

An inventory model with shortages for imperfect items using substitution of two products

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Abstract

Inventory models with imperfect quality items are studied by researchers in past two decades. Till now none of them have considered the effect of substitutions to cope up with shortage and avoid lost sales. This paper presents an EOQ approach for inventory system with shortages and two types of products with imperfect quality by one way substitution. Our model provides significant advantage for substitution case while maintaining its inherent simplicity. We have provided numerical example and sensitivity analysis to justify the effectiveness of our model. It is observed that, presence of imperfect items affect the lot size of minor and major products differently. Under certain conditions, our model generalizes the previous existing models in this direction.

keywords- inventory, EOQ, imperfect quality, substitution, shortages

1 Introduction

Inventories are essentially required everywhere . From production to distribution and then to final customer, the role of inventory is incomparable. But absence of inventory irate customers and becomes catastrophic in the sense of future profit or goodwill. The behavior of a customer is unpredictable when a product becomes out of stock. The customer may often quit the store and get the same product from other places; or he may wait for the item; or he may take a similar product from the store itself. The phenomenon in the third case is known as product substitution. For example, consider a customer who want to purchase a tourist bag of a particular brand. If it is not available in the shop the customer may be willing to purchase the other brand rather than going to other shop. The reason may be either shopkeeper insistence by offering similar product, or reputation of the shop or the other brand or it may be the customer's likeliness. This example demonstrates that how the inventory of a product will affect not only its own demand but also the demand of other products. Similar situations may frequently occur in the cases of food products, clothes, hard-drives, furniture, spare-parts, pharmaceuticals(drugs of various brands with identical composition). Therefore, a decision maker should consider the effect of substitution between products in determining the order quantities. In product substitution scenario, a collection of multiple products are available with the possibility to use an alternate

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product from the collection as a substitute whenever the original product runs out of stock. In brief, substitution means demand for a certain quantity of a product is fulfilled by another product. Generally, there is a preferred product for satisfying a specific demand, which can optionally be substituted by certain alternative products (substitutes). The products between which substitution take place may be either different products manufactured by the organization itself or products replenished from one or more suppliers. Classical deterministic inventory models usually do not include product substitutions. However, neglecting substitution option in inventory systems may result in reduced efficiency in terms of customer satisfaction and costs. Sometimes in an inventory model, the products can be indexed in such a manner that a lower-index product may be used as substitute in place of a higher-index product. The product used to meet the demand of another product may be physically brought from another place and incur a transformation cost or it may be used in its original form. There are several advantages of substitution between products in inventory systems. Firstly, stockouts can be managed by the alternative manner using substitutes. It needs to store less amount of inventory of different variety of same type of product, leading to less amount of total holding cost. By ordering larger lot sizes of a smaller number of products (that can substitute for other products), the total ordering cost and lead time can be reduced. Cheaper substitute is profitable in terms of purchase cost also. Substitutions can also be used to reduce the amount of outdated inventory of perishable products, e.g., by consuming substitute stocks first if they have an earlier expiry date. Also it may initiate revenue sharing contract among various systems which allows one to use other's inventory. This increases the possibility of inventory pooling to hedge against demand uncertainties and to help in reducing the safety stocks. Sometimes this concept is also used as backup strategy, when the common product acts as a backup for the standard product. Demand may initially be satisfied by the standard product, and the common product will only be used to fulfill the demand if the standard product becomes out of stock. Many cases are there when two or more generations (batch number, lot number or product version) of same product can be observed with different willingness of purchase by customers. This can also be seen as the two different products with different demands (e.g. Operating Systems like Windows7 and Windows8; Alpha and Beta versions of a software like MATLAB). In this way, we can observe that substitution between products is very much practical and it is thus useful to include its effects on classical inventory models too. Now we provide some literature survey which is related to our paper to grasp the matter more effectively.

2 Literature Survey

Classical Economic order quantity type models are used to determine the optimal inventory policy when the demand is deterministic and the optimal ordering or production quantity are influenced by two opposite types of costs. These models are used to study the optimal inventory lot size that minimizes inventory storage costs and maximizes systems' efficiency. Although,

these models have been successfully applied in the area of inventory management from early decades of previous century; yet they bear few unrealistic assumptions. The fundamental EOQ model developed by Harris (1913) involved the assumption of perfect quality items and no shortages ; both of these conditions fail to cope up with the realistic situations in the business scenario. In reality, the production process not always produce perfect quality items. Imperfect quality items are ineluctable in an inventory system due to imperfect production process, natural disasters, damages, or many other reasons. During last three decades, lot of research work were published in the area of EOQ and EPQ of imperfect quality items. Rosenblatt and Lee (1986) and Porteus (1986) discussed deterministic inventory models where they assumed that the defective items could be reworked at a cost and found that the presence of a fraction of defective items motivate smaller lot sizes. Schwaller (1988) presented a procedure and assumed that imperfect quality items are present in the lot in known proportions. He considered fixed and variable inspection costs for finding and removing the defective items. Zhang and Gerchak (1990) considered a joint lot sizing and inspection policy with the assumption that a random proportion of lot size are defective. They assumed that defective items are not reworkable and used the concept of replacement of defective items by good quality items. Salameh and Jaber (2000) assumed that the defective items could be sold at a discounted price in a single batch by the end of the 100 % screening process and found that the economic lot size quantity tends to increase as the average percentage of imperfect quality items increase. Goyal and Cardenas-Barron (2002) made some modifications in the model of Salameh and Jaber (2000) to calculate actual cost and actual order quantity. From a different viewpoint, Sheu (2004) developed optimal production and preventive maintenance policy for an imperfect production system with two types of out of control states. Papachristos and Konstantaras (2006) pointed out that the sufficient conditions to prevent shortages given in Salameh and Jaber (2000) may not really prevent their occurrence and considering the withdraw time of the imperfect quality items from stock, they clarified a point not clearly stated in Salameh and Jaber (2000). Wee et al. (2007) developed the optimal inventory policy for an inventory system items with imperfect quantity where shortages were backordered. They allowed 100% screening of items in which screening rate is greater than the demand rate. Chung et al. (2009) considered an inventory model with imperfect quality items under the condition of two warehouses for storing items. Jaber et al. (2008) made certain assumption on learning curve and shown that percentage of defective lot size reduces according to the learning curve. A detailed survey of the recent inventory models with imperfect items are provided by Khan and Guiffrida (2011). Roy et al. (2011) developed an EOQ model for imperfect items where a portion of demand were partially backlogged. Amalgamating the effect of product deterioration, imperfect quality, permissible delay and inflation, Jaggi et al. (2011) investigated an inventory model when demand is a function of selling price. Yu et al. (2012) extended Salameh and Jaber (2000) when a portion of the defectives can be utilized as perfect quality and the utilization of the acceptable defective part will affect the consumption of the remaining perfect quality items in the stock. They have considered two cases for presence of imperfect items as constant fraction and random fraction for their study. In another model, Jaber

et al. (2013) investigated Economic Order Quantity (EOQ) model for imperfect quality items under the push-and-pull effect of purchase and repair option, when the option when defectives are repaired at some cost or it is replaced by good items at some higher cost. They presented two mathematical models; one for each case and discussed optimal policies for each case.

