we can solve (2.8) by solving its equivalent semidefinite optimization formulation (2.9) with any SDP solver. We are now ready to study some properties of optimal solutions of (2.8). We have: $\|\mathbf{X}\|_{k,2}^* + \theta \|\mathbf{X}\|_1$ is a norm for $\theta \geq 0$ and we denote it by $\|\mathbf{X}\|_{k,2,\theta}$. According to Proposition 1, the dual norm $\|\mathbf{X}\|_{k,2,\theta}^*$,

$$\begin{aligned} \|\mathbf{A}\|_{k,2,\theta}^* &= \max \langle \mathbf{A}, \mathbf{X} \rangle \\ \text{s.t. } &\|\mathbf{X}\|_{k,2,\theta} \leq 1, \end{aligned} \tag{2.12}$$

can be calculated by solving the following optimization problem given $\theta > 0$:

$$\begin{aligned} \|\mathbf{A}\|_{k,2,\theta}^* &= \min \max \{ \|\mathbf{Y}\|_{k,2}, \theta^{-1} \|\mathbf{Z}\|_\infty \} \\ \text{s.t. } &\mathbf{Y} + \mathbf{Z} = \mathbf{A}. \end{aligned} \tag{2.13}$$

Similar to Proposition 2, we can provide the optimality conditions for (2.8) in the following proposition.

Proposition 4. *Consider a feasible solution \mathbf{X} of Problem (2.8). If there exists (\mathbf{Y}, \mathbf{Z}) that satisfies the conditions below,*

$$(i) \quad \mathbf{Y} + \mathbf{Z} = \mathbf{A} \text{ and } \|\mathbf{Y}\|_{k,2} = \theta^{-1} \|\mathbf{Z}\|_\infty,$$

$$(ii) \mathbf{X} \in \alpha \partial \|\mathbf{Y}\|_{k,2}, \alpha \geq 0,$$

$$(iii) \mathbf{X} \in \beta \partial \|\mathbf{Z}\|_{\infty}, \beta \geq 0,$$

$$(iv) \alpha + \theta\beta = (\|\mathbf{Y}\|_{k,2})^{-1},$$

then \mathbf{X} is an optimal solution of Problem (2.3). In addition, if

$$(v) \|\cdot\|_{k,2} \text{ is differentiable at } \mathbf{Y} \text{ or } \|\cdot\|_{\infty} \text{ is differentiable at } \mathbf{Z},$$

then \mathbf{X} is the unique optimal solution.

The optimality conditions presented in Proposition 4 indicate that some properties of optimal solutions of (2.8) can be derived from the structure of $\partial \|\cdot\|_{k,2}$. We shall characterize the subgradient $\partial \|\cdot\|_{k,2}$ next. According to Watson [14], since $\|\cdot\|_{k,2}$ is a unitarily invariant norm, $\partial \|\mathbf{A}\|_{k,2}$ is related to $\partial \|\sigma(\mathbf{A})\|_{k,2}$, where $\sigma(\mathbf{A})$ is the vector of singular values of \mathbf{A} . Let $\mathbf{A} \neq \mathbf{0}$ be a matrix with singular values that satisfy

$$\sigma_1 \geq \dots > \sigma_{k-t+1} = \dots = \sigma_k = \dots = \sigma_{k+s} > \dots \geq \sigma_p,$$

where $p = \min\{m, n\}$, so that the multiplicity of σ_k is $s + t$. The subgradient $\partial \|\sigma\|_{k,2}$ is characterized in the following lemma.

Lemma 1. $\mathbf{v} \in \partial \|\sigma\|_{k,2}$ if and only \mathbf{v} satisfies the following conditions:

$$(i) v_i = \frac{\sigma_i}{\|\sigma\|_{k,2}} \text{ for all } i = 1, \dots, k - t.$$

$$(ii) v_i = \tau_i \frac{\sigma_k}{\|\sigma\|_{k,2}}, 0 \leq \tau_i \leq 1 \text{ for all } i = k - t + 1, \dots, k + s, \text{ and } \sum_{i=k-t+1}^{k+s} \tau_i = t.$$

$$(iii) v_i = 0 \text{ for all } i = k + s + 1, \dots, p.$$

Proof. Let \mathcal{N}_k be the collection of all subsets with k elements of $\{1, \dots, p\}$, we have:

$$\|\sigma\|_{k,2} = \max_{N \in \mathcal{N}_k} f_N(\sigma),$$

where $f_N(\sigma) = \left(\sum_{i \in N} \sigma_i^2 \right)^{\frac{1}{2}}$ for all $N \in \mathcal{N}_k$. The subgradient of $\|\cdot\|_{k,2}$ is computed as follows:

$$\partial \|\sigma\|_{k,2} = \text{conv} \left\{ \partial f_N(\sigma) : N \in \mathcal{N}_k, f_N(\sigma) = \|\sigma\|_{k,2} \right\}.$$

With the structure of $\boldsymbol{\sigma}$, clearly $\{1, \dots, k-t\} \in N$ for all $N \in \mathcal{N}_k$ such that $f_N(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}\|_{k,2}$. The remaining t elements of N are chosen from $s+t$ values from $\{k-t+1, \dots, k+s\}$. Since $\boldsymbol{\sigma} \neq \mathbf{0}$, all f_N that satisfy $f_N(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}\|_{k,2}$ is differentiable at $\boldsymbol{\sigma}$ (even in the case $\sigma_k = 0$) and

$$\frac{\partial f_N(\boldsymbol{\sigma})}{\partial \sigma_i} = \frac{\sigma_i}{\|\boldsymbol{\sigma}\|_{k,2}}, \forall i \in N, \quad \frac{\partial f_N(\boldsymbol{\sigma})}{\partial \sigma_i} = 0, \quad i \notin N.$$

Thus if $\mathbf{v} \in \partial \|\boldsymbol{\sigma}\|_{k,2}$, for all $i = 1, \dots, k-t$, we have: $v_i = \frac{\sigma_i}{\|\boldsymbol{\sigma}\|_{k,2}}$ and $v_i = 0$ for all $i = k+s+1, \dots, p$.

We now have: counting arguments for the appearance of each index in $\{k-t+1, \dots, k+s\}$ with respect to all subsets $N \in \mathcal{N}_k$ that satisfy $f_N(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}\|_{k,2}$ allow us to characterize v_i for $i = k-t+1, \dots, k+s$ as $v_i = \tau_i \frac{\sigma_k}{\|\boldsymbol{\sigma}\|_{k,2}}$, $0 \leq \tau_i \leq 1$ and $\sum_{i=k-t+1}^{k+s} \tau_i = t$. \square

We are ready to characterize the subgradient of $\|\cdot\|_{k,2}$ with the following proposition.

Proposition 5. *Consider $\mathbf{A} \neq \mathbf{0}$. Let $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ be a particular singular value decomposition of \mathbf{A} and assume that $\sigma(\mathbf{A})$ satisfies $\sigma_1 \geq \dots > \sigma_{k-t+1} = \dots = \sigma_k = \dots = \sigma_{k+s} > \dots \geq \sigma_p$.*

(i) *If $\sigma_k > 0$, then $\mathbf{G} \in \partial \|\mathbf{A}\|_{k,2}$ if and only if there exists $\mathbf{T} \in \mathbb{R}^{(s+t) \times (s+t)}$ such that*

$$\mathbf{G} = \frac{1}{\|\mathbf{A}\|_{k,2}} \left(\mathbf{U}_{[:,1:k-t]} \boldsymbol{\Sigma}_{[1:k-t,1:k-t]} \mathbf{V}_{[:,1:k-t]}^T + \sigma_k \mathbf{U}_{[:,k-t+1:k+s]} \mathbf{T} \mathbf{V}_{[:,k-t+1:k+s]}^T \right),$$

where \mathbf{T} is symmetric positive semidefinite, $\|\mathbf{T}\| \leq 1$ and $\|\mathbf{T}\|_* = t$.

(ii) *If $\sigma_k = 0$, then*

$$\mathbf{G} \in \partial \|\mathbf{A}\|_{k,2} \Leftrightarrow \mathbf{G} = \frac{1}{\|\mathbf{A}\|_{k,2}} \mathbf{U}_{[:,1:k-t]} \boldsymbol{\Sigma}_{[1:k-t,1:k-t]} \mathbf{V}_{[:,1:k-t]}^T.$$

Proof. According to Watson [14], we have:

$$\partial \|\mathbf{A}\|_{k,2} = \left\{ \mathbf{U} \text{Diag}(\mathbf{g}) \mathbf{V}^T : \mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T \text{ is any SVD of } \mathbf{A}, \mathbf{g} \in \partial \|\sigma(\mathbf{A})\|_{k,2} \right\}.$$

Let $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ be a particular singular value decomposition of \mathbf{A} and assume that a singular value $\sigma_i > 0$ has the multiplicity of r with corresponding singular vectors $\mathbf{U}_i \in \mathbb{R}^{m \times r}$ and $\mathbf{V}_i \in \mathbb{R}^{n \times r}$. Then for any singular value decomposition of \mathbf{A} , $\mathbf{A} = \bar{\mathbf{U}}\bar{\boldsymbol{\Sigma}}\bar{\mathbf{V}}^T$, there exists an orthonormal matrix $\mathbf{W} \in \mathbb{R}^{r \times r}$, $\mathbf{W}\mathbf{W}^T = \mathbf{I}$, such that $\bar{\mathbf{U}}_i = \mathbf{U}_i\mathbf{W}$ and $\bar{\mathbf{V}}_i = \mathbf{V}_i\mathbf{W}$ (for example, see Ziętak [15]).

Combining these results with Lemma 1, the proof is straightforward with a singular value (or eigenvalue) decomposition of matrix \mathbf{T} . \square

Corollary 1. $\|\cdot\|_{k,2}$ is differentiable at any $\mathbf{A} \neq \mathbf{0}$ such that $\sigma_k > \sigma_{k+1}$ ($\sigma_{p+1} = 0$) or $\sigma_k = 0$.

Proof. The case $\sigma_k = 0$ has been shown in Proposition 5. If $\sigma_k > \sigma_{k+1}$, we have: $s = 0$, thus $\mathbf{T} = \mathbf{I}$ is unique since $\mathbf{T} \in \mathcal{S}^t$, $\|\mathbf{T}\|_* = t$, and $\|\mathbf{T}\| \leq 1$. Thus $\partial\|\mathbf{A}\|_{k,2}$ is a singleton, which implies $\|\cdot\|_{k,2}$ is differentiable at \mathbf{A} . \square

Proposition 5 shows that the problem (2.8) with $\theta = 0$ is a convex optimization problem that finds k -approximation of a matrix \mathbf{A} . It also shows that intuitively, the problem (2.8) can be used to recover k largest approximately rank-one submatrices with $\theta > 0$. In the next section, we shall study the recovery of these submatrices under the presence of random noise.

3 Recovery with Block Diagonal Matrices and Random Noise

We consider $\mathbf{A} = \mathbf{B} + \mathbf{R}$, where \mathbf{B} is a block diagonal matrix, each block having rank one, while \mathbf{R} is a noise matrix. The main theorem shows that under certain assumptions concerning the noise, the positions of the blocks can be recovered from the solution of (2.8). As mentioned in the introduction, this corresponds to solving a special case of the approximate NMF problem, that is, a factorization $\mathbf{A} \approx \mathbf{W}\mathbf{H}^T$, where \mathbf{W} and \mathbf{H} are nonnegative matrices. The special case solved is that \mathbf{W} and \mathbf{H} each consist of nonnegative columns with nonzeros in disjoint positions (so that \mathbf{A} is approximately a matrix with disjoint blocks each of rank one). Even this special case of NMF is NP-hard unless further restrictions are placed on the data model given the fact that the (exact) LAROS problem is NP-hard (see [5] for details).

Before starting the proof of the theorem, we need to consider some properties of subgaussian random variables. A random variable x is *b-subgaussian* if $\mathbb{E}[x] = 0$ and there exists a $b > 0$ such that for all $t \in \mathbb{R}$,

$$\mathbb{E}[e^{tx}] \leq e^{\frac{b^2 t^2}{2}}. \quad (3.1)$$

We can apply the Markov inequality for the b -subgaussian random variable x and obtain the following inequalities:

$$\mathbb{P}(x \geq t) \leq \exp(-t^2/(2b^2)) \text{ and } \mathbb{P}(x \leq -t) \leq \exp(-t^2/(2b^2)), \quad \forall t > 0. \quad (3.2)$$

The next three lemmas, which show several properties of random matrices and vectors with independent subgaussian entries, are adopted from Doan and Vavasis [5] and references therein.

Lemma 2. *Let x_1, \dots, x_k be independent b -subgaussian random variables and let a_1, \dots, a_k be scalars that satisfy $\sum_{i=1}^k a_i^2 = 1$. Then $x = \sum_{i=1}^k a_i x_i$ is a b -subgaussian random variable.*

Lemma 3. Let $\mathbf{B} \in \mathbb{R}^{m \times n}$ be a random matrix, where b_{ij} are independent b -subgaussian random variables for all $i = 1, \dots, m$, and $j = 1, \dots, n$. Then for any $u > 0$,

$$\mathbb{P}(\|\mathbf{B}\| \geq u) \leq \exp\left(-\left(\frac{8u^2}{81b^2} - (\log 7)(m+n)\right)\right).$$

Lemma 4. Let \mathbf{x}, \mathbf{y} be two vectors in \mathbb{R}^n with i.i.d. b -subgaussian entries. Then for any $t > 0$,

$$\mathbb{P}(\mathbf{x}^T \mathbf{y} \geq t) \leq \exp\left(-\min\left\{\frac{t^2}{(4eb^2)^2 n}, \frac{t}{4eb^2}\right\}\right), \text{ and } \mathbb{P}(\mathbf{x}^T \mathbf{y} \leq -t) \leq \exp\left(-\min\left\{\frac{t^2}{(4eb^2)^2 n}, \frac{t}{4eb^2}\right\}\right).$$

With these properties of subgaussian variables presented, we are now able to state and prove the main theorem, which gives sufficient conditions for optimization problem (2.8) to recover k blocks in the presence of noise.

Theorem 2. Suppose $\mathbf{A} = \mathbf{B} + \mathbf{R}$, where \mathbf{B} is a block diagonal matrix with k_0 blocks, that is, $\mathbf{B} = \text{diag}(\mathbf{B}_1, \dots, \mathbf{B}_{k_0})$, where $\mathbf{B}_i = \bar{\sigma}_i \bar{\mathbf{u}}_i \bar{\mathbf{v}}_i^T$, $\bar{\mathbf{u}}_i \in \mathbb{R}^{m_i}$, $\bar{\mathbf{v}}_i \in \mathbb{R}^{n_i}$, $\|\bar{\mathbf{u}}_i\|_2 = \|\bar{\mathbf{v}}_i\|_2 = 1$, $\bar{\mathbf{u}}_i > \mathbf{0}$, $\bar{\mathbf{v}}_i > \mathbf{0}$ for all $i = 1, \dots, k_0$. Assume the blocks are ordered so that $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \dots \geq \bar{\sigma}_{k_0} > 0$. Matrix \mathbf{R} is a random matrix composed of blocks in which each entry is a translated b -subgaussian variable, i.e., there exists $\mu_{ij} \geq 0$ such that elements of the matrix block $\mathbf{R}_{ij}/(\phi_i \phi_j)^{1/2} - \mu_{ij} \mathbf{e}_{m_i} \mathbf{e}_{n_j}^T$ are independent b -subgaussian random variables for all $i, j = 1, \dots, k_0$. Here $\phi_i = \bar{\sigma}_i / \sqrt{m_i n_i}$, $i = 1, \dots, k_0$, is a scaling factor to match the scale of \mathbf{R}_{ij} with that of \mathbf{B}_i and \mathbf{B}_j , and \mathbf{e}_m denotes the m -vector of all 1's.

