Stochastic Quasi-Fejér Block-Coordinate Fixed Point Iterations with Random Sweeping*

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Abstract

This work proposes block-coordinate fixed point algorithms with applications to nonlinear analysis and optimization in Hilbert spaces. The asymptotic analysis relies on a notion of stochastic quasi-Fejér monotonicity, which is thoroughly investigated. The iterative methods under consideration feature random sweeping rules to select the blocks of variables that are activated over the course of the iterations and they allow for stochastic errors in the evaluation of the operators. Algorithms using quasinonexpansive operators or compositions of averaged nonexpansive operators are constructed. The results are shown to yield novel block-coordinate operator splitting methods for solving structured monotone inclusion and convex minimization problems. In particular, the proposed framework leads to random block-coordinate versions of the Douglas-Rachford and forward-backward algorithms and of some of their variants.

Keywords. Averaged nonexpansive operator, block-coordinate algorithm, Douglas-Rachford algorithm, fixed-point algorithm, forward-backward algorithm, primal-dual operator splitting, stochastic quasi Fejér sequence, structured minimization problem.

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1 Introduction

The main advantage of block-coordinate algorithms is to result in implementations with reduced complexity and memory requirements per iteration. These benefits have long been recognized [3, 15, 45] and have become increasingly important in very large-scale problems. In addition, block-coordinate strategies may lead to faster [17] or distributed [37] implementations. In this paper, we propose a block-coordinate fixed point algorithmic framework to solve a variety of problems in Hilbertian nonlinear numerical analysis and optimization. Algorithmic fixed point theory in Hilbert spaces provides a unifying and powerful framework for the analysis and the construction of a wide array of solution methods in such problems [5, 7, 16, 19, 59]. Although several block-coordinate algorithms exist for solving specific optimization problems in Euclidean spaces, a framework for dealing with general fixed point methods in Hilbert spaces and which guarantees the convergence of the iterates does not seem to exist at present. In the proposed constructs, a random sweeping strategy is employed for selecting the blocks of coordinates which are activated over the iterations. Furthermore, the algorithms tolerate stochastic errors in the implementation of the operators.

A main ingredient for proving the convergence of many fixed point algorithm is the fundamental concept of (quasi-)Fejér monotonicity [20, 18, 30, 51]. In Section 2, following the seminal work of [31, 32, 33], we revisit this concept from a stochastic standpoint. By exploiting properties of almost super-martingales [53], we establish novel almost sure convergence results for an abstract stochastic iteration scheme. In Section 3, this scheme is applied to the design of block-coordinate algorithms for relaxed iterations of quasinonexpansive operators. A simple instance of such iterations is the Krasnosel'skiĭ–Mann method, which has found numerous applications [7, 14]. In Section 4, we design block-coordinate algorithms involving compositions of averaged nonexpansive operators. The results are used in Section 5 to construct block-coordinate algorithms for structured monotone inclusion and convex minimization problems. Splitting algorithms have recently becomes tools of choice in signal processing and machine learning; see, e.g., [14, 24, 26, 28, 50, 54]. Providing versatile block-coordinate versions of these algorithms is expected to benefit these emerging areas, as well as more traditional fields of applications of splitting methods, e.g., [36]. One of the offsprings of our work is an original block-coordinate primal-dual algorithm which can be employed to solve a large class of variational problems.

2 Stochastic quasi-Fejér monotonicity

Fejér monotonicity has been exploited in various areas of nonlinear analysis and optimization to unify the convergence proofs of deterministic algorithms; see, e.g., [7, 20, 30, 51]. In the late 1960s, this notion was revisited in a stochastic setting in Euclidean spaces [31, 32, 33]. In this section, we investigate a notion of stochastic quasi-Fejér monotone sequence in Hilbert spaces and apply the results to a general stochastic iterative method. Throughout the paper, the following notation will be used.

Notation 2.1 H is a separable real Hilbert space with scalar product $\langle\cdot \mid \cdot\rangle$, associated norm $\|\cdot\|$, and Borel σ-algebra $\mathcal{B}$. Id denotes the identity operator on H and $\rightharpoonup$ and $\to$ denote, respectively, weak and strong convergence in H. The set of weak sequential cluster points of a sequence $(\mathsf{x}_n)_{n\in\mathbb{N}}$ in H is denoted by $\mathfrak{W}(\mathsf{x}_n)_{n\in\mathbb{N}}$. The underlying probability space is $(\Omega, \mathcal{F}, \mathsf{P})$. A H-valued random variable is a measurable map $x\colon(\Omega,\mathcal{F})\to(\mathsf{H},\mathcal{B})$. The smallest σ-algebra generated by a family Φ of random variables is denoted by $\sigma(\Phi)$. Let $\mathscr{F}=(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of sub-sigma algebras of $\mathcal{F}$ such that $(\forall n\in\mathbb{N})\ \mathcal{F}_n\subset\mathcal{F}_{n+1}$. We denote by $\ell_+(\mathscr{F})$ the set of sequences of $[0,+\infty[$-valued random variables $(\xi_n)_{n\in\mathbb{N}}$ such that, for every $n\in\mathbb{N}$, ξ_n is $\mathcal{F}_n$-measurable. We set

$$(\forall p\in\,]0,+\infty[)\quad \ell_+^p(\mathscr{F})=\left\{(\xi_n)_{n\in\mathbb{N}}\in\ell_+(\mathscr{F})\;\middle|\;\sum_{n\in\mathbb{N}}\xi_n^p<+\infty\ \text{P-a.s.}\right\} \tag{2.1}$$

and

$$\ell_+^\infty(\mathscr{F})=\left\{(\xi_n)_{n\in\mathbb{N}}\in\ell_+(\mathscr{F})\;\middle|\;\sup_{n\in\mathbb{N}}\xi_n<+\infty\ \text{P-a.s.}\right\}. \tag{2.2}$$

Given a sequence $(x_n)_{n\in\mathbb{N}}$ of H-valued random variables, we define

$$\mathscr{X}=(\mathcal{X}_n)_{n\in\mathbb{N}},\quad \text{where}\quad (\forall n\in\mathbb{N})\quad \mathcal{X}_n=\sigma(x_0,\ldots,x_n). \tag{2.3}$$

Equalities and inequalities involving random variables will always be understood to hold P-almost surely, even if the expression "P-a.s." is not explicitly written. For background on probability in Hilbert spaces, see [34, 38].

Lemma 2.2 [53, Theorem 1] *Let $\mathscr{F}=(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of sub-sigma algebras of $\mathcal{F}$ such that $(\forall n\in\mathbb{N})\ \mathcal{F}_n\subset\mathcal{F}_{n+1}$. Let $(\alpha_n)_{n\in\mathbb{N}}\in\ell_+(\mathscr{F})$, $(\vartheta_n)_{n\in\mathbb{N}}\in\ell_+(\mathscr{F})$, $(\eta_n)_{n\in\mathbb{N}}\in\ell_+^1(\mathscr{F})$, and $(\chi_n)_{n\in\mathbb{N}}\in\ell_+^1(\mathscr{F})$ be such that*

$$(\forall n\in\mathbb{N})\quad \mathsf{E}(\alpha_{n+1}\,|\,\mathcal{F}_n)+\vartheta_n\leqslant(1+\chi_n)\alpha_n+\eta_n\quad \text{P-a.s.} \tag{2.4}$$

Then $(\vartheta_n)_{n\in\mathbb{N}}\in\ell_+^1(\mathscr{F})$ and $(\alpha_n)_{n\in\mathbb{N}}$ converges P-a.s. *to a $[0,+\infty[$-valued random variable.*

Proposition 2.3 *Let F be a nonempty closed subset of H, let $\phi\colon[0,+\infty[\,\to[0,+\infty[$ be a strictly increasing function such that $\lim_{t\to+\infty}\phi(t)=+\infty$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of H-valued random variables. Suppose that, for every $\mathsf{z}\in\mathsf{F}$, there exist $(\chi_n(\mathsf{z}))_{n\in\mathbb{N}}\in\ell_+^1(\mathscr{X})$, $(\vartheta_n(\mathsf{z}))_{n\in\mathbb{N}}\in\ell_+(\mathscr{X})$, and $(\eta_n(\mathsf{z}))_{n\in\mathbb{N}}\in\ell_+^1(\mathscr{X})$ such that the following is satisfied* P-a.s.*:*

$$(\forall n\in\mathbb{N})\quad \mathsf{E}(\phi(\|x_{n+1}-\mathsf{z}\|)\,|\,\mathcal{X}_n)+\vartheta_n(\mathsf{z})\leqslant(1+\chi_n(\mathsf{z}))\phi(\|x_n-\mathsf{z}\|)+\eta_n(\mathsf{z}). \tag{2.5}$$

Then the following hold:

(i) $(\forall\mathsf{z}\in\mathsf{F})\ \big[\sum_{n\in\mathbb{N}}\vartheta_n(\mathsf{z})<+\infty$ P-a.s.$\big]$

(ii) $(x_n)_{n\in\mathbb{N}}$ *is bounded* P-a.s.