The effect of substitution in inventory systems has been studied by various researchers. In particular, the behavior of two-product substitution, where one product can substitute for the other, has been investigated by numerous researchers pioneered by McGillivray and Silver (1978). They assumed identical cost structures and substitution is stockout based but probabilistic. They obtained optimal policy using the simulation and heuristic approach. Parlar and Goyal (1984) considered two item inventory system with substitution under newsvendor framework and obtained optimal order size. Drezner et al. (1995) were the first known researchers who considered the effect of shortage based substitution in deterministic inventory system for two types of items. They have shown that substitution is a profitable policy subject to a condition involving costs and demands, otherwise no substitution is optimal. They obtained the analytical expression of the optimal order quantities of both type items. Goyal (1995) suggested that it may be useful to consider time as a decision variable for the model of Drezner et al. (1995) and suggested an algorithm to obtain optimal policy. Gurnani and Drezner (2000) generalized the model of Drezner et al. (1995) under hierarchical substitution scenarios. Since their formulation became intractable and was not providing the optimal solution, they have remodified their cost function in terms of optimal run out times and obtained the optimal inventory policy numerically. After that, no research work is available in the literature in deterministic inventory system with substitution. But many authors like Nagarajan and Rajagopalan (2008), Yadavalli et al. (2005), Karakul and Chan (2008) etc. have considered substitution in inventory control policy under newsvendor approaches. Pineyro and Viera (2010) applied tabu search procedure to obtain maximizing remanufacturing quantity for economic lot-sizing problem with product returns and one-way substitution. Vaagen et al. (2011) considered stochastic programming and simulation-based efficient optimization heuristics which can be applied to various models. They found that customer driven substitution are more costlier than retailer driven substitution. Liu et al. (2013) studied effect of substitution when two loss averse retailers are competing for substitutable product under stochastic demand rate and deterministic substitution rate. They have developed game theoretic model obtained unique optimal policy for Nash equilibrium under certain conditions. Tan and Karabati (2013) obtained the optimal service level for the profit maximizing policy of a retail product with Poisson arrival processes, stock-out based dynamic demand substitution, and lost sales using genetic algorithm. Recently, Salameh et al. (2014) have solved joint economic lot size model with substitution and observed that substitution is highly beneficial in saving cost. Thus we observe that, previous researchers did not consider the effect of substitution in the EOQ with imperfect items in inventory management studies. The effect of substitute present in a deterministic inventory model can be used to reduce the inventory level. In the larger lot sizes due to imperfect production a substitution effect comes into play that reduces the lot size which is the interesting part of this paper. The main purpose of this

present study is to develop a mathematical model with shortages for computing the economic order quantities where imperfect production process and substitution effect are taken into account.

The organization of the paper is as follows. In Section 3, the basic model of Drezner et al. (1995) is reviewed and solved using time parameters. In Section 4, we modified above model to formulate a new model that considers the effect of imperfect quality items and substitution. In Section 5, numerical example is presented and sensitivity analysis is performed to observe the effect of change of various parameters in the optimal policy. Section 6 gives conclusions and discusses various possible extension of this model. An appendix is also provided for some clarification.

3 Brief review of the Drezner et al's model: Basic Model

Drezner et al. (1995) modified the EOQ model for two items by assuming a fraction of shortage of minor category item is met from major category item. Upon replenishing the inventory, both items are subject to face their own demands during the selling season. Minor items exhaust first and major items are delivered when the demand of minor items arises. The cycle ends when inventory of major items reaches to zero. All other assumptions of this model remain to be the same as for the EOQ model; which include constant demand rates, no shortages, zero-lead time, etc. The behavior of inventory of minor and major items w.r.t. time in Drezner et al. (1995) model is depicted in Figure - 1.

In this model, it is assumed that first item is the major item and second item is the minor item which is substituted by the first item. According to the requirement, if necessary, the first product is converted to second one, item-wise. Thus there is no holding cost for the second item after it is converted into first item. There are two inventory cycles for each product shown in figure 1. We follow the hints provided by Goyal (1995) modified formulation of the model but explain it along the views of original model to adhere some level of simplicity.

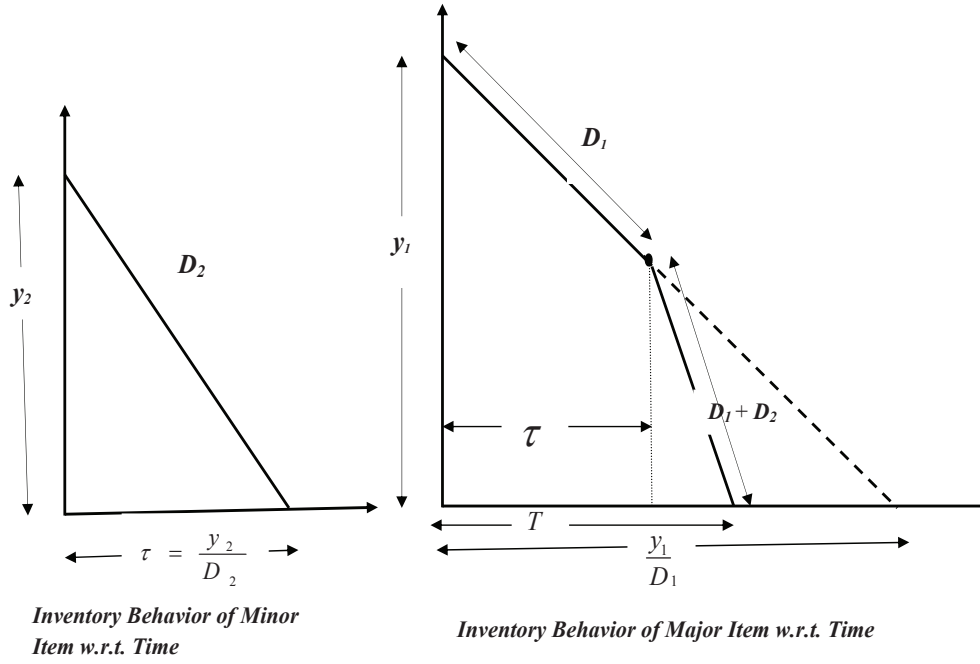


Figure 1-Behavior of Inventory with time in basic model

Throughout this paper, we shall follow the notations given below. Let us assume following notations for reviewing the modified Drezner et al.sDrezner et al. (1995) model:

D_i = the annual demand for the product $i = 1, 2$;

y_i = the order quantity for the product $i = 1, 2$;

c_{hi} = the holding cost for the product $i = 1, 2$;

c_t = the transfer cost for product into product 2

c_o = the ordering cost for both product

τ = the time interval during which no substitution is required.

T = cycle time.

