

Preprint: Considering Copositivity Locally*

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Abstract

We say that a symmetric matrix A is copositive if $\mathbf{v}^T A \mathbf{v} \geq 0$ for all nonnegative vectors \mathbf{v} . The main result of this paper is a characterization of the cone of feasible directions at a copositive matrix A , i.e., the convex cone of symmetric matrices B such that there exists $\delta > 0$ satisfying $A + \delta B$ being copositive. This cone is described by a set of linear inequalities on the elements of B constructed from the so called set of (minimal) zeros of A . This characterization is used to furnish descriptions of the minimal (exposed) face of the copositive cone containing A in a similar manner. In particular, we can check whether A lies on an extreme ray of the copositive cone by examining the solution set of a system of linear equations. In addition, we deduce a simple necessary and sufficient condition for the irreducibility of A with respect to a copositive matrix C .

Keywords: copositive matrix, face, irreducibility, extreme rays

AMS Subject Classification: 15A48, 52A20.

1 Introduction

Let \mathcal{S}^n be the vector space of real symmetric $n \times n$ matrices. A matrix $A \in \mathcal{S}^n$ is called *copositive* if $\mathbf{x}^T A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^n$, where \mathbb{R}_+^n denotes the set of element-wise nonnegative n -vectors. The set of copositive matrices forms a convex cone, the *copositive cone*, \mathcal{COP}^n . This matrix cone is of interest for combinatorial optimization, for surveys see [7, 10, 12, 17]. It is a classical result by Diananda [8, Theorem 2] that for $n \leq 4$ the copositive cone can be described as the sum of the cone of positive semi-definite matrices \mathcal{S}_+^n and the cone of element-wise nonnegative symmetric matrices \mathcal{N}^n . In general, this sum is a subset of the copositive cone, $\mathcal{S}_+^n + \mathcal{N}^n \subset \mathcal{COP}^n$ (we use $\mathcal{A} \subset \mathcal{B}$ to denote that \mathcal{A} is a subset of \mathcal{B} , but not necessarily a strict subset). Prof. Alfred Horn showed that for $n \geq 5$ the inclusion is strict [8, p.25].

In this contribution we investigate properties of the copositive cone related to convex analysis. In particular, we consider the minimal faces of copositive matrices and irreducibility of copositive matrices with respect to other copositive matrices, and relate these to the set of their zeros or their minimal zeros.

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A vector $\mathbf{u} \in \mathbb{R}_+^n$ whose elements sum up to one is called a *zero* of a copositive matrix A if $\mathbf{u}^T A \mathbf{u} = 0$. Note that for all $\lambda > 0$ we have $\mathbf{u}^T A \mathbf{u} = 0$ if and only if $(\lambda \mathbf{u})^T A (\lambda \mathbf{u}) = 0$, and thus the requirement that the elements of \mathbf{u} sum up to one is simply a normalization. In the literature, sometimes other normalizations are used or no normalization is used or instead of a normalization the authors require that $\mathbf{u} \neq \mathbf{0}$. However, it is a relatively trivial matter to transfer between these definitions.

A zero \mathbf{u} of A is called *minimal* if for no other zero \mathbf{v} of A , the index set of positive entries of \mathbf{v} is a strict subset of the index set of positive entries of \mathbf{u} .

A copositive matrix A is called *irreducible* with respect to another copositive matrix C if for every $\delta > 0$, we have $A - \delta C \notin \mathcal{COP}^n$, and it is called irreducible with respect to a subset $\mathcal{M} \subset \mathcal{COP}^n$ if it is irreducible with respect to all nonzero elements $C \in \mathcal{M}$.

It has been recognised early that the zero set of a copositive matrix is a useful tool in the study of the structure of the cone \mathcal{COP}^n [8, 14]. In [3] Baumert considered the possible zero sets of matrices in \mathcal{COP}^5 . He provided a partial classification of the zero sets of matrices $A \in \mathcal{COP}^5$ which are irreducible with respect to the cone \mathcal{N}^5 . In [11] this classification was completed, and a necessary and sufficient condition for irreducibility of a copositive matrix $A \in \mathcal{COP}^n$ with respect to the cone \mathcal{N}^n was given in terms of its zero set. In [15], a similar condition in terms of the minimal zero set was given for irreducibility of a copositive matrix $A \in \mathcal{COP}^n$ with respect to the cone \mathcal{S}_+^n .

In [9], the facial structure and the extreme rays of the copositive cone \mathcal{COP}^n and its dual, the completely positive cone, have been investigated. It has been shown that not every extreme ray of \mathcal{COP}^n is exposed.

Our main result in this paper is a necessary and sufficient condition on a pair (A, B) , where $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$, for the existence of a scalar $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$. For fixed A , the set of all such matrices $B \in \mathcal{S}^n$ forms a convex cone \mathcal{K}^A , which is referred to as the *cone of feasible directions* [18]. We express this cone in terms of the zeros of A and their supports.

The obtained description of the cone \mathcal{K}^A is a powerful tool. It will allow us to compute the minimal face and the minimal exposed face of A . In particular, we obtain a simple test of extremality of A , which amounts to checking the rank of a certain matrix constructed from the minimal zeros of A . The necessary and sufficient conditions for the irreducibility of A with respect to a nonnegative matrix $C \in \mathcal{N}^n$ or a positive semi-definite matrix $C \in \mathcal{S}_+^n$, which have been given in [11] and [15], respectively, are generalized to the case of arbitrary matrices $C \in \mathcal{COP}^n$. The conditions in [11] and [15] follow as particular cases.

The remainder of the paper is structured as follows. In the next section we provide necessary definitions and notations, and in the following section we collect some results from the literature and provide some preliminary results. In Section 4 we provide our main result, the description of the cone \mathcal{K}^A of feasible directions of \mathcal{COP}^n at A . We also compute its closure, the *tangent cone* $\text{cl}(\mathcal{K}^A)$, and the *tangent space* $\text{cl}(\mathcal{K}^A) \cap -\text{cl}(\mathcal{K}^A)$ [18]. In Section 5 we deduce the descriptions of the minimal face and the minimal exposed face of a copositive matrix. In Section 6 we consider irreducibility of a copositive matrix with respect to another arbitrary copositive matrix. Finally, we give a summary in the last section.

2 Notations

We shall denote vectors with bold lower-case letters and matrices with upper-case letters. Individual entries of a vector \mathbf{u} and a matrix A will be denoted by u_i and a_{ij} respectively.

For a matrix A and a vector \mathbf{u} of compatible dimension, the i -th element of the matrix-vector product $A\mathbf{u}$ will be denoted by $(A\mathbf{u})_i$. Inequalities $\mathbf{u} \geq \mathbf{0}$ on vectors will be meant element-wise, where we denote by $\mathbf{0} = (0, \dots, 0)^T$ the all-zeros vector. Similarly we denote by $\mathbf{1} = (1, \dots, 1)^T$ the all-ones vector. We further let \mathbf{e}_i be the unit vector with i -th entry equal to one and all other entries equal to zero. Let $\Delta^n = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{1}^T \mathbf{x} = 1\}$ be the standard simplex in \mathbb{R}^n . For a subset $\mathcal{I} \subset \{1, \dots, n\}$ we denote by $A_{\mathcal{I}}$ the principal submatrix of A whose elements have row and column indices in \mathcal{I} , i.e. $A_{\mathcal{I}} = (a_{ij})_{i,j \in \mathcal{I}} \in \mathcal{S}^{|\mathcal{I}|}$. Similarly for a vector $\mathbf{u} \in \mathbb{R}^n$ we define the subvector $\mathbf{u}_{\mathcal{I}} = (u_i)_{i \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$.