We define the following positive scalars that control the degree of heterogeneity among the first k blocks:

$$\delta_u \leq \min_{i=1, \dots, k} \|\bar{\mathbf{u}}_i\|_1 / \sqrt{m_i}, \quad (3.3)$$

$$\delta_v \leq \min_{i=1, \dots, k} \|\bar{\mathbf{v}}_i\|_1 / \sqrt{n_i}, \quad (3.4)$$

$$\xi_u \leq \min_{i=1, \dots, k} \left(\min_{j=1, \dots, m_i} \bar{u}_{i,j} \right) \sqrt{m_i}, \quad (3.5)$$

$$\xi_v \leq \min_{i=1, \dots, k} \left(\min_{j=1, \dots, n_i} \bar{v}_{i,j} \right) \sqrt{n_i}, \quad (3.6)$$

$$\pi_u \geq \max_{i=1, \dots, k} \left(\max_{j=1, \dots, m_i} \bar{u}_{i,j} \right) \sqrt{m_i}, \quad (3.7)$$

$$\pi_v \geq \max_{i=1, \dots, k} \left(\max_{j=1, \dots, n_i} \bar{v}_{i,j} \right) \sqrt{n_i}, \quad (3.8)$$

$$\rho_m \geq \max_{i,j=1, \dots, k} m_i / m_j, \quad (3.9)$$

$$\rho_n \geq \max_{i,j=1, \dots, k} n_i / n_j, \quad (3.10)$$

$$\rho_\sigma \geq \bar{\sigma}_1 / \bar{\sigma}_k. \quad (3.11)$$

We also assume that the blocks do not diverge much from being square; more precisely we assume that $m_i \leq O(n_j^2)$ and $n_i \leq O(m_j^2)$ for $i, j = 1, \dots, k$. Let $\mathbf{p} = (k, \delta_u, \delta_v, \pi_u, \pi_v, \xi_u, \xi_v, \rho_\sigma, \rho_m, \rho_n)$ denote the vector of parameters controlling the heterogeneity.

For the remaining noise blocks $i = k + 1, \dots, k_0$, we assume that their dominant singular values are substantially smaller than those of the first k :

$$\bar{\sigma}_{k+1} \leq \frac{0.23\bar{\sigma}_k}{k+1}, \quad (3.12)$$

that their scale is bounded:

$$\phi_i \leq c_0\phi_j, \quad (3.13)$$

for all $i = k + 1, \dots, k_0$ and $j = 1, \dots, k$. where c_0 is a constant, and that their size is bounded:

$$\sum_{i=k+1}^{k_0} (m_i + n_i) \leq c_1(\mathbf{p}, c_0, b) \min_{i=1, \dots, k} m_i n_i, \quad (3.14)$$

where $c_1(\mathbf{p}, c_0, b)$ is given by (3.119) below. Assume that

$$\mu_{ij} \leq c_2(\mathbf{p}, c_0), \quad (3.15)$$

for all $i, j = 1, \dots, k_0$, where $c_2(\mathbf{p}, c_0)$ is given by (3.114) below. Then provided that

$$c_3(\mathbf{p}) \left(\sum_{i=1}^k m_i n_i \right)^{-1/2} \leq \theta \leq 2c_3(\mathbf{p}) \left(\sum_{i=1}^k m_i n_i \right)^{-1/2}, \quad (3.16)$$

where $c_3(\mathbf{p})$ is given by (3.111) below, the optimization problem (2.8) will return \mathbf{X} with nonzero entries precisely in the positions of $\mathbf{B}_1, \dots, \mathbf{B}_k$ with probability exponentially close to 1 as $m_i, n_i \rightarrow \infty$ for all $i = 1, \dots, k_0$.

Remarks.

1. Note that the theorem does not recover the exact values of $(\bar{\sigma}_i, \bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i)$; it is clear that this is impossible in general under the assumptions made.
2. The theorem is valid under arbitrary permutation of the rows and columns (i.e., the block structure may be ‘concealed’) since (2.8) is invariant under such transformations.
3. Given the fact that for all $i = 1, \dots, k$,

$$0 < \left(\min_{j=1, \dots, m_i} \bar{u}_{i,j} \right) \sqrt{m_i} \leq \|\bar{\mathbf{u}}_i\|_1 / \sqrt{m_i} \leq 1 \leq \left(\max_{j=1, \dots, m_i} \bar{u}_{i,j} \right) \sqrt{m_i},$$

we can always choose ξ_u , δ_u , and π_u such that $0 < \xi_u \leq \delta_u \leq 1 \leq \pi_u$. Similarly, we assume $0 < \xi_v \leq \delta_v \leq 1 \leq \pi_v$. These parameters measure how much $\bar{\mathbf{u}}_i$ and $\bar{\mathbf{v}}_i$ diverge from \mathbf{e}_{m_i} and \mathbf{e}_{n_i} after normalization respectively. The best case for our theory (i.e., the least restrictive values of parameters) occurs when all of these scalars are equal to 1. Similarly $\rho_\sigma, \rho_m, \rho_n \geq 1$, and the best case for the theory is when they are all equal to 1.

4. It is an implicit assumption of the theorem that the scalars contained in \mathbf{p} as well as b , which controls the subgaussian random variables, stay fixed as $m_i, n_i \rightarrow \infty$.
5. Note that (3.14) means that the total size of the noise blocks can be much larger (approximately the square) than the size of the data blocks. Thus, the theorem shows that the k blocks can be found even though they are hidden in a much larger matrix.

In order to simplify the proof, we first consolidate all blocks $i = k+1, \dots, k_0$ into a single block and call it block $(k+1)$ of size $\bar{m}_{k+1} \times \bar{n}_{k+1}$ where $\bar{m}_{k+1} = \sum_{i=k+1}^{k_0} m_i$ and $\bar{n}_{k+1} = \sum_{i=k+1}^{k_0} n_i$. The only difference is that the new block $\bar{\mathbf{B}}_{k+1, k+1} \in \mathbb{R}^{\bar{m}_{k+1} \times \bar{n}_{k+1}}$ is now a block diagonal matrix with $k_0 - k$ blocks instead of a rank-one block. Similarly, new blocks $\bar{\mathbf{R}}_{i, k+1}$ and $\bar{\mathbf{R}}_{k+1, i}$, $i = 1, \dots, k_0$, now have more than one subblock with different parameters μ instead of a single one. This new block structure helps us derive the optimality conditions more concisely. Clearly, we would like to achieve the optimal solution \mathbf{X} with the following structure

$$\mathbf{X} = \begin{pmatrix} \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & & \mathbf{0} \\ \vdots & \ddots & \ddots & \mathbf{0} & \vdots \\ \vdots & & \mathbf{0} & \sigma_k \mathbf{u}_k \mathbf{v}_k^T & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\|\mathbf{u}_i\|_2 = \|\mathbf{v}_i\|_2 = 1$ for $i = 1, \dots, k$. Padding appropriate zeros to \mathbf{u}_i and \mathbf{v}_i to construct $\mathbf{u}_i^0 \in \mathbb{R}_+^m$ and $\mathbf{v}_i^0 \in \mathbb{R}_+^n$ for $i = 1, \dots, k$, we obtain the optimality conditions based on Proposition 4 as follows:

There exist \mathbf{Y} and \mathbf{Z} such that $\mathbf{Y} + \mathbf{Z} = \mathbf{A}$ and

$$\mathbf{Y} = \|\mathbf{A}\|_{k,2,\theta}^* \left[\sum_{i=1}^k \sigma_i \mathbf{u}_i^0 (\mathbf{v}_i^0)^T + \mathbf{W} \right], \quad \mathbf{Z} = \theta \|\mathbf{A}\|_{k,2,\theta}^* \mathbf{V},$$

where $\sigma_i > 0$ for $i = 1, \dots, k$, $\sum_{i=1}^k \sigma_i^2 = 1$, $\|\mathbf{W}\| \leq \min_{i=1, \dots, k} \{\sigma_i\}$, $\mathbf{W} \mathbf{v}_i^0 = \mathbf{0}$, $\mathbf{W}^T \mathbf{u}_i^0 = \mathbf{0}$, for $i = 1, \dots, k$, and $\|\mathbf{V}\|_\infty \leq 1$, $\mathbf{V}_{ii} = \mathbf{e}_{m_i} \mathbf{e}_{n_i}^T$, for $i = 1, \dots, k$.

Since \mathbf{A} has the block structure, we can break these optimality conditions into appropriate conditions for each block. Starting with diagonal (i, i) blocks, $i = 1, \dots, k$, the detailed conditions are:

$$\sigma_i \mathbf{u}_i \mathbf{v}_i^T + \mathbf{W}_{ii} = \lambda (\bar{\sigma}_i \bar{\mathbf{u}}_i \bar{\mathbf{v}}_i^T + \mathbf{R}_{ii}) - \theta \mathbf{e}_{m_i} \mathbf{e}_{n_i}^T, \quad (3.17)$$

$$\mathbf{W}_{ii}^T \mathbf{u}_i = \mathbf{0}, \quad (3.18)$$

$$\mathbf{W}_{ii} \mathbf{v}_i = \mathbf{0}, \quad (3.19)$$

where $\lambda = 1/\|\mathbf{A}\|_{k,2,\theta}^*$. For non-diagonal (i, j) blocks, $i \neq j$ and $i, j = 1, \dots, k$, we obtain the following conditions:

$$\mathbf{W}_{ij} + \theta \mathbf{V}_{ij} = \lambda \mathbf{R}_{ij}, \quad (3.20)$$

$$\mathbf{W}_{ij}^T \mathbf{u}_i = \mathbf{0}, \quad (3.21)$$

$$\mathbf{W}_{ij} \mathbf{v}_j = \mathbf{0}, \quad (3.22)$$

$$\|\mathbf{V}_{ij}\|_\infty \leq 1. \quad (3.23)$$

For $(i, k+1)$ blocks, $i = 1, \dots, k$, we have:

$$\mathbf{W}_{i,k+1} + \theta \mathbf{V}_{i,k+1} = \lambda \bar{\mathbf{R}}_{i,k+1}, \quad (3.24)$$

$$\mathbf{W}_{i,k+1}^T \mathbf{u}_i = \mathbf{0}, \quad (3.25)$$

$$\|\mathbf{V}_{i,k+1}\|_\infty \leq 1. \quad (3.26)$$

Similarly, for $(k+1, j)$ blocks, $j = 1, \dots, k$, the conditions are:

$$\mathbf{W}_{k+1,j} + \theta \mathbf{V}_{k+1,j} = \lambda \bar{\mathbf{R}}_{k+1,j}, \quad (3.27)$$

$$\mathbf{W}_{k+1,j} \mathbf{v}_j = \mathbf{0}, \quad (3.28)$$

$$\|\mathbf{V}_{k+1,j}\|_\infty \leq 1. \quad (3.29)$$

Finally, the $(k+1, k+1)$ block needs the following conditions:

$$\mathbf{W}_{k+1,k+1} + \theta \mathbf{V}_{k+1,k+1} = \lambda (\bar{\mathbf{B}}_{k+1,k+1} + \bar{\mathbf{R}}_{k+1,k+1}), \quad (3.30)$$

$$\|\mathbf{V}_{k+1,k+1}\|_\infty \leq 1. \quad (3.31)$$

The remaining conditions are not block separable. We still need $\sigma_i > 0$, $i = 1, \dots, k$, and $\sum_{i=1}^k \sigma_i^2 = 1$.

The last condition, which is $\|\mathbf{W}\| \leq \min_{i=1,\dots,k} \{\sigma_i\}$, can be replaced by the following sufficient conditions that are block separable:

$$\|\mathbf{W}_{ij}\| \leq \frac{1}{k+1} \min_{i=1,\dots,k} \{\sigma_i\}, \quad i, j = 1, \dots, k+1. \quad (3.32)$$

With these sufficient block separable conditions, in order to construct (\mathbf{V}, \mathbf{W}) , we now need to construct $(\mathbf{V}_{ij}, \mathbf{W}_{ij})$ for different pairs (i, j) block by block. The block by block details are shown in the following analysis.

3.1 Analysis for block (i, i) , $i = 1, \dots, k$

We begin with the proof of the existence of a $\lambda > 0$ that satisfies the optimality conditions. We then show the sufficient condition (3.32) for block (i, i) , $i = 1, \dots, k$. The final condition that needs to be proved for these blocks is the positivity of \mathbf{u}_i and \mathbf{v}_i , $i = 1, \dots, k$.

3.1.1 Existence of λ^*

The conditions for (i, i) block, $i = 1, \dots, k$, namely, (3.17)–(3.19), indicate that $(\sigma_i, \mathbf{u}_i, \mathbf{v}_i)$ is the dominant singular triple of $\mathbf{L}_i = \lambda(\bar{\sigma}_i \bar{\mathbf{u}}_i \bar{\mathbf{v}}_i^T + \mathbf{R}_{ii}) - \theta \mathbf{e}_{m_i} \mathbf{e}_{n_i}^T$. They also indicate that

$$\|\mathbf{W}_{ii}\| = \sigma_2(\mathbf{L}_i) \quad (3.33)$$

since (3.17)–(3.19) are equivalent to the first step of a singular value decomposition of \mathbf{L}_i .

For the rest of this analysis, it is more convenient notationally work with $\tau = \lambda/\theta$ rather than with λ directly. The condition $\sum_{i=1}^k \sigma_i^2 = 1$ becomes

$$f(\tau) = \sum_{i=1}^k \left\| \tau(\bar{\sigma}_i \bar{\mathbf{u}}_i \bar{\mathbf{v}}_i^T + \mathbf{R}_{ii}) - \mathbf{e}_{m_i} \mathbf{e}_{n_i}^T \right\|^2 - \theta^{-2} = 0. \quad (3.34)$$

We will prove that there exists $\tau^* > 0$ such that $f(\tau^*) = 0$. More precisely, we will focus our analysis of $f(\tau)$ for $\tau \in [\tau_\ell, \tau_u]$, where τ_ℓ is given by (3.106) and τ_u is given by (3.112) below and prove that there exists $\tau^* \in [\tau_\ell, \tau_u]$ such that $f(\tau^*) = 0$.

Letting $\mathbf{Q}_{ij} = \mathbf{R}_{ij}/\sqrt{\phi_i \phi_j} - \mu_{ij} \mathbf{e}_{m_i} \mathbf{e}_{n_j}^T$ for $i, j = 1, \dots, k$, we have: \mathbf{Q}_{ij} are b -subgaussian random matrices with independent elements. The function f can be rewritten as follows:

$$\begin{aligned} f(\tau) &= \sum_{i=1}^k \left\| \tau \bar{\sigma}_i \bar{\mathbf{u}}_i \bar{\mathbf{v}}_i^T - (1 - \tau \phi_i \mu_{ii}) \mathbf{e}_{m_i} \mathbf{e}_{n_i}^T + \tau \phi_i \mathbf{Q}_{ii} \right\|^2 - \theta^{-2} \\ &= \sum_{i=1}^k \left\| \mathbf{P}_i(\tau) + \tau \phi_i \mathbf{Q}_{ii} \right\|^2 - \theta^{-2}, \end{aligned} \quad (3.35)$$

where $\mathbf{P}_i(\tau) = \tau \bar{\sigma}_i \bar{\mathbf{u}}_i \bar{\mathbf{v}}_i^T - (1 - \tau \phi_i \mu_{ii}) \mathbf{e}_{m_i} \mathbf{e}_{n_i}^T$. Applying triangle inequality, we have:

$$\|\mathbf{P}_i(\tau)\| - \tau \phi_i \|\mathbf{Q}_{ii}\| \leq \|\mathbf{P}_i(\tau) + \tau \phi_i \mathbf{Q}_{ii}\| \leq \|\mathbf{P}_i(\tau)\| + \tau \phi_i \|\mathbf{Q}_{ii}\|. \quad (3.36)$$

We start the analysis with $\|\mathbf{P}_i(\tau)\|$. We first define the following function

$$g_i(\tau; a) = \phi_i^2 \tau^2 - 2a\phi_i\tau(1 - \mu_{ii}\phi_i\tau) + (1 - \mu_{ii}\phi_i\tau)^2, \quad (3.37)$$

which is a quadratic function in τ with any fixed parameter a . Note by (3.115) below that $\tau_u \leq 0.3/(\phi_i\mu_{ii})$ for all $i = 1, \dots, k$, so $1 - \mu_{ii}\phi_i\tau \geq 0$ and $\tau \geq 0$ for $\tau \in [\tau_\ell, \tau_u]$. Therefore, provided $a \leq 1$ and $\tau \in [\tau_\ell, \tau_u]$,

$$\begin{aligned} g_i(\tau; a) &= (\phi_i\tau - (1 - \mu_{ii}\phi_i\tau))^2 + 2(1 - a)\phi_i\tau(1 - \mu_{ii}\phi_i\tau) \\ &\geq (\phi_i\tau - (1 - \mu_{ii}\phi_i\tau))^2 \\ &= (\phi_i(1 + \mu_{ii})\tau - 1)^2 \end{aligned} \quad (3.38)$$

$$\geq 0. \quad (3.39)$$

We now analyze the dominant singular triple of $\mathbf{P}_i(\tau) = \tau\bar{\sigma}_i\bar{\mathbf{u}}_i\bar{\mathbf{v}}_i^T - (1 - \tau\phi_i\mu_{ii})\mathbf{e}_{m_i}\mathbf{e}_{n_i}^T$ for a fixed $\tau \in [\tau_\ell, \tau_u]$. It is clear that dominant right singular vector lies in $\text{span}\{\bar{\mathbf{v}}_i, \mathbf{e}_{n_i}\}$ since this is the range of $(\mathbf{P}_i(\tau))^T$. Letting $\zeta_i = \|\mathbf{P}_i(\tau)\|^2$ be the square of the dominant singular value, we have: ζ_i is a solution of the following eigenvector problem:

$$(\mathbf{P}_i(\tau))^T \mathbf{P}_i(\tau)(\alpha\bar{\mathbf{v}}_i + \beta\mathbf{e}_{n_i}) = \zeta_i(\alpha\bar{\mathbf{v}}_i + \beta\mathbf{e}_{n_i}).$$