(iii) *There exists $\widetilde{\Omega}\in\mathcal{F}$ such that $\mathsf{P}(\widetilde{\Omega})=1$ and, for every $\omega\in\widetilde{\Omega}$ and every $\mathsf{z}\in\mathsf{F}$, $(\|x_n(\omega)-\mathsf{z}\|)_{n\in\mathbb{N}}$ converges.*

(iv) *Suppose that $\mathfrak{W}(x_n)_{n\in\mathbb{N}}\subset\mathsf{F}$* P-a.s. *Then $(x_n)_{n\in\mathbb{N}}$ converges weakly* P-a.s. *to an* F*-valued random variable.*

Proof. (i): Fix $\mathsf{z} \in \mathsf{F}$. It follows from (2.5) and Lemma 2.2 that $\sum_{n\in\mathbb{N}} \vartheta_n(\mathsf{z}) < +\infty$ P-a.s.

(ii): Let $\mathsf{z} \in \mathsf{F}$ and set $(\forall n \in \mathbb{N})\ \xi_n = \|x_n - \mathsf{z}\|$. We derive from (2.5) and Lemma 2.2 that $(\phi(\xi_n))_{n\in\mathbb{N}}$ converges P-a.s., say $\phi(\xi_n) \to \alpha$ P-a.s., where α is a $[0,+\infty[$-valued random variable. In turn, since $\lim_{t\to+\infty} \phi(t) = +\infty$, $(\xi_n)_{n\in\mathbb{N}}$ is bounded P-a.s. and so is $(x_n)_{n\in\mathbb{N}}$. For subsequent use, let us also note that

$$(\|x_n - \mathsf{z}\|)_{n\in\mathbb{N}} \text{ converges to a } [0,+\infty[\text{-valued random variable P-a.s.} \tag{2.6}$$

Indeed, take $\omega \in \Omega$ such that $(\xi_n(\omega))_{n\in\mathbb{N}}$ is bounded. Suppose that there exist $\tau(\omega) \in [0,+\infty[$, $\zeta(\omega) \in [0,+\infty[$, and subsequences $(\xi_{k_n}(\omega))_{n\in\mathbb{N}}$ and $(\xi_{l_n}(\omega))_{n\in\mathbb{N}}$ such that $\xi_{k_n}(\omega) \to \tau(\omega)$ and $\xi_{l_n}(\omega) \to \zeta(\omega) > \tau(\omega)$, and let $\delta(\omega) \in\ ]0,(\zeta(\omega)-\tau(\omega))/2[$. Then, for n sufficiently large, $\xi_{k_n}(\omega) \leqslant \tau(\omega)+\delta(\omega) < \zeta(\omega)-\delta(\omega) \leqslant \xi_{l_n}(\omega)$ and, since ϕ is strictly increasing, $\phi(\xi_{k_n}(\omega)) \leqslant \phi(\tau(\omega)+\delta(\omega)) < \phi(\zeta(\omega)-\delta(\omega)) \leqslant \phi(\xi_{l_n}(\omega))$. Taking the limit as $n \to +\infty$ yields $\alpha(\omega) \leqslant \phi(\tau(\omega)+\delta(\omega)) < \phi(\zeta(\omega)-\delta(\omega)) \leqslant \alpha(\omega)$, which is impossible. It follows that $\tau(\omega) = \zeta(\omega)$ and, in turn, that $\xi_n(\omega) \to \tau(\omega)$. Thus, $\xi_n \to \tau$ P-a.s.

(iii): Since H is separable, there exists a countable set Z such that $\overline{\mathsf{Z}} = \mathsf{F}$. According to (2.6), for every $\mathsf{z} \in \mathsf{F}$, there exists a set $\Omega_{\mathsf{z}} \in \mathcal{F}$ such that $\mathsf{P}(\Omega_{\mathsf{z}}) = 1$ and, for every $\omega \in \Omega_{\mathsf{z}}$, the sequence $(\|x_n(\omega) - \mathsf{z}\|)_{n\in\mathbb{N}}$ converges. Now set $\widetilde{\Omega} = \bigcap_{\mathsf{z}\in\mathsf{Z}} \Omega_{\mathsf{z}}$. Then, since Z is countable, $\mathsf{P}(\widetilde{\Omega}) = 1 - \mathsf{P}(\complement\widetilde{\Omega}) = 1 - \mathsf{P}(\bigcup_{\mathsf{z}\in\mathsf{Z}} \complement\Omega_{\mathsf{z}}) \geqslant 1 - \sum_{\mathsf{z}\in\mathsf{Z}} \mathsf{P}(\complement\Omega_{\mathsf{z}}) = 1$, hence $\mathsf{P}(\widetilde{\Omega}) = 1$. We now fix $\mathsf{z} \in \mathsf{F}$. Since $\overline{\mathsf{Z}} = \mathsf{F}$, there exists a sequence $(\mathsf{z}_k)_{k\in\mathbb{N}}$ in Z such that $\mathsf{z}_k \to \mathsf{z}$. As just seen, (2.6) yields

$$(\forall k \in \mathbb{N})(\exists\, \tau_k \colon \Omega \to [0,+\infty[)(\forall \omega \in \Omega_{\mathsf{z}_k}) \quad \|x_n(\omega) - \mathsf{z}_k\| \to \tau_k(\omega). \tag{2.7}$$

Now let $\omega \in \widetilde{\Omega}$. We have

$$(\forall k \in \mathbb{N})(\forall n \in \mathbb{N}) \quad -\|\mathsf{z}_k - \mathsf{z}\| \leqslant \|x_n(\omega) - \mathsf{z}\| - \|x_n(\omega) - \mathsf{z}_k\| \leqslant \|\mathsf{z}_k - \mathsf{z}\|. \tag{2.8}$$

Therefore

$$\begin{aligned}(\forall k \in \mathbb{N}) \quad -\|\mathsf{z}_k - \mathsf{z}\| &\leqslant \varliminf_{n\to+\infty} \|x_n(\omega) - \mathsf{z}\| - \lim_{n\to+\infty} \|x_n(\omega) - \mathsf{z}_k\| \\ &= \varliminf_{n\to+\infty} \|x_n(\omega) - \mathsf{z}\| - \tau_k(\omega) \\ &\leqslant \varlimsup_{n\to+\infty} \|x_n(\omega) - \mathsf{z}\| - \tau_k(\omega) \\ &= \varlimsup_{n\to+\infty} \|x_n(\omega) - \mathsf{z}\| - \lim_{n\to+\infty} \|x_n(\omega) - \mathsf{z}_k\| \\ &\leqslant \|\mathsf{z}_k - \mathsf{z}\|. \end{aligned} \tag{2.9}$$

Hence, taking the limit as $k \to +\infty$ in (2.9), we obtain that $(\|x_n(\omega) - \mathsf{z}\|)_{n\in\mathbb{N}}$ converges; more precisely, $\lim_{n\to+\infty} \|x_n(\omega) - \mathsf{z}\| = \lim_{k\to+\infty} \tau_k(\omega)$.

(iv): By assumption, there exists $\widehat{\Omega} \in \mathcal{F}$ such that $\mathsf{P}(\widehat{\Omega}) = 1$ and $(\forall \omega \in \widehat{\Omega})\ \mathfrak{W}(x_n(\omega))_{n\in\mathbb{N}} \subset \mathsf{F}$. Now define $\widetilde{\Omega}$ as in the proof of (iii), let $\omega \in \widehat{\Omega} \cap \widetilde{\Omega}$, and let $x(\omega)$ and $y(\omega)$ be two points in $\mathfrak{W}(x_n(\omega))_{n\in\mathbb{N}}$, say $x_{k_n}(\omega) \rightharpoonup x(\omega)$ and $x_{l_n}(\omega) \rightharpoonup y(\omega)$. Then (iii) implies that $(\|x_n(\omega) - x(\omega)\|)_{n\in\mathbb{N}}$ and $(\|x_n(\omega) - y(\omega)\|)_{n\in\mathbb{N}}$ converge. In turn, since

$$(\forall n \in \mathbb{N}) \quad \langle x_n(\omega) \mid x(\omega) - y(\omega)\rangle = \frac{1}{2}\big(\|x_n(\omega) - y(\omega)\|^2 - \|x_n(\omega) - x(\omega)\|^2 + \|x(\omega)\|^2 - \|y(\omega)\|^2\big), \tag{2.10}$$

the sequence $(\langle x_n(\omega) \mid x(\omega) - y(\omega)\rangle)_{n\in\mathbb{N}}$ converges, say

$$\langle x_n(\omega) \mid x(\omega) - y(\omega)\rangle \to \varrho(\omega). \tag{2.11}$$