We call a vector $\mathbf{u} \in \Delta^n$ a *zero* of a matrix $A \in \mathcal{S}^n$ if $\mathbf{u}^T A \mathbf{u} = 0$, and we denote the set of zeros of A by $\mathcal{V}^A = \{\mathbf{u} \in \Delta^n \mid \mathbf{u}^T A \mathbf{u} = 0\}$. For a vector $\mathbf{u} \in \mathbb{R}^n$ we define its *support* as $\text{supp } \mathbf{u} = \{i \in \{1, \dots, n\} \mid u_i \neq 0\}$. We also define $\text{supp}_+ \mathbf{u} = \{i \in \{1, \dots, n\} \mid u_i > 0\}$, and note that for zeros of copositive matrices these two notions are equivalent. For a matrix $A \in \mathcal{COP}^n$ and a zero \mathbf{v} of A we define the set $\mathcal{J}(\mathbf{v}, A)$ of indices i such that there exists $\mathbf{u} \in \mathcal{V}^A$ satisfying $\{i\} \cup \text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{u})$. In other words, $\mathcal{J}(\mathbf{v}, A)$ is the union of $\text{supp}(\mathbf{u})$ over all $\mathbf{u} \in \mathcal{V}^A$ with $\text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{u})$.

A zero \mathbf{u} of a copositive matrix A is called *minimal* if there exists no zero \mathbf{v} of A such that the inclusion $\text{supp } \mathbf{v} \subset \text{supp } \mathbf{u}$ holds strictly. We shall denote the set of minimal zeros of a copositive matrix A by \mathcal{V}_{\min}^A . From [15], for all $A \in \mathcal{COP}^n$, the set \mathcal{V}_{\min}^A is a finite set, and there are algorithmic methods for finding this set.

In fact we will see in Corollary 4 that if two different zeros have the same support then the line connecting them, intersected with Δ^n , consists of zeros. The end points of such a segment are zeros with supports strictly contained in those of the original two zeros. Therefore different minimal zeros must have different supports, and their number must be finite.¹

3 Preliminary Results

We start by considering the following preliminary results on sets of zeros.

Lemma 1 ([2, p.200]). *Let $A \in \mathcal{COP}^n$ and $\mathbf{u} \in \mathcal{V}^A$. Then $A\mathbf{u} \geq \mathbf{0}$.*

Lemma 2 ([11, Lemma 2.5]). *Let $A \in \mathcal{COP}^n$ and $\mathbf{u} \in \mathcal{V}^A$. Then $(A\mathbf{u})_i = 0$ for all $i \in \text{supp}(\mathbf{u})$.*

Lemma 3 ([15, Corollary 3.4]). *Let $A \in \mathcal{COP}^n$ and $\mathbf{u} \in \mathcal{V}^A$. Then \mathbf{u} can be represented as a convex combination of minimal zeros of A .*

We now consider the following corollary which generalizes Lemma 2.

Corollary 4. *For $A \in \mathcal{COP}^n$ and $\mathbf{u}, \mathbf{v} \in \mathcal{V}^A$ such that $\text{supp}(\mathbf{u}) \subset \text{supp}(\mathbf{v})$, we have*

$$\emptyset = \text{supp}(\mathbf{u}) \cap \text{supp}(A\mathbf{v}) = \text{supp}(A\mathbf{u}) \cap \text{supp}(\mathbf{v}),$$

and thus $(\theta\mathbf{u} + \lambda\mathbf{v})^T A(\theta\mathbf{u} + \lambda\mathbf{v}) = 0$ for all $\theta, \lambda \in \mathbb{R}$.

¹This limits there to be at most $2^n - 1$ minimal zeros. By noting that the system of supports of minimal zeros forms an antichain with respect to set inclusion, we can use Sperner's theorem [20] to get a tighter upper bound of $\binom{n}{\lfloor n/2 \rfloor} \sim 2^n \sqrt{\frac{2}{\pi n}}$ as $n \rightarrow \infty$. This bound already appears in [5] for the number of (strict local) minimizers of the quadratic form $\mathbf{x}^T A \mathbf{x}$. See also [6] for relations to game theory and other optimization concepts.

Proof. From Lemma 2 we have $\text{supp}(\mathbf{v}) \subset \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$, and as $\text{supp}(\mathbf{u}) \subset \text{supp}(\mathbf{v})$ this implies that $\emptyset = \text{supp}(\mathbf{u}) \cap \text{supp}(A\mathbf{v})$. This in turn implies that

$$0 = \sum_{i=1}^n u_i (A\mathbf{v})_i = \mathbf{u}^T A\mathbf{v} = \sum_{i=1}^n v_i (A\mathbf{u})_i.$$

From this observation and the fact that $A\mathbf{u} \geq \mathbf{0}$ we then get $\emptyset = \text{supp}(\mathbf{v}) \cap \text{supp}(A\mathbf{u})$, which completes the proof. \square

The set $\mathcal{J}(\mathbf{v}, A)$ will be considered later in the paper. Due to its applications we would like to have an algorithmic method to find it. Based on the fact that the finite set \mathcal{V}_{\min}^A can be found algorithmically [15], such a method is provided by the following lemma.

Lemma 5. *For $A \in \mathcal{COP}^n$, $\mathbf{v} \in \mathcal{V}^A$ and $i \in \{1, \dots, n\}$, we have $i \in \mathcal{J}(\mathbf{v}, A)$ if and only if there exists $\mathbf{w} \in \mathcal{V}_{\min}^A$ such that $i \in \text{supp}(\mathbf{w}) \subset \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$.*

Proof. We begin by considering the forward implication. If $i \in \mathcal{J}(\mathbf{v}, A)$ then there exists $\mathbf{u} \in \mathcal{V}^A$ such that $\{i\} \cup \text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{u})$, and from Corollary 4 we have $\text{supp}(\mathbf{u}) \subset \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$. By Lemma 3 there exists $\mathbf{w} \in \mathcal{V}_{\min}^A$ such that $i \in \text{supp}(\mathbf{w}) \subset \text{supp}(\mathbf{u})$, which completes the proof of the forward implication.

For the reverse implication we let $\mathbf{u} = \frac{1}{2}(\mathbf{v} + \mathbf{w})$, and note that $\{i\} \cup \text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{u})$. We have $0 = \mathbf{v}^T A\mathbf{v} = \mathbf{w}^T A\mathbf{w} = \mathbf{w}^T A\mathbf{v}$ and thus $\mathbf{u}^T A\mathbf{u} = 0$. Therefore $\mathbf{u} \in \mathcal{V}^A$ and $i \in \mathcal{J}(\mathbf{v}, A)$. \square

4 Main result

For a matrix $A \in \mathcal{COP}^n$ we consider the set $\mathcal{K}^A = \{B \in \mathcal{S}^n \mid \exists \delta > 0 \text{ s.t. } A + \delta B \in \mathcal{COP}^n\}$. This set is trivially a cone. As \mathcal{COP}^n is a convex cone, we have $\mathcal{COP}^n \subset \mathcal{K}^A$ and we have that \mathcal{K}^A is a convex set. The convex cone \mathcal{K}^A is not pointed, unless $A = 0$, as we always have $\pm A \in \mathcal{K}^A$. It is also in general not closed, as we shall see in Example 7.