Now, holding cost of each type items:

$$H_1 = c_{h1} \left[\int_0^T (D_1 t) dt + \int_0^\tau (D_2 t) dt \right]$$

$$= c_{h1} \left[D_1 \frac{T^2}{2} + D_2 \frac{(T^2 - \tau^2)}{2} \right]$$

$$\text{and, } H_2 = c_{h2} \left[\int_0^\tau D_2 t dt \right] = c_{h2} D_2 \frac{\tau^2}{2}$$

Transfer cost of substituted item:

$$CT = D_2 c_t (T - \tau)$$

Total average cost of system:

$$TAC = \frac{c_o + H_1 + H_2 + CT}{T}$$

$$TAC = \frac{c_o}{T} + c_{h1} \left[\frac{D_1}{2} T + \frac{D_2}{2} \left(T - \frac{\tau^2}{T} \right) \right] + c_{h2} \frac{D_2 \tau^2}{2T} + D_2 c_t \left(1 - \frac{\tau}{T} \right) \quad (1)$$

In above formulation there will be three cases :

(i) *Partial Substitution*, (ii) *Full Substitution*, and (iii) *No Substitution*.

We first consider the case of **partial substitution case** in which $0 < \tau < T$ and $y_2 > 0$.

We have from $\frac{\partial TAC}{\partial \tau} = 0$,

$$\frac{-c_{h1}D_2\tau + c_{h2}D_2\tau - D_2c_t}{T} = 0 \quad (2)$$

which gives the optimal value of τ as,

$$\tau^* = \frac{c_t}{c_{h2} - c_{h1}} \quad (3)$$

Also, $\frac{\partial^2 TAC}{\partial \tau^2} = D_2(c_{h2} - c_{h1})$

which is positive since it is assumed that $c_{h2} > c_{h1}$. Thus TAC is convex.

But if $\frac{c_t}{c_{h2} - c_{h1}} > T$, we observe that time interval during which no substitution is necessary, is more than the cycle time T which is absurd. Thus we have $0 \leq \tau \leq T$ for all practical cases.

Now from $\frac{\partial TAC}{\partial T} = 0$, we obtain:

$$c_{h1} \left(\frac{D_1 + D_2}{2} + \frac{D_2\tau^2}{2T^2} \right) - c_{h2} \frac{D_2\tau^2}{2T^2} + D_2c_t \frac{\tau}{T^2} - \frac{c_o}{T^2} = 0 \quad (4)$$

From equation (4), the optimal value of T^* is obtained as:

$$T^* = \sqrt{\frac{c_o + \frac{D_2\tau^2}{2}(c_{h2} - c_{h1}) - D_2c_t}{c_{h1}(\frac{D_1+D_2}{2})}}$$

Substituting the value of τ from equation (3), we obtain T^* as follows:

$$T^* = \sqrt{\frac{2c_o - \frac{D_2c_t^2}{c_{h2} - c_{h1}}}{c_{h1}(D_1 + D_2)}} \quad (5)$$

and the corresponding optimal order quantity y_2^* becomes:

$$y_2^* = \tau^* D_2 = \frac{c_t D_2}{c_{h2} - c_{h1}} \quad (6)$$

The optimal order quantity of type-1 items y_1^* is clearly the required order quantity for satisfying demand for both items during whole season T_1 minus the optimal order quantity y_2^* .

$$y_1^* = T^*(D_1 + D_2) - y_2^* = T^*(D_1 + D_2) - \frac{c_t D_2}{c_{h2} - c_{h1}} \quad (7)$$

where T^* is given by (5).

Theorem 3.1 *There exist unique optimal substitutional economic order quantity for all possible parameter values if $c_t < \sqrt{\frac{2c_o(c_{h2}-c_{h1})}{D_2}}$.*

Proof : We know that for a convex function local optimum is global optimum. In order to prove the unique optimal solution, it suffice to show the convexity of total cost function. Let us re-write the equation (1) as,

$$TAC = \frac{c_o}{T} + c_{h1} \left[\frac{D_1}{2}T + \frac{D_2}{2} \left(T - \frac{\tau^2}{T} \right) \right] + c_{h2} \frac{D_2}{2} \frac{\tau^2}{T} + D_2 c_t \left(1 - \frac{\tau}{T} \right)$$

Simple calculation gives:

$$\frac{\partial^2 TAC(\tau, T)}{\partial \tau^2} = \frac{D_2(c_{h2}-c_{h1})}{T}$$

$$\frac{\partial^2 TAC(\tau, T)}{\partial T^2} = \frac{1}{4T^3} \left[2c_o - \frac{D_2 c_t^2}{c_{h2}-c_{h1}} \right]$$

$$\frac{\partial^2 TAC(\tau, T)}{\partial \tau \partial T} = D_2 \frac{c_t - (c_{h2}-c_{h1})}{T^2}$$

Now we have by definition,

$$\begin{aligned} \text{Hessian } H_0(\tau, T) &= \frac{\partial^2 TAC(\tau, T)}{\partial \tau^2} \frac{\partial^2 TAC(\tau, T)}{\partial T^2} - \left(\frac{\partial^2 TAC(\tau, T)}{\partial \tau \partial T} \right)^2 \\ &= \frac{D_2 \left(\frac{1}{2} (c_{h2}-c_{h1}) c_o - \left(\frac{3}{4} (c_{h2}-c_{h1})^2 \tau^2 - \frac{3}{2} (c_{h2}-c_{h1}) \tau c_t + c_t^2 \right) D_2 \right)}{T^4} \end{aligned}$$

Putting the value of τ from (3), we obtain:

$$H_0(\tau, T) = \frac{2D_2}{T^4} \left[c_o(c_{h2} - c_{h1}) - \frac{1}{2} D_2 c_t^2 \right]$$

$$\text{Clearly, } H_0(\tau, T) > 0 \text{ if } c_t < \sqrt{\frac{2c_o(c_{h2}-c_{h1})}{D_2}}$$

Hence the proof follows.

Now we study the special cases of above model to incorporate the situations when all the items of minor products are substituted completely by major products and also the case when substitution is not required.

SPECIAL CASES

Case 1: Full Substitution :

In this case, $\tau = 0$, $y_2 = 0$ and $T = \frac{y_1}{D_1+D_2}$

We have, holding cost of major items $H_1 = \frac{c_{h1}}{2}(D_1 + D_2)T^2$, and

holding cost of minor items $H_2 = 0$

Transformation cost $CT = c_t(D_2T)$.

Now the total cost becomes,

$$TAC = \frac{c_{h1}}{2}(D_1 + D_2)T + D_2 c_t + \frac{c_o}{T} \quad (8)$$

$\frac{\partial TAC}{\partial T} = 0$ gives,

$$T^* = \sqrt{\frac{2c_o}{c_{h1}(D_1 + D_2)}} \quad (9)$$

Also $\frac{\partial^2 TAC}{\partial T^2} = \frac{2c_o}{T^3} > 0$

Thus we have the only optimal order quantity of major item:

$$y_2 = D_2 T = D_2 \sqrt{\frac{2c_o}{c_{h1}(D_1 + D_2)}} \quad (10)$$

Case 2: No Substitution :

In this case we have, $\tau = T$ and $\frac{y_1}{D_1} = \frac{y_2}{D_2} = T$

The holding costs of major items and minor items become

$$H_1 = \frac{c_{h1}D_1T^2}{2}, \quad H_2 = \frac{c_{h2}D_2T^2}{2} \quad \text{respectively.}$$

Also, transformation cost $CT = 0$

Now the total cost becomes,

$$TAC = (c_{h1}D_1 + c_{h2}D_2)\frac{T}{2} + \frac{c_o}{T} \quad (11)$$

First order condition $\frac{\partial TAC}{\partial T} = 0$ gives,

$$T^* = \sqrt{\frac{2c_o}{c_{h1}D_1 + c_{h1}D_2}} \quad (12)$$

In this case, also $\frac{\partial^2 TAC}{\partial T^2} = \frac{2c_o}{T^3} > 0$

-which means that convexity criteria of total average cost TAC is satisfied.