However, since $x_{k_n}(\omega) \rightharpoonup x(\omega)$, we have $\langle x(\omega) \mid x(\omega) - y(\omega)\rangle = \varrho(\omega)$. Likewise, passing to the limit along the subsequence $(x_{l_n}(\omega))_{n\in\mathbb{N}}$ in (2.11) yields $\langle y(\omega) \mid x(\omega) - y(\omega)\rangle = \varrho(\omega)$. Thus,

$$0 = \langle x(\omega) \mid x(\omega) - y(\omega)\rangle - \langle y(\omega) \mid x(\omega) - y(\omega)\rangle = \|x(\omega) - y(\omega)\|^2. \tag{2.12}$$

This shows that $x(\omega) = y(\omega)$. Since $\omega \in \widetilde{\Omega}$, $(x_n(\omega))_{n\in\mathbb{N}}$ is bounded and we invoke [7, Lemma 2.38] to conclude that $x_n(\omega) \rightharpoonup x(\omega) \in \mathsf{F}$. Altogether, since $\mathsf{P}(\widehat{\Omega} \cap \widetilde{\Omega}) = 1$, $x_n \rightharpoonup x$ P-a.s. and the measurability of x follows from [49, Corollary 1.13]. □

Remark 2.4 Suppose that $\phi\colon t \mapsto t^2$ in (2.5). Then special cases of Proposition 2.3 are stated in several places in the literature. Thus, stochastic quasi-Fejér sequences were first discussed in [31] in the case when H is a Euclidean space and for every $n \in \mathbb{N}$, $\vartheta_n = \chi_n = 0$ and η_n is deterministic. A Hilbert space version of the results of [31] appears in [4] without proof. Finally, the case when all the processes are deterministic in (2.5) is discussed in [18].

The analysis of our main algorithms will rely on the following key illustration of Proposition 2.3. This result involves a general stochastic iterative process and it should also be of interest in the analysis of the asymptotic behavior of a broad class of stochastic algorithms, beyond those discussed in the present paper.

Theorem 2.5 *Let* F *be a nonempty closed subset of* H *and let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,1]$. *In addition, let* $(t_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, *and* $(e_n)_{n\in\mathbb{N}}$ *be sequences of* H*-valued random variables. Suppose that the following hold:*

(i) $(\forall n \in \mathbb{N})\ x_{n+1} = x_n + \lambda_n(t_n + e_n - x_n)$.

(ii) $\sum_{n\in\mathbb{N}} \lambda_n \sqrt{\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n)} < +\infty$ P-a.s.

(iii) *For every* $\mathsf{z} \in \mathsf{F}$, *there exist* $(\theta_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+(\mathscr{X})$, $(\mu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^\infty(\mathscr{X})$, *and* $(\nu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^\infty(\mathscr{X})$ *such that* $(\lambda_n\mu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X})$, $(\lambda_n\nu_n(\mathsf{z}))_{n\in\mathbb{N}} \in \ell_+^{1/2}(\mathscr{X})$, *and the following is satisfied* P-a.s.*:*

$$(\forall n \in \mathbb{N}) \quad \mathsf{E}(\|t_n - \mathsf{z}\|^2 \,|\, \mathfrak{X}_n) + \theta_n(\mathsf{z}) \leqslant (1 + \mu_n(\mathsf{z}))\|x_n - \mathsf{z}\|^2 + \nu_n(\mathsf{z}). \tag{2.13}$$

Then

$$(\forall \mathsf{z} \in \mathsf{F}) \quad \Bigg[\sum_{n\in\mathbb{N}} \lambda_n\theta_n(\mathsf{z}) < +\infty \text{ P-a.s.}\Bigg] \tag{2.14}$$

and

$$\sum_{n\in\mathbb{N}} \lambda_n(1-\lambda_n)\mathsf{E}(\|t_n - x_n\|^2 \,|\, \mathfrak{X}_n) < +\infty \text{ P-a.s.} \tag{2.15}$$

Furthermore, suppose that:

(iv) $\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset \mathsf{F}$ P-a.s.

Then $(x_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F*-valued random variable.*

Proof. Let $\mathsf{z} \in \mathsf{F}$ and set

$$(\forall n \in \mathbb{N}) \qquad \varepsilon_n = \lambda_n \sqrt{\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n)}. \tag{2.16}$$

It follows from Jensen's inequality and (iii) that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \mathsf{E}(\|t_n - \mathsf{z}\| \,|\, \mathfrak{X}_n) &\leqslant \sqrt{\mathsf{E}(\|t_n - \mathsf{z}\|^2 \,|\, \mathfrak{X}_n)} \\ &\leqslant \sqrt{(1+\mu_n(\mathsf{z}))\|x_n - \mathsf{z}\|^2 + \nu_n(\mathsf{z})} \\ &\leqslant \sqrt{1+\mu_n(\mathsf{z})}\|x_n - \mathsf{z}\| + \sqrt{\nu_n(\mathsf{z})} \\ &\leqslant (1+\mu_n(\mathsf{z})/2)\|x_n - \mathsf{z}\| + \sqrt{\nu_n(\mathsf{z})}. \end{aligned} \tag{2.17}$$

On the other hand, (i) and the triangle inequality yield

$$(\forall n \in \mathbb{N}) \quad \|x_{n+1} - \mathsf{z}\| \leqslant (1-\lambda_n)\|x_n - \mathsf{z}\| + \lambda_n\|t_n - \mathsf{z}\| + \lambda_n\|e_n\|. \tag{2.18}$$

Consequently,

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \mathsf{E}(\|x_{n+1} - \mathsf{z}\| \,|\, \mathfrak{X}_n) &\leqslant (1-\lambda_n)\|x_n - \mathsf{z}\| + \lambda_n\mathsf{E}(\|t_n - \mathsf{z}\| \,|\, \mathfrak{X}_n) + \lambda_n\mathsf{E}(\|e_n\| \,|\, \mathfrak{X}_n) \\ &\leqslant \Big(1 + \frac{\lambda_n\mu_n(\mathsf{z})}{2}\Big)\|x_n - \mathsf{z}\| + \lambda_n\sqrt{\nu_n(\mathsf{z})} + \lambda_n\sqrt{\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n)} \\ &= \Big(1 + \frac{\lambda_n\mu_n(\mathsf{z})}{2}\Big)\|x_n - \mathsf{z}\| + \sqrt{\lambda_n\nu_n(\mathsf{z})} + \varepsilon_n. \end{aligned} \tag{2.19}$$

Upon applying Proposition 2.3(ii) with $\phi\colon t \mapsto t$, we deduce from (2.19) that $(x_n)_{n\in\mathbb{N}}$ is almost surely bounded and, by virtue of assumption (iii), that $(\mathsf{E}(\|t_n - \mathsf{z}\|^2 \,|\, \mathfrak{X}_n))_{n\in\mathbb{N}}$ is likewise. Thus,

$$\rho_1(\mathsf{z}) = \operatorname*{ess\,sup}_{n\in\mathbb{N}} \|x_n - \mathsf{z}\| < +\infty \quad \text{and} \quad \rho_2(\mathsf{z}) = \operatorname*{ess\,sup}_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|t_n - \mathsf{z}\|^2 \,|\, \mathfrak{X}_n)} < +\infty. \tag{2.20}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \chi_n(\mathsf{z}) = \lambda_n\mu_n(\mathsf{z}) \\ \xi_n(\mathsf{z}) = 2\lambda_n(1-\lambda_n)\|x_n - \mathsf{z}\|\,\|e_n\| + 2\lambda_n^2\|t_n - \mathsf{z}\|\,\|e_n\| + \lambda_n^2\|e_n\|^2 \\ \vartheta_n(\mathsf{z}) = \lambda_n\theta_n(\mathsf{z}) + \lambda_n(1-\lambda_n)\mathsf{E}(\|t_n - x_n\|^2 \,|\, \mathfrak{X}_n) \\ \eta_n(\mathsf{z}) = \mathsf{E}(\xi_n(\mathsf{z}) \,|\, \mathfrak{X}_n) + \lambda_n\nu_n(\mathsf{z}). \end{cases} \tag{2.21}$$

On the one hand, it follows from (2.21), the Cauchy-Schwarz inequality, and (2.16) that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \mathsf{E}(\xi_n(\mathsf{z}) \,|\, \mathfrak{X}_n) &= 2\lambda_n(1-\lambda_n)\|x_n - \mathsf{z}\|\,\mathsf{E}(\|e_n\| \,|\, \mathfrak{X}_n) \\ &\quad + 2\lambda_n^2\mathsf{E}(\|t_n - \mathsf{z}\|\,\|e_n\| \,|\, \mathfrak{X}_n) + \lambda_n^2\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n) \\ &\leqslant 2\lambda_n\|x_n - \mathsf{z}\|\,\sqrt{\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n)} \\ &\quad + 2\lambda_n\sqrt{\mathsf{E}(\|t_n - \mathsf{z}\|^2 \,|\, \mathfrak{X}_n)}\sqrt{\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n)} + \lambda_n^2\mathsf{E}(\|e_n\|^2 \,|\, \mathfrak{X}_n) \\ &\leqslant 2(\rho_1(\mathsf{z}) + \rho_2(\mathsf{z}))\varepsilon_n + \varepsilon_n^2. \end{aligned} \tag{2.22}$$