We now present the following main result.

Theorem 6. *For $A \in \mathcal{COP}^n$ we have:*

$$\mathcal{K}^A = \left\{ B \in \mathcal{S}^n \mid \begin{array}{l} \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A, \\ (B\mathbf{v})_i \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B, i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v}) \end{array} \right\}$$

Proof. We shall prove the forward and reverse inclusions separately.

\subset : Suppose that $A + \delta B \in \mathcal{COP}^n$ for some $\delta > 0$.

Then for all $\mathbf{v} \in \mathcal{V}^A$ we have $0 \leq \frac{1}{\delta} \mathbf{v}^T (A + \delta B)\mathbf{v} = \mathbf{v}^T B\mathbf{v}$.

Also, for all $\mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B$, $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$ and $\varepsilon > 0$ we have $(\mathbf{v} + \varepsilon \mathbf{e}_i) \in \mathbb{R}_+^n$ and thus $0 \leq \frac{1}{2\varepsilon\delta} (\mathbf{v} + \varepsilon \mathbf{e}_i)^T (A + \delta B)(\mathbf{v} + \varepsilon \mathbf{e}_i) = (B\mathbf{v})_i + \frac{\varepsilon}{2\delta} (a_{ii} + \delta b_{ii})$. Letting $\varepsilon \rightarrow 0$ this implies that $(B\mathbf{v})_i \geq 0$, which completes this part of the proof.

\supset : Suppose for the sake of contradiction that the conditions on the right hand side hold for a given $B \in \mathcal{S}^n \setminus \mathcal{K}^A$.

For $\delta > 0$ let

$$\mathbf{v}_\delta \in \arg \min_{\mathbf{v}} \{ \mathbf{v}^T (A + \delta B)\mathbf{v} \mid \mathbf{v} \in \Delta^n \}, \quad (1)$$

and note that for all $\delta > 0$ we would have $\mathbf{v}_\delta^T(A + \delta B)\mathbf{v}_\delta < 0$ and thus $\mathbf{v}_\delta^T B\mathbf{v}_\delta < -\frac{1}{\delta}\mathbf{v}_\delta^T A\mathbf{v}_\delta \leq 0$.

As Δ^n is a compact set, there exists a sequence $\{\delta_k \mid k \in \mathbb{N}\} \subset \mathbb{R}_+ \setminus \{0\}$ and $\mathbf{v}^* \in \Delta^n$ such that $\lim_{k \rightarrow \infty} \delta_k = 0$ and $\lim_{k \rightarrow \infty} \mathbf{v}_{\delta_k} = \mathbf{v}^*$. Furthermore, without loss of generality (by possibly omitting leading δ_k 's), for all $k \in \mathbb{N}$ we have

$$\text{supp}(\mathbf{v}^*) \subset \text{supp}(\mathbf{v}_{\delta_k}), \quad \text{supp}_+(A\mathbf{v}^*) \subset \text{supp}_+((A + \delta_k B)\mathbf{v}_{\delta_k}).$$

We have $0 \leq \mathbf{v}^{*T} A\mathbf{v}^* = \lim_{k \rightarrow \infty} \mathbf{v}_{\delta_k}^T (A + \delta_k B)\mathbf{v}_{\delta_k} \leq 0$ and thus $\mathbf{v}^* \in \mathcal{V}^A$ and $\text{supp}(A\mathbf{v}^*) = \text{supp}_+(A\mathbf{v}^*)$ by Lemma 1. Therefore, by the assumptions, we have $0 \leq \mathbf{v}^{*T} B\mathbf{v}^* = \lim_{k \rightarrow \infty} \mathbf{v}_{\delta_k}^T B\mathbf{v}_{\delta_k} \leq 0$. This implies that $\mathbf{v}^* \in \mathcal{V}^A \cap \mathcal{V}^B$ and thus again by the assumptions we have $(B\mathbf{v}^*)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v}^*)$.

From now on we consider an arbitrary fixed $k \in \mathbb{N}$.

Having only linear constraints and a differentiable objective function, problem (1) fulfills Abadie's constraint qualification [1],[13, p.52], and by the Karush-Kuhn-Tucker optimality conditions there exist $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ and $\mu \in \mathbb{R}$ such that $\text{supp}(\mathbf{v}_{\delta_k}) \subset \{1, \dots, n\} \setminus \text{supp}(\boldsymbol{\lambda})$ and $(A + \delta_k B)\mathbf{v}_{\delta_k} = \boldsymbol{\lambda} - \mu \mathbf{1}$. We then have

$$\mu = -\mathbf{v}_{\delta_k}^T (A + \delta_k B)\mathbf{v}_{\delta_k} + \mathbf{v}_{\delta_k}^T \boldsymbol{\lambda} = -\mathbf{v}_{\delta_k}^T (A + \delta_k B)\mathbf{v}_{\delta_k} > 0$$

and thus $\lambda_i > [(A + \delta_k B)\mathbf{v}_{\delta_k}]_i$ for all i , so that

$$\text{supp}(\boldsymbol{\lambda}) \supset \text{supp}_+((A + \delta_k B)\mathbf{v}_{\delta_k}) \supset \text{supp}_+(A\mathbf{v}^*) = \text{supp}(A\mathbf{v}^*).$$

Therefore $(B\mathbf{v}^*)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(\boldsymbol{\lambda})$. Furthermore, from $\text{supp}(\mathbf{v}_{\delta_k}) \subset \{1, \dots, n\} \setminus \text{supp}(\boldsymbol{\lambda})$, we have $(B\mathbf{v}^*)_i \geq 0$ for all $i \in \text{supp}(\mathbf{v}_{\delta_k})$. This implies that $\mathbf{v}^{*T} B\mathbf{v}_{\delta_k} \geq 0$ and we get the contradiction

$$\begin{aligned} 0 &= \mathbf{v}^{*T} \mathbf{0} \\ &= \mathbf{v}^{*T} ((A + \delta_k B)\mathbf{v}_{\delta_k} - \boldsymbol{\lambda} + \mu \mathbf{1}) \\ &= \underbrace{\mathbf{v}^{*T} A}_{\geq 0} \underbrace{\mathbf{v}_{\delta_k}}_{\geq 0} + \underbrace{\delta_k \mathbf{v}^{*T} B\mathbf{v}_{\delta_k}}_{\geq 0} - \underbrace{\mathbf{v}^{*T} \boldsymbol{\lambda}}_{=0} + \underbrace{\mu}_{>0} \underbrace{\mathbf{v}^{*T} \mathbf{1}}_{=1} \\ &> 0. \end{aligned} \quad \square$$

Combining this result with Lemma 2, we get that if $A \in \mathcal{COP}^n$ and $B \in \mathcal{K}^A$, i.e., there exists $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$, then for all $\mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B \subset \mathcal{V}^{A+\delta B}$ and all $i \in \text{supp}(\mathbf{u})$ we have $0 = \frac{1}{\delta}[(A + \delta B)\mathbf{u}]_i = (B\mathbf{v})_i$. This observation then provides the following alternative characterization:

$$\mathcal{K}^A = \left\{ B \in \mathcal{S}^n \left| \begin{array}{l} \mathbf{v}^T B\mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A, \\ (B\mathbf{v})_i = 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B, \ i \in \text{supp}(\mathbf{v}), \\ (B\mathbf{v})_i \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A \cap \mathcal{V}^B, \ i \in \{1, \dots, n\} \setminus (\text{supp}(\mathbf{v}) \cup \text{supp}(A\mathbf{v})) \end{array} \right. \right\}$$

We now consider a ready example, from which we see that \mathcal{K}^A is not in general closed.