Thus we have:

the optimal order quantity of minor item:

$$y_1 = D_1 T^* = D_1 \sqrt{\frac{2c_o}{c_{h1}D_1 + c_{h1}D_2}}$$

the optimal order quantity of major item:

$$y_2 = \frac{D_2}{D_1} y_1 = D_2 \sqrt{\frac{2c_o}{c_{h1}D_1 + c_{h1}D_2}}$$

It can be observed that above formulae are identical with corresponding results in Drezner et al. (1995) model.

4 Our Mathematical Model: EOQ model for imperfect items with shortage and substitution(EOQISS)

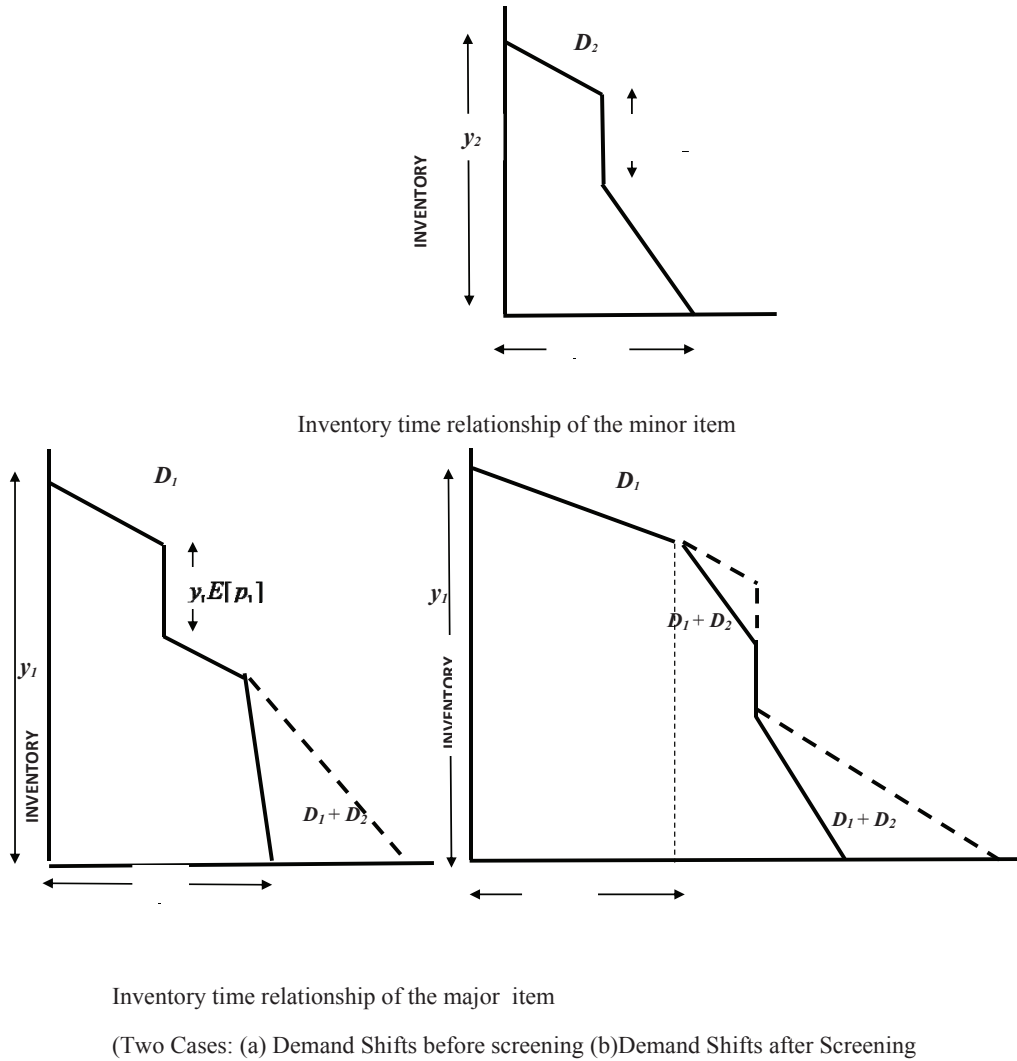


Figure 2-Behavior of Inventory with time in our model

In this section, we incorporate the presence of imperfect items to the problem discussed in aforementioned model. The behavior of inventory of minor and major items w.r.t. time in our model is shown in Figure - 2. We need few more notations and assumptions to develop our model which are explicitly provided in forthcoming subsections.

4.1 Notations

In order to incur such modification we use following additional notations :

p_i = the fraction of imperfect items in the lot of the product $i = 1, 2$;

t_{si} = the time epoch at which the screening finishes for the product $i = 1, 2$;

x_i = the screening (rate)speed for the product $i = 1, 2$;

$E[\cdot]$ denote probabilistic expectation operator.

We have $t_{si} = \frac{y_i}{x_i}$

4.2 Assumptions

The following assumptions are used to establish our mathematical model:

1. Replenishment is instantaneous, Lead time is negligible for both type of items.
2. The demand rate of each type of items are constant and unequal in general.
3. Substitution of first item by second item is allowed only when first item is out of stock (Stockout based substitution).
4. Substitute is abundant whose shortage never occur.
5. A fraction of both types of items received contain imperfect items; which are screened before sending to customers. Such fraction are random variables whose probability distribution follow uniform distribution.
6. Screening rates of both type of defective items greater than their demand rates $x_i > D_i$ for $i = 1, 2$
7. The following relation holds between expected value of defective items:
 $E[p_i] < 1 - \frac{D_i}{x_i}$; where $i = 1, 2$.
8. The inventory reaches zero level at the end of each cycle.
9. The following relation holds between holding cost parameters:

$$c_{h2} > c_{h1}$$

The model we are developing are subjected to two scenarios:

$$(1). \tau \leq t_{s1} \quad \text{and} \quad (2). \tau \geq t_{s1}$$

Note that in both cases, number of perfect quality items of major items remain same. This imply when shortage of minor items will occur, it will be meet from perfect quality major type items. Thus in effect, there is no change in the corresponding optimal policies.

The differential Equations governing the inventory level at time $t \in [0, t_{s2}]$

$$\frac{dI_2}{dt} = -D_2 \tag{13}$$

subject to the initial condition: $I_2(0) = y_2$.

Solving and applying the initial condition,

$$I_2(t) = y_2 - D_2t$$

At time t_{s2} , the initial inventory $= y_2 - D_2t_{s2}$

At time t_{s2} , the final inventory $I_{s2} = y_2(1 - E[p_2]) - D_2t_{s2}$

The differential equation governing the inventory level at time $t \in [t_{s2}, \tau]$ is

$$\frac{dI_2}{dt} = -D_2 \tag{14}$$

subject to $I_2(t_s) = I_{s2}$

The differential equation governing the inventory level at time $t \in [0, \tau]$ is

$$\frac{dI_1}{dt} = -D_1 \quad (15)$$

subject to the initial condition: $I_1(0) = y_1$.

