Technical Report

A Note on Linear On/Off Constraints

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Abstract This note studies compact representations of linear on/off constraints in mixed-integer linear optimization. A characterization of the convex hull of linear disjunctions is given in the space of original variables. This result can improve formulations of mixed-integer linear programs featuring on/off constraints by reducing the integrality gap in a Branch and Bound approach.

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1 Introduction

Given convex functions $f : \mathbb{R}^{n+m} \to \mathbb{R}$, $\mathbf{h} : \mathbb{R}^{n+m} \to \mathbb{R}^q$ and $g_k : \mathbb{R}^n \to \mathbb{R}$, $\forall k \in K$, we are interested in optimisation problems of the form

$$\begin{aligned}
\min\ & f(\mathbf{x}, \mathbf{z}) \\
\text{s.t. } & \mathbf{h}(\mathbf{x}, \mathbf{z}) \leqslant \mathbf{0}, \\
& g_k(\mathbf{x}) \leqslant 0 \text{ if } z_k = 1,\ \forall k \in K \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{Z}^m.
\end{aligned} \tag{Pr}$$

Each $g_k(\mathbf{x}) \leqslant 0$ represents an "on/off" constraint, with z_k as its corresponding indicator variable. $\mathbf{h}(\mathbf{x}, \mathbf{z}) \leqslant \mathbf{0}$ gathers the remaining constraints of the problem. Bounds on variables are assumed to be finite. (Pr) can be reformulated as a disjunctive program

$$\begin{aligned}
\min\ & f(\mathbf{x}, \mathbf{z}) \\
\text{s.t. } & \mathbf{h}(\mathbf{x}, \mathbf{z}) \leqslant \mathbf{0}, \\
& (\mathbf{x}, z_k) \in \Gamma_0^k \cup \Gamma_1^k,\ \forall k \in K \\
& \Gamma_0^k = \{(\mathbf{x}, z_k) : z_k = 0,\ \mathbf{l}^0 \leqslant \mathbf{x} \leqslant \mathbf{u}^0\} \\
& \Gamma_1^k = \{(\mathbf{x}, z_k) : z_k = 1,\ g_k(\mathbf{x}) \leqslant 0,\ \mathbf{l}^1 \leqslant \mathbf{x} \leqslant \mathbf{u}^1\}. \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{Z}^m.
\end{aligned}$$

With this approach, one can define the best convex relaxation of each disjunctive constraint g_k, considered separately, to be the convex hull $conv\left(\Gamma_0^k \cup \Gamma_1^k\right)$. When the set Γ_0^k reduces to a single point ($\mathbf{l}^0 = \mathbf{u}^0 = 0$), $conv\left(\Gamma_0^k \cup \Gamma_1^k\right)$ can be formulated in the space of original variables [1]:

$$conv\left(\Gamma_0 \cup \Gamma_1\right) = closure\left(\Gamma_c\right), \text{ where}$$

$$\Gamma_c = \left\{ \begin{array}{l} (\mathbf{x}, z) \in \mathbb{R}^{n+1} : \\ z g(\mathbf{x}/z) \leqslant 0, \\ z\mathbf{l}^1 \leqslant \mathbf{x} \leqslant z\mathbf{u}^1,\ 0 < z \leqslant 1 \end{array} \right\} \tag{1}$$

This result can be extended to the general case ($\mathbf{l}^0 \neq \mathbf{u}^0$) when functions g are monotonic [2]. Specifically, in the linear case where $g(\mathbf{x}) = \boldsymbol{a}^T\mathbf{x} - b$, the convex hull is given by the following corollary:

Corollary 1 *Let:*

$$\Gamma_0 = \{\ (x,\ z) \in \mathbb{R}^{n+1} : z = 0,\ l_i^0 \leqslant x_i \leqslant u_i^0,\ \forall i \in \{1, \ldots, n\}\ \},$$

$$\Gamma_1 = \{\ (x,\ z) \in \mathbb{R}^{n+1} : z = 1,\ \sum_{i=1}^{n} a_i x_i \leqslant b,\ l_i^1 \leqslant x_i \leqslant u_i^1,\ \forall i \in \{1, \ldots, n\}\ \},\ \textit{non empty.}$$

Then: $\boldsymbol{conv}\,(\boldsymbol{\Gamma_0} \cup \boldsymbol{\Gamma_1}) = \boldsymbol{\Gamma^*}$,

$$\Gamma^* = \left\{ \begin{array}{l} (\mathbf{x}, z) \in \mathbb{R}^{n+1} : \\ \sum\limits_{i \notin S} a_i x_i \leqslant z \left(b - \sum\limits_{\substack{i \in S,\\ a_i < 0}} a_i u_i^1 - \sum\limits_{\substack{i \in S,\\ a_i > 0}} a_i l_i^1 \right) \\ +(1-z) \left(\sum\limits_{\substack{i \notin S,\\ a_i < 0}} a_i l_i^0 + \sum\limits_{\substack{i \notin S,\\ a_i > 0}} a_i u_i^0 \right), \ \forall S \subset I, \\ z l_i^1 + (1-z) l_i^0 \leqslant x_i \leqslant z u_i^1 + (1-z) u_i^0, \ \forall i \in \{1, \dots, n\}, \\ 0 \leqslant z \leqslant 1. \end{array} \right\}$$

where $I = \{i \in \{1, \dots, n\} \mid a_i \neq 0\}$, *denoting the set of variable indices appearing in the linear constraint.*

Proof We only show that $conv\,(\Gamma_0 \cup \Gamma_1) \subseteq \Gamma^*$, the other direction can be obtained by applying Theorem 2 in [2].