Example 7. Let $A = \sum_{i=2}^n \mathbf{e}_i \mathbf{e}_i^T$, then we have $\mathcal{V}^A = \{\mathbf{e}_1\}$ and $\text{supp}(A\mathbf{e}_1) = \emptyset$. Therefore

$$\mathcal{K}^A = \{B \in \mathcal{S}^n \mid b_{11} > 0\} \cup \{B \in \mathcal{S}^n \mid b_{11} = 0, \ b_{1i} \geq 0 \text{ for all } i\},$$

which is not closed.

We now consider another example which again has a simple matrix for A , but this time with a much more complicated expression for \mathcal{K}^A .

Example 8. Letting $A = \sum_{i=3}^n \mathbf{e}_i \mathbf{e}_i^T$ we have $\mathcal{V}^A = \{\theta \mathbf{e}_1 + (1 - \theta) \mathbf{e}_2 \mid 0 \leq \theta \leq 1\}$ and

$$\begin{aligned} \mathcal{K}^A = & \left\{ B \in \mathcal{S}^n \mid b_{11}, b_{22} > 0, b_{12} > -\sqrt{b_{11}b_{22}} \right\} \\ & \cup \{B \in \mathcal{S}^n \mid b_{11} = 0, b_{22} > 0, b_{1i} \geq 0 \text{ for all } i\} \\ & \cup \{B \in \mathcal{S}^n \mid b_{22} = 0, b_{11} > 0, b_{2i} \geq 0 \text{ for all } i\} \\ & \cup \{B \in \mathcal{S}^n \mid b_{11} = b_{22} = 0, b_{1i}, b_{2i} \geq 0 \text{ for all } i\} \\ & \cup \{B \in \mathcal{S}^n \mid b_{11}, b_{22} > 0, b_{12} = -\sqrt{b_{11}b_{22}}, b_{12}b_{1i} \leq b_{11}b_{2i} \text{ for all } i\}. \end{aligned}$$

Similarly to what we have seen in the previous example, using the characterizations of \mathcal{K}^A so far presented, it is in general difficult to give such an explicit expression for \mathcal{K}^A . Later in the paper, in Corollary 18, we shall present yet another characterization, which will allow for an alternative explicit expression (using Theorem 17). For the moment, however, we shall content ourselves by considering its closure, denoted $\text{cl}(\mathcal{K}^A)$, which admits a simpler characterization. In the literature this cone is sometimes referred to as the tangent cone [16, 18].

Theorem 9. *For $A \in \mathcal{COP}^n$ we have $\text{cl}(\mathcal{K}^A) = \{B \in \mathcal{S}^n \mid \mathbf{v}^T B \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A\}$.*

Proof. We let \mathcal{M} be the set on the right-hand side of the equation above. From Theorem 6 we have $\mathcal{K}^A \subset \mathcal{M}$. Furthermore, the set \mathcal{M} is the intersection of closed sets, and thus is itself closed. From this we then get $\text{cl}(\mathcal{K}^A) \subset \mathcal{M}$.

We now consider an arbitrary $B \in \mathcal{M}$. Letting I be the identity matrix we have

$$\mathbf{v}^T (B + \varepsilon I) \mathbf{v} = \underbrace{\mathbf{v}^T B \mathbf{v}}_{\geq 0} + \underbrace{\varepsilon \mathbf{v}^T \mathbf{v}}_{> 0} > 0 \quad \text{for all } \mathbf{v} \in \mathcal{V}^A, \varepsilon > 0.$$

Therefore by Theorem 6 we have $B + \varepsilon I \in \mathcal{K}^A$ for all $\varepsilon > 0$, and thus $B \in \text{cl}(\mathcal{K}^A)$, which completes the proof. \square

From the work of [9, 15] it can be seen that for $A \in \mathcal{COP}^n$ we have that \mathcal{V}^A is the union of finitely many polyhedra contained in Δ^n . We can then characterise $\text{cl}(\mathcal{K}^A)$ by noting the following two trivial results.

Lemma 10. *For a polyhedron $\mathcal{M} = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ (where ‘conv’ denotes the convex hull), letting $V = [\mathbf{v}_1 \ \dots \ \mathbf{v}_m] \in \mathbb{R}^{n \times m}$, we have*

$$\{B \in \mathcal{S}^n \mid \mathbf{v}^T B \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{M}\} = \{B \in \mathcal{S}^n \mid V^T B V \in \mathcal{COP}^m\}.$$

Lemma 11. *For sets $\mathcal{M}_1, \dots, \mathcal{M}_p \subset \mathbb{R}^n$ we have*

$$\left\{ B \in \mathcal{S}^n \mid \mathbf{v}^T B \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \bigcup_{i=1}^p \mathcal{M}_i \right\} = \bigcap_{i=1}^p \{B \in \mathcal{S}^n \mid \mathbf{v}^T B \mathbf{v} \geq 0 \text{ for all } \mathbf{v} \in \mathcal{M}_i\}.$$

In fact, for $A \in \mathcal{COP}^n \setminus \{0\}$, all the polyhedra composing \mathcal{V}^A are of dimension less than or equal to $n - 2$. This can be seen from Corollary 4, as all the polyhedra must have dimension strictly less than that of Δ^n , otherwise we would get $\mathcal{V}^A = \Delta^n$ and thus $A = 0$. We can partition these polyhedra into simplices, each with at most $n - 1$ vertices. This then means that \mathcal{K}^A can be characterized using cones \mathcal{COP}^m where $m \leq n - 1$. Noting that for $m \leq 4$ we have $\mathcal{COP}^m = \mathcal{S}_+^m + \mathcal{N}^m$, we then get the following result:

Lemma 12. For $A \in \mathcal{COP}^5 \setminus \{0\}$ we have that $\text{cl}(\mathcal{K}^A)$ is a semi-definite representable set, i.e., it is linearly isomorphic to the linear projection of a linear section of the positive semi-definite matrix cone \mathcal{S}_+^m for some integer m .

This is of interest as it is still an open question whether \mathcal{COP}^5 itself is a semi-definite representable set.

From Theorem 9 we can also obtain an expression for $\text{cl}(\mathcal{K}^A) \cap -\text{cl}(\mathcal{K}^A)$, which is sometimes referred to as the tangent space [18].

Theorem 13. For $A \in \mathcal{COP}^n$ we have

$$\begin{aligned} \text{cl}(\mathcal{K}^A) \cap -\text{cl}(\mathcal{K}^A) &= \{B \in \mathcal{S}^n \mid \mathcal{V}^A \subset \mathcal{V}^B\} = \{B \in \mathcal{S}^n \mid \mathbf{v}^T B \mathbf{v} = 0 \text{ for all } \mathbf{v} \in \mathcal{V}^A\} \\ &= \{B \in \mathcal{S}^n \mid \mathbf{u}^T B \mathbf{v} = 0 \text{ for all } \{\mathbf{u}, \mathbf{v}\} \subset \mathcal{V}_{\min}^A \text{ s.t. } \mathbf{u}^T A \mathbf{v} = 0\}. \end{aligned}$$

Proof. The first two equalities follow directly from Theorem 9.