Expanding and gathering multiples of $\bar{\mathbf{v}}_i$ and \mathbf{e}_{n_i} , we obtain the following 2×2 eigenvalue problem

$$\mathbf{M}_i \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \zeta_i \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad (3.40)$$

where

$$\mathbf{M}_i = \begin{pmatrix} \tau^2\bar{\sigma}_i^2 - \tau\bar{\sigma}_i h_i(\tau) \|\bar{\mathbf{u}}_i\|_1 \|\bar{\mathbf{v}}_i\|_1 & \tau^2\bar{\sigma}_i^2 - \tau\bar{\sigma}_i h_i(\tau) \|\bar{\mathbf{u}}_i\|_1 n_i \\ (h_i(\tau))^2 \|\bar{\mathbf{v}}_i\|_1 m_i - \tau\bar{\sigma}_i h_i(\tau) \|\bar{\mathbf{u}}_i\|_1 & (h_i(\tau))^2 m_i n_i - \tau\bar{\sigma}_i h_i(\tau) \|\bar{\mathbf{u}}_i\|_1 \|\bar{\mathbf{v}}_i\|_1 \end{pmatrix}, \quad (3.41)$$

and $h_i(\tau) = 1 - \tau\phi_i\mu_{ii}$, $i = 1, \dots, k$. Thus, ζ_i is a root of the equation

$$\zeta_i^2 - \text{trace}(\mathbf{M}_i)\zeta_i + \det(\mathbf{M}_i) = 0, \quad (3.42)$$

where

$$\begin{aligned} \text{trace}(\mathbf{M}_i) &= \tau^2\bar{\sigma}_i^2 - 2\tau\bar{\sigma}_i(1 - \tau\phi_i\mu_{ii}) \|\bar{\mathbf{u}}_i\|_1 \|\bar{\mathbf{v}}_i\|_1 + (1 - \tau\phi_i\mu_{ii})^2 m_i n_i \\ &= m_i n_i [\tau^2\phi_i^2 - 2\tau\phi_i(1 - \tau\phi_i\mu_{ii})\delta_{u,i}\delta_{v,i} + (1 - \tau\phi_i\mu_{ii})^2] \\ &= m_i n_i g_i(\tau; \delta_{u,i}\delta_{v,i}). \end{aligned} \quad (3.43)$$

and

$$\begin{aligned}
\det(\mathbf{M}_i) &= \tau^2 \bar{\sigma}_i^2 (1 - \tau \phi_i \mu_{ii})^2 (m_i - \|\bar{\mathbf{u}}_i\|_1^2) (n_i - \|\bar{\mathbf{v}}_i\|_1^2) \\
&= m_i^2 n_i^2 \tau^2 \phi_i^2 (1 - \tau \phi_i \mu_{ii})^2 (1 - \delta_{u,i}^2) (1 - \delta_{v,i}^2) \\
&\geq 0.
\end{aligned} \tag{3.44}$$

Here, we have introduced notation

$$\begin{aligned}
\delta_{u,i} &= \|\mathbf{u}_i\|_1 / \sqrt{m_i}, \\
\delta_{v,i} &= \|\mathbf{v}_i\|_1 / \sqrt{n_i},
\end{aligned}$$

that we will continue to use for the remainder of the proof. It is apparent that $\delta_{u,i} \in [\delta_u, 1]$ by (3.3) and similarly $\delta_{v,i} \in [\delta_v, 1]$.

Let Δ be the discriminant of the quadratic equation (3.42), that is,

$$\Delta = \text{trace}(\mathbf{M}_i)^2 - 4 \det(\mathbf{M}_i). \tag{3.45}$$

We have:

$$\begin{aligned}
\Delta &= m_i n_i \left[\tau^2 \phi_i^2 - 2\tau \phi_i (1 - \tau \phi_i \mu_{ii}) \left(\delta_{u,i} \delta_{v,i} + \sqrt{(1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)} \right) + (1 - \tau \phi_i \mu_{ii})^2 \right] \\
&\quad \cdot m_i n_i \left[\tau^2 \phi_i^2 - 2\tau \phi_i (1 - \tau \phi_i \mu_{ii}) \left(\delta_{u,i} \delta_{v,i} - \sqrt{(1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)} \right) + (1 - \tau \phi_i \mu_{ii})^2 \right] \\
&= (m_i n_i)^2 g_i \left(\tau; \delta_{u,i} \delta_{v,i} + \sqrt{(1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)} \right) \cdot g_i \left(\tau; \delta_{u,i} \delta_{v,i} - \sqrt{(1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)} \right). \tag{3.46}
\end{aligned}$$

Note that $1 - \left(\delta_{u,i} \delta_{v,i} + \sqrt{(1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)} \right)^2 = \left(\delta_{u,i} \sqrt{1 - \delta_{v,i}^2} - \delta_{v,i} \sqrt{1 - \delta_{u,i}^2} \right)^2 \geq 0$. Therefore, the second argument to each invocation of g_i in the previous equation is less than or equal to 1. Since $\tau \in [\tau_\ell, \tau_u]$, it follows that both evaluations of g_i yield nonnegative numbers, and therefore $\Delta \geq 0$.

We next claim that

$$\Delta = m_i^2 n_i^2 g_i(\tau; p_i(\tau))^2 \tag{3.47}$$

for a continuous $p_i(\tau) \in [\delta_{u,i} \delta_{v,i}, 1]$ for all $\tau \in [\tau_\ell, \tau_u]$. In other words, there exists a continuous $p_i(\tau)$ in the range $[a, 1]$ satisfying the equation

$$g_i(\tau; p_i(\tau))^2 = g_i(\tau; a + c) g_i(\tau; a - c), \tag{3.48}$$

where, for this paragraph, $a = \delta_{u,i} \delta_{v,i}$ and $c = \sqrt{(1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)}$. This is proved by first treating p_i as an unknown and expanding (3.48). After simplification, the result is a quadratic equation for p_i .

The facts that $0 \leq a, c \leq 1$ and $a + c \leq 1$ allow one to argue that the quadratic equation has a sign change over the interval $[a, 1]$ for all $\tau \in [0, 1/(\phi_i \mu_{ii})]$ (hence for all $\tau \in [\tau_\ell, \tau_u]$). Thus, the quadratic has a unique root in this interval, which may be taken to be p_i ; it must vary continuously with the coefficients of the quadratic and hence with τ . The details are left to the reader. In addition to τ , $p_i(\tau)$ depends on μ_{ii} , ϕ_i , $\delta_{u,i}$ and $\delta_{v,i}$.

Thus, by the quadratic formula applied to (3.42), we can obtain ζ_i as the larger root

$$\zeta_i = \frac{1}{2}(\text{trace}(\mathbf{M}_i(\tau)) + \sqrt{\Delta}) = m_i n_i g_i(\tau; a_i(\tau)), \quad (3.49)$$

where the second equation comes from adding (3.43) to the square root of (3.47) and noting that for any τ, a, b , $(g_i(\tau; a) + g_i(\tau; b))/2 = g_i(\tau; (a+b)/2)$. Here, $a_i(\tau) = \frac{1}{2}(\delta_{u,i} \delta_{v,i} + p_i(\tau))$. By the earlier bound on $p_i(\tau)$, this implies $a_i(\tau) \in [\underline{a}_i, \bar{a}_i]$, where

$$\underline{a}_i = \delta_{u,i} \delta_{v,i}; \quad \bar{a}_i = \frac{1}{2} + \frac{1}{2} \delta_{u,i} \delta_{v,i}. \quad (3.50)$$

Clearly $0 \leq \underline{a}_i \leq \bar{a}_i \leq 1$ for all i since $\delta_{u,i}, \delta_{v,i} \in [0, 1]$. Note that tighter bounds are possible by a more careful analysis of Δ .

Since $\zeta_i = \|\mathbf{P}_i(\tau)\|^2$, we can then express $\|\mathbf{P}_i(\tau)\|$ as follows:

$$\|\mathbf{P}_i(\tau)\| = \sqrt{\zeta_i} = \sqrt{m_i n_i g_i(\tau; a_i(\tau))}. \quad (3.51)$$

Next, consider again (3.38); the right-hand side is a convex quadratic function of τ with minimizer at $1/(\phi_i(1 + \mu_{ii}))$. It follows from (3.108) that $\tau_\ell \geq 2/\phi_i$ for all i . Thus, for $\tau \in [\tau_\ell, \tau_u]$, we have:

$$\tau \geq \frac{2}{\phi_i} \geq \frac{1}{\phi_i(1/2 + \mu_{ii})} > \frac{1}{\phi_i(1 + \mu_{ii})}.$$

Thus, the right-hand side of (3.38) is an increasing function of τ for $\tau \in [\tau_\ell, \tau_u]$. We then have, for any $\tau \in [\tau_\ell, \tau_u]$

$$\begin{aligned} g_i(\tau; a) &\geq \left(\phi_i(1 + \mu_{ii}) \left(\frac{1}{\phi_i(1/2 + \mu_{ii})} \right) - 1 \right)^2 \\ &= \left(\frac{1}{1 + 2\mu_{ii}} \right)^2. \end{aligned}$$

Thus,

$$\|\mathbf{P}_i(\tau)\| \geq \frac{\sqrt{m_i n_i}}{1 + 2\mu_{ii}} \quad (3.52)$$

for any $\tau \in [\tau_\ell, \tau_u]$. We also have a second lower bound that grows linearly with τ :

$$\sqrt{g_i(\tau; a)} \geq \phi_i(1 + \mu_{ii})\tau - 1 \quad (3.53)$$

$$\begin{aligned} &= \phi_i\tau/2 + (\phi_i(1/2 + \mu_{ii})\tau - 1) \\ &\geq \phi_i\tau/2, \end{aligned} \quad (3.54)$$

where the first inequality follows from (3.38) and the other inequality is due to the fact that $\tau\phi_i \geq 2$ as noted above. This implies

$$\begin{aligned} \|\mathbf{P}_i(\tau)\| &\geq \sqrt{m_i n_i} \phi_i \tau / 2 \\ &= \bar{\sigma}_i \tau / 2. \end{aligned} \quad (3.55)$$

Next, we combine this linear lower bound on $\|\mathbf{P}_i(\tau)\|$ with an upper bound on $\|\mathbf{Q}_{ii}\|$ in order to be able to take advantage of (3.36).

Claim 1. $\|\mathbf{Q}_{ij}\| \leq (m_i n_j)^{\frac{3}{8}}$ with probability exponentially close to 1 as $m_i, n_j \rightarrow \infty$ for all $i, j = 1, \dots, k$.

To establish the claim observe that \mathbf{Q}_{ij} is random with i.i.d. elements that are b -subgaussian. Thus by Lemma 3(i),

$$\mathbb{P}\left(\|\mathbf{Q}_{ij}\| \geq (m_i n_j)^{3/8}\right) \leq \exp\left(-\left(\frac{(m_i n_j)^{3/4}}{81b^2} - (\log 7)(m_i + n_j)\right)\right), \quad (3.56)$$

where u is set to be $(m_i n_j)^{3/8}$. The right-hand side tends to zero exponentially fast since $(m_i n_j)^{3/4}$ asymptotically dominates $m_i + n_j$ under the assumption that $m_i^{1/4} \leq O(n_j^{1/2})$ and $n_j^{1/4} \leq O(m_i^{1/2})$, which was stated as a hypothesis in the theorem.

Then with a probability exponentially close to 1, the event in (3.56) does not happen, hence we assume $\|\mathbf{Q}_{ij}\| \leq (m_i n_j)^{\frac{3}{8}}$. Focusing on the $i = j$ case for now, this implies

$$\|\mathbf{Q}_{ii}\| \leq \sqrt{m_i n_i} / 40, \quad (3.57)$$

for large $m_i n_i$; since the theorem applies to the asymptotic range, we assume this inequality holds true as well. Combining the inequality (3.57) with (3.55) and (3.36), we obtain

$$\begin{aligned} \|\mathbf{P}_i(\tau) + \tau\phi_i\mathbf{Q}_{ii}\| &= (1 + \gamma_i(\tau))\|\mathbf{P}_i(\tau)\| \\ &= (1 + \gamma_i(\tau))\sqrt{m_i n_i g_i(\tau; a_i(\tau))}. \end{aligned} \quad (3.58)$$

In the first line, we have introduced scalar $\gamma_i(\tau)$ to stand for a quantity in the range $[-1/20, 1/20]$ that varies continuously with τ . This notation will be used throughout the remainder of the proof. The second line follows from (3.51). Combining (3.58) and (3.55), we conclude

$$\|\mathbf{P}_i(\tau) + \tau\phi_i\mathbf{Q}_{ii}\| \geq 0.47\bar{\sigma}_i\tau. \quad (3.59)$$

Finally, because $\mathbf{P}_i(\tau) + \tau\phi_i\mathbf{Q}_{ii}$ is a rescaling of the right-hand side of (3.17) by θ , we conclude that

$$\sigma_i \geq 0.47\bar{\sigma}_i\tau\theta. \quad (3.60)$$

Applying (3.58) to the formulation of $f(\tau)$ in (3.35), we have, for $\tau \in [\tau_\ell, \tau_u]$:

$$\begin{aligned} f(\tau) &= \sum_{i=1}^k \|\mathbf{P}_i(\tau) + \tau\phi_i\mathbf{Q}_{ii}\|^2 - \theta^{-2} \\ &= \sum_{i=1}^k m_i n_i (1 + \gamma_i(\tau))^2 g_i(\tau; a_i(\tau)) - \theta^{-2} \\ &= A(\tau)\tau^2 - 2B(\tau)\tau - C(\tau). \end{aligned}$$

The third line is obtained by expanding the quadratic formula for $g_i(\tau; a_i(\tau))$, which results in

$$\begin{aligned} A(\tau) &= \sum_{i=1}^k (1 + \gamma_i(\tau))^2 \bar{\sigma}_i^2 (1 + 2\mu_{ii}a_i(\tau) + \mu_{ii}^2), \\ B(\tau) &= \sum_{i=1}^k (1 + \gamma_i(\tau))^2 \sqrt{m_i n_i} \bar{\sigma}_i (a_i(\tau) + \mu_{ii}), \\ C(\tau) &= \theta^{-2} - \sum_{i=1}^k (1 + \gamma_i(\tau))^2 m_i n_i. \end{aligned}$$

We will now prove that there exists $\tau^* \in [\tau_\ell, \tau_u]$ such that $f(\tau^*) = 0$ by applying the following lemma, which is a specific form of intermediate theorem for ‘pseudo-quadratic’ functions.

Lemma 5. *Consider a real-valued function $\hat{f}(\tau)$ of the form*

$$\hat{f}(\tau) = A(\tau)\tau^2 - 2B(\tau)\tau - C(\tau),$$

where $A(\tau), B(\tau), C(\tau)$ are continuous functions of τ . Suppose there are two triples of positive numbers $(\underline{A}, \underline{B}, \underline{C}) < (\bar{A}, \bar{B}, \bar{C})$ (where ‘<’ is understood element-wise). Define

$$\tau'_\ell = \frac{\underline{B} + \sqrt{\underline{B}^2 + \underline{A} \cdot \underline{C}}}{\bar{A}}, \quad (3.61)$$

and

$$\tau'_u = \frac{\bar{B} + \sqrt{\bar{B}^2 + \bar{A} \cdot \bar{C}}}{\underline{A}}. \quad (3.62)$$

(Clearly $\tau'_\ell < \tau'_u$.) Suppose further that there is an interval $[\tau_\ell, \tau_u]$ such that $\tau_\ell \leq \tau'_\ell \leq \tau'_u \leq \tau_u$ and such that for all $\tau \in [\tau_\ell, \tau_u]$,

$$(\underline{A}, \underline{B}, \underline{C}) \leq (A(\tau), B(\tau), C(\tau)) \leq (\bar{A}, \bar{B}, \bar{C}).$$

Then there exists a root $\tau^* \in [\tau'_\ell, \tau'_u]$ (and therefore also in $[\tau_\ell, \tau_u]$) such that $\hat{f}(\tau^*) = 0$.