One can check that the constraints defined in Γ^* are indeed valid for any subset $S \subset I$.

(i) Suppose that (x, z) is such that $z = 1$, $\mathbf{a}^T x \leqslant b$, $\mathbf{l}^1 \leqslant \mathbf{x} \leqslant \mathbf{u}^1$, then

$$\sum_{\substack{i \in S,\\ a_i < 0}} a_i u_i^1 + \sum_{\substack{i \in S,\\ a_i > 0}} a_i l_i^1 + \sum_{i \notin S} a_i x_i \leqslant \mathbf{a}^T x \leqslant b$$

(ii) Suppose that (x, z) is such that $z = 0$, $\mathbf{l}^0 \leqslant \mathbf{x} \leqslant \mathbf{u}^0$ then

$$-\sum_{\substack{i \notin S,\\ a_i < 0}} a_i l_i^0 - \sum_{\substack{i \notin S,\\ a_i > 0}} a_i u_i^0 + \sum_{i \notin S} a_i x_i = \sum_{\substack{i \notin S,\\ a_i < 0}} a_i (x_i - l_i^0) + \sum_{\substack{i \notin S,\\ a_i > 0}} a_i (x_i - u_i^0) \leqslant 0$$

where the last inequality comes from $\mathbf{l}^0 \leqslant \mathbf{x} \leqslant \mathbf{u}^0$.

□

We can make two remarks. First, if we take $S = \emptyset$ we obtain a big-M-like constraint:

$$\sum_{i \in I} a_i x_i \leqslant bz + (1-z) \left(\sum_{\substack{i \in I\\ a_i < 0}} a_i l_i^0 + \sum_{\substack{i \in I\\ a_i > 0}} a_i u_i^0 \right) \tag{2}$$

Clearly this constraint is needed and is enough to obtain a valid formulation (the other constraints are dominated when $z = 1$). Therefore, a natural question is whether this constraint dominates the others for all values of z. This seems actually not to be the case. (2) would dominate the other constraints if for any S, one has:

$$z\left(\sum_{\substack{i\in S,\\a_i<0}} a_i u_i^1 + \sum_{\substack{i\in S,\\a_i>0}} a_i l_i^1\right) + (z-1)\left(\sum_{\substack{i\notin S,\\a_i<0}} a_i l_i^0 + \sum_{\substack{i\notin S,\\a_i>0}} a_i u_i^0\right) + \sum_{i\notin S} a_i x_i \leqslant$$

$$\sum_{i\in I} a_i x_i + (z-1)\left(\sum_{\substack{i\in I\\a_i<0}} a_i l_i^0 + \sum_{\substack{i\in I\\a_i>0}} a_i u_i^0\right) \quad (3)$$

Re-arranging the terms, this is equivalent to:

$$\sum_{\substack{i\in S,\\a_i<0}} a_i \left(z u_i^1 + (1-z) l_i^0 - x_i\right) + \sum_{\substack{i\in S,\\a_i>0}} a_i \left(z l_i^1 + (1-z) u_i^0 - x_i\right) \leqslant 0 \quad (4)$$

If (4) would hold for any $S \neq \varnothing$ then constraint (2) would dominate all the others. Take for instance $S = \{i\}$ for some i such that $a_i > 0$, then (4) becomes

$$z l_i^1 + (1-z) u_i^0 - x_i \leqslant 0 \quad (5)$$

In general, there is no reason for this inequality to be always valid, note that for $z = 0$, (5) is never satisfied.

The following example shows that in dimension 3, constraint (2) does not dominate the others.

Example 1 Take $X = \left\{(x, z) \in [0,1]^3 \times \{0,1\} : z = 1 \Rightarrow x_1 + x_2 + x_3 \leqslant 1\right\}$. Then $\text{conv}(X)$ is given by the following inequalities:

$$x_1 + x_2 + x_3 \leqslant z + 3(1-z) = 3 - 2z \quad (6)$$

$$x_1 + x_2 \leqslant z + 2(1-z) = 2 - z \quad (7)$$

$$x_1 + x_3 \leqslant z + 2(1-z) = 2 - z \quad (8)$$

$$x_2 + x_3 \leqslant z + 2(1-z) = 2 - z \quad (9)$$

$$x \in [0,1]^3, z \in [0,1] \quad (10)$$

(6) does not dominate (7), (8) and (9). Take for instance the point $x = (1, 1, 0), z = 0.5$, it satisfies (6) but not (7).

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