We now consider an arbitrary $B \in \mathcal{S}^n$ such that $\mathcal{V}^A \subset \mathcal{V}^B$. For all $\{\mathbf{u}, \mathbf{v}\} \subset \mathcal{V}_{\min}^A \subset \mathcal{V}^B$ such that $\mathbf{u}^T A \mathbf{v} = 0$ we have $\frac{1}{2}(\mathbf{u} + \mathbf{v}) \in \mathcal{V}^A \subset \mathcal{V}^B$ and thus $0 = (\mathbf{u} + \mathbf{v})^T B (\mathbf{u} + \mathbf{v}) = 2\mathbf{u}^T B \mathbf{v}$. Therefore

$$\{B \in \mathcal{S}^n \mid \mathcal{V}^A \subset \mathcal{V}^B\} \subset \{B \in \mathcal{S}^n \mid \mathbf{u}^T B \mathbf{v} = 0 \text{ for all } \{\mathbf{u}, \mathbf{v}\} \subset \mathcal{V}_{\min}^A \text{ s.t. } \mathbf{u}^T A \mathbf{v} = 0\}.$$

To prove that the reverse inclusion relation also holds, we consider an arbitrary $\mathbf{w} \in \mathcal{V}^A$ and $B \in \mathcal{S}^n$ such that $\mathbf{u}^T B \mathbf{v} = 0$ for all $\{\mathbf{u}, \mathbf{v}\} \subset \mathcal{V}_{\min}^A$ with $\mathbf{u}^T A \mathbf{v} = 0$. By Lemma 3, there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{V}_{\min}^A$ and $\theta_1, \dots, \theta_m > 0$ such that $\mathbf{w} = \sum_{i=1}^m \theta_i \mathbf{v}_i$. Using Lemma 1 we note that

$$0 = \mathbf{w}^T A \mathbf{w} = \sum_{i,j=1}^m \underbrace{\theta_i \theta_j}_{>0} \underbrace{\mathbf{v}_i^T}_{\geq 0} \underbrace{A \mathbf{v}_j}_{\geq 0},$$

and thus $\mathbf{v}_i^T A \mathbf{v}_j = 0$ for all i, j . By the assumptions, this implies that $\mathbf{v}_i^T B \mathbf{v}_j = 0$ for all i, j , and thus $\mathbf{w} \in \mathcal{V}^B$. \square

Note that the latter characterization of the tangent space is as a linear space described by finitely many linear equality relations. It is for this reason that \mathcal{V}_{\min}^A is preferable to \mathcal{V}^A in characterizations, and we will extend this idea further in the next section.

5 Minimal Faces

In this section we apply Theorem 6 to determine the minimal face and the minimal exposed face of the copositive cone containing a matrix $A \in \mathcal{COP}^n$. We begin by giving basic definitions and results for minimal (exposed) faces of the copositive cone. For good discussions of faces for general convex cones we recommend [4, 19].

Definition 14. A convex subset $\mathcal{F} \subset \mathcal{COP}^n$ is a *face* of \mathcal{COP}^n if every closed line segment in \mathcal{COP}^n with a relative interior point in \mathcal{F} must have both end points in \mathcal{F} . For $A \in \mathcal{COP}^n$ we let \mathcal{F}^A equal the intersection of all faces of \mathcal{COP}^n containing A . This is itself a face, and is referred to as the *minimal face* of \mathcal{COP}^n containing A .

We say that $\widehat{\mathcal{F}}$ is an exposed face of a convex set $\mathcal{C} \subset \mathcal{S}^n$ if there exists $\beta \in \mathbb{R}$ and $B \in \mathcal{S}^n$ such that $\widehat{\mathcal{F}} = \{X \in \mathcal{C} \mid \langle X, B \rangle = \beta\}$ and $\mathcal{C} \subset \{X \in \mathcal{S}^n \mid \langle X, B \rangle \geq \beta\}$. Here $\langle A, B \rangle = \text{trace}(AB)$ is the Frobenius scalar product on \mathcal{S}^n . For $\mathcal{C} \subset \mathcal{S}^n$ being a nonempty closed convex cone we have that $\widehat{\mathcal{F}} \neq \emptyset$ is an exposed face of \mathcal{C} if and only if there exists $B \in \mathcal{C}^* \setminus \{0\}$ such that $\widehat{\mathcal{F}} = \{X \in \mathcal{C} \mid \langle X, B \rangle = 0\}$, where $\mathcal{C}^* := \{Y \in \mathcal{S}^n \mid \langle X, Y \rangle \geq 0 \text{ for all } X \in \mathcal{C}\}$ is the dual cone of \mathcal{C} . For \mathcal{C} being the copositive cone this translates to the following definition.

Definition 15. A nonempty set $\widehat{\mathcal{F}} \subset \mathcal{COP}^n$ is an *exposed face* of \mathcal{COP}^n if there exists $\mathcal{W} \subset \Delta^n$ such that $\widehat{\mathcal{F}} = \{X \in \mathcal{COP}^n \mid \mathcal{W} \subset \mathcal{V}^X\}$. For $A \in \mathcal{COP}^n$ we let $\widehat{\mathcal{F}}^A = \{X \in \mathcal{COP}^n \mid \mathcal{V}^A \subset \mathcal{V}^X\}$. This is an exposed face which is the intersection of all exposed faces of \mathcal{COP}^n containing A , and it is referred to as the *minimal exposed face* of \mathcal{COP}^n containing A .

Note that for $A \in \mathcal{COP}^n$ we have $\mathcal{F}^A \subset \widehat{\mathcal{F}}^A$, as an exposed face is itself a face. Additionally, A is in the interior of the copositive cone if and only if $\mathcal{F}^A = \widehat{\mathcal{F}}^A = \mathcal{COP}^n$. Also note that for $A \in \mathcal{COP}^n$ we always have $\{\lambda A \mid \lambda \geq 0\} \subset \mathcal{F}^A \subset \widehat{\mathcal{F}}^A \subset \mathcal{COP}^n$.

If $A \in \mathcal{COP}^n \setminus \{0\}$ and \mathcal{F}^A is of dimension equal to one then we say that A gives an *extreme ray* of the copositive cone. We in fact then have $\mathcal{F}^A = \{\lambda A \mid \lambda \geq 0\}$.

If $A \in \mathcal{COP}^n \setminus \{0\}$ and $\widehat{\mathcal{F}}^A$ is of dimension equal to one then we say that A gives an *exposed ray* of the copositive cone, which is a special type of extreme ray. Similarly to before we then have $\widehat{\mathcal{F}}^A = \{\lambda A \mid \lambda \geq 0\}$.

For $A \in \mathcal{COP}^n$ we now let

$$\begin{aligned}\mathcal{L}^A &= \{B \in \mathcal{S}^n \mid \exists \delta > 0 \text{ s.t. } A + \delta B \in \mathcal{F}^A\} \\ \widehat{\mathcal{L}}^A &= \{B \in \mathcal{S}^n \mid \exists \delta > 0 \text{ s.t. } A + \delta B \in \widehat{\mathcal{F}}^A\}.\end{aligned}$$

We then have

$$\dim(\mathcal{F}^A) = \dim(\mathcal{L}^A), \quad \dim(\widehat{\mathcal{F}}^A) = \dim(\widehat{\mathcal{L}}^A).$$

Therefore A gives an extreme (resp. exposed) ray of the copositive cone if and only if \mathcal{L}^A (resp. $\widehat{\mathcal{L}}^A$) is of dimension equal to one.