Proof. Some simple algebra shows that $\hat{f}(\tau'_\ell) = A(\tau'_\ell)(\tau'_\ell)^2 - 2B(\tau'_\ell)(\tau'_\ell) - C(\tau'_\ell) \leq 0$ while $\hat{f}(\tau'_u) = A(\tau'_u)(\tau'_u)^2 - 2B(\tau'_u)\tau'_u - C(\tau'_u) \geq 0$, so there is a $\tau^* \in [\tau'_\ell, \tau'_u]$ such that $\hat{f}(\tau^*) = 0$ by the intermediate value theorem. \square

In order to apply Lemma 5, we now define the following scalars:

$$\begin{aligned} \bar{A} &= (10/9) \sum_{i=1}^k \bar{\sigma}_i^2 (1 + 2\mu_{ii}\bar{a}_i + \mu_{ii}^2), \\ \underline{A} &= 0.90 \sum_{i=1}^k \bar{\sigma}_i^2 (1 + 2\mu_{ii}\underline{a}_i + \mu_{ii}^2), \\ \bar{B} &= (10/9) \sum_{i=1}^k \sqrt{m_i n_i} \bar{\sigma}_i (\bar{a}_i + \mu_{ii}), \\ \underline{B} &= 0.90 \sum_{i=1}^k \sqrt{m_i n_i} \bar{\sigma}_i (\underline{a}_i + \mu_{ii}), \\ \bar{C} &= (10/9) \left(\theta^{-2} - \sum_{i=1}^k m_i n_i \right), \\ \underline{C} &= (9/10) \left(\theta^{-2} - \sum_{i=1}^k m_i n_i \right). \end{aligned}$$

It is obvious that $(0, 0) < (\underline{A}, \underline{B}) < (\bar{A}, \bar{B})$. It follows from (3.16), (3.107), and (3.111) below that the parenthesized quantity in the definitions of \bar{C}, \underline{C} is positive,

$$\theta^{-2} - \sum_{i=1}^k m_i n_i \geq \left(\frac{1}{4c_3(\mathbf{p})^2} - 1 \right) \sum_{i=1}^k m_i n_i = 1.2^4 (c_4(\mathbf{p}))^2 \left(\frac{\rho_m \rho_n k}{k + \rho_m \rho_n - 1} \right) \sum_{i=1}^k m_i n_i > 0,$$

and hence we also have $0 < \underline{C} < \bar{C}$.

In addition, given the fact that $a_i(\tau) \in [\underline{a}_i; \bar{a}_i]$ and $\gamma_i(\tau) \in [-1/20; 1/20]$ for $\tau \in [\tau_\ell, \tau_u]$, so this establishes for this interval that $(\underline{A}, \underline{B}, \underline{C}) \leq (A(\tau), B(\tau), C(\tau)) \leq (\bar{A}, \bar{B}, \bar{C})$.

We now show that $\tau'_\ell \geq \tau_\ell$ and $\tau'_u \leq \tau_u$. We have

$$\tau'_\ell = \frac{\underline{B} + \sqrt{\underline{B}^2 + \underline{A} \cdot \underline{C}}}{\bar{A}} \geq \frac{\sqrt{\underline{A} \cdot \underline{C}}}{\bar{A}}.$$

Using the facts that $0 \leq \bar{a}_i \leq 1$, $0 \leq \mu_{ii} \leq c_2(\mathbf{p}, c_0) \leq 0.08$ (see (3.114) below), and $\theta^{-2} - \sum_{i=1}^k m_i n_i \geq$

$1.2^4 (c_4(\mathbf{p}))^2 \left(\frac{\rho_m \rho_n k}{k + \rho_m \rho_n - 1} \right) \sum_{i=1}^k m_i n_i > 0$ as above, we have:

$$\tau'_\ell \geq \left(\frac{\rho_m \rho_n k}{k + \rho_m \rho_n - 1} \right)^{1/2} c_4(\mathbf{p}) \left(\sum_{i=1}^k m_i n_i \right)^{1/2} \left(\sum_{i=1}^k \bar{\sigma}_i^2 \right)^{-1/2}.$$

Next, observe that

$$\left(\sum_{i=1}^k \bar{\sigma}_i^2 \right)^{-1/2} \geq k^{-1/2} \sigma_1^{-1}$$

while

$$\left(\sum_{i=1}^k m_i n_i \right)^{1/2} \geq \left(1 + \frac{k-1}{\rho_m \rho_n} \right)^{1/2} (m_1 n_1)^{1/2}.$$

Since $\phi_1 = \sigma_1 / \sqrt{m_1 n_1}$, we conclude that

$$\tau'_\ell \geq c_4(\mathbf{p}) \phi_1^{-1} = \tau_\ell,$$

given the definition of τ_ℓ in (3.106).

We now consider the condition for τ'_u . We have:

$$\tau'_u = \frac{\bar{B} + \sqrt{\bar{B}^2 + \bar{A} \cdot \bar{C}}}{\underline{A}} = \frac{\bar{B}}{\underline{A}} + \sqrt{\frac{\bar{B}^2}{\underline{A}^2} + \frac{\bar{A} \cdot \bar{C}}{\underline{A}^2}}.$$

Using the fact that $0 \leq \bar{a}_i \leq 1$, $0 \leq \mu_{ii} \leq 0.08$, we have

$$\begin{aligned} \frac{\bar{B}}{\underline{A}} &\leq \frac{100}{81} \cdot \left(1.08 \sum_{i=1}^k \bar{\sigma}_i \sqrt{m_i n_i} \right) \left(\sum_{i=1}^k \bar{\sigma}_i^2 \right)^{-1} \\ &\leq \frac{4}{3} \cdot \left(\frac{1 + (k-1) \sqrt{\rho_m \rho_n}}{1 + (k-1) \rho_\sigma^{-2}} \right) \phi_1^{-1}, \end{aligned}$$

and, using also (3.16),

$$\begin{aligned} \frac{\bar{A} \cdot \bar{C}}{\underline{A}^2} &\leq \frac{16}{9} \cdot (c_3(\mathbf{p})^{-2} - 1) \left(\sum_{i=1}^k m_i n_i \right) \left(\sum_{i=1}^k \bar{\sigma}_i^2 \right)^{-1} \\ &\leq \frac{16}{9} \cdot (c_3(\mathbf{p})^{-2} - 1) \left(\frac{1 + (k-1) \rho_m \rho_n}{1 + (k-1) \rho_\sigma^{-2}} \right) \phi_1^{-2}. \end{aligned}$$

Note that $0 < c_3(\mathbf{p}) < 1$ given its definition in (3.111). Now, combining these terms and we conclude that

$$\tau'_u \leq c_5(\mathbf{p}) \phi_1^{-1} = \tau_u,$$

given the definition of τ_u in (3.112) below with $c_5(\mathbf{p})$ defined in (3.113). Thus applying Lemma 5, we prove that there exists $\tau^* \in [\tau_\ell, \tau_u]$ such that $f(\tau^*) = 0$. This also means the existence of $\lambda^* = \theta\tau^*$. For the remainder of this proof, we will drop the asterisks and simply write these selected values as τ and λ .

3.1.2 Upper bound on $\|W_{ii}\|$

Now consider the condition (3.32) for block (i, i) , $i = 1, \dots, k$. By (3.33), it suffices to show

$$\sigma_2(\mathbf{P}_i(\tau) + \tau\phi_i\mathbf{Q}_{ii}) \leq \frac{1}{k+1}\sigma_1(\mathbf{P}_j(\tau) + \tau\phi_j\mathbf{Q}_{jj}) \quad (3.63)$$

for all $j = 1, \dots, k$. In order to analyze $\sigma_2(\mathbf{P}_i(\tau) + \tau\phi_i\mathbf{Q}_{ii})$, we start with $\sigma_2(\mathbf{P}_i(\tau))$. Since $\mathbf{P}_i(\tau)$ has the rank of at most two, $\bar{\zeta}_i = \sigma_2^2(\mathbf{P}_i(\tau))$ can be computed as the smaller root of the quadratic equation (3.42), $i = 1, \dots, k$. Using the fact that $\zeta_i\bar{\zeta}_i = \det(\mathbf{M}_i)$, we have

$$\bar{\zeta}_i = \frac{m_i n_i \tau^2 \phi_i^2 (1 - \tau\phi_i \mu_{ii})^2 (1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)}{g_i(\tau; a_i(\tau))} \quad (3.64)$$

from (3.44) and (3.49).

Now we note that from standard singular value perturbation theory that

$$\begin{aligned} \sigma_2(\mathbf{P}_i + \tau\phi_i\mathbf{Q}_{ii}) &\leq \sigma_2(\mathbf{P}_i) + \tau\phi_i \|\mathbf{Q}_{ii}\| \\ &\leq \left(\frac{m_i n_i \tau^2 \phi_i^2 (1 - \tau\phi_i \mu_{ii})^2 (1 - \delta_{u,i}^2)(1 - \delta_{v,i}^2)}{g_i(\tau; a_i(\tau))} \right)^{1/2} + \tau\phi_i (m_i n_i)^{3/8} \\ &\equiv T_1 + T_2. \end{aligned}$$

We handle the two terms separately. Since we are interested in the asymptotic case of $m_i, n_i \rightarrow \infty$, we will assume

$$(m_i n_i)^{-1/8} \leq \frac{1}{10(k+1)\rho_\sigma}, \quad (3.65)$$

for all $i, j = 1, \dots, k$.

First, we have:

$$\begin{aligned}
T_1 &= \left(\frac{m_i n_i \tau^2 \phi_i^2 (1 - \tau \phi_i \mu_{ii})^2 (1 - \delta_{u,i}^2) (1 - \delta_{v,i}^2)}{g_i(\tau; a_i(\tau))} \right)^{1/2} \\
&\leq \left(\frac{m_i n_i \tau^2 \phi_i^2}{\phi_i^2 \tau^2 / 4} \right)^{1/2} \\
&= 2\sqrt{m_i n_i} \\
&\leq \frac{2\phi_j \tau \sqrt{m_j n_j}}{6(k+1)} \\
&= \frac{\tau \bar{\sigma}_j}{3(k+1)}
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
&\leq \frac{\sigma_1(\mathbf{P}_j(\tau) + \tau \phi_j \mathbf{Q}_{jj})}{3 \cdot 0.47(k+1)}.
\end{aligned} \tag{3.67}$$

The inequality in the second line follows from the fact that $0 < 1 - \tau \phi_i \mu_{ii} \leq 1$ for $\tau \in [\tau_\ell, \tau_u]$ for the numerator and (3.54) for the denominator. The inequality in the fourth line follows from $\tau \phi_j \geq 6(k+1)\sqrt{\rho_m \rho_n}$, which follows from (3.108). The last line follows from (3.59). Next, we have:

$$\begin{aligned}
T_2 &= \tau \phi_i (m_i n_i)^{3/8} \\
&= \tau \bar{\sigma}_i / (m_i n_i)^{1/8} \\
&\leq \tau \bar{\sigma}_j / (10(k+1)) \\
&\leq \frac{\sigma_1(\mathbf{P}_j(\tau) + \tau \phi_j \mathbf{Q}_{jj})}{10 \cdot 0.47(k+1)},
\end{aligned}$$

where the third line follows from (3.65) and the fourth again from (3.59). This inequality and (3.67) together establish (3.63).

3.1.3 Positivity of u_i and v_i

The final condition for the (i, i) block is the positivity of singular vectors. We will show that with high probability, the matrix $\mathbf{S}_i = (\mathbf{P}_i(\tau) + \tau \phi_i \mathbf{Q}_{ii})^T (\mathbf{P}_i(\tau) + \tau \phi_i \mathbf{Q}_{ii})$ is positive, which implies the positivity of the right singular vector. The same analysis can be applied for the left singular vector with the matrix $\mathbf{T}_i = (\mathbf{P}_i(\tau) + \tau \phi_i \mathbf{Q}_{ii})(\mathbf{P}_i(\tau) + \tau \phi_i \mathbf{Q}_{ii})^T$. We have: $\mathbf{S}_i = \mathbf{S}_i^1 + \mathbf{S}_i^2 + \mathbf{S}_i^3 + \mathbf{S}_i^4$, where $\mathbf{S}_i^1 = (\mathbf{P}_i(\tau))^T \mathbf{P}_i(\tau)$, $\mathbf{S}_i^2 = \tau \phi_i (\mathbf{P}_i(\tau))^T \mathbf{Q}_{ii}$, $\mathbf{S}_i^3 = \tau \phi_i \mathbf{Q}_{ii}^T \mathbf{P}_i(\tau)$, and $\mathbf{S}_i^4 = \tau^2 \phi_i^2 \mathbf{Q}_{ii}^T \mathbf{Q}_{ii}$. Start with \mathbf{S}_i^1 . Recall $\delta_{u,i} = \mathbf{e}_{m_i}^T \bar{\mathbf{u}}_i / \sqrt{m_i}$ and $\bar{\sigma}_i = \phi_i \sqrt{m_i n_i}$. Then we have:

$$\begin{aligned}
S_i^1(l, j) &= (1 - \tau \phi_i \mu_{ii})^2 m_i \left(\sqrt{n_i} \max\{\bar{v}_{i,l}, \bar{v}_{i,j}\} \psi_i \left(\sqrt{n_i} \min\{\bar{v}_{i,l}, \bar{v}_{i,j}\} \psi_i - \frac{\delta_{u,i}(\bar{v}_{i,l} + \bar{v}_{i,j})}{\max\{\bar{v}_{i,l}, \bar{v}_{i,j}\}} \right) + 1 \right) \\
&\geq (1 - \tau \phi_i \mu_{ii})^2 m_i [\sqrt{n_i} \max\{\bar{v}_{i,l}, \bar{v}_{i,j}\} \psi_i (\xi_v \psi_i - 2) + 1],
\end{aligned} \tag{3.68}$$

where we let ψ_i denote $\tau\phi_i/(1 - \tau\phi_i\mu_{ii})$ for the remainder of the analysis of the positivity condition, and where ξ_v was defined by (3.6).

From (3.109), $\tau_\ell \geq 4/(\xi_v\phi_i)$. Since $\tau \in [\tau_\ell, \tau_u]$, we have $\tau\phi_i \geq 4/\xi_v$ (and similarly, $\tau\phi_i \geq 4/\xi_u$) for all $i = 1, \dots, k$. Thus, we have:

$$\begin{aligned} \xi_v\psi_i - 2 &= \frac{\tau\phi_i\xi_v}{1 - \mu_{ii}\tau\phi_i} - 2 \\ &\geq \tau\phi_i\xi_v - 2 \\ &\geq \tau\phi_i\xi_v/2 \\ &\geq \frac{\tau\phi_i\xi_v}{20(1 - \mu_{ii}\phi_i\tau)} \\ &= \psi_i\xi_v/20. \end{aligned}$$

To obtain the second and fourth line we use the fact that $0.1 < 0.7 \leq 1 - \mu_{ii}\tau\phi_i \leq 1$ since $\tau \in [\tau_\ell, \tau_u]$. Substituting into (3.68) yields

$$\begin{aligned} S_i^1(l, j) &\geq (1 - \tau\phi_i\mu_{ii})^2 m_i [\sqrt{n_i} \max\{\bar{v}_{i,l}, \bar{v}_{i,j}\} \psi_i^2 \xi_v/20 + 1] \\ &\geq (1 - \tau\phi_i\mu_{ii})^2 m_i (\psi_i^2 \xi_v^2/20 + 1), \end{aligned} \quad (3.69)$$

for $l, j = 1, \dots, n_i$.