In turn, we deduce from (2.16), (2.21), (ii), and (iii) that

$$\big(\eta_n(\mathsf{z})\big)_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X}) \quad \text{and} \quad \big(\chi_n(\mathsf{z})\big)_{n\in\mathbb{N}} \in \ell_+^1(\mathscr{X}). \tag{2.23}$$

On the other hand, we derive from (i), [7, Corollary 2.14], and (2.21) that

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \|x_{n+1}-\mathsf{z}\|^2 &= \|(1-\lambda_n)(x_n-\mathsf{z})+\lambda_n(t_n-\mathsf{z})\|^2\\ &\quad + 2\lambda_n\big\langle(1-\lambda_n)(x_n-\mathsf{z})+\lambda_n\big(t_n-\mathsf{z}\big)\mid e_n\big\rangle+\lambda_n^2\|e_n\|^2\\ &\leqslant (1-\lambda_n)\|x_n-\mathsf{z}\|^2+\lambda_n\|t_n-\mathsf{z}\|^2\\ &\quad -\lambda_n(1-\lambda_n)\|t_n-x_n\|^2+\xi_n(\mathsf{z}).\end{aligned} \tag{2.24}$$

Hence, (iii), (2.21), and (2.22) imply that

$$\begin{aligned}&(\forall n\in\mathbb{N})\quad \mathsf{E}(\|x_{n+1}-\mathsf{z}\|^2\,|\,\mathcal{X}_n)\\ &\quad\leqslant (1-\lambda_n)\|x_n-\mathsf{z}\|^2+\lambda_n\mathsf{E}(\|t_n-\mathsf{z}\|^2\,|\,\mathcal{X}_n)-\lambda_n(1-\lambda_n)\mathsf{E}(\|t_n-x_n\|^2\,|\,\mathcal{X}_n)+\mathsf{E}(\xi_n(\mathsf{z})\,|\,\mathcal{X}_n)\\ &\quad\leqslant (1-\lambda_n)\|x_n-\mathsf{z}\|^2+\lambda_n\big((1+\mu_n(\mathsf{z}))\|x_n-\mathsf{z}\|^2+\nu_n(\mathsf{z})-\theta_n(\mathsf{z})\big)\\ &\quad\quad -\lambda_n(1-\lambda_n)\mathsf{E}(\|t_n-x_n\|^2\,|\,\mathcal{X}_n)+\mathsf{E}(\xi_n(\mathsf{z})\,|\,\mathcal{X}_n)\\ &\quad\leqslant (1+\chi_n(\mathsf{z}))\|x_n-\mathsf{z}\|^2-\vartheta_n(\mathsf{z})+\eta_n(\mathsf{z}).\end{aligned} \tag{2.25}$$

Thus, in view of (2.23), applying Proposition 2.3(i) with $\phi\colon t\mapsto t^2$ yields $\sum_{n\in\mathbb{N}}\vartheta_n(\mathsf{z})<+\infty$ P-a.s. and it follows from (2.21) that (2.14) and (2.15) are established. Finally, the weak convergence assertion follows from (iv) and Proposition 2.3(iv) applied with $\phi\colon t\mapsto t^2$. □

Although our primary objective is to apply Theorem 2.5 to block-coordinate methods, it also yields new results for classical methods. As an illustration, the following application describes a Krasnosel'skiĭ–Mann iteration with stochastic errors.

Corollary 2.6 *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ such that $\sum_{n\in\mathbb{N}}\lambda_n(1-\lambda_n)=+\infty$ and let $\mathsf{T}\colon\mathsf{H}\to\mathsf{H}$ be a nonexpansive operator such that $\operatorname{Fix}\mathsf{T}\neq\varnothing$. Let x_0 and $(e_n)_{n\in\mathbb{N}}$ be H-valued random variables. Iterate*

$$\begin{array}{l}\textit{for } n=0,1,\ldots\\ \big\lfloor\; x_{n+1}=x_n+\lambda_n\big(\mathsf{T}x_n+e_n-x_n\big).\end{array} \tag{2.26}$$

In addition, assume that $\sum_{n\in\mathbb{N}}\lambda_n\sqrt{\mathsf{E}(\|e_n\|^2\,|\,\mathcal{X}_n)}<+\infty$ P-a.s. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly P-a.s. to a $(\operatorname{Fix}\mathsf{T})$-valued random variable.

Proof. Set $\mathsf{F}=\operatorname{Fix}\mathsf{T}$. Since T is continuous, T is measurable and F is closed. Now let $\mathsf{z}\in\mathsf{F}$ and set $(\forall n\in\mathbb{N})$ $t_n=\mathsf{T}x_n$. Then, using the nonexpansiveness of T, we obtain

$$(\forall n\in\mathbb{N})\quad \begin{cases}x_{n+1}=x_n+\lambda_n(t_n+e_n-x_n)\\ \mathsf{E}(\|t_n-x_n\|^2\,|\,\mathcal{X}_n)=\|\mathsf{T}x_n-x_n\|^2\\ \mathsf{E}(\|t_n-\mathsf{z}\|^2\,|\,\mathcal{X}_n)=\|\mathsf{T}x_n-\mathsf{T}\mathsf{z}\|^2\leqslant\|x_n-\mathsf{z}\|^2.\end{cases} \tag{2.27}$$

It follows that properties (i)–(iii) in Theorem 2.5 are satisfied with $(\forall n\in\mathbb{N})$ $\theta_n=0$, $\mu_n=0$, and $\nu_n=0$. Hence, (2.15) and (2.27) imply the existence of $\widetilde{\Omega}\in\mathcal{F}$ such that $\mathsf{P}(\widetilde{\Omega})=1$ and

$$(\forall\omega\in\widetilde{\Omega})\quad \sum_{n\in\mathbb{N}}\lambda_n(1-\lambda_n)\|\mathsf{T}x_n(\omega)-x_n(\omega)\|^2<+\infty. \tag{2.28}$$

To conclude, let us establish property (iv) of Theorem 2.5. We have

$$(\forall n\in\mathbb{N})\quad \begin{aligned}\|\mathsf{T}x_{n+1}-x_{n+1}\| &= \|\mathsf{T}x_{n+1}-\mathsf{T}x_n+(1-\lambda_n)(\mathsf{T}x_n-x_n)-\lambda_n e_n\|\\ &\leqslant \|\mathsf{T}x_{n+1}-\mathsf{T}x_n\|+(1-\lambda_n)\|\mathsf{T}x_n-x_n\|+\lambda_n\|e_n\|\\ &\leqslant \|x_{n+1}-x_n\|+(1-\lambda_n)\|\mathsf{T}x_n-x_n\|+\lambda_n\|e_n\|\\ &\leqslant \lambda_n\|\mathsf{T}x_n-x_n\|+(1-\lambda_n)\|\mathsf{T}x_n-x_n\|+2\lambda_n\|e_n\|\\ &= \|\mathsf{T}x_n-x_n\|+2\lambda_n\|e_n\|\end{aligned}\tag{2.29}$$

and therefore

$$(\forall n\in\mathbb{N})\quad \begin{aligned}\mathsf{E}(\|\mathsf{T}x_{n+1}-x_{n+1}\|\,|\,\mathcal{X}_n) &\leqslant \|\mathsf{T}x_n-x_n\|+2\lambda_n\mathsf{E}(\|e_n\|\,|\,\mathcal{X}_n)\\ &\leqslant \|\mathsf{T}x_n-x_n\|+2\lambda_n\sqrt{\mathsf{E}(\|e_n\|^2\,|\,\mathcal{X}_n)}.\end{aligned}\tag{2.30}$$

In turn, Lemma 2.2 implies that there exists $\widehat{\Omega}\subset\widetilde{\Omega}$ such that $\widehat{\Omega}\in\mathcal{F}$, $\mathsf{P}(\widehat{\Omega})=1$, and $(\forall\omega\in\widehat{\Omega})$ $(\|\mathsf{T}x_n(\omega)-x_n(\omega)\|)_{n\in\mathbb{N}}$ converges. Now let $\omega\in\widehat{\Omega}$ and let $\mathsf{x}\in\mathfrak{W}(x_n(\omega))_{n\in\mathbb{N}}$, say $x_{k_n}(\omega)\rightharpoonup\mathsf{x}$. In view of (2.28), since $\sum_{n\in\mathbb{N}}\lambda_n(1-\lambda_n)=+\infty$, we have $\underline{\lim}\,\|\mathsf{T}x_n(\omega)-x_n(\omega)\|=0$ and therefore $\|\mathsf{T}x_n(\omega)-x_n(\omega)\|\to 0$. Altogether, $x_{k_n}(\omega)\rightharpoonup\mathsf{x}$ and $\mathsf{T}x_{k_n}(\omega)-x_{k_n}(\omega)\to 0$. Since T is nonexpansive, the demiclosed principle [7, Corollary 4.18] asserts that $\mathsf{x}\in\mathsf{F}$. □

Remark 2.7 Corollary 2.6 extends [18, Theorem 5.5], which is restricted to deterministic processes and therefore less realistic error models. As shown in [7, 16, 18], the Krasnosel'skiĭ–Mann iteration process is at the core of many algorithms in variational problems and optimization. Corollary 2.6 therefore provides stochastically perturbed versions of these algorithms.