Solving and applying the initial condition, we obtain:

$$I_1(t) = y_1 - D_1 t \quad (16)$$

The differential equation governing the inventory level at time $t \in [\tau, t_{s1}]$ is

$$\frac{dI_1}{dt} = -(D_1 + D_2) \forall t \in [\tau, t_{s1}] \quad (17)$$

Solving (17), we get

$$I_1(t) = y_1 - D_1 t - D_2(t - \tau)$$

At time t_s , the final inventory $I_{s1} = y_1(1 - E[p_1]) - D_1 t_{s1} - D_2(t_{s1} - \tau)$

The differential equation governing the inventory level at time $t \in (t_s, T]$

$$\frac{dI_1}{dt} = -(D_1 + D_2) \forall t \in [t_{s1}, T] \quad (18)$$

subject to the initial condition: $I_2(t_s) = I_{s1}$.

Solution of (18) is

$$I_1(t) = y_1(1 - E[p_1]) - D_1 t + D_2(t - \tau) \quad (19)$$

Now the various cost items are calculated as follows:

$$OC = c_o$$

$$HC = c_{h2} \int_0^\tau I_2(t) dt + c_{h1} \int_0^T I_1(t) dt \quad \text{and} \quad CT = c_t D_2 (T - \tau)$$

The expression of total cost is calculated as:

$$\begin{aligned} TC = & c_{h1} \left[\frac{D_1 + D_2}{2} T^2 + \frac{((T - \tau)D_2 + T D_1)^2 E[p_1]}{(1 - E[p_1])^2 x_1} - \frac{1}{2} D_2 \tau^2 \right] + \\ & c_{h2} \left[\frac{1}{2} D_2 \tau^2 + \frac{E[p_2] D_2 \tau^2}{(1 - E[p_2])^2 x_2} \right] + c_o + D_2 c_t (T - \tau) \end{aligned} \quad (20)$$

We have, Total Average Cost $TAC = \frac{TC}{T}$

Hence,

$$\begin{aligned} TAC = & c_{h1} \left[\frac{D_1 + D_2}{2} T + \frac{((T - \tau)D_2 + T D_1)^2 E[p_1]}{(1 - E[p_1])^2 x_1 T} - \frac{D_2 \tau^2}{2T} \right] + c_{h2} \left[\frac{D_2 \tau^2}{2T} + \frac{E[p_2] D_2 \tau^2}{(1 - E[p_2])^2 x_2 T} \right] \\ & + \frac{c_o}{T} + D_2 c_t \left[1 - \frac{\tau}{T} \right] \end{aligned} \quad (21)$$

The convexity of TAC guarantees the unique optimal solution in terms of τ and T .

We now state the following theorem for convexity of cost function as well optimality and uniqueness of the optimal solution:

Theorem 4.1 *If the parameters of the cost function given by equation (21) satisfy*

$c_o > \frac{c_t^2}{2(c_{h2}-c_{h1})} + \tau(T-1)c_t D_2$, then the optimal cost can be obtained uniquely with optimally unique parameters τ^ and T^* .*

For Proof, see **Appendix-I**.

We obtain first order condition for τ , by differentiating TAC w.r.t. τ , and equating to zero, that

is, $\frac{\partial TAC}{\partial \tau} = 0$, which gives,

$$c_{h1} \left(-2 \frac{((T-\tau)D_2 + TD_1)E[p_1]D_2}{(1-E[p_1])^2 x_1} - D_2 \tau \right) + c_{h2} \left(D_2 \tau + 2 \frac{E[p_2]D_2 \tau}{(-(1-E[p_2]))^2 x_2} \right) - D_2 c_t = 0 \quad (22)$$

Simplification of equation(22) gives

$$\tau = 2 \frac{(1-E[p_2])^2 (\frac{1}{2}c_t(1-E[p_1])^2 x_1 + Tc_{h1}E[p_1](D_1+D_2))x_2}{((1-E[p_2])^2(c_{h2}-c_{h1})x_2 + 2c_{h2}E[p_2])(1-E[p_1])^2 x_1 + 2c_{h1}x_2 D_2 E[p_1](1-E[p_2])^2} \quad (23)$$

Substituting $E[p_1] = E[p_2] = 0$ in equation (23), we obtain,

$$\tau = \frac{c_t}{c_{h2}-c_{h1}}$$

which is same as equation(3).

Hence the optimal order quantity of minor items y_2^* is given by:

$$y_2^* = \frac{D_2 \tau}{1-E[p_2]} \quad (24)$$

where τ is given by equation (23).

In order to obtain optimal order quantity of major item y_1 , we require to find the expression for the optimal value T^* .

We differentiate (21) w.r.t. T , and equating to zero, we obtain the value of T^* as:

$$T^* = \frac{\sqrt{A_1}}{c_{h1}x_2(D_1+D_2)(1-E[p_2])\left(\frac{x_1}{2}(1-E[p_1])^2 + (D_1+D_2)E[p_1]\right)} \quad (25)$$

where A_1 is given by:

$$A_1 = c_{h1}x_2(D_1+D_2)\left(-\frac{1}{2}(1-E[p_1])^2\left(\tau((c_{h1}-h_2)\tau + 2c_t)D_2 - 2c_o\right)(1-E[p_2])^2x_2 + 2c_{h2}D_2\tau^2E[p_2]x_1 + c_{h1}x_2E[p_1]D_2^2\tau^2(1-E[p_2])^2\left(\frac{1}{2}(1-E[p_1])^2x_1 + (D_1+D_2)E[p_1]\right)\right)$$

Now the optimal order quantity of major item is given by:

$$y_1 = \frac{(D_1+D_2)T^* - D_2\tau^*}{1-E[p_1]} \quad (26)$$

The equation (26) can be visualized as the extension of equation (5) in terms of imperfect quality items.

By substituting $E[p_1] = E[p_2] = 0$ and $\tau = \frac{c_t}{c_{h2}-c_{h1}}$, we obtain T^* which corresponds to the same value as in equation (5).

SPECIAL CASES

Case 1: Full Substitution :

In this case, $\tau = 0$, $y_2 = 0$ and $T = \frac{y_1(1-E(p_1))}{D_1+D_2}$

We have $H_1 = \frac{c_{h1}}{2}(D_1 + D_2)T^2$, $H_2 = 0$ and $CT = D_2c_tT$.

Thus we obtain

$$TAC = c_{h1}\left(\frac{1}{2}(D_1)T + \frac{TD_1^2E[p_1]}{(1-E[p_1])^2x_1}\right) + c_{h2}\left(\frac{1}{2}D_2T + \frac{E[p_2]D_2T}{(1-E[p_2])^2x_2T}\right) + \frac{c_o}{T} \quad (27)$$

Above cost function consists of single variable T only and it is obviously convex because:

$$\frac{\partial^2 TAC(\tau, T)}{\partial T^2} = \frac{2c_o}{T^3} > 0$$

Now the first order optimality condition w.r.t. variable T , that is, $\frac{\partial TAC}{\partial T} = 0$ gives,

$$T^* = \sqrt{\frac{2c_o x_1 x_2 (1-E[p_1])^2 (1-E[p_2])^2}{x_1 (1-E[p_1])^2 ((c_{h1}D_1 + c_{h2}D_2)(1-E[p_2])^2 + 2c_{h2}D_2E[p_2]) + 2x_2 c_{h1}D_1^2 E[p_1](1-E[p_2])^2}} \quad (28)$$

Now, we obtain the value of optimal order quantity of major item y_1^* using equation (26) by substituting the value of T^* calculated above and putting $\tau^* = 0$ in equation (26).

$$\text{Thus } y_1^* = \frac{(D_1+D_2)T^*}{1-E[p_1]}$$

Case 2: No Substitution :

In this case we have, $\tau = T$.