Now consider the matrix \mathbf{S}_i^2 , we have:

$$S_i^2(l, j) = \tau\phi_i \sum_{s=1}^{m_i} (\tau\bar{\sigma}_i \bar{v}_{i,l} \bar{u}_{i,s} - (1 - \tau\phi_i\mu_{ii})) Q_{ii}(s, j).$$

According to Lemma 2, $S_i^2(l, j)$ is a subgaussian random variable with the parameter $b\tau\phi_i \cdot \|\tau\bar{\sigma}_i \bar{v}_{i,l} \bar{\mathbf{u}}_i - (1 - \tau\phi_i\mu_{ii})\mathbf{e}_{m_i}\|$. Applying (3.2) with $t = (1 - \tau\phi_i\mu_{ii})^2 m_i \psi_i^2 \xi_v^2/60$ (i.e., less than a third of the lower bound given by (3.69)) and again making substitutions, we have:

$$\mathbb{P}(S_i^2(l, j) \leq -(1 - \tau\phi_i\mu_{ii})^2 m_i \psi_i^2 \xi_v^2/60) \leq \exp\left(\frac{-m_i^2 \psi_i^4 \xi_v^4}{7200b^2 \psi_i^2 \|\psi_i \bar{v}_{i,l} \sqrt{m_i n_i} \bar{\mathbf{u}}_i - \mathbf{e}_{m_i}\|^2}\right). \quad (3.70)$$

We have:

$$\begin{aligned} \|\psi_i \bar{v}_{i,l} \sqrt{m_i n_i} \bar{\mathbf{u}}_i - \mathbf{e}_{m_i}\|^2 &= m_i (\psi_i \bar{v}_{i,l} \sqrt{n_i} (\psi_i \bar{v}_{i,l} \sqrt{n_i} - 2\delta_{u,i}) + 1) \\ &\leq m_i (\psi_i^2 \pi_v^2 + 1). \end{aligned}$$

Substituting into (3.70) and canceling $m_i \psi_i^4$ from the numerator and denominator yields

$$\begin{aligned} \mathbb{P}\left(S_i^2(l, j) \leq \frac{-(1 - \tau\phi_i\mu_{ii})^2 m_i \psi_i^2 \xi_v^2}{60}\right) &\leq \exp\left(\frac{-m_i \xi_v^4}{7200b^2 (\pi_v^2 + 1/\psi_i^2)}\right) \\ &\leq \exp\left(\frac{-m_i \xi_v^4}{7200b^2 (\pi_v^2 + 1)}\right), \end{aligned} \quad (3.71)$$

where the second line follows from the fact that $1/\psi_i \leq 1/(\tau\phi_i) \leq 1/(6(k+1)) \leq 1$ since $\tau \geq \tau_\ell$. Note that $\mathbf{S}_i^3 = (\mathbf{S}_i^2)^T$, which means the analysis is the same.

For the matrix \mathbf{S}_i^4 , we have $S_i^4(l, j) = \tau^2 \phi_i^2 [(\mathbf{Q}_{ii}(:, l))^T \mathbf{Q}_{ii}(:, j)]$, where the square-bracketed factor is the inner product of two independent b -subgaussian random vector for all $l \neq j$. (Note that when $l = j$, $S_i^4(l, j) \geq 0$ so there is nothing to analyze.) We can bound the probability such that $S_i^4(l, j) \leq -(1 - \tau\phi\mu_{ii})^2 m_i \psi_i^2 \xi_v / 60$ using Lemma 4 with $t = (1 - \tau\phi\mu_{ii})^2 m_i \psi_i^2 \xi_v / 60$, “ b ” in the lemma taken to be $\tau\phi_i b$, and “ n ” in the lemma taken to be m_i ; note that the common factor “ $t/(4eb)$ ” simplifies to $m_i \xi_v^2 / (240eb^2)$ since $\psi_i = \tau\phi_i / (1 - \mu_{ii}\phi_i\tau)$. Hence by the lemma,

$$\mathbb{P} \left(S_i^4(l, j) \leq \frac{-(1 - \tau\phi_i\mu_{ii})^2 m_i \psi_i^2 \xi_v^2}{60} \right) \leq \exp \left(-m_i \cdot \min \left(\frac{\xi_v^4}{240^2 e^2 b^4}, \frac{\xi_v^2}{240eb^2} \right) \right). \quad (3.72)$$

Combining (3.69), (3.71), and (3.72), we have:

$$\begin{aligned} \mathbb{P} \left(\min_{l, j} S(l, j) \leq 0 \right) &\leq n_i(n_i - 1) \cdot \left[\exp \left(\frac{-m_i \xi_v^4}{7200b^2(\pi_v^2 + 1)} \right) \right. \\ &\quad \left. + (1/2) \exp \left(-m_i \cdot \min \left(\frac{\xi_v^4}{240^2 e^2 b^4}, \frac{\xi_v^2}{240eb^2} \right) \right) \right]. \end{aligned} \quad (3.73)$$

Similarly, we also have:

$$\begin{aligned} \mathbb{P} \left(\min_{l, j} T(l, j) \leq 0 \right) &\leq m_i(m_i - 1) \cdot \left[\exp \left(\frac{-n_i \xi_u^4}{7200b^2(\pi_u^2 + 1)} \right) \right. \\ &\quad \left. + (1/2) \exp \left(-n_i \cdot \min \left(\frac{\xi_u^4}{240^2 e^2 b^4}, \frac{\xi_u^2}{240eb^2} \right) \right) \right]. \end{aligned} \quad (3.74)$$

3.2 Analysis for block (i, j) , $i \neq j$, $i, j = 1, \dots, k$

We now consider the off-diagonal (i, j) block, $i \neq j$, $i, j = 1, \dots, k$. Recall our notation: \mathbf{u}_i , \mathbf{v}_i stand for the unit-norm dominant left and right singular vectors respectively of the right-hand side of (3.17), or, equivalently, of $\mathbf{P}_i(\tau) + \tau\phi_i \mathbf{Q}_{ii}$.

Let us consider the following construction

$$\mathbf{V}_{ij} = \tau \left(\frac{\mathbf{e}_{m_i}(\mathbf{u}_i^T \mathbf{R}_{ij})}{\|\mathbf{u}_i\|_1} + \frac{(\mathbf{R}_{ij} \mathbf{v}_j) \mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1} - \frac{\mathbf{u}_i^T \mathbf{R}_{ij} \mathbf{v}_j}{\|\mathbf{u}_i\|_1 \|\mathbf{v}_j\|_1} \mathbf{e}_{m_i} \mathbf{e}_{n_j}^T \right).$$

The matrix $\mathbf{W}_{ij} = \lambda \mathbf{R}_{ij} - \theta \mathbf{V}_{ij}$ clearly satisfies two orthogonal requirements, (3.21) and (3.22). We now just need to find the conditions so that $\|\mathbf{W}_{ij}\| \leq \frac{1}{k+1} \min_{i=1, \dots, k} \sigma_i$ and $\|\mathbf{V}_{ij}\|_\infty \leq 1$.

3.2.1 Upper bound on $\|\mathbf{V}_{ij}\|_\infty$

We have:

$$|V_{ij}(s, t)| \leq \tau \left(\left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij}(:, t)}{\|\mathbf{u}_i\|_1} \right| + \left| \frac{\mathbf{R}_{ij}(s, :)\mathbf{v}_j}{\|\mathbf{v}_j\|_1} \right| + \left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij}\mathbf{v}_j}{\|\mathbf{u}_i\|_1 \|\mathbf{v}_j\|_1} \right| \right).$$

In order to show $\|\mathbf{V}_{ij}\|_\infty \leq 1$ with high probability, we will show the sufficient condition that all probabilities,

$$\mathbb{P} \left(\tau \left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij}(:, t)}{\|\mathbf{u}_i\|_1} \right| > \frac{1}{3} \right), \quad (3.75)$$

$$\mathbb{P} \left(\tau \left| \frac{\mathbf{R}_{ij}(s, :)\mathbf{v}_j}{\|\mathbf{v}_j\|_1} \right| > \frac{1}{3} \right), \quad (3.76)$$

$$\mathbb{P} \left(\tau \left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij}\mathbf{v}_j}{\|\mathbf{u}_i\|_1 \|\mathbf{v}_j\|_1} \right| > \frac{1}{3} \right), \quad (3.77)$$

are exponentially small.

Since $\tau \in [\tau_\ell, \tau_u]$, we have $\tau\sqrt{\phi_i\phi_j} \leq 0.3/\mu_{ij}$ by (3.116). Thus we have:

$$\mathbb{P} \left(\tau \left| \frac{\mathbf{R}_{ij}(s, :)\mathbf{v}_j}{\|\mathbf{v}_j\|_1} \right| > \frac{1}{3} \right) \leq \mathbb{P} \left(\tau \left| \frac{(\mathbf{R}_{ij}(s, :) - \mu_{ij}\sqrt{\phi_i\phi_j}\mathbf{e}_{n_j}^T)\mathbf{v}_j}{\|\mathbf{v}_j\|_1} \right| > \frac{1}{30} \right).$$

Thus, to analyze (3.76), it suffices to show that the probability on the right-hand side of the preceding inequality is exponentially small. Since $\|\mathbf{v}_j\| = 1$, $((\phi_i\phi_j)^{-1/2}\mathbf{R}_{ij}(s, :) - \mu_{ij}\mathbf{e}_{n_j}^T)\mathbf{v}_j$ is a b -subgaussian random variable. By (3.2), we have:

$$\mathbb{P} \left(\tau \left| \frac{(\mathbf{R}_{ij}(s, :) - \mu_{ij}\sqrt{\phi_i\phi_j}\mathbf{e}_{n_j}^T)\mathbf{v}_j}{\|\mathbf{v}_j\|_1} \right| > \frac{1}{30} \right) \leq 2 \exp \left(-\frac{0.1^2 \|\mathbf{v}_j\|_1^2}{18b^2\tau^2\phi_i\phi_j} \right). \quad (3.78)$$

We now must show that the the probability on the right-hand side of (3.78) is exponentially small.

First, we observe that

$$\begin{aligned} \tau^2\phi_i\phi_j &\leq (\tau_u')^2\phi_i\phi_j \\ &\leq c_5(\mathbf{p})^2\phi_i\phi_j/\phi_1^2 \\ &\leq c_6(\mathbf{p}), \end{aligned}$$

where $c_6(\mathbf{p}) = c_5(\mathbf{p})^2\rho_\sigma^2\rho_m\rho_n$ with $c_5(\mathbf{p})$ defined in (3.113) below. This follows from the fact that, for any $i, j = 1, \dots, k$,

$$\phi_i/\phi_j = (\bar{\sigma}_i/\bar{\sigma}_j)\sqrt{m_j/m_i}\sqrt{n_j/n_i} \leq \rho_\sigma\rho_m^{1/2}\rho_n^{1/2}. \quad (3.79)$$

We now provide a lower bound on $\|\mathbf{v}_j\|_1$. We start with the right singular vector $\hat{\mathbf{v}}(\mathbf{P}_j(\tau))$ of the matrix $\mathbf{P}_j(\tau)$. As noted prior to (3.40), this singular vector may be written as $\hat{\alpha}_j\bar{\mathbf{v}}_j + \hat{\beta}_j\mathbf{e}_{n_j}$. Let

$\mathbf{v}(\mathbf{P}_j(\tau))$ be the rescaling of $\hat{\mathbf{v}}(\mathbf{P}_j(\tau))$ with the scale chosen so that $\mathbf{v}(\mathbf{P}_j(\tau)) = \alpha_j \bar{\mathbf{v}}_j + \mathbf{e}_{n_j}$ (i.e., $\beta_j = 1$). Then we can obtain the value of α_j using the second equation obtained from (3.40) (see also Lemma 8 in [5]), and simplifying by substituting (3.37) yields

$$\alpha_j = \frac{\sqrt{n_j}}{\eta_j} \cdot \frac{\tau\phi_j[\tau\phi_j - (2a_j - \delta_{u,i}\delta_{v,i})\eta_j]}{\delta_{v,j}\eta_j - \tau\phi_j\delta_{u,j}}, \quad (3.80)$$

where η_j stands for $1 - \tau\phi_j\mu_{jj}$, which lies in $[0.7, 1]$ since $\tau \leq \tau_u$. (Note that the scaling $\beta_j = 1$ is valid only if the denominator of the above fraction is nonzero, which we shall show next.) Observe that the square-bracketed quantity in the second numerator is nonnegative and at least $\tau\phi_j - 2$ since $a_j \leq 1$ and $\tau \leq \tau_u$.

Using the facts that $\xi_u \leq \delta_u \leq \delta_{u,i}$ and $\xi_v \leq \delta_v \leq \delta_{v,i}$ we conclude from (3.110) that $\tau\phi_j \geq 2 + 2\delta_{u,j}/\delta_{v,j}$ and $\tau\phi_j \geq 2 + 2\delta_{v,j}/\delta_{u,j}$ for all $j = 1, \dots, k$ whenever $\tau \geq \tau_\ell$.

Now, ignoring the additive term of 2 for a moment, this assumption implies that the second denominator is negative and no more than $\tau\phi_j\delta_{u,j}$ in absolute value. Thus we have:

$$\alpha_j \leq -\frac{\sqrt{n_j}(\tau\phi_j - 2)}{\delta_{u,j}}.$$

We can argue that $\tau\phi_j - 2 \geq 2\delta_{u,j}/\delta_{v,j}$ to obtain inequality

$$\alpha_j \leq -2\sqrt{n_j}/\delta_{v,j}. \quad (3.81)$$

Now we write the 1- and 2-norms of $\mathbf{v}(P_j)$ in terms of α_j and the other parameters. Starting with the 1-norm,

$$\begin{aligned} \|\mathbf{v}(P_j)\|_1 &= \|\alpha_j \bar{\mathbf{v}}_j + \mathbf{e}_{n_j}\|_1 \\ &\geq \|\alpha_j \bar{\mathbf{v}}_j\|_1 - n_j \\ &= |\alpha_j| \sqrt{n_j} \delta_{v,j} - n_j \\ &\geq |\alpha_j| \sqrt{n_j} \delta_{v,j} / 2, \end{aligned}$$

where, to obtain the last line, we used the fact that $|\alpha_j| \sqrt{n_j} \delta_{v,j} / 2 \geq n_j$, a consequence of (3.81). Also,

$\|\mathbf{v}(P_j)\| \leq |\alpha_j| + \sqrt{n_j}$ by the triangle inequality. Thus, we conclude that

$$\begin{aligned}
\|\hat{\mathbf{v}}(P_j)\|_1 &= \frac{\|\mathbf{v}(P_j)\|_1}{\|\mathbf{v}(P_j)\|} \\
&\geq \frac{|\alpha_j|\sqrt{n_j}\delta_{v,j}/2}{|\alpha_j| + \sqrt{n_j}} \\
&= \frac{\sqrt{n_j}\delta_{v,j}}{2(1 + \sqrt{n_j}/|\alpha_j|)} \\
&\geq \frac{\sqrt{n_j}\delta_{v,j}}{2(1 + \delta_{v,j}/2)} \\
&\geq \frac{\sqrt{n_j}\delta_{v,j}}{3}
\end{aligned} \tag{3.82}$$

Next, we observe by the triangle inequality that

$$\begin{aligned}
\|\mathbf{v}_j\|_1 &\geq \|\hat{\mathbf{v}}(\mathbf{P}_j)\|_1 - \|\hat{\mathbf{v}}(\mathbf{P}_j) - \mathbf{v}_j\|_1 \\
&\geq \|\hat{\mathbf{v}}(\mathbf{P}_j)\|_1 - \sqrt{n_j}\|\hat{\mathbf{v}}(\mathbf{P}_j) - \mathbf{v}_j\|.
\end{aligned} \tag{3.83}$$

We will use Wedin's theorem on perturbation of singular vectors (see Doan and Vavasis [5] and references therein for details) to analyze the final norm in the above inequality since \mathbf{v}_j is the leading singular vector of $\mathbf{P}_j(\tau) + \tau\phi\mathbf{Q}_{jj}$ while $\hat{\mathbf{v}}(\mathbf{P}_j)$ is the leading singular vector of $\mathbf{P}_j(\tau)$.