3 Single-layer random block-coordinate fixed point algorithms

In the remainder of the paper, the following notation will be used.

Notation 3.1 $\mathsf{H}_1,\ldots,\mathsf{H}_m$ are separable real Hilbert spaces and $\mathbf{H}=\mathsf{H}_1\oplus\cdots\oplus\mathsf{H}_m$ is their direct Hilbert sum. The scalar products and associated norms of these spaces are all denoted by $\langle\cdot\mid\cdot\rangle$ and $\|\cdot\|$, respectively, and $\mathbf{x}=(\mathsf{x}_1,\ldots,\mathsf{x}_m)$ denotes a generic vector in $\mathbf{H}$. Given a sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}=(x_{1,n},\ldots,x_{m,n})_{n\in\mathbb{N}}$ of $\mathbf{H}$-valued random variables, we set $(\forall n\in\mathbb{N})$ $\boldsymbol{\mathcal{X}}_n=\sigma(\boldsymbol{x}_0,\ldots,\boldsymbol{x}_n)$.

We recall that an operator $\mathbf{T}\colon\mathbf{H}\to\mathbf{H}$ with fixed point set $\operatorname{Fix}\mathbf{T}$ is quasinonexpansive if [7]

$$(\forall\mathbf{z}\in\operatorname{Fix}\mathbf{T})(\forall\mathbf{x}\in\mathbf{H})\quad \|\mathbf{T}\mathbf{x}-\mathbf{z}\|\leqslant\|\mathbf{x}-\mathbf{z}\|.\tag{3.1}$$

Theorem 3.2 *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\inf_{n\in\mathbb{N}}\lambda_n>0$ and $\sup_{n\in\mathbb{N}}\lambda_n<1$ and set $\mathsf{D}=\{0,1\}^m\smallsetminus\{\mathbf{0}\}$. For every $n\in\mathbb{N}$, let $\mathbf{T}_n\colon\mathbf{H}\to\mathbf{H}\colon\mathbf{x}\mapsto(\mathsf{T}_{i,n}\,\mathbf{x})_{1\leqslant i\leqslant m}$ be a quasinonexpansive operator where, for every $i\in\{1,\ldots,m\}$, $\mathsf{T}_{i,n}\colon\mathbf{H}\to\mathsf{H}_i$ is measurable. Let $\boldsymbol{x}_0$ and $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ be $\mathbf{H}$-valued random variables, and let $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ be identically distributed D-valued random variables. Iterate*

$$\begin{array}{l}\textit{for } n=0,1,\ldots\\ \left\lfloor\begin{array}{l}\textit{for } i=1,\ldots,m\\ \left\lfloor\; x_{i,n+1}=x_{i,n}+\varepsilon_{i,n}\lambda_n\big(\mathsf{T}_{i,n}\,(x_{1,n},\ldots,x_{m,n})+a_{i,n}-x_{i,n}\big),\right.\end{array}\right.\end{array}\tag{3.2}$$

and set $(\forall n\in\mathbb{N})$ $\boldsymbol{\mathcal{E}}_n=\sigma(\boldsymbol{\varepsilon}_n)$. In addition, assume that the following hold:

(i) $\mathsf{F} = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} \mathsf{T}_n \neq \varnothing$.

(ii) $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2 \,|\, \mathfrak{X}_n)} < +\infty$.

(iii) *For every* $n \in \mathbb{N}$, $\mathfrak{E}_n$ *and* $\mathfrak{X}_n$ *are independent.*

(iv) $(\forall i \in \{1,\ldots,m\})\ \mathsf{p}_i = \sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=1} \mathsf{P}[\boldsymbol{\varepsilon}_0 = \boldsymbol{\epsilon}] > 0.$

Then $\mathsf{T}_n\boldsymbol{x}_n - \boldsymbol{x}_n \to \mathbf{0}$ P-a.s. *Furthermore, suppose that:*

(v) $\mathfrak{W}(\boldsymbol{x}_n)_{n\in\mathbb{N}} \subset \mathsf{F}$ P-a.s.

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* F*-valued random variable.*

Proof. We define a norm $|||\cdot|||$ on H by

$$(\forall \mathsf{x} \in \mathsf{H}) \quad |||\mathsf{x}|||^2 = \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i}\|\mathsf{x}_i\|^2. \tag{3.3}$$

We are going to apply Theorem 2.5 in $(\mathsf{H}, |||\cdot|||)$. Let us set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{t}_n = (t_{i,n})_{1\leqslant i\leqslant m} \\ \boldsymbol{e}_n = (\varepsilon_{i,n}a_{i,n})_{1\leqslant i\leqslant m}, \end{cases} \quad \text{where } (\forall i \in \{1,\ldots,m\})\ t_{i,n} = x_{i,n} + \varepsilon_{i,n}(\mathsf{T}_{i,n}\,\boldsymbol{x}_n - x_{i,n}). \tag{3.4}$$

Then it follows from (3.2) that

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \lambda_n\big(\boldsymbol{t}_n + \boldsymbol{e}_n - \boldsymbol{x}_n\big), \tag{3.5}$$

while (ii) implies that

$$\sum_{n\in\mathbb{N}} \lambda_n \mathsf{E}(|||\boldsymbol{e}_n|||^2 \,|\, \mathfrak{X}_n) \leqslant \sum_{n\in\mathbb{N}} \mathsf{E}(|||\boldsymbol{a}_n|||^2 \,|\, \mathfrak{X}_n) < +\infty. \tag{3.6}$$

Since the operators $(\mathsf{T}_n)_{n\in\mathbb{N}}$ are quasinonexpansive, F is closed [6, Section 2]. Now let $\mathsf{z} \in \mathsf{F}$ and set

$$(\forall n \in \mathbb{N})(\forall i \in \{1,\ldots,m\}) \quad \mathsf{q}_{i,n} \colon \mathsf{H} \times \mathsf{D} \to \mathbb{R} \colon (\mathsf{x}, \boldsymbol{\epsilon}) \mapsto \|\mathsf{x}_i - \mathsf{z}_i + \epsilon_i(\mathsf{T}_{i,n}\,\mathsf{x} - \mathsf{x}_i)\|^2. \tag{3.7}$$

Note that, for every $n \in \mathbb{N}$ and every $i \in \{1,\ldots,m\}$, since $\mathsf{T}_{i,n}$ is measurable, so are the functions $(\mathsf{q}_{i,n}(\cdot,\boldsymbol{\epsilon}))_{\boldsymbol{\epsilon}\in\mathsf{D}}$. Consequently, since, for every $n \in \mathbb{N}$, (iii) asserts that the events $([\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}])_{\boldsymbol{\epsilon}\in\mathsf{D}}$ form an almost sure partition of Ω and are independent from $\mathfrak{X}_n$, and since the random variables $(\mathsf{q}_{i,n}(\boldsymbol{x}_n,\boldsymbol{\epsilon}))_{\substack{1\leqslant i\leqslant m\\ \boldsymbol{\epsilon}\in\mathsf{D}}}$ are $\mathfrak{X}_n$-measurable, we obtain [40, Section 28.2]

$$\begin{aligned}(\forall n \in \mathbb{N})(\forall i \in \{1,\ldots,m\}) \quad \mathsf{E}(\|t_{i,n} - \mathsf{z}_i\|^2 \,|\, \mathfrak{X}_n) &= \mathsf{E}\bigg(\mathsf{q}_{i,n}(\boldsymbol{x}_n,\boldsymbol{\varepsilon}_n)\sum_{\boldsymbol{\epsilon}\in\mathsf{D}} 1_{[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]} \,\bigg|\, \mathfrak{X}_n\bigg) \\ &= \sum_{\boldsymbol{\epsilon}\in\mathsf{D}} \mathsf{E}(\mathsf{q}_{i,n}(\boldsymbol{x}_n,\boldsymbol{\epsilon})1_{[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]} \,|\, \mathfrak{X}_n) \\ &= \sum_{\boldsymbol{\epsilon}\in\mathsf{D}} \mathsf{E}(1_{[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]} \,|\, \mathfrak{X}_n)\mathsf{q}_{i,n}(\boldsymbol{x}_n,\boldsymbol{\epsilon}) \\ &= \sum_{\boldsymbol{\epsilon}\in\mathsf{D}} \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}]\mathsf{q}_{i,n}(\boldsymbol{x}_n,\boldsymbol{\epsilon}). \end{aligned} \tag{3.8}$$