Total average cost function becomes

$$TAC = c_{h1}\left(\frac{1}{2}(D_1 + D_2)T + \frac{T(D_1 + D_2)^2 E[p_1]}{(1-E[p_1])^2 x_1}\right) + \frac{c_o}{T} + D_2c_t$$

In this case also, the convexity of TAC holds because $\frac{\partial^2 TAC}{\partial T^2} = \frac{2c_o}{T^3} > 0$

The first order condition gives: $\frac{\partial TAC}{\partial T} = 0$ gives,

$$T^* = \sqrt{\frac{2c_o x_1}{c_{h1}(D_1 + D_2)\left((D_1 + D_2)E[p_1] + \frac{x_1(1-E[p_1])^2}{2}\right)}} \quad (29)$$

Now, from equation (24) using the value of T^* calculated above and $\tau^* = T^*$, we derive the value of y_1^* as

$$y_1^* = \frac{D_1 T^*}{1-E[p_1]}$$

Also we have from equation (24)

$$y_2^* = \frac{D_2 T^*}{1-E[p_2]}$$

Above expression of y_1^* and y_2^* shows that each of the major and minor items satisfies their demand independently and there is no demand interactions- which is the consequence of no substitution assumption.

Remark:

Since some moderate cost is associated with the transformation of minor item to major item and the holding cost of major item is higher than that of minor item; it becomes obvious that full substitution is never optimal. This was also true for Drezner et al. (1995) model. This will help us to develop simple solution procedure which is provided in next subsection.

4.3 Solution Procedure

Due to suboptimality of full substitution, we have developed an algorithm resembling the algorithm of Goyal (1995) that will determine the optimal policy of our model for the case of partial substitution and no substitution. The algorithm is straightforward and provides optimal policy in simple way. We now state the algorithm as follows:

Algorithm:

Step-1: Calculate $T_0 = \sqrt{\frac{2c_o x_1}{c_{h1}(D_1+D_2)\left((D_1+D_2)E[p_1] + \frac{x_1(1-E[p_1])^2}{2}\right)}}$ and $\tau^* = 2 \frac{(1-E[p_2])^2\left(\frac{1}{2}c_t(1-E[p_1])^2 x_1 + T c_{h1} E[p_1](D_1+D_2)\right)x_2}{((1-E[p_2])^2(c_{h2}-c_{h1})x_2 + 2c_{h2} E[p_2])(1-E[p_1])^2 x_1 + 2c_{h1} x_2 D_2 E[p_1](1-E[p_2])^2)}$

Step-2: If $T_0 > \tau$, go to next step (partial substitution case); otherwise, assign the optimal values(no substitution case) as follows:

$$T^* = T_0 \quad y_1^* = \frac{(D_1)T^*}{1-E[p_1]} \quad y_2^* = \frac{(D_2)T^*}{1-E[p_2]} \text{ and stop.}$$

Step-3: Evaluate optimal values of T^* using equation (25).

With this value of T^* , optimal order quantities are then obtained as $y_1^* = \frac{(D_1+D_2)T^* - D_2\tau^*}{1-E[p_1]}$ and $y_2^* = \frac{(D_2)\tau^*}{1-E[p_2]}$.

5 Numerical Example and Sensitivity Analysis**5.1 Numerical Example**

In order to compare our work with that of Drezner et al. (1995), we have taken the data from their paper and solved the basic model in terms of τ and T and obtained identical result for order quantities. The parameters are as follows:

$$D_1 = D_2 = 1000 \text{ units/year, } c_o = \$4500/\text{cycle and } c_{h1} = \$1/\text{unit/year.}$$

We include following new parameters in our model $x_1 = 175200$, $x_2 = 175100$,

Let p_1 and p_2 be uniformly distributed random variables whose expected values are $E[p_1]$ and $E[p_2]$. We consider following three cases: (a) $E[p_2] - E[p_1]$ is very small, (b) $E[p_2] - E[p_1]$ is high. (c) $E[p_2] < E[p_1]$ Beside that, we consider the cases of partial substitution, full substitu-

tion and no substitution situations in basic model as well as our (EOQISS) model.

Numerical results of basic model which matches with Drezner et al. (1995) model is given in Table 1. Results for our(EOQISS) model for three different pair of values of $E[p_1]$ and $E[p_2]$ is shown in Table 2.

	Partial Substitution			Full Substitution			No Substitution		
	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC
2	(1,2)	(3000,1000)	5000.00	(0,2.12)	(2121.32,0)	5242.64	(1.732,1.732)	(1732.05,1732.05)	5196.15
11	(0.1,2.11)	(4119,100)	5219.00	(0,2.12)	(2121.32,0)	5242.64	(0.866,0.866)	(866.02,866.02)	10392.30
1001	(0.001,2.12)	(4241.4,1)	5242.40	(0,2.12)	(2121.32,0)	5242.64	(0.0947,0.0947)	(94.77,94.77)	94963.15

Table 1: Table for basic model: EOQ model with substitution

c_{h2}	Partial Substitution			Full Substitution			No Substitution		
	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC
2	(1.001,1.999)	(1021.43,3059.18)	5000.53	(0,2.12)	(0,2380.49)	5243.65	(1.732,1.732)	(1767.40,1767.40)	5196.36
11	(0.101,2.109)	(4204.28,103.06)	5219.96	(0,2.12)	(0,2380.49)	5243.65	(0.866,0.866)	(795.30,795.30)	10392.40
1001	(0.001,2.121)	(4325.51,1.02)	5243.41	(0,2.12)	(0,2380.49)	5243.65	(0.095,0.095)	(96.94,96.94)	94963.19

(a) $E[p_1]=0.02, E[p_2]=0.05$

c_{h2}	Partial Substitution			Full Substitution			No Substitution		
	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC
2	(.20,2.097)	(4075.56,222.433)	5196.14	(0,2.12)	(0,2380.49)	5243.65	(1.732,1.732)	(1767.35,1924.50)	5196.36
11	(0.10,2.109)	(4201.94,111.22)	5219.96	(0,2.12)	(0,2380.49)	5243.65	(0.866,0.866)	(883.67,962.22)	10392.41
1001	(0.001,2.121)	(4326.93,1.112)	5243.41	(0,2.12)	(0,2380.49)	5243.65	(0.094,0.094)	(95.92,105.56)	94963.19