For Wedin's theorem, we choose $\mathbf{A} = \mathbf{P}_j(\tau)$, $\mathbf{T} = \tau\phi_j\mathbf{Q}_{jj}$, and $\mathbf{B} = \mathbf{A} + \mathbf{T}$. We have: $\|\mathbf{T}\| \leq \tau\phi_j(m_j n_j)^{3/8}$. In addition,

$$\begin{aligned}
\sigma_1(\mathbf{B}) &\geq \sigma_1(\mathbf{A}) - \sigma_1(\mathbf{T}) \\
&\geq \sqrt{m_j n_j g_j(\tau; a_j)} - \tau\phi_j(m_j n_j)^{3/8},
\end{aligned}$$

where the second line is obtained from (3.51). Finally, using (3.64),

$$\sigma_2(\mathbf{A}) = \tau\phi_j\eta_j \left(\frac{m_j n_j (1 - \delta_{u,j}^2)(1 - \delta_{v,j}^2)}{g_j(\tau; a_j)} \right)^{1/2}.$$

Therefore,

$$\begin{aligned}
\sin \theta(\mathbf{v}_j, \hat{\mathbf{v}}(\mathbf{P}_j(\tau))) &\leq \frac{\tau\phi_j(m_j n_j)^{3/8}}{\sqrt{m_j n_j g_j(\tau; a_j)} - \tau\phi_j(m_j n_j)^{3/8} - \tau\phi_j\eta_j \left(\frac{m_j n_j (1 - \delta_{u,j}^2)(1 - \delta_{v,j}^2)}{g_j(\tau; a_j)} \right)^{1/2}} \\
&= \frac{(m_j n_j)^{-1/8}}{\sqrt{g_j(\tau; a_j)}/(\tau\phi_j) - (m_j n_j)^{-1/8} - \eta_j \left(\frac{(1 - \delta_{u,j}^2)(1 - \delta_{v,j}^2)}{g_j(\tau; a_j)} \right)^{1/2}}.
\end{aligned}$$

Observe that the numerator tends to zero like $(m_j n_j)^{-1/8}$ while the denominator does not depend on $m_j n_j$ (except for a vanishing term). Furthermore, the denominator is positive; this follows from the fact that the first term in the denominator is at least 0.5 by (3.54) whereas the last term is at most $1/\sqrt{5}$ again by (3.54) and the fact that $\phi_i \tau \geq 5$ thanks to (3.108).

This shows that

$$\|\mathbf{v}_j - \hat{\mathbf{v}}(\mathbf{P}_j(\tau))\|_2 \leq O\left((m_j n_j)^{-\frac{1}{8}}\right). \quad (3.84)$$

Combining (3.82), (3.83) and (3.84), we can then pick a constant less than 1/3, say 0.3, and claim that

$$\|\mathbf{v}_j\|_1 \geq 0.3\delta_{v,j}\sqrt{n_j} \geq 0.3\delta_v\sqrt{n_j}, \quad (3.85)$$

as long as m_j, n_j are large. Combining this bound with (3.78), we can claim that the probability (3.76) is exponential small:

$$\mathbb{P}\left(\tau \left| \frac{\mathbf{R}_{ij}(s, \cdot) \mathbf{v}_j}{\|\mathbf{v}_j\|_1} \right| > \frac{1}{3}\right) \leq 2 \exp\left(-\frac{0.1^2 0.3^2 \delta_v^2 n_j}{18b^2 c_6(\mathbf{p})}\right). \quad (3.86)$$

Similarly, the first probability (3.75) can also be proved to be exponentially small using the analogous lower bound of $\|\mathbf{u}_i\|_1$:

$$\|\mathbf{u}_i\|_1 \geq 0.3\delta_u\sqrt{m_i}. \quad (3.87)$$

The bound for the first probability can therefore be written as follows:

$$\mathbb{P}\left(\tau \left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij}(:, t)}{\|\mathbf{u}_i\|_1} \right| > \frac{1}{3}\right) \leq 2 \exp\left(-\frac{0.1^2 0.3^2 \delta_u m_i}{18b^2 c_6(\mathbf{p})}\right). \quad (3.88)$$

For the third probability (3.77), we again use the fact that $\tau\sqrt{\phi_i\phi_j} \leq 0.3/\mu_{ij}$ since $\tau \leq \tau_u$:

$$\mathbb{P}\left(\tau \left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij} \mathbf{v}_j}{\|\mathbf{u}_i\|_1 \|\mathbf{v}_j\|_1} \right| > \frac{1}{3}\right) \leq \mathbb{P}\left(\tau \left| \frac{\mathbf{u}_i^T (\mathbf{R}_{ij} - \mu_{ij} \sqrt{\phi_i \phi_j} \mathbf{e}_{m_i} \mathbf{e}_{n_j}^T) \mathbf{v}_j}{\|\mathbf{u}_i\|_1 \|\mathbf{v}_j\|_1} \right| > \frac{1}{30}\right),$$

where $\mathbf{u}_i^T (\mathbf{R}_{ij} / \sqrt{\phi_i \phi_j} - \mu_{ij} \mathbf{e}_{m_i} \mathbf{e}_{n_j}^T) \mathbf{v}_j$ is a b -subgaussian random variable since $\|\mathbf{u}_i\|_2 = \|\mathbf{v}_j\|_2 = 1$. We again can bound this probability using the lower bounds of $\|\mathbf{u}_i\|_1$ and $\|\mathbf{v}_j\|_1$ as follows:

$$\mathbb{P}\left(\tau \left| \frac{\mathbf{u}_i^T \mathbf{R}_{ij} \mathbf{v}_j}{\|\mathbf{u}_i\|_1 \|\mathbf{v}_j\|_1} \right| > \frac{1}{3}\right) \leq 2 \exp\left(-\frac{0.1^2 0.3^4 \delta_u^2 \delta_v^2 m_i n_j}{18b^2 c_6(\mathbf{p})}\right). \quad (3.89)$$

Combining (3.86), (3.88), (3.89), we obtain the following tail bound:

$$\begin{aligned} \mathbb{P}(\|\mathbf{V}_{ij}\|_\infty > 1) &\leq 2 \exp\left(-\frac{0.1^2 0.3^2 \delta_v^2 n_j}{18b^2 c_6(\mathbf{p})}\right) \\ &\quad + 2 \exp\left(-\frac{0.1^2 0.3^2 \delta_u^2 m_j}{18b^2 c_6(\mathbf{p})}\right) \\ &\quad + 2 \exp\left(-\frac{0.1^2 0.3^4 \delta_u^2 \delta_v^2 m_i n_j}{18b^2 c_6(\mathbf{p})}\right). \end{aligned} \quad (3.90)$$

3.2.2 Upper bound on $\|\mathbf{W}_{ij}\|$

The second constraint for this type of block is $\|\mathbf{W}_{ij}\| \leq \frac{1}{k+1} \min_{i=1,\dots,k} \sigma_i$. Using the fact that $\mathbf{Q}_{ij} = \mathbf{R}_{ij}/\sqrt{\phi_i\phi_j} - \mu_{ij}\mathbf{e}_{m_i}\mathbf{e}_{n_j}^T$, we have:

$$\mathbf{W}_{ij} = \tau\theta\sqrt{\phi_i\phi_j} \left(\mathbf{Q}_{ij} - \frac{\mathbf{e}_{m_i}\mathbf{u}_i^T\mathbf{Q}_{ij}}{\|\mathbf{u}_i\|_1} - \frac{\mathbf{Q}_{ij}\mathbf{v}_j\mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1} + \frac{\mathbf{u}_i^T\mathbf{Q}_{ij}\mathbf{v}_j}{\|\mathbf{u}_i\|_1\|\mathbf{v}_j\|_1}\mathbf{e}_{m_i}\mathbf{e}_{n_j}^T \right).$$

We will establish that $\|\mathbf{W}_{ij}\| \leq \frac{1}{k+1} \min_{i=1,\dots,k} \sigma_i$ by showing that

$$\tau\theta\sqrt{\phi_i\phi_j} \left\| \frac{\mathbf{Q}_{ij}}{2} - \frac{\mathbf{e}_{m_i}\mathbf{u}_i^T\mathbf{Q}_{ij}}{\|\mathbf{u}_i\|_1} \right\| \leq \frac{1}{3(k+1)} \min_{i=1,\dots,k} \sigma_i, \quad (3.91)$$

$$\tau\theta\sqrt{\phi_i\phi_j} \left\| \frac{\mathbf{Q}_{ij}}{2} - \frac{\mathbf{Q}_{ij}\mathbf{v}_j\mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1} \right\| \leq \frac{1}{3(k+1)} \min_{i=1,\dots,k} \sigma_i, \quad (3.92)$$

$$\tau\theta\sqrt{\phi_i\phi_j m_i n_j} \left| \frac{\mathbf{u}_i^T\mathbf{Q}_{ij}\mathbf{v}_j}{\|\mathbf{u}_i\|_1\|\mathbf{v}_j\|_1} \right| \leq \frac{1}{3(k+1)} \min_{i=1,\dots,k} \sigma_i. \quad (3.93)$$

Given that $m_i, n_i \rightarrow \infty$ for all $i = 1, \dots, k$, we make the following assumption:

$$(m_j n_i)^{-1/8} \leq \frac{.47}{3(k+1)(\bar{\rho}_m \bar{\rho}_n)^{1/8} \rho_\sigma c_\tau(\mathbf{p})}, \quad (3.94)$$

for all $i, j = 1, \dots, k$, where we introduce

$$c_\tau(\mathbf{p}) = \max \left\{ \frac{1}{2} + \frac{1}{0.3\delta_u}, \frac{1}{2} + \frac{1}{0.3\delta_v}, \frac{1}{0.3^2\delta_u\delta_v} \right\}. \quad (3.95)$$

Now, inequality (3.91) is derived as follows:

$$\begin{aligned} \left\| \frac{\mathbf{Q}_{ij}}{2} - \frac{\mathbf{e}_{m_i}\mathbf{u}_i^T\mathbf{Q}_{ij}}{\|\mathbf{u}_i\|_1} \right\| &\leq \|\mathbf{Q}_{ij}\| \cdot \left\| \frac{\mathbf{I}}{2} - \frac{\mathbf{e}_{m_i}\mathbf{u}_i^T}{\|\mathbf{u}_i\|_1} \right\| \\ &\leq (m_i n_j)^{3/8} \cdot \left(\frac{1}{2} + \frac{\sqrt{m_i}}{\|\mathbf{u}_i\|_1} \right) \\ &\leq (m_i n_j)^{3/8} \cdot \left(\frac{1}{2} + \frac{1}{0.3\delta_{u,i}} \right). \end{aligned}$$

The first line uses submultiplicativity of the 2-norm since we have:

$$\mathbf{Q}_{ij}/2 - \mathbf{e}_{m_i}\mathbf{u}_i^T\mathbf{Q}_{ij}/\|\mathbf{u}_i\|_1 = (\mathbf{I}/2 - \mathbf{e}_{m_i}\mathbf{u}_i^T/\|\mathbf{u}_i\|_1)\mathbf{Q}_{ij}.$$

The second uses the triangle inequality, and the third uses (3.87). Multiply by the scalar $\tau\theta\sqrt{\phi_i\phi_j}$ and

let $l = 1, \dots, k$ be arbitrary:

$$\begin{aligned}
\tau\theta\sqrt{\phi_i\phi_j}\left\|\frac{\mathbf{Q}_{ij}}{2}-\frac{\mathbf{e}_{m_i}\mathbf{u}_i^T\mathbf{Q}_{ij}}{\|\mathbf{u}_i\|_1}\right\| &\leq \tau\theta\sqrt{\phi_i\phi_j}(m_in_j)^{3/8}\cdot\left(\frac{1}{2}+\frac{1}{0.3\delta_{u,i}}\right) \\
&= \tau\theta\sqrt{\bar{\sigma}_i\bar{\sigma}_j}\frac{m_i^{1/8}n_j^{1/8}}{m_j^{1/4}n_i^{1/4}}\cdot\left(\frac{1}{2}+\frac{1}{0.3\delta_{u,i}}\right) \\
&\leq \frac{.47\tau\theta}{3(k+1)}\bar{\sigma}_l \\
&\leq \frac{\sigma_l}{3(k+1)}.
\end{aligned}$$

The third line follows from (3.94) and the last from (3.60). Inequality (3.92) is established using the same argument. Finally, (3.93) is established by a similar argument starting from the inequality $|\mathbf{u}_i^T\mathbf{Q}_{ij}\mathbf{v}_j| \leq \|\mathbf{Q}_{ij}\| \leq (m_in_j)^{3/8}$.

3.3 Analysis for block $(k+1, j)$, $j = 1, \dots, k$

We now consider the $(k+1, j)$ block. Similar to the above approach, we will construct the following matrix $\mathbf{V}_{k+1, j}$:

$$\mathbf{V}_{k+1, j} = \tau \frac{\bar{\mathbf{R}}_{k+1, j} \mathbf{v}_j \mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1}.$$

3.3.1 Upper bound on $\|\mathbf{V}_{k+1, j}\|_\infty$

The condition $\|\mathbf{V}_{k+1, j}\|_\infty \leq 1$ can be dealt with using the same approach as before. We have:

$$V_{k+1, j}(s, t) = \tau \frac{\bar{\mathbf{R}}_{k+1, j}(s, :) \mathbf{v}_j}{\|\mathbf{v}_j\|_1}.$$

Since $\tau \leq \tau_u$, we can conclude from (3.117) that $\tau \leq 0.9/(\mu_{ij}\sqrt{\phi_i\phi_j})$ for all $i = k+1, \dots, k_0$. Thus, we have

$$\mathbb{P}\left(\tau\left|\frac{\bar{\mathbf{R}}_{k+1, j}(s, :) \mathbf{v}_j}{\|\mathbf{v}_j\|_1}\right| > 1\right) \leq \mathbb{P}\left(\tau\left|\frac{(\bar{\mathbf{R}}_{k+1, j}(s, :) - \mu_{i(s), j}\sqrt{\phi_{i(s)}\phi_j}\mathbf{e}_{n_j}^T)\mathbf{v}_j}{\|\mathbf{v}_j\|_1}\right| > 0.1\right),$$

where $i(s)$ is the corresponding original block (row) index for the s th row of $\bar{\mathbf{R}}_{k+1, j}$. Since $\|\mathbf{v}_j\| = 1$, $(\bar{\mathbf{R}}_{k+1, j}(s, :)/\sqrt{\phi_{i(s)}\phi_j} - \mu_{i(s), j}\mathbf{e}_{n_j}^T)\mathbf{v}_j$ is a b -subgaussian random variable. Thus, by (3.2), we have:

$$\mathbb{P}\left(\tau\left|\frac{(\bar{\mathbf{R}}_{k+1, j}(s, :) - \mu_{i(s), j}\sqrt{\phi_{i(s)}\phi_j}\mathbf{e}_{n_j}^T)\mathbf{v}_j}{\|\mathbf{v}_j\|_1}\right| > 0.1\right) \leq 2\exp\left(-\frac{0.1^2\|\mathbf{v}_j\|_1^2}{2b^2\tau^2\phi_{i(s)}\phi_j}\right).$$

To show this is exponentially small, we first analyze the denominator. We start by noting that

$$\begin{aligned}
\tau^2 \phi_{i(s)} \phi_j &\leq (\tau'_u)^2 \phi_{i(s)} \phi_j \\
&\leq c_5(\mathbf{p})^2 \phi_{i(s)} \phi_j / \phi_1^2 \\
&\leq c_5(\mathbf{p})^2 c_0 \phi_j / \phi_1 \\
&\leq c_5(\mathbf{p})^2 c_0 \rho_\sigma (\rho_m \rho_n)^{1/2} \\
&\equiv c_8(\mathbf{p}).
\end{aligned}$$

The second line was obtained from (3.113) and the third from (3.13), and the last line introduces another constant. Combining with (3.85) for the numerator, we obtain the following tail bound:

$$\mathbb{P}(\|\mathbf{V}_{k+1,j}\|_\infty > 1) \leq 2 \exp\left(-\frac{0.1^2 0.3^2 \delta_v^2 n_j}{2b^2 c_8(\mathbf{p})}\right). \quad (3.96)$$

3.3.2 Upper bound on $\|\mathbf{W}_{k+1,j}\|$

Now consider $\mathbf{W}_{k+1,j}$. It is clear that $\mathbf{W}_{k+1,j} \mathbf{v}_j = \mathbf{0}$. In addition, we have:

$$\mathbf{W}_{k+1,j} = \tau \theta \Phi \bar{\mathbf{Q}}_{k+1,j} \left(\mathbf{I} - \frac{\mathbf{v}_j \mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1} \right), \quad (3.97)$$

where $\bar{\mathbf{Q}}_{k+1,j} \in \mathbb{R}^{\bar{m}_{k+1} \times n_j}$ is a b -subgaussian matrix that is a concatenation of $\mathbf{Q}_{l,j}$, $l = k+1, \dots, k_0$ and

$$\Phi = \begin{pmatrix} \sqrt{\phi_{k+1} \phi_j} \mathbf{I}_{m_{k+1}} & & \\ & \ddots & \\ & & \sqrt{\phi_{k_0} \phi_j} \mathbf{I}_{m_{k_0}} \end{pmatrix}.$$

By the same argument as before,

$$\left\| \mathbf{I} - \frac{\mathbf{v}_j \mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1} \right\| \leq 1 + \frac{1}{0.3 \delta_{v,j}} \leq c_7(\mathbf{p}) + 1/2. \quad (3.98)$$

where $c_7(\mathbf{p})$ was defined by (3.95). Also,

$$\|\Phi\| = \sqrt{\bar{\phi}_{k+1} \phi_j}, \quad (3.99)$$

where $\bar{\phi}_{k+1} = \max_{i=k+1, \dots, k_0} \phi_i$. Now suppose

$$\|\bar{\mathbf{Q}}_{k+1,j}\| \leq c_9(\mathbf{p}) (m_j n_j)^{1/2} / \sqrt{c_0}, \quad (3.100)$$

where

$$c_9(\mathbf{p}) = \frac{0.47}{\rho_\sigma(c_7(\mathbf{p}) + 1/2)(k+1)} \quad (3.101)$$

and c_0 was defined in (3.13). (Below we will argue that (3.100) happens with high probability.)