In turn, (3.3), (3.7), (3.4), (iv), and (3.1) yield

$$
\begin{aligned}
(\forall n \in \mathbb{N}) \quad & \mathsf{E}(|||\boldsymbol{t}_n - \mathbf{z}|||^2 \,|\, \boldsymbol{\mathfrak{X}}_n) \\
&= \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i} \mathsf{E}(\|t_{i,n} - \mathsf{z}_i\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n) \\
&= \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i} \sum_{\boldsymbol{\epsilon} \in \mathsf{D}} \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}] \|x_{i,n} - \mathsf{z}_i + \epsilon_{i,n}(\mathsf{T}_{i,n}\,\boldsymbol{x}_n - x_{i,n})\|^2 \\
&= \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i} \left(\sum_{\boldsymbol{\epsilon} \in \mathsf{D}, \epsilon_i = 1} \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}] \, \|\mathsf{T}_{i,n}\,\boldsymbol{x}_n - \mathsf{z}_i\|^2 + \sum_{\boldsymbol{\epsilon} \in \mathsf{D},\, \epsilon_i = 0} \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}] \, \|x_{i,n} - \mathsf{z}_i\|^2 \right) \\
&= \|\mathbf{T}_{\boldsymbol{n}} \boldsymbol{x}_n - \mathbf{z}\|^2 + \sum_{i=1}^{m} \frac{1 - \mathsf{p}_i}{\mathsf{p}_i} \|x_{i,n} - \mathsf{z}_i\|^2 \\
&= |||\boldsymbol{x}_n - \mathbf{z}|||^2 + \|\mathbf{T}_n \boldsymbol{x}_n - \mathbf{z}\|^2 - \|\boldsymbol{x}_n - \mathbf{z}\|^2 \\
&\leqslant |||\boldsymbol{x}_n - \mathbf{z}|||^2.
\end{aligned} \tag{3.9}
$$

Altogether, properties (i)–(iii) of Theorem 2.5 are satisfied with $(\forall n \in \mathbb{N})\ \theta_n = \mu_n = \nu_n = 0$. We therefore derive from (2.15) that $\sum_{n \in \mathbb{N}} \lambda_n (1 - \lambda_n) \mathsf{E}(|||\boldsymbol{t}_n - \boldsymbol{x}_n|||^2 \,|\, \boldsymbol{\mathfrak{X}}_n) < +\infty$ P-a.s. In view of our conditions on $(\lambda_n)_{n \in \mathbb{N}}$, this yields

$$
\mathsf{E}(|||\boldsymbol{t}_n - \boldsymbol{x}_n|||^2 \,|\, \boldsymbol{\mathfrak{X}}_n) \to 0 \ \text{ P-a.s.} \tag{3.10}
$$

On the other hand, proceeding as in (3.8) leads to

$$
(\forall n \in \mathbb{N})(\forall i \in \{1, \ldots, m\}) \quad \mathsf{E}(\|t_{i,n} - x_{i,n}\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n) = \sum_{\boldsymbol{\epsilon} \in \mathsf{D}} \epsilon_i \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}] \|\mathsf{T}_{i,n}\,\boldsymbol{x}_n - x_{i,n}\|^2. \tag{3.11}
$$

Hence, it follows from (3.3), (3.4), and (iv) that

$$
\begin{aligned}
(\forall n \in \mathbb{N}) \quad \mathsf{E}(|||\boldsymbol{t}_n - \boldsymbol{x}_n|||^2 \,|\, \boldsymbol{\mathfrak{X}}_n) &= \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i} \mathsf{E}(\|t_{i,n} - x_{i,n}\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n) \\
&= \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i} \sum_{\boldsymbol{\epsilon} \in \mathsf{D}} \epsilon_i \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}] \|\mathsf{T}_{i,n}\,\boldsymbol{x}_n - x_{i,n}\|^2 \\
&= \sum_{i=1}^{m} \frac{1}{\mathsf{p}_i} \sum_{\boldsymbol{\epsilon} \in \mathsf{D}, \epsilon_i = 1} \mathsf{P}[\boldsymbol{\varepsilon}_n = \boldsymbol{\epsilon}] \|\mathsf{T}_{i,n}\,\boldsymbol{x}_n - x_{i,n}\|^2 \\
&= \|\mathbf{T}_n \boldsymbol{x}_n - \boldsymbol{x}_n\|^2.
\end{aligned} \tag{3.12}
$$

Accordingly, (3.10) yields $\mathbf{T}_n \boldsymbol{x}_n - \boldsymbol{x}_n \to \mathbf{0}$ P-a.s. We conclude by observing that the weak convergence assertion is a consequence of Theorem 2.5. □

Remark 3.3 Let us make a few observations on Theorem 3.2.

(i) The binary variable $\varepsilon_{i,n}$ signals whether the i-th coordinate $\mathsf{T}_{i,n}$ of the operator $\mathbf{T}_n$ is activated or not at iteration n.

(ii) Assumption (iv) guarantees that each operator in $(\mathsf{T}_{i,n})_{1\leqslant i\leqslant m}$ is activated with a nonzero probability at each iteration n of Algorithm (4.1). The simplest scenario corresponds to the case when the block sweeping process assigns nonzero probabilities to multivariate indices $\boldsymbol{\epsilon} \in \mathsf{D}$ having a single component equal to 1. Then only one of the operators in $(\mathsf{T}_{i,n})_{1\leqslant i\leqslant m}$ is activated randomly.

Our first corollary is a random block-coordinate version of the Krasnosel'skiĭ–Mann iteration.

Corollary 3.4 *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\inf_{n\in\mathbb{N}}\lambda_n > 0$ and $\sup_{n\in\mathbb{N}}\lambda_n < 1$, set $\mathsf{D} = \{0,1\}^m \smallsetminus \{\mathbf{0}\}$, and let $\mathbf{T}\colon \mathbf{H} \to \mathbf{H}\colon \mathbf{x} \mapsto (\mathsf{T}_i\,\mathbf{x})_{1\leqslant i\leqslant m}$ be a nonexpansive operator such that $\operatorname{Fix}\mathbf{T} \neq \varnothing$ where, for every $i \in \{1,\ldots,m\}$, $\mathsf{T}_i\colon \mathbf{H} \to \mathsf{H}_i$. Let $\boldsymbol{x}_0$ and $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ be $\mathbf{H}$-valued random variables, and let $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ be identically distributed D-valued random variables. Iterate*

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for } i = 1,\ldots,m\\
\left\lfloor\; x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\lambda_n\big(\mathsf{T}_i\,(x_{1,n},\ldots,x_{m,n}) + a_{i,n} - x_{i,n}\big),\right.
\end{array}
\right.
\end{array}
\tag{3.13}
$$

and set $(\forall n\in\mathbb{N})$ $\boldsymbol{\mathcal{E}}_n = \sigma(\boldsymbol{\varepsilon}_n)$. In addition, assume that properties (ii)–(iv) of Theorem 3.2 hold. Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ converges weakly P-a.s. *to a* $(\operatorname{Fix}\mathbf{T})$*-valued random variable.*

Proof. This is an application of Theorem 3.2 with $(\forall n \in \mathbb{N})$ $\mathbf{T}_n = \mathbf{T}$. Indeed, Theorem 3.2 asserts that $\mathbf{T}\boldsymbol{x}_n - \boldsymbol{x}_n \to \mathbf{0}$ P-a.s. Now take $\omega \in \Omega$ such that $\mathbf{T}\boldsymbol{x}_n(\omega) - \boldsymbol{x}_n(\omega) \to \mathbf{0}$ and $\mathbf{x} \in \mathfrak{W}(\boldsymbol{x}_n(\omega))$, say $\boldsymbol{x}_{k_n}(\omega) \to \mathbf{x}$. Then it follows from the demiclosed principle [7, Corollary 4.18] that $\mathbf{x} \in \operatorname{Fix}\mathbf{T}$. Hence property (v) of Theorem 3.2 is satisfied. □

Remark 3.5 A special case of Corollary 3.4 appears in [37]. It corresponds to the scenario in which $\mathbf{H}$ is finite-dimensional, $\mathbf{T}$ is firmly nonexpansive, and, for every $n \in \mathbb{N}$, $\lambda_n = 1$, $\boldsymbol{a}_n = \mathbf{0}$, and only one block is activated as in Remark 3.3(ii). Let us also note that a renorming similar to that performed in (3.3) was employed in [44].

Next, we consider the construction of a fixed point of a family of averaged operators.

Definition 3.6 Let $\mathsf{T}\colon \mathsf{H} \to \mathsf{H}$ be nonexpansive and let $\alpha \in]0,1[$. Then T is averaged with constant α, or α-averaged, if there exists a nonexpansive operator $\mathsf{R}\colon \mathsf{H} \to \mathsf{H}$ such that $\mathsf{T} = (1-\alpha)\mathrm{Id} + \alpha\mathsf{R}$.