(b) $E[p_1]=0.02, E[p_2]=0.1$

c_{h2}	Partial Substitution			Full Substitution			No Substitution		
	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC	(τ, T)	(y_1, y_2)	TAC
2	(1,1.997)	(3326.67,1020.41)	5003.16	(0,2.118)	(0,2588.67)	5248.61	(1.732,1.732)	(1924.44, 1767.35)	5197.37
11	(0.100,2.100)	(4555.56,102.04)	5254.67	(0,2.118)	(0,2588.67)	5248.61	(0.866,0.866)	(962.22, 883.67)	10392.91
1001	(0.001,2.120)	(4710.00,1.02)	5248.38	(0,2.118)	(0,2588.67)	5248.61	(0.095,0.095)	(105.56,95.56)	94963.23

(c) $E[p_1]=0.1, E[p_2]=0.02$

Table 2: Table for EOQIS model under various assumptions on imperfect fractions (a) $E[p_1] \approx E[p_2]$ (b) $E[p_1] \ll E[p_2]$ (c) $E[p_1] \gg E[p_2]$

From the above tables, we infer following observations for the inventory model with imperfect quality items:

- For the partial substitution case, we observe that:

- The order quantity y_2 of the minor item obtained by our model is very nearly equal to the corresponding order quantity in the basic model without imperfect items if the expected value of imperfect fraction of minor item is very less. But with small difference w.r.t. expected imperfect fraction value of the major item, the order quantity of major item increases moderately when holding cost of the second item increases gradually. The total average cost in above situation is very nearly equal to the corresponding total average cost for perfect items because we have assumed that screening cost is negligible.
- Order quantity y_2 of the minor item obtained by our model decreases when its lot contains higher expected fraction of defective items. This is quite logical in the sense that it reduces the total expected number of defective items in the inventory. Consequently the order quantity of the major item increases and total cost also increases.
- Order quantity y_1 of the major item obtained by our model increases rapidly than that of minor items when its lot contains higher expected fraction of defectives. The increase in the lot size of minor item is very less (about 2% in our case). Corresponding costs have also minor variation compared to basic model.
- For the case of no substitution, we observe that time at which shortages of minor and major item occur are their corresponding cycle times. Both items are satisfied by their own demand and have independent inventory policies. Also we notice the following:
 - When the expected fraction of defectives in both type of items are small, the order quantity of each item increases as the holding cost of the second items increases gradually. Corresponding optimal cost increases negligibly.
 - When the expected fraction of minor items' defectives are higher than that of major items, the optimal lot size increases moderately but the corresponding optimal cost increase negligibly.
 - When the expected fraction of major items' defectives are higher than that of minor items, the optimal lot size increases moderately but the cost increase marginally.
- For the case of full substitution, we observe that order quantity $y_2 = 0$ and run-out time $\tau = 0$. But the lot sizes and costs remain identical for variation in holding costs. Since there is no minor item, therefore $E[p_2] = 0$. Also we notice that:
 - When the expected fraction of major items' defectives increases, the order quantity of major items increases rapidly corresponding to other cases. Associated optimal cost also increases rapidly.
 - Thus we can say that presence of defective items play major role in full substitution case and order quantity and optimal cost are most sensitive to expected fraction of defectives.

It can be stated from the above discussion that other than holding costs, the presence of defective items also play major role in the decision making regarding optimal inventory and costs in all cases of our model.

6 Conclusion

In this paper, we attempt to investigate a two-product deterministic inventory system with imperfect quality items under shortages and one way substitution. We have used an effective method which considers time period viz. run-out time and cycle time as decision variables. Using this approach, we have obtained a simplified closed form expression for the optimal order quantities which is comparable with the model of Drezner et al. (1995). Our work differs from previous research works in the sense that we consider the presence of a fraction of imperfect items in the system, which is screened out and sold at some negligible salvage price at the end of the selling season. Thus, the contribution of this paper is two-fold. First, we modify the condition for optimality (in terms of cycle time) of cost function of the basic model of Drezner et al. (1995) without imperfect items. Another contribution of our paper is to incorporate the effect of imperfect quality items in both type of items when the substitution of a product is permitted. We observed the different behavior of the optimal order quantities and corresponding optimal cost when expected defective fraction is more in major item and otherwise. In particular, it is observed that when there is large imperfect items in minor(substitutable) products, the decision maker has to reduce its lot size whereas if there is a larger quantity of imperfect items in major(substituting) products, he has to reduce its lot size. We have provided an effective algorithm to solve our model. Numerical example and sensitivity analysis are provided to test the validity and effectiveness of our model and to recommend some useful insights from a managerial viewpoint. The areas where our model can be applied include cement, metallic sheets, clothes, electronics items etc. This work can be extended to investigate a general multi-item inventory system. This model can be also modified to consider a problem with multi-level substitution in-spite of single level substitution. It should also be noticed that the system described in this paper assume continuous review inventory policy, which can be extended to a more complex policy in future work. One can incorporate the effect of inspection error, process shifting, learning and forgetting etc. in our model. In addition, we have assumed that the lead-times of the products are negligible in this study, which enabled us to work with simplified model of the the aforementioned inventory system consisting of both products and formulate the standard EOQ type equations. This condition can also be relaxed using positive or stochastic lead time in future. Another useful future modification can be to develop a model for a supply chain system with wholesaler and retailer with substitution in a more general sense. More research is required with wide experimental data set to improve the present analysis herein and to obtain more generalized results, that can be applied to complex inventory systems.

Appendix-I

We have obtained the expression for total average cost as follows:

$$TAC(\tau, T) = c_{h1} \left[\frac{D_1 + D_2}{2} T + \frac{((T - \tau)D_2 + TD_1)^2 E[p_1]}{(1 - E[p_1])^2 x_1 T} - \frac{D_2 \tau^2}{2T} \right] + c_{h2} \left[\frac{D_2 \tau^2}{2T} + \frac{E[p_2] D_2 \tau^2}{(1 - E[p_2])^2 x_2 T} \right] + \frac{c_o}{T} + D_2 c_t \left[1 - \frac{\tau}{T} \right] \quad (30)$$

In order to prove the convexity of $TAC(\tau, T)$, we have to prove the following:

$$H(\tau, T) = \frac{\partial^2 TAC(\tau, T)}{\partial \tau^2} - \frac{\partial^2 TAC(\tau, T)}{\partial T^2} - \frac{\partial^2 TAC(\tau, T)}{\partial \tau \partial T} > 0$$

After calculation we obtain:

$$H(\tau, T) = \frac{1}{x_1 x_2 T^4 (1 - E[p_1])^2 (1 - E[p_2])^2} \left[2x_1 (1 - E[p_1])^2 \left[\left(-\frac{c_t^2}{2} - \tau(c_{h2} - c_{h1})(T - 1)c_t D_2 + c_o(c_{h2} - c_{h1}) \right) (1 - E[p_2])^2 x_2 + 2E[p_2]c_{h2}(\tau c_t(T - 1)D_2 - c_o) \right] + D_2 x_2 c_{h1} (1 - E[p_2])^2 (c_o - \tau c_t D_2(T - 1)) \right]$$

Consider an auxiliary function:

$$G(\tau, T) = 2x_1 (1 - E[p_1])^2 \left[\left(-\frac{c_t^2}{2} - \tau(c_{h2} - c_{h1})(T - 1)c_t D_2 + c_o(c_{h2} - c_{h1}) \right) (1 - E[p_2])^2 x_2 + 2E[p_2]c_{h2}(\tau c_t(T - 1)D_2 - c_o) \right] + D_2 x_2 c_{h1} (1 - E[p_2])^2 (c_o - \tau c_t D_2(T - 1))$$

We have to obtain the condition under which the function $G(\tau, T) > 0$

Now by inspection on the function G , we observe that if the condition $c_o > \frac{c_t^2}{2(c_{h2} - c_{h1})} + \tau(T - 1)c_t D_2$ holds, $G(\tau, T)$ will always positive.