Using the hypothesis (3.100),

$$\begin{aligned} \|\mathbf{W}_{k+1,j}\| &\leq \tau\theta \|\Phi\| \cdot \left\| \mathbf{I} - \frac{\mathbf{v}_j \mathbf{e}_{n_j}^T}{\|\mathbf{v}_j\|_1} \right\| \cdot \|\bar{\mathbf{Q}}_{k+1,j}\| \\ &\leq \tau\theta \sqrt{\bar{\phi}_{k+1} \phi_j (c_7(\mathbf{p}) + 1/2) c_9(\mathbf{p}) (m_j n_j)^{1/2} / \sqrt{c_0}} \\ &\leq \tau\theta \phi_j (c_7(\mathbf{p}) + 1/2) c_9(\mathbf{p}) (m_j n_j)^{1/2} \\ &= \tau\theta \bar{\sigma}_j \frac{0.47}{\rho_\sigma(k+1)} \\ &\leq \frac{0.47\tau\theta}{k+1} \bar{\sigma}_k \\ &\leq \frac{1}{k+1} \cdot \min_{i=1,\dots,k} \sigma_k. \end{aligned}$$

The first line follows from (3.97), the second from (3.99), (3.98), and (3.100). The third and fifth follow from (3.13) and (3.11) respectively, and the last from (3.60).

Now we show that the hypothesis (3.100) holds with high probability using Lemma 3. As mentioned above, \bar{m}_{k+1} denotes the number of rows of $\bar{\mathbf{Q}}_{k+1,j}$, i.e., $m_{k+1} + \dots + m_{k_0}$.

$$\begin{aligned} \mathbb{P}\left(\|\bar{\mathbf{Q}}_{k+1,j}\| > c_9(\mathbf{p})(m_j n_j)^{1/2} / \sqrt{c_0}\right) &\leq \exp\left(-\frac{8c_9(\mathbf{p})^2}{81b^2 c_0} m_j n_j + (\log 7)(\bar{m}_{k+1} + n_j)\right) \\ &= \exp\left(-\frac{4c_9(\mathbf{p})^2}{81b^2 c_0} m_j n_j + (\log 7)\bar{m}_{k+1}\right) \\ &\quad \cdot \exp\left(-\frac{4c_9(\mathbf{p})^2 c_0}{81b^2} m_j n_j + (\log 7)n_j\right). \end{aligned}$$

The second exponent in the second line tends to $-\infty$ linearly with m_j ; the first exponent also tends to $-\infty$ linearly provided that

$$\frac{\bar{m}_{k+1}}{m_j n_j} \leq K < \frac{4c_9(\mathbf{p})^2}{81b^2 c_0 (\log 7)}, \quad (3.102)$$

where K is some constant (independent of m_i, n_i for any i), which holds under the assumption (3.14).

The analysis of $(i, k+1)$ block is similar for $i = 1, \dots, k$.

3.4 Analysis for block $(k+1, k+1)$

3.4.1 Upper bound on $\|\mathbf{V}_{k+1, k+1}\|_\infty$

For the last block $(k+1, k+1)$, we will simply construct $\mathbf{V}_{k+1, k+1} \in \mathbb{R}^{\bar{m}_{k+1} \times \bar{n}_{k+1}}$ from $(k_0 - k)^2$ sub-blocks $\mathbf{V}_{st}^{(k+1)} \in \mathbb{R}^{m_s \times n_t}$,

$$\mathbf{V}_{st}^{(k+1)} = \tau \mu_{st} \sqrt{\phi_s \phi_t} \mathbf{e}_{m_s} \mathbf{e}_{n_t}^T, \quad s, t = k+1, \dots, k_0.$$

Since $\tau \leq \tau_u$, by (3.118) we have: $\tau \leq 0.9/(\mu_{st} \sqrt{\phi_s \phi_t})$ for all $s, t = k+1, \dots, k_0$. Thus, $\|\mathbf{V}_{k+1, k+1}\|_\infty \leq 1$.

3.4.2 Upper bound on $\|\mathbf{W}_{k+1, k+1}\|$

We have, $\mathbf{W}_{k+1, k+1}$ is composed of blocks: $\mathbf{W}_{s,t}^{(k+1)} = \tau \theta (\sqrt{\phi_s \phi_t} \bar{\mathbf{Q}}_{st} + \bar{\mathbf{B}}_{s,t})$, where $\bar{\mathbf{Q}}_{s,t} \in \mathbb{R}^{m_s \times n_t}$. We will write the sum as:

$$\mathbf{W}_{k+1, k+1} = \tau \theta (\Phi_2 \bar{\mathbf{Q}}_{k+1, k+1} \Phi_3 + \bar{\mathbf{B}}_{k+1, k+1})$$

where $\bar{\mathbf{Q}}_{k+1, k+1}$ contains entries chosen from a b -subgaussian distribution, and

$$\Phi_2 = \begin{pmatrix} \sqrt{\phi_{k+1}} \mathbf{I}_{m_{k+1}} & & \\ & \ddots & \\ & & \sqrt{\phi_{k_0}} \mathbf{I}_{m_{k_0}} \end{pmatrix},$$

and

$$\Phi_3 = \begin{pmatrix} \sqrt{\phi_{k+1}} \mathbf{I}_{n_{k+1}} & & \\ & \ddots & \\ & & \sqrt{\phi_{k_0}} \mathbf{I}_{n_{k_0}} \end{pmatrix}.$$

We have: $\|\bar{\mathbf{B}}_{k+1, k+1}\| = \max_{l=k+1, \dots, k_0} \bar{\sigma}_l = \bar{\sigma}_{k+1}$ and $\|\Phi_2\| = \|\Phi_3\| = (\bar{\phi}_{k+1})^{1/2}$, where $\bar{\phi}_{k+1}$ was defined as in (3.99). Thus

$$\|\mathbf{W}_{k+1, k+1}\| \leq \tau \theta \bar{\phi}_{k+1} \|\bar{\mathbf{Q}}_{k+1, k+1}\| + \tau \theta \bar{\sigma}_{k+1}. \quad (3.103)$$

Applying the assumption (3.12) to the second term of (3.103), we have:

$$\begin{aligned} \tau \theta \bar{\sigma}_{k+1} &\leq \frac{0.47 \tau \theta \bar{\sigma}_k}{2(k+1)} \\ &\leq \frac{1}{2(k+1)} \cdot \min_{i=1, \dots, k} \sigma_i. \end{aligned}$$

The second line follows from (3.60).

Now turning to the first term, let us suppose that

$$\|\bar{\mathbf{Q}}_{k+1,k+1}\| \leq \frac{0.23\sqrt{m_l n_l}}{(k+1)c_0}, \quad (3.104)$$

where c_0 is from (3.13) and l is the index of the min of $\sigma_1, \dots, \sigma_k$. (Below we will argue that this holds with probability exponentially close to 1.) Then

$$\begin{aligned} \tau\theta\bar{\phi}_{k+1}\|\bar{\mathbf{Q}}_{k+1,k+1}\| &\leq \frac{0.23\tau\theta\phi_{k+1}\sqrt{m_l n_l}}{(k+1)c_0} \\ &\leq \frac{0.23\tau\theta\phi_l\sqrt{m_l n_l}}{k+1} \\ &= \frac{0.46\tau\theta\bar{\sigma}_l}{2(k+1)} \\ &\leq \frac{1}{2(k+1)} \cdot \min_{i=1,\dots,k} \sigma_i. \end{aligned}$$

The second line uses (3.104) and the last line uses (3.60) and the choice of l . Thus, we have analyzed both of the terms of (3.103) and established $\|\mathbf{W}_{k+1,k+1}\| \leq \frac{1}{k+1} \min_{i=1,\dots,k} \sigma_i$ as required.

We now analyze the probability that (3.104) fails. According to Lemma 3

$$\mathbb{P}\left(\|\bar{\mathbf{Q}}_{k+1,k+1}\| > \frac{0.23\sqrt{m_l n_l}}{(k+1)c_0}\right) \leq \exp\left(-\frac{8 \cdot 0.23^2 m_l n_l}{81b^2(k+1)c_0} + (\log 7)(\bar{m}_{k+1} + \bar{n}_{k+1})\right).$$

This quantity tends to zero exponentially fast as long as

$$\frac{\bar{m}_{k+1} + \bar{n}_{k+1}}{\min_{i=1,\dots,k} m_i n_i} \leq K < \frac{8 \cdot 0.23^2}{81b^2(k+1)c_0(\log 7)}, \quad (3.105)$$

where K is some constant (independent of the matrix size), which holds under the assumption (3.14).

3.5 Definitions of the scalars

The definitions of the scalars appearing in the theorem and the proof can now be provided based on the inequalities developed during the proof.

We start by defining τ_ℓ as follows:

$$\tau_\ell = c_4(\mathbf{p})\phi_1^{-1}, \quad (3.106)$$

where $c_4(\mathbf{p})$ is defined as

$$c_4(\mathbf{p}) = \rho_\sigma \sqrt{\rho_m \rho_n} \max\left\{6(k+1)\sqrt{\rho_m \rho_n}, \frac{4}{\xi_u}, \frac{4}{\xi_v}, 2 + \frac{2}{\delta_u}, 2 + \frac{2}{\delta_v}\right\}. \quad (3.107)$$

Applying inequality (3.79), the following inequalities indeed hold:

$$\tau_\ell \geq 6(k+1)\sqrt{\rho_m\rho_n} \max_{i=1,\dots,k} \phi_i^{-1}, \quad (3.108)$$

$$\tau_\ell \geq \max\left\{\frac{4}{\xi_u}, \frac{4}{\xi_v}\right\} \max_{i=1,\dots,k} \phi_i^{-1}, \quad (3.109)$$

$$\tau_\ell \geq \left(2 + \max\left\{\frac{2}{\delta_u}, \frac{2}{\delta_v}\right\}\right) \max_{i=1,\dots,k} \phi_i^{-1}. \quad (3.110)$$

The constant $c_3(\mathbf{p})$ is then defined as follows:

$$c_3(\mathbf{p}) = \frac{1}{2} \left(1.2^4 (c_4(\mathbf{p}))^2 \left(\frac{k\rho_m\rho_n}{k + \rho_m\rho_n - 1}\right) + 1\right)^{-1/2}. \quad (3.111)$$

Next, we define

$$\tau_u = c_5(\mathbf{p})\phi_1^{-1}, \quad (3.112)$$

where

$$c_5(\mathbf{p}) = \frac{4}{3} \left(\frac{1 + (k-1)\sqrt{\rho_m\rho_n}}{1 + (k-1)\rho_\sigma^2} + \sqrt{\left(\frac{1 + (k-1)\sqrt{\rho_m\rho_n}}{1 + (k-1)\rho_\sigma^2}\right)^2 + \frac{(1 + (k-1)\rho_m\rho_n)(c_3(\mathbf{p})^{-2} - 1)}{1 + (k-1)\rho_\sigma^2}} \right). \quad (3.113)$$

Note that $c_4(\mathbf{p}) = \frac{25}{36} \cdot \sqrt{\frac{k + \rho_m\rho_n - 1}{k\rho_m\rho_n}} \cdot \sqrt{c_3(\mathbf{p})^{-2}/4 - 1}$ from (3.111), which implies $c_4(\mathbf{p}) < c_5(\mathbf{p})$ or $\tau_\ell < \tau_u$.

We now define

$$c_2(\mathbf{p}, c_0) = \frac{1}{c_5(\mathbf{p})} \cdot \min\left\{\frac{0.3}{\rho_\sigma\sqrt{\rho_m\rho_n}}, \frac{0.9}{(c_0\rho_\sigma\sqrt{\rho_m\rho_n})^{1/2}}, \frac{0.9}{\sqrt{c_0}}\right\}. \quad (3.114)$$

Clearly, since $c_5(\mathbf{p}) \geq (4/3)(1 + (c_3(\mathbf{p}))^{-1}) \geq 4$ and $\rho_\sigma, \rho_m, \rho_n \geq 1$, we have: $c_2(\mathbf{p}, c_0) \leq 0.075 < 0.08$.

Now, using (3.79) and the upper bound $\phi_i/\phi_j \leq c_0$ for all $i = 1, \dots, k+1$ and all $j = 1, \dots, k$ (a restatement of (3.13)), the following inequalities indeed hold:

$$\tau_u \leq \min_{i=1,\dots,k} \frac{0.3}{\mu_{ii}\phi_i}, \quad (3.115)$$

$$\tau_u \leq \min_{i,j=1,\dots,k} \frac{0.3}{\mu_{ij}\sqrt{\phi_i\phi_j}}, \quad (3.116)$$

$$\tau_u \leq \min_{i=k+1,\dots,k_0; j=1,\dots,k} \frac{0.9}{\max\{\mu_{ij}, \mu_{ji}\}\sqrt{\phi_i\phi_j}}, \quad (3.117)$$

$$\tau_u \leq \min_{i,j=k+1,\dots,k_0} \frac{0.9}{\mu_{ij}\sqrt{\phi_i\phi_j}}. \quad (3.118)$$

The last scalar to define is $c_1(\mathbf{p}, c_0, b)$. We define it as follows:

$$c_1(\mathbf{p}, c_0, b) = \min\left(\frac{4c_9(\mathbf{p})^2}{81b^2c_0(\log 7)}, \frac{8 \cdot 0.23^2}{81b^2(k+1)c_0(\log 7)}\right), \quad (3.119)$$

where $c_9(\mathbf{p})$ was defined by (3.101).

3.6 Examples

We consider a simple example that involves a bipartite graph $G = (U, V, E)$ with two non-overlapping bicliques given by $U_1 \times V_1$ and $U_2 \times V_2$, where $U_1 \cap U_2 = \emptyset$ and $V_1 \cap V_2 = \emptyset$. The remaining edges in E are inserted at random with probability p . The U -to- V adjacency matrix can be written in the form $\mathbf{A} = \mathbf{B} + \mathbf{R}$, where \mathbf{B} is a block diagonal matrix with $k_0 = 3$ diagonal blocks, the last of which is a block of all zeros while the other two of which are blocks of all ones. If $U_1 \cup U_2 = U$ and $V_1 \cup V_2 = V$, we can consider \mathbf{B} with just $k_0 = 2$ diagonal blocks. We also assume that $|U_1| = |U_2| = 1/2 |U| = m/2$ and $|V_1| = |V_2| = 1/2 |V| = n/2$. We would like to find these $k = 2$ planted bicliques within the graph G under the presence of random noise simultaneously.