Proposition 3.7 [7, Proposition 4.25] *Let $\mathsf{T}\colon \mathsf{H} \to \mathsf{H}$ be nonexpansive and let $\alpha \in]0,1[$. Then T is α-averaged if and only if* $(\forall \mathsf{x} \in \mathsf{H})(\forall \mathsf{y} \in \mathsf{H})$ $\|\mathsf{T}\mathsf{x} - \mathsf{T}\mathsf{y}\|^2 \leqslant \|\mathsf{x}-\mathsf{y}\|^2 - \dfrac{1-\alpha}{\alpha}\|(\mathrm{Id}-\mathsf{T})\mathsf{x} - (\mathrm{Id}-\mathsf{T})\mathsf{y}\|^2$.

Corollary 3.8 *Let $\chi \in]0,1[$, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$, and set $\mathsf{D} = \{0,1\}^m \smallsetminus \{\mathbf{0}\}$. For every $n \in \mathbb{N}$, let $\lambda_n \in [\chi/\alpha_n, (1-\chi)/\alpha_n]$ and let $\mathbf{T}_n\colon \mathbf{H} \to \mathbf{H}\colon \mathbf{x} \mapsto (\mathsf{T}_{i,n}\,\mathbf{x})_{1\leqslant i\leqslant m}$ be an α_n-averaged operator, where, for every $i \in \{1,\ldots,m\}$, $\mathsf{T}_{i,n}\colon \mathbf{H} \to \mathsf{H}_i$. Furthermore, let $\boldsymbol{x}_0$ and $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ be $\mathbf{H}$-valued random variables, and let $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ be identically distributed D-valued random variables. Iterate*

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for } i = 1,\ldots,m\\
\left\lfloor\; x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\lambda_n\big(\mathsf{T}_{i,n}\,(x_{1,n},\ldots,x_{m,n}) + a_{i,n} - x_{i,n}\big),\right.
\end{array}
\right.
\end{array}
\tag{3.14}
$$

and set $(\forall n\in\mathbb{N})$ $\boldsymbol{\mathcal{E}}_n = \sigma(\boldsymbol{\varepsilon}_n)$. In addition, assume that the following hold:

(i) $\mathbf{F} = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} \mathbf{T}_n \neq \varnothing$.

(ii) $\sum_{n\in\mathbb{N}} \alpha_n^{-1}\sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n)} < +\infty$.

(iii) *For every* $n \in \mathbb{N}$*,* $\boldsymbol{\mathcal{E}}_n$ *and* $\boldsymbol{\mathfrak{X}}_n$ *are independent.*

(iv) $(\forall i \in \{1,\ldots,m\}) \displaystyle\sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=1} \mathsf{P}[\boldsymbol{\varepsilon}_0 = \boldsymbol{\epsilon}] > 0$.

(v) *There exists* $\widehat{\Omega} \in \mathcal{F}$ *such that* $\mathsf{P}(\widehat{\Omega}) = 1$ *and*

$$(\forall \omega \in \widehat{\Omega}) \quad \Big[\alpha_n^{-1}\big(\mathbf{T}_n\boldsymbol{x}_n(\omega) - \boldsymbol{x}_n(\omega)\big) \to \mathbf{0} \quad \Rightarrow \quad \mathfrak{W}(\boldsymbol{x}_n(\omega))_{n\in\mathbb{N}} \subset \mathbf{F}\Big]. \tag{3.15}$$

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* $\mathbf{F}$*-valued random variable.*

Proof. Set $(\forall n \in \mathbb{N})$ $\mathbf{R}_n = (1-\alpha_n^{-1})\mathbf{Id} + \alpha_n^{-1}\mathbf{T}_n$ and $(\forall i \in \{1,\ldots,m\})$ $\mathsf{R}_{i,n} = (1-\alpha_n^{-1})\mathrm{Id} + \alpha_n^{-1}\mathsf{T}_{i,n}$. Moreover, set $(\forall n \in \mathbb{N})$ $\mu_n = \alpha_n\lambda_n$ and $\boldsymbol{b}_n = \alpha_n^{-1}\boldsymbol{a}_n$. Then $(\forall n \in \mathbb{N})$ $\operatorname{Fix}\mathbf{R}_n = \operatorname{Fix}\mathbf{T}_n$ and $\mathbf{R}_n$ is nonexpansive. In addition, we derive from (3.14) that

$$(\forall n \in \mathbb{N})(\forall i \in \{1,\ldots,m\}) \quad x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\mu_n\big(\mathsf{R}_{i,n}\,\boldsymbol{x}_n + b_{i,n} - x_{i,n}\big). \tag{3.16}$$

Since $(\mu_n)_{n\in\mathbb{N}}$ lies in $[\chi, 1-\chi]$ and $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n)} = \sum_{n\in\mathbb{N}} \alpha_n^{-1}\sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n)} < +\infty$, the result follows from Theorem 3.2. □

Remark 3.9 In the special case of a single-block (i.e., $m = 1$) and of deterministic errors, Corollary 3.8 reduces to a scenario found in [19, Theorem 4.2].

4 Double-layer random block-coordinate fixed point algorithms

The algorithm analyzed in this section comprises two successive applications of nonexpansive operators at each iteration. We recall that Notation 3.1 is in force.

Theorem 4.1 *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}}$ *be sequences in* $]0,1[$ *such that* $\sup_{n\in\mathbb{N}} \alpha_n < 1$ *and* $\sup_{n\in\mathbb{N}} \beta_n < 1$*, let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,1]$ *such that* $\inf_{n\in\mathbb{N}} \lambda_n > 0$*, and set* $\mathsf{D} = \{0,1\}^m \smallsetminus \{\mathbf{0}\}$*. Let* $\boldsymbol{x}_0$*,* $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$*, and* $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ *be* $\mathbf{H}$*-valued random variables, and let* $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ *be identically distributed* D*-valued random variables. For every* $n \in \mathbb{N}$*, let* $\mathbf{R}_n \colon \mathbf{H} \to \mathbf{H}$ *be* β_n*-averaged and let* $\mathbf{T}_n \colon \mathbf{H} \to \mathbf{H} \colon \mathbf{x} \mapsto (\mathsf{T}_{i,n}\mathbf{x})_{1\leqslant i\leqslant m}$ *be* α_n*-averaged, where,* $(\forall i \in \{1,\ldots,m\})$ $\mathsf{T}_{i,n} \colon \mathbf{H} \to \mathsf{H}_i$*. Iterate*

$$\begin{array}{l}\text{for } n = 0,1,\ldots\\ \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \mathbf{R}_n\boldsymbol{x}_n + \boldsymbol{b}_n\\ \text{for } i = 1,\ldots,m\\ \left\lfloor\; x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\lambda_n\big(\mathsf{T}_{i,n}\boldsymbol{y}_n + a_{i,n} - x_{i,n}\big),\right.\end{array}\right.\end{array} \tag{4.1}$$

and set $(\forall n \in \mathbb{N})$ $\boldsymbol{\mathcal{E}}_n = \sigma(\boldsymbol{\varepsilon}_n)$*. In addition, assume that the following hold:*

(i) $\mathbf{F} = \bigcap_{n\in\mathbb{N}} \operatorname{Fix}(\mathbf{T}_n \circ \mathbf{R}_n) \neq \varnothing$.

(ii) $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n)} < +\infty$ *and* $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2 \,|\, \boldsymbol{\mathfrak{X}}_n)} < +\infty$.

(iii) *For every* $n \in \mathbb{N}$*,* $\boldsymbol{\mathcal{E}}_n$ *and* $\boldsymbol{\mathcal{X}}_n$ *are independent.*

(iv) $(\forall i \in \{1,\ldots,m\})$ $\mathsf{p}_i = \sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=1} \mathsf{P}[\boldsymbol{\varepsilon}_0 = \boldsymbol{\epsilon}] > 0.$

Then

$$\Big[\,(\forall \mathbf{z} \in \mathbf{F})\;\; \mathbf{T}_n(\mathbf{R}_n\boldsymbol{x}_n) - \mathbf{R}_n\boldsymbol{x}_n + \mathbf{R}_n\mathbf{z} \to \mathbf{z}\,\Big] \text{ P-a.s.} \tag{4.2}$$

and

$$\Big[\,(\forall \mathbf{z} \in \mathbf{F})\;\; \boldsymbol{x}_n - \mathbf{R}_n\boldsymbol{x}_n + \mathbf{R}_n\mathbf{z} \to \mathbf{z}\,\Big] \text{ P-a.s.} \tag{4.3}$$

Furthermore, suppose that:

(v) $\mathfrak{W}(\boldsymbol{x}_n)_{n\in\mathbb{N}} \subset \mathbf{F}$ P-a.s.