The advantage of using the sets \mathcal{L}^A and $\widehat{\mathcal{L}}^A$ comes from Theorems 17 and 19 below, in which we see that the characterizations of \mathcal{L}^A and $\widehat{\mathcal{L}}^A$ are relatively simple. This then gives us a method for checking whether a copositive matrix gives an extreme/exposed ray.

Before presenting these theorems, we first recall the following result relating \mathcal{F}^A and \mathcal{K}^A :

Lemma 16 ([18, Lemma 3.2.1]). *For $A \in \mathcal{COP}^n$ we have $\mathcal{K}^A = \mathcal{COP}^n + \text{span}(\mathcal{F}^A)$.*

We are now ready to present the first of our results on minimal faces.

Theorem 17. *For $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$ the following are equivalent:*

1. $B \in \mathcal{L}^A$,
2. $B \in \text{span}(\mathcal{F}^A)$,
3. $\exists \delta > 0$ such that $A \pm \delta B \in \mathcal{COP}^n$ (equivalently $B \in \mathcal{K}^A \cap (-\mathcal{K}^A)$),

4. $(B\mathbf{v})_i = 0$ for all $\mathbf{v} \in \mathcal{V}^A$, $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$,
5. $(B\mathbf{v})_i = 0$ for all $\mathbf{v} \in \mathcal{V}_{\min}^A$, $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$.

Proof. We shall split this proof into the following parts:

(1) \Rightarrow (2): This follows trivially from the definitions.

(2) \Rightarrow (3): This follows directly from Lemma 16.

(3) \Rightarrow (1): Suppose that (3) holds and consider the set $\mathcal{M} = \{A + \theta B \mid -\delta \leq \theta \leq \delta\}$. This is a closed line segment in \mathcal{COP}^n with A in its relative interior. Therefore, from the definition of a face, we have $A + \delta B \in \mathcal{F}^A$, and thus $B \in \mathcal{L}^A$.

(3) \Leftrightarrow (4): This follows from Theorem 6 and the fact that $\text{supp}(\mathbf{v}) \subset \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$.

(4) \Rightarrow (5): This follows trivially from the fact that $\mathcal{V}_{\min}^A \subset \mathcal{V}^A$.

(5) \Rightarrow (4): Suppose that statement (5) holds and consider an arbitrary $\mathbf{u} \in \mathcal{V}^A$. By Lemma 3, there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{V}_{\min}^A$ and $\theta_1, \dots, \theta_m > 0$ such that $\mathbf{u} = \sum_{j=1}^m \theta_j \mathbf{v}_j$. For all j we have $A\mathbf{v}_j \geq \mathbf{0}$ and thus $\text{supp}(A\mathbf{u}) \supset \text{supp}(A\mathbf{v}_j)$. Therefore for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{u})$ we have $(B\mathbf{v}_j)_i = 0$ for all j , and thus $(B\mathbf{u})_i = (B\mathbf{v}_1)_i + \dots + (B\mathbf{v}_m)_i = 0$. \square

Note that from this theorem $\mathcal{L}^A = \text{span}(\mathcal{F}^A)$ is a linear subspace of \mathcal{S}^n , and as \mathcal{V}_{\min}^A is a finite set, the system of linear equations in Theorem 17 (5) is finite. We can thus algorithmically compute the dimension of \mathcal{L}^A by finding the rank of the coefficient matrix of this system of linear equations. This then allows us to determine if the copositive matrix A gives an extreme ray.

Also note that using Lemma 16 we get the following alternative characterization for \mathcal{K}^A . In comparison to the previous characterizations in Section 4, in general it is easier to give this one explicitly.

Corollary 18. *For $A \in \mathcal{COP}^n$ we have*

$$\mathcal{K}^A = \mathcal{COP}^n + \{B \in \mathcal{S}^n \mid (B\mathbf{v})_i = 0 \text{ for all } \mathbf{v} \in \mathcal{V}_{\min}^A, i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})\}.$$

In comparison to the characterization of $\text{cl}(\mathcal{K}^A)$ in Section 4, our latest characterization of \mathcal{K}^A involves one cone of copositive matrices of order n , whereas the characterization of $\text{cl}(\mathcal{K}^A)$ involves multiple cones of copositive matrices of orders strictly less than n .

We now present the following result for minimal exposed faces, which is closely related to Theorem 17.

Theorem 19. *For $A \in \mathcal{COP}^n$ and $B \in \mathcal{S}^n$ the following are equivalent:*

1. $B \in \widehat{\mathcal{L}}^A$,
2. $\exists \delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$ and $\mathcal{V}^A \subset \mathcal{V}^{A+\delta B}$,
3. For all $\mathbf{v} \in \mathcal{V}^A$ we have $\mathbf{v}^T B \mathbf{v} = 0$ and $(B\mathbf{v})_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$,
4. For all $\mathbf{v} \in \mathcal{V}_{\min}^A$ we have

$$\begin{aligned} (B\mathbf{v})_i &= 0 && \text{for all } i \in \mathcal{J}(\mathbf{v}, A), \\ (B\mathbf{v})_i &\geq 0 && \text{for all } i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v}). \end{aligned}$$

Proof. We shall split this proof into the following parts:

(1) \Leftrightarrow (2): This equivalence follows directly from the definition of $\widehat{\mathcal{L}}^A$.

(2) \Rightarrow (3): Let $\mathbf{v} \in \mathcal{V}^A$. Then $\mathbf{v} \in \mathcal{V}^{A+\delta B}$ and hence $\mathbf{v} \in \mathcal{V}^B$. Moreover, $B \in \mathcal{K}^A$ and hence by Theorem 6 we have $(B\mathbf{v})_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$.

(3) \Rightarrow (2): By assumption $\mathcal{V}^A \subset \mathcal{V}^B$ and hence also $\mathcal{V}^A \subset \mathcal{V}^{A+\delta B}$ for all δ . Moreover, from Theorem 6 we have $B \in \mathcal{K}^A$ and hence there exists $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$.

(3) \Rightarrow (4): By assumption we have $\mathcal{V}^A \subset \mathcal{V}^B$. Since $\mathcal{V}_{\min}^A \subset \mathcal{V}^A$, the inequalities in (4) hold. Consider an arbitrary $\mathbf{v} \in \mathcal{V}_{\min}^A \subset \mathcal{V}^B$ and $i \in \mathcal{J}(\mathbf{v}, A)$. By definition of $\mathcal{J}(\mathbf{v}, A)$ there exists $\mathbf{u} \in \mathcal{V}^A \subset \mathcal{V}^B$ such that $\{i\} \cup \text{supp}(\mathbf{v}) \subset \text{supp}(\mathbf{u})$. From Corollary 4 we have $\text{supp}(\mathbf{u}) \subset \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$ and $\frac{1}{2}(\mathbf{u} + \mathbf{v}) \in \mathcal{V}^A \subset \mathcal{V}^B$. Hence by assumption $(B\mathbf{v})_j \geq 0$ for all $j \in \text{supp}(\mathbf{u})$ and

$$0 = (\mathbf{u} + \mathbf{v})^T B(\mathbf{u} + \mathbf{v}) = 2\mathbf{u}^T B\mathbf{v} = 2 \sum_{j \in \text{supp}(\mathbf{u})} \underbrace{u_j}_{>0} \underbrace{(B\mathbf{v})_j}_{\geq 0} \geq 0.$$

It follows that $(B\mathbf{v})_j = 0$ for all $j \in \text{supp}(\mathbf{u})$, and in particular $(B\mathbf{v})_i = 0$, which yields (4).