For this example, $\bar{\mathbf{u}}_i = \mathbf{e}_{m_i}/\sqrt{m_i}$ and $\bar{\mathbf{v}}_i = \mathbf{e}_{n_i}/\sqrt{n_i}$ for $i = 1, 2$. In addition, $\bar{\sigma}_i = \sqrt{m_i n_i}$, $i = 1, 2$, which means $\phi_1 = \phi_2 = 1$. We can then choose $\rho_u = \rho_v = 1$, $\xi_u = \xi_v = 1$, $\pi_u = \pi_v = 1$, $\rho_m = \rho_n = 1$, and $\rho_\sigma = 1$. Under the random setting described above, $\mu_{ij} = p$ for all $i \neq j = 1, 2$. Given that $k = k_0$, we can set $c_0 = 0$ and there is no need to consider the conditions related to noise blocks. With $\bar{\mathbf{u}}_i = \mathbf{e}_{m_i}/\sqrt{m_i}$ and $\bar{\mathbf{v}}_i = \mathbf{e}_{n_i}/\sqrt{n_i}$ for $i = 1, 2$, the analysis is simpler and we only need $c_4(\mathbf{p}) = 2$ since (3.108) and (3.109) are not needed while (3.110) can be relaxed to $\tau_\ell \geq 2 \max_{i=1, \dots, k} \phi_i^{-1}$. The constant $c_3(\mathbf{p})$ has a better approximation:

$$c_3(\mathbf{p}) = \frac{1}{2} \left(\frac{36}{25} c_4(\mathbf{p}) - 1 \right)^{-1} = \frac{25}{94} \approx 0.266.$$

We then can compute $c_2(\mathbf{p}, c_0)$ as follows:

$$c_2(\mathbf{p}, c_0) = 0.3/c_5(\mathbf{p}) = (0.9/4) \left(1 + (c_3(\mathbf{p}))^{-1} \right)^{-1} \approx 0.047,$$

which means with $p \leq 0.047$, we are able to recover two planted cliques using the proposed convex formulation in (2.8) with $0.376 \cdot (mn)^{-1/2} \leq \theta \leq 0.752 \cdot (mn)^{-1/2}$ with high probability. The results are quite restricted given the way how we construct the dual solutions solely based on matrices of all ones. Having said that, these conditions are theoretical sufficient conditions. Practically, the convex formulation (2.8) with a wider range of θ can recover planted bicliques under the presence of more random noise, i.e., higher probability p . The numerical computation is performed with CVX [8] for the biclique example discussed here with $m = n = 50$. We test the problem with 10 values of p ranging from 0.05 to 0.5. For each value of p , we construct a random matrix \mathbf{A} and solve (2.8) with 20 different values of θ ranging from 0.005 to 1.0. The solution \mathbf{X} is scaled so that the maximum value of its entries is 1. We compare \mathbf{X} and \mathbf{B} by taking the maximum differences between their entries in diagonal blocks

of \mathbf{B} , $\delta_1 = \max_{(i,j):B(i,j)=1} \{1 - X(i, j)\}$, and that of off-diagonal blocks, $\delta_0 = \max_{(i,j):B(i,j)=0} \{X(i, j)\}$. For this example, we are not able to recover two planted bicliques, i.e., the block diagonal structure of the matrix \mathbf{B} , with $\theta = 0.005$ for any p given large values for δ_0 and δ_1 . Figure 1 shows the minimum values of θ with which (2.8) can be used to recover the planted bicliques when there is a significant reduction in the values of δ_0 and δ_1 . The graph indicates that we need larger θ for the settings with more random noise. Figure 2 plots these differences (in log scale) for $p = 0.30$ and we can see that δ_0 and δ_1 change from 10^{-2} to 10^{-10} between $\theta = 0.03$ and $\theta = 0.04$. Figure 3 and 4 plot the actual values of these differences, δ_0 and δ_1 , for different values of θ and p when the planted bicliques can be recovered. We can see that all of these values are in the order of 10^{-6} or less, which indicates the recovery ability of our proposed formulation for this example under the presence of noise.

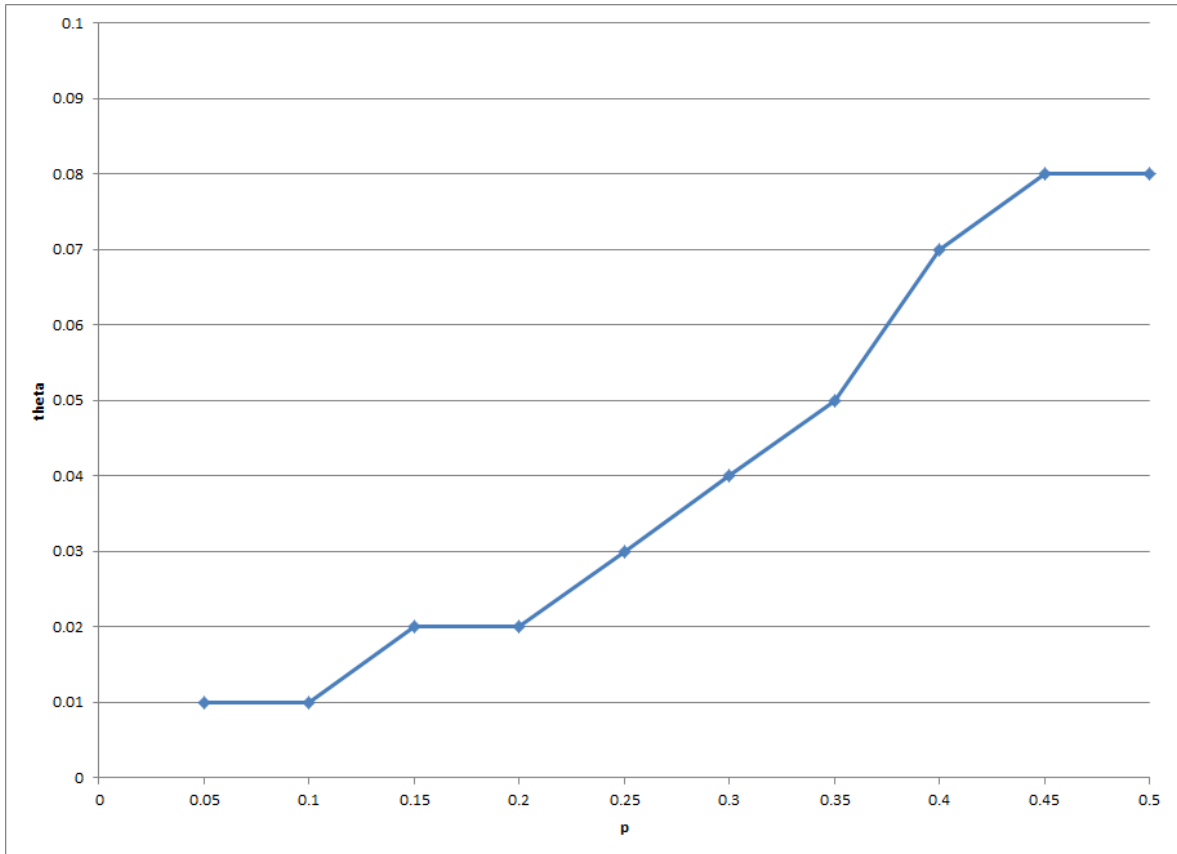


Figure 1: Minimum values of θ to recover two planted bicliques for different values of p

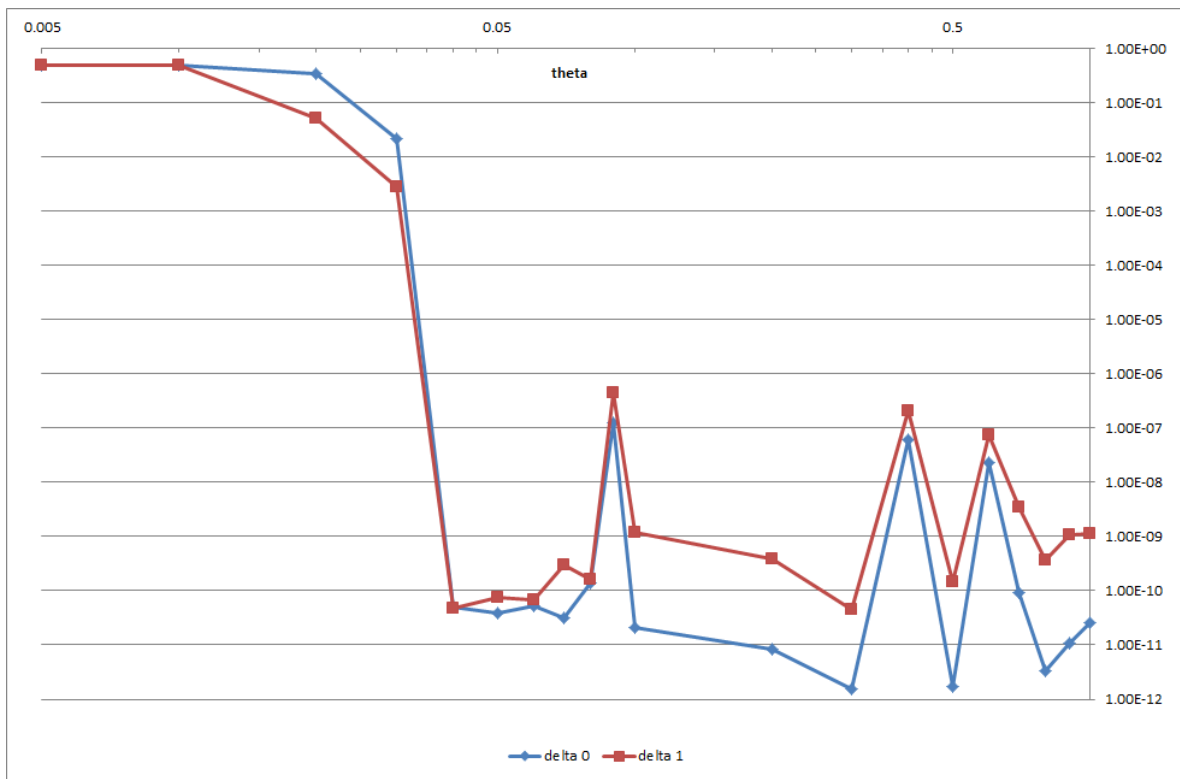


Figure 2: Maximum differences between entries in diagonal blocks and off-diagonal blocks for $p = 0.30$

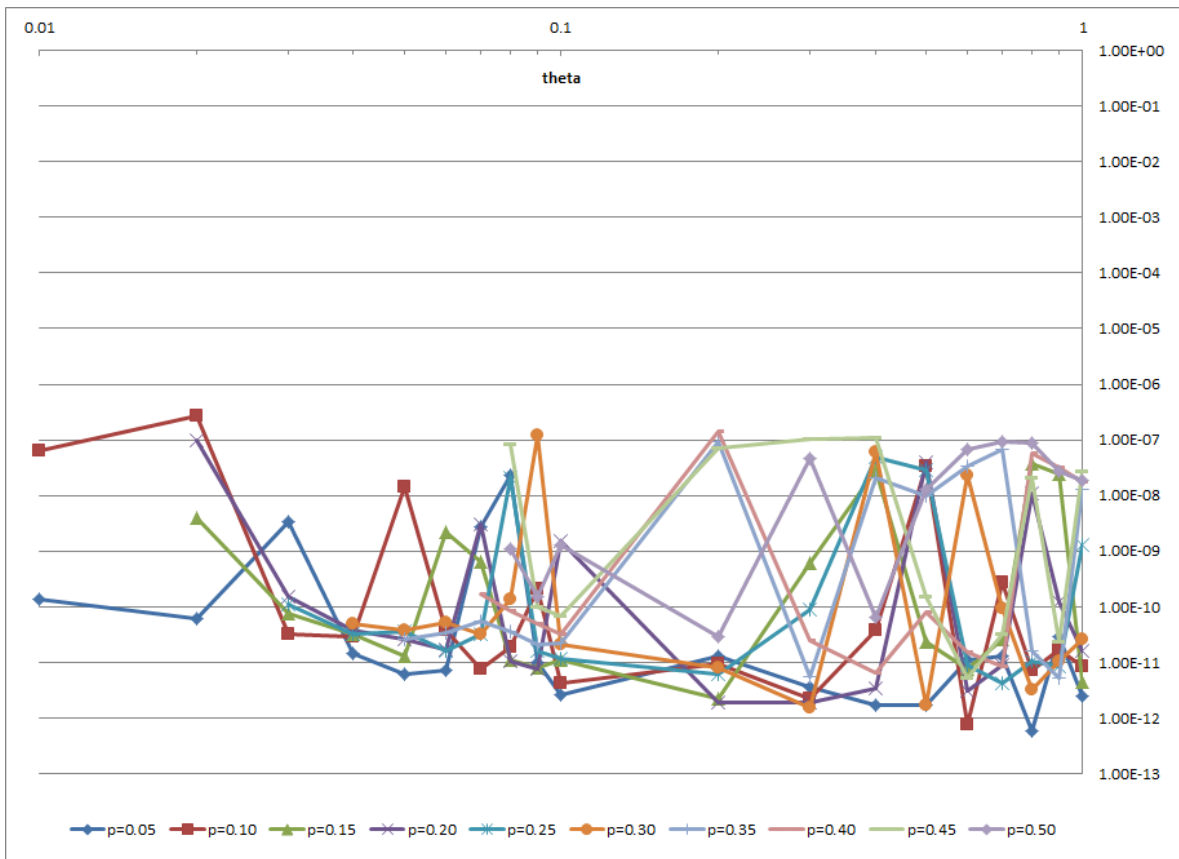


Figure 3: Maximum difference between entries in off-diagonal blocks for different values of p and θ

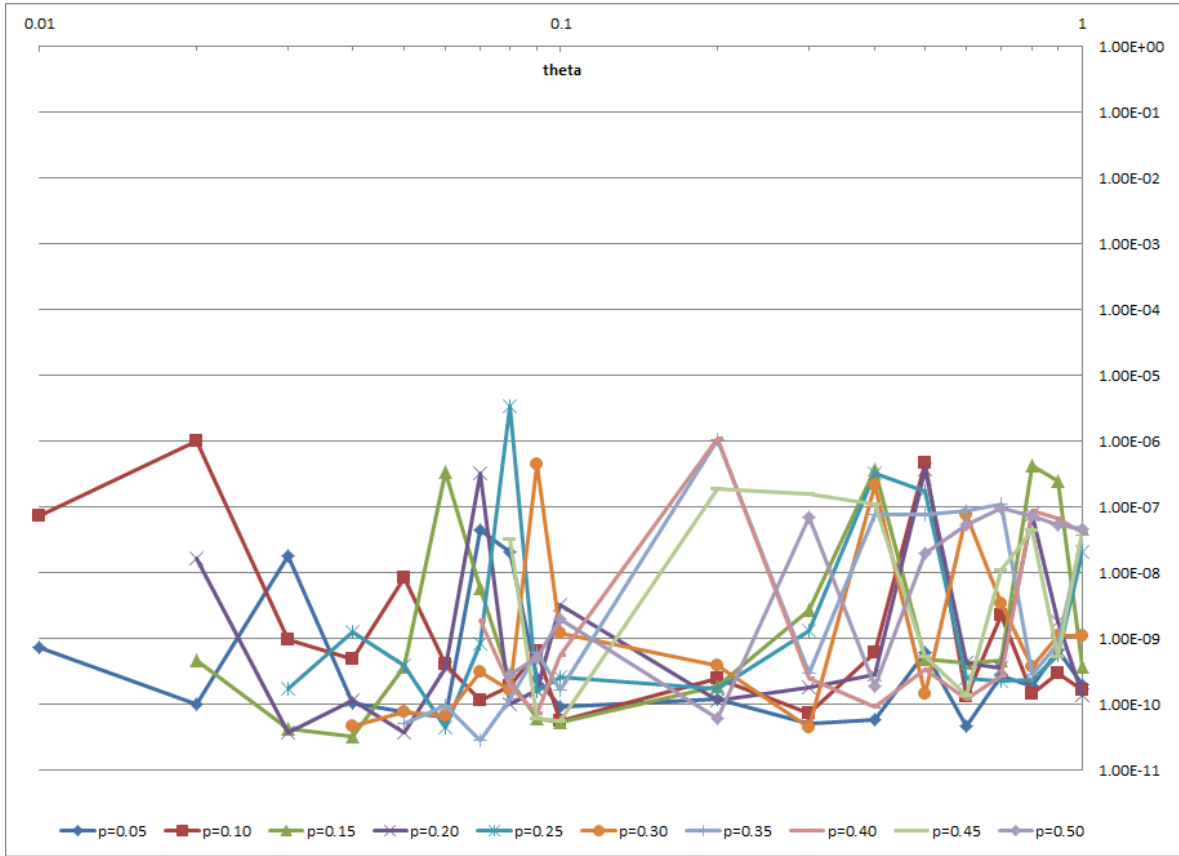


Figure 4: Maximum difference between entries in diagonal blocks for different values of p and θ

4 Conclusions

We have shown that a convex optimization problem with Ky Fan 2 - k -norm and ℓ_1 -norm can recover the k largest blocks of nonnegative block diagonal matrices under the presence of noise under certain conditions. This is an extension of the work in [5] and it could be used in biclustering applications.

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