Then so will be $H(\tau, T)$.

Hence the proof of convexity of total cost follows.

References

- Harris, F. W. "How many parts to make at once?", *Factory Magazine*, p.p. 947–950.
- Rosenblatt, M. J. and Lee, H. L. 1986. "Economic production cycles with imperfect production processes", *IIE Transactions* 18: 48–55 .
- Porteus, E. L. 1986. "Optimal lot sizing, process quality improvement and setup cost reduction", *Operations Research* 34: 137–144.
- Schwaller R. L. 1988; "EOQ under inspection costs". *Production and Inventory Management Journal* 29, 22–24.
- Zhang X. and Gerchak Y. 1990. "Joint lot sizing and inspection policy in an EOQ model with random yield", *IIE Transactions* 22: 1, 41–47.
- Salameh M. K. and Jaber M. Y. 2000 . "Economic production quantity model for items with imperfect quality", *International Journal of Production Economics* 64: 59–64.

- Goyal , S. K. 2002 and Cardenas-Barron L. E. “Note on: economic production quantity model for items with imperfect quality a practical approach”, *International Journal of Production Economics* 77: 85–87.
- Shey-H. S., Chen, J. , 2004 “Optimal lot-sizing problem with imperfect maintenance and imperfect production”, *International Journal of Systems Science*, 35:1,pp. 69-77,2004.
- Papachristos, S . and Konstantaras, I. 2006. “Economic ordering quantity models for items with imperfect quality.”, *International Journal of Production Economics* 100: 148–154 .
- Wee H. M. ,Yu J. and Chen M. C. 2007. “Optimal Inventory Model for Items with Imperfect Quality and Shortage Backordering.”, *Omega* 35: 7–11.
- Konstantaras I. , Goyal S. K. and Papachristos S. 2007, Economic ordering policy for an item with imperfect quality subject to the in-house inspection, *International Journal of Systems Science*, 38:6, 473–482,
- Maddaha B. , Jaber M. Y. 2008. “Economic order quantity for items with imperfect quality: Revisited” , *International Journal of Production Economics* 112 : 808-815
- Jaber M. Y. . Goyal S.K. and Imran, M. 2008 “Economic production quantity model for items with imperfect quality subject to learning effects“, *International Journal of Production Economics* vol.115, pp.143–150, 2008.
- Wang P.-C. , Lin Y.-H. , Chen Y. C.& Chen J. M. ,2009. An optimal production lot-sizing problem for an imperfect process with imperfect maintenance and inspection time length, *International Journal of Systems Science*, 40:10, 1051-1061, DOI: 10.1080/00207720902974645
- Chung. K.J. Her ,C. C. and S. D. Lin, “A two-warehouse inventory model with imperfect quality production processes”, *Computers & Industrial Engineering*, vol.56, pp.193–197, 2009.
- Khan M. , Jaber M Y and Guiffrida A. L . 2011. “A review of the extensions of a modified EOQ model for imperfect quality items”, *International Journal of Production Economics* 132: 1–12.
- Roy M. D. , Sana S. S. and Chaudhuri K. S. .2011. “An economic order quantity model of imperfect quality items with partial backlogging”, *International Journal of Systems Science*, vol. 42:8, 1409-1419, 2011.
- Jaggi, C. K. Goel S. K. and Mittal, M.. 2011 “Pricing and Replenishment Policies for Imperfect Quality Deteriorating Items Under Inflation and Permissible Delay in Payments”, *International Journal of Strategic Decision Sciences* , vol. 2 :2, pp. 237–248, 2011.

- Yu H.-F. , Hsu W.-K.& Chang,W.-J. 2012. “EOQ model where a portion of the defectives can be used as perfect quality”, *International Journal of Systems Science*, vol. 43:9, 1689–1698, 2012.
- Jaber Mohamad Y. , Zanoni Simone, Zavanella Lucio E. 2013.“ Economic order quantity models for imperfect items with buy and repair options”, *International Journal of Production Economics*.<http://dx.doi.org/10.1016/j.ijpe.2013.10.014i>
- McGillivray A. R. and Silver E. A., 1978.“International Journal of Production Research”, *INFOR: Information System and Operation Research* , vol. 51:6 , pp. 47–63, 1978.
- Parlar. M. and Goyal. S. K., 1984.“ Optimal ordering decisions for two substitutable products with stochastic demands”, *OPSEARCH* , vol. 21:1, pp. 1–15.1984.
- Drezner Z, Gurnani H and Pasternack B. A, 1995 “An EOQ model with substitutions between products.’, *Journal of the Operational Research Society* vol. 46:7, pp. 887–89, 1995.
- Goyal ,S. K. “A viewpoint on an EOQ model with substitutions between products. 1995”, *Journal of the Operational Research Society* vol. 47:3, pp. 476–477, 1995.
- Gurnani H. and Drezner Z. , “Deterministic hierarchical substitution inventory models”,2000 *Journal of the Operational Research Society*, vol. 51:1, pp. 129-133., 2000.
- Yadavalli V. S. S., De W. Van Schoor C. and Udayabaskaran S, 2005, “ A substitutable two-product inventory system with joint-ordering policy and common demand”, *Applied Mathematics and Computation*, vol. 172:2, 1257–1271. 2005.
- Nagarajan M. and Rajagopalan S, 2008.“ Inventory models for substitutable products: optimal policies and heuristics”, *Management Science* , vol. 54:8: pp.1453–1466.2008.
- Karakul,M. Chan, L. M. A. 2008. “ Analytical and managerial implications of integrating product substitutability in the joint pricing and procurement problem”, *European Journal of Operations Research* vol. 116, pp. 179–204. 2008.
- Pedro Pineyro, Omar Viera; 2010. “The economic lot-sizing problem with remanufacturing and one-way substitution”, *International Journal of Production Economics*, 124, 482-488.
- Hajnalka Vaagen, Stein W. Wallace, Michal Kaut;2011. “Modelling consumer-directed substitution”, *International Journal of Production Economics*, 134, 388–397.
- Baris Tan, Selcuk Karabati;2013. “Retail inventory management with stock-out based dynamic demand substitution”, *International Journal of Production Economics*, 145, 78–87.
- Wei Liu, Shiji Song, Cheng Wu; 2013. “Impact of loss aversion on the newsvendor game with product substitution”, *International Journal of Production Economics*, 141, 352–359.

Salameh M. K. , Yassine A. A., Maddah B. , Ghaddar L. , 2014.“Joint replenishment model with substitution” , *Applied Mathematical Modelling*, DOI: <http://dx.doi.org/10.1016/j.apm.2013.12.008>, 2014.