(4) \Rightarrow (3): Consider an arbitrary $\mathbf{u} \in \mathcal{V}^A$. By Lemma 3, there exist $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathcal{V}_{\min}^A$ and $\theta_1, \dots, \theta_m > 0$ such that $\mathbf{u} = \sum_{j=1}^m \theta_j \mathbf{v}_j$. For all $j = 1, \dots, m$ we have $\text{supp}(\mathbf{v}_j) \subset \text{supp}(\mathbf{u})$ and hence $\text{supp}(\mathbf{u}) \subset \mathcal{J}(\mathbf{v}_j, A)$. Moreover, we have $A\mathbf{u} = \sum_{j=1}^m \theta_j A\mathbf{v}_j$ and hence $\text{supp}(A\mathbf{v}_j) \subset \text{supp}(A\mathbf{u})$ by Lemma 1. Therefore by assumption for all $j = 1, \dots, m$ we have $(B\mathbf{v}_j)_i = 0$ for all $i \in \text{supp}(\mathbf{u})$ and $(B\mathbf{v}_j)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{u}) \subseteq \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v}_j)$. This implies that $(B\mathbf{u})_i = \sum_{j=1}^m \theta_j (B\mathbf{v}_j)_i \geq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{u})$ and

$$\mathbf{u}^T B\mathbf{u} = \sum_{i \in \text{supp}(\mathbf{u})} u_i (B\mathbf{u})_i = \sum_{i \in \text{supp}(\mathbf{u})} \sum_{j=1}^m \theta_j u_i (B\mathbf{v}_j)_i = 0. \quad \square$$

Example 20. Consider $A = \mathbf{e}_1 \mathbf{e}_1^T$. We have

$$\begin{aligned} \mathcal{V}^A &= \text{conv}\{\mathbf{e}_j \mid j \in \{2, \dots, n\}\}, \\ \mathcal{V}_{\min}^A &= \{\mathbf{e}_j \mid j \in \{2, \dots, n\}\}, \\ \text{supp}(A\mathbf{e}_j) &= \emptyset \quad \text{for all } j \in \{2, \dots, n\}, \\ \mathcal{J}(\mathbf{e}_j, A) &= \{2, \dots, n\} \quad \text{for all } j \in \{2, \dots, n\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{L}^A &= \{B \in \mathcal{S}^n \mid (B)_{ij} = 0 \text{ for all } (i, j) \in \{1, \dots, n\} \times \{2, \dots, n\}\} = \text{span}(\{A\}), \\ \widehat{\mathcal{L}}^A &= \left\{ B \in \mathcal{S}^n \mid \begin{array}{l} (B)_{ij} = 0 \text{ for all } (i, j) \in \{2, \dots, n\} \times \{2, \dots, n\}, \\ (B)_{1j} \geq 0 \text{ for all } j \in \{2, \dots, n\} \end{array} \right\} \end{aligned}$$

We thus have $\dim(\mathcal{L}^A) = 1$ and $\dim(\widehat{\mathcal{L}}^A) = n$. This implies that A gives an extreme but not exposed ray of the copositive cone.

This example also shows that the index set $\mathcal{J}(\mathbf{v}, A)$ in Theorem 19 cannot in general be replaced by $\{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$.

6 Irreducibility

In this subsection we describe irreducibility of a copositive matrix A with respect to another copositive matrix C . This allows us to recover the results on irreducibility from [11] and [15] as special cases.

Recall that we are using the following definition of irreducibility:

Definition 21 ([11, Definition 1.1]). For a matrix $A \in \mathcal{COP}^n$ and a subset $\mathcal{M} \subset \mathcal{COP}^n$, we say that A is *irreducible with respect to \mathcal{M}* if there do not exist $\delta > 0$ and $M \in \mathcal{M} \setminus \{0\}$ such that $A - \delta M \in \mathcal{COP}^n$.

Note that this definition differs from the concept of an irreducible matrix that is normally used in matrix theory. For simplicity we speak about irreducibility with respect to M when $\mathcal{M} = \{M\}$. We also note that as \mathcal{COP}^n is a convex cone, the following result holds.

Lemma 22 ([15, Lemma 2.2]). *Let $A \in \mathcal{COP}^n$ and $\mathcal{M} \subset \mathcal{COP}^n$. Then the following are equivalent:*

1. A is irreducible with respect to \mathcal{M} ,
2. A is irreducible with respect to M for all $M \in \mathcal{M}$,
3. A is irreducible with respect to $\mathbb{R}_+ \mathcal{M}$,
4. A is irreducible with respect to the convex conic hull of \mathcal{M} .

We now present the following result on when irreducibility does not occur.

Theorem 23. *For $\{A, C\} \subset \mathcal{COP}^n$ the following are equivalent:*

1. A is not irreducible with respect to C ,
2. For all $\mathbf{v} \in \mathcal{V}^A$ we have $\text{supp}(C\mathbf{v}) \subset \text{supp}(A\mathbf{v})$ (and thus $\mathbf{v} \in \mathcal{V}^C$),
3. For all $\mathbf{v} \in \mathcal{V}_{\min}^A$ we have $\text{supp}(C\mathbf{v}) \subset \text{supp}(A\mathbf{v})$ (and thus $\mathbf{v} \in \mathcal{V}^C$).

Proof. In order to prove this this we add the following two statements:

4. $\exists \delta > 0$ such that $A - \delta C \in \mathcal{COP}^n$ and $\mathcal{V}^A \subset \mathcal{V}^{A - \delta C}$,
5. For all $\mathbf{v} \in \mathcal{V}^A$ we have $\mathbf{v}^T C \mathbf{v} = 0$ and $(C\mathbf{v})_i \leq 0$ for all $i \in \{1, \dots, n\} \setminus \text{supp}(A\mathbf{v})$.

We shall show that all of the statements (1) to (5) are equivalent.

From Lemmas 1, 2 and 3 it is trivial to see that statements (2), (3) and (5) are equivalent, and from Theorem 19 (considering $B = -C$) we have that statements (4) and (5) are equivalent. Furthermore, it follows directly from the definitions that statement (4) implies statement (1). We are thus left to show that statement (1) implies statement (4). To show this we note that if (1) holds, then there exists $\delta > 0$ such that $A - \delta C \in \mathcal{COP}^n$. For every $\mathbf{v} \in \mathcal{V}^A$ we then have $0 \leq \mathbf{v}^T (A - \delta C) \mathbf{v} = -\delta \mathbf{v}^T C \mathbf{v} \leq 0$, and hence $\mathbf{v} \in \mathcal{V}^C$ and $\mathbf{v} \in \mathcal{V}^{A - \delta C}$. \square

Alternatively we could have stated this theorem as follows:

Theorem 24. *For $\{A, C\} \subset \mathcal{COP}^n$ the following are equivalent:*

1. A is irreducible with respect to C ,
2. There exists $\mathbf{v} \in \mathcal{V}^A$, $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i$.
3. There exists $\mathbf{v} \in \mathcal{V}_{\min}^A$, $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i$.

We will now recover the results from [11] and [15].

Corollary 25 ([11, Theorem 2.6]). *Let $A \in \mathcal{COP}^n$ and $C = \mathbf{e}_k \mathbf{e}_l^T + \mathbf{e}_l \mathbf{e}_k^T$ where $k, l \in \{1, \dots, n\}$. Then A is irreducible with respect to C if and only if there exists $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_k = (A\mathbf{v})_l = 0 < v_k + v_l$.*

Proof. By Theorem 24, A is irreducible with respect to C if and only if there exists $\mathbf{v} \in \mathcal{V}^A$ and $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i = \delta_{ik}v_l + \delta_{il}v_k$, where δ_{ik}, δ_{il} are Kronecker deltas, i.e., $\delta_{ik} = \mathbf{e}_i^T \mathbf{e}_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$. We shall now show that this is equivalent to the condition in the corollary.

To show the reverse implication we suppose that $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_k = (A\mathbf{v})_l = 0 < v_k + v_l$. Without loss of generality we have $v_l > 0$, and taking $i = k$ we get $(A\mathbf{v})_i = 0 < v_l + \delta_{il}v_k = (C\mathbf{v})_i$.

To prove the forward implication we suppose that $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_i = 0 \neq \delta_{ik}v_l + \delta_{il}v_k$. Without loss of generality we have $i = k$, and thus $(A\mathbf{v})_k = 0 \neq (1 + \delta_{kl})v_l$. Therefore $v_l > 0$ and thus $(A\mathbf{v})_l = 0 < v_k + v_l$. \square

From this, together with Lemma 22, we get that A is irreducible with respect to \mathcal{N}^n if and only if for all $k, l \in \{1, \dots, n\}$ there exists $\mathbf{v} \in \mathcal{V}^A$ such that $(A\mathbf{v})_k = (A\mathbf{v})_l = 0 < v_k + v_l$.

Corollary 25 still holds if we replace \mathcal{V}^A by \mathcal{V}_{\min}^A [15, Corollary 4.2]. The short proof of Corollary 25 provided above remains valid for this strengthened version by replacing \mathcal{V}^A by \mathcal{V}_{\min}^A .

Corollary 26 ([15, Corollary 4.4]). *Let $A \in \mathcal{COP}^n$, $\mathbf{c} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and $C = \mathbf{c}\mathbf{c}^T$. Then A is irreducible with respect to C if and only if there exists $\mathbf{v} \in \mathcal{V}_{\min}^A$ such that $\mathbf{v}^T \mathbf{c} \neq 0$.*

Proof. Suppose that A is irreducible with respect to C . Then from Theorem 24, there exist $\mathbf{v} \in \mathcal{V}_{\min}^A$, $i \in \{1, \dots, n\}$ such that $(A\mathbf{v})_i = 0 \neq (C\mathbf{v})_i = c_i \mathbf{c}^T \mathbf{v}$, and thus $\mathbf{v}^T \mathbf{c} \neq 0$.

Now suppose that there exists $\mathbf{v} \in \mathcal{V}_{\min}^A$ such that $\mathbf{v}^T \mathbf{c} \neq 0$. Then there exists $i \in \text{supp}(\mathbf{v}) \cap \text{supp}(\mathbf{c})$ and by Lemma 2 we have $(A\mathbf{v})_i = 0 \neq c_i \mathbf{c}^T \mathbf{v} = (C\mathbf{v})_i$. Therefore, by Theorem 24, A is irreducible with respect to C . \square

From this, together with Lemma 22, we get that A is irreducible with respect \mathcal{S}_+^n if and only if $\text{span}(\mathcal{V}_{\min}^A) = \mathbb{R}^n$.

7 Conclusions

In this paper we have given necessary and sufficient conditions on a pair $(A, B) \in \mathcal{COP}^n \times \mathcal{S}^n$ for the existence of $\delta > 0$ such that $A + \delta B \in \mathcal{COP}^n$. For fixed $A \in \mathcal{COP}^n$, the set of matrices B satisfying this condition forms a convex cone \mathcal{K}^A . We have described this cone by a set of linear inequalities constructed from the set of zeros of A . This description allowed us to compute the linear span of the minimal face of A in \mathcal{COP}^n . In particular, we devised a simple test for the extremality of A . The result can also be applied for

checking irreducibility of A with respect to an arbitrary matrix $C \in \mathcal{COP}^n$. This result covers previous results from [11] and [15] on irreducibility as special cases.

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References

- [1] Jean Abadie. On the Kuhn-Tucker theorem. In Jean Abadie, editor, *Nonlinear programming*, pages 21–36. North Holland, Amsterdam, 1967.
- [2] Leonard D. Baumert. Extreme copositive quadratic forms. *Pacific J. Math.*, 19(2):197–204, 1966.
- [3] Leonard D. Baumert. Extreme copositive quadratic forms. II. *Pacific J. Math.*, 20(1):1–20, 1967.
- [4] Abraham Berman. *Cones, Matrices and Mathematical Programming*, volume 79 of *Lecture Notes in Economics and Mathematical Systems*. Springer Verlag, 1973.
- [5] Immanuel M. Bomze. Non-cooperative two-person games in biology: A classification. *International Journal of Game Theory*, 15(1):31–57, 1986.
- [6] Immanuel M. Bomze. Regularity versus degeneracy in dynamics, games, and optimization: A unified approach to different aspects. *SIAM review*, 44(3):394–414, 2002.
- [7] Immanuel M. Bomze, Werner Schachinger, and Gabriele Uchida. Think co(mpletely)positive ! – matrix properties, examples and a clustered bibliography on copositive optimization. *J. Global Optim.*, 52:423–445, 2012.
- [8] Palahenedi H. Diananda. On nonnegative forms in real variables some or all of which are nonnegative. *Proc. Cambridge Philos. Soc.*, 58:17–25, 1962.
- [9] Peter J.C. Dickinson. Geometry of the copositive and completely positive cones. *J. Math. Anal. Appl.*, 380(1):377–395, 2011.
- [10] Peter J.C. Dickinson. *The Copositive Cone, the Completely Positive Cone and their Generalisations*. PhD thesis, University of Groningen, 2013.
- [11] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben, and Roland Hildebrand. Irreducible elements of the copositive cone. *Linear Algebra Appl.*, 439:1605–1626, 2013.
- [12] Mirjam Dür. Copositive programming — a survey. In Moritz Diehl, François Glineur, Elias Jarlebring, and Wim Michiels, editors, *Recent advances in optimization and its applications in engineering*, pages 3–20. Springer, Berlin, Heidelberg, 2010.

- [13] Graham C. Goodwin, María M. Seron, and José A. de Doná. *Constrained Control and Estimation: an optimisation approach*. Springer, London, 2005.
- [14] Marshall Hall Jr. and Morris Newman. Copositive and completely positive quadratic forms. *Proc. Cambridge Philos. Soc.*, 59:329–339, 1963.
- [15] Roland Hildebrand. Minimal zeros of copositive matrices. *Linear Algebra Appl.*, 459:154–174, 2014.
- [16] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Convex Analysis and Minimization Algorithms I*, volume 305 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, Heidelberg, New York, 1993.
- [17] Jean-Baptiste Hiriart-Urruty and Alberto Seeger. A variational approach to copositive matrices. *SIAM Rev.*, 52(4):593–629, 2010.
- [18] Gábor Pataki. The geometry of semidefinite programming. In Henry Wolkowicz, Romesh Saigal, and Lieven Vandenbergh, editors, *Handbook of Semidefinite Programming*, pages 29–65. Kluwer Academic Publishers, 2000.
- [19] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970.
- [20] Emanuel Sperner. Ein Satz über Untermengen einer endlichen Menge. *Mathematische Zeitschrift*, 27(1):544–548, 1928.