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* $\mathbf{F}$*-valued random variable.*

Proof. Let us prove that the result is an application of Theorem 2.5 in the renormed Hilbert space $(\mathbf{H}, |||\cdot|||)$, where $|||\cdot|||$ is defined in (3.3). Note that

$$(\forall \mathbf{x} \in \mathbf{H}) \quad \|\mathbf{x}\|^2 \leqslant |||\mathbf{x}|||^2 \leqslant \frac{1}{\min\limits_{1\leqslant i\leqslant m} \mathsf{p}_i}\|\mathbf{x}\|^2 \tag{4.4}$$

and that, since the operators $(\mathbf{R}_n \circ \mathbf{T}_n)_{n\in\mathbb{N}}$ are nonexpansive, the sets $(\operatorname{Fix}(\mathbf{T}_n \circ \mathbf{R}_n))_{n\in\mathbb{N}}$ are closed [7, Corollary 4.15], and so is $\mathbf{F}$. Next, for every $n \in \mathbb{N}$, set $\boldsymbol{r}_n = \mathbf{R}_n\boldsymbol{x}_n$, and define $\boldsymbol{t}_n$, $\boldsymbol{c}_n$, $\boldsymbol{d}_n$, and $\boldsymbol{e}_n$ coordinate-wise by

$$(\forall i \in \{1,\ldots,m\}) \quad \begin{cases} t_{i,n} = x_{i,n} + \varepsilon_{i,n}(\mathsf{T}_{i,n}\boldsymbol{r}_n - x_{i,n}) \\ c_{i,n} = \varepsilon_{i,n}a_{i,n} \\ d_{i,n} = \varepsilon_{i,n}(\mathsf{T}_{i,n}\boldsymbol{y}_n - \mathsf{T}_{i,n}\boldsymbol{r}_n) \end{cases} \quad \text{and} \quad e_{i,n} = c_{i,n} + d_{i,n}. \tag{4.5}$$

Then (4.1) implies that

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \lambda_n\big(\boldsymbol{t}_n + \boldsymbol{e}_n - \boldsymbol{x}_n\big). \tag{4.6}$$

On the other hand, we derive from (4.5) that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \sqrt{\mathsf{E}(|||\boldsymbol{e}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n)} &\leqslant \sqrt{\mathsf{E}(|||\boldsymbol{c}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n)} + \sqrt{\mathsf{E}(|||\boldsymbol{d}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n)} \\ &\leqslant \sqrt{\mathsf{E}(|||\boldsymbol{a}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n)} + \sqrt{\mathsf{E}(|||\boldsymbol{d}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n)}. \end{aligned} \tag{4.7}$$

However, it follows from (4.5), (4.4), and the nonexpansiveness of the operators $(\mathbf{T}_n)_{n\in\mathbb{N}}$ that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \mathsf{E}(|||\boldsymbol{d}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n) &\leqslant \frac{1}{\min\limits_{1\leqslant i\leqslant m} \mathsf{p}_i}\mathsf{E}\bigg(\sum_{i=1}^m \|\varepsilon_{i,n}(\mathsf{T}_{i,n}\boldsymbol{y}_n - \mathsf{T}_{i,n}\boldsymbol{r}_n)\|^2\bigg|\,\boldsymbol{\mathcal{X}}_n\bigg) \\ &\leqslant \frac{1}{\min\limits_{1\leqslant i\leqslant m} \mathsf{p}_i}\mathsf{E}(\|\mathbf{T}_n\boldsymbol{y}_n - \mathbf{T}_n\boldsymbol{r}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n) \\ &\leqslant \frac{1}{\min\limits_{1\leqslant i\leqslant m} \mathsf{p}_i}\mathsf{E}(\|\boldsymbol{y}_n - \boldsymbol{r}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n) \\ &= \frac{1}{\min\limits_{1\leqslant i\leqslant m} \mathsf{p}_i}\mathsf{E}(\|\boldsymbol{b}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n). \end{aligned} \tag{4.8}$$

Consequently (4.4), (4.7), and (ii) yield

$$\sum_{n\in\mathbb{N}}\lambda_n\sqrt{\mathsf{E}(|||\boldsymbol{e}_n|||^2\,|\,\boldsymbol{\mathcal{X}}_n)}\leqslant\frac{1}{\min\limits_{1\leqslant i\leqslant m}\sqrt{\mathsf{p}_i}}\left(\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n)}+\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n)}\right)$$
$$<+\infty. \tag{4.9}$$

Now let $\mathbf{z}\in\mathbf{F}$, and set

$$(\forall n\in\mathbb{N})(\forall i\in\{1,\ldots,m\})\quad \mathsf{q}_{i,n}\colon\mathbf{H}\times\mathsf{D}\to\mathbb{R}\colon(\mathbf{x},\boldsymbol{\epsilon})\mapsto\|\mathsf{x}_i-\mathsf{z}_i+\epsilon_i(\mathsf{T}_{i,n}(\mathbf{R}_n\mathbf{x})-\mathsf{x}_i)\|^2. \tag{4.10}$$

Observe that, for every $n\in\mathbb{N}$ and every $i\in\{1,\ldots,m\}$, by continuity of $\mathbf{R}_n$ and $\mathsf{T}_{i,n}$, $\mathsf{T}_{i,n}\circ\mathbf{R}_n$ is measurable, and the functions $(\mathsf{q}_{i,n}(\cdot,\boldsymbol{\epsilon}))_{\boldsymbol{\epsilon}\in\mathsf{D}}$ are therefore likewise. Consequently, using (iv) and arguing as in (3.8) leads to

$$\begin{aligned}(\forall n\in\mathbb{N})(\forall i\in\{1,\ldots,m\})\quad &\mathsf{E}(\|t_{i,n}-\mathsf{z}_i\|^2\,|\,\boldsymbol{\mathcal{X}}_n)\\ &=\sum_{\boldsymbol{\epsilon}\in\mathsf{D}}\mathsf{E}(\mathsf{q}_{i,n}(\boldsymbol{x}_n,\boldsymbol{\epsilon})1_{[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]}\,|\,\boldsymbol{\mathcal{X}}_n)\\ &=\sum_{\boldsymbol{\epsilon}\in\mathsf{D}}\mathsf{P}[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]\|x_{i,n}-\mathsf{z}_i+\epsilon_i(\mathsf{T}_{i,n}\boldsymbol{r}_n-x_{i,n})\|^2.\end{aligned} \tag{4.11}$$

Hence, recalling (3.3) and (iv), we obtain

$$\begin{aligned}(\forall n\in\mathbb{N})\quad &\mathsf{E}(|||\boldsymbol{t}_n-\mathbf{z}|||^2\,|\,\boldsymbol{\mathcal{X}}_n)\\ &=\sum_{i=1}^m\frac{1}{\mathsf{p}_i}\mathsf{E}(\|t_{i,n}-\mathsf{z}_i\|^2\,|\,\boldsymbol{\mathcal{X}}_n)\\ &=\sum_{i=1}^m\frac{1}{\mathsf{p}_i}\sum_{\boldsymbol{\epsilon}\in\mathsf{D}}\mathsf{P}[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]\|x_{i,n}-\mathsf{z}_i+\epsilon_i(\mathsf{T}_{i,n}\boldsymbol{r}_n-x_{i,n})\|^2\\ &=\sum_{i=1}^m\frac{1}{\mathsf{p}_i}\left(\sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=1}\mathsf{P}[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]\,\|\mathsf{T}_{i,n}\boldsymbol{r}_n-\mathsf{z}_i\|^2+\sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=0}\mathsf{P}[\boldsymbol{\varepsilon}_n=\boldsymbol{\epsilon}]\,\|x_{i,n}-\mathsf{z}_i\|^2\right)\\ &=\|\mathbf{T}_n\boldsymbol{r}_n-\mathbf{z}\|^2+\sum_{i=1}^m\frac{1-\mathsf{p}_i}{\mathsf{p}_i}\|x_{i,n}-\mathsf{z}_i\|^2\\ &=|||\boldsymbol{x}_n-\mathbf{z}|||^2+\|\mathbf{T}_n\boldsymbol{r}_n-\mathbf{z}\|^2-\|\boldsymbol{x}_n-\mathbf{z}\|^2.\end{aligned} \tag{4.12}$$

However, we deduce from (i) and Proposition 3.7 that

$$(\forall n\in\mathbb{N})\quad \|\mathbf{T}_n\boldsymbol{r}_n-\mathbf{z}\|^2+\frac{1-\alpha_n}{\alpha_n}\|\boldsymbol{r}_n-\mathbf{T}_n\boldsymbol{r}_n-\mathbf{R}_n\mathbf{z}+\mathbf{z}\|^2\leqslant\|\boldsymbol{r}_n-\mathbf{R}_n\mathbf{z}\|^2. \tag{4.13}$$

Combining (4.12) with (4.13) yields

$$\begin{aligned}(\forall n\in\mathbb{N})\quad &\mathsf{E}(|||\boldsymbol{t}_n-\mathbf{z}|||^2\,|\,\boldsymbol{\mathcal{X}}_n)+\frac{1-\alpha_n}{\alpha_n}\|\boldsymbol{r}_n-\mathbf{T}_n\boldsymbol{r}_n-\mathbf{R}_n\mathbf{z}+\mathbf{z}\|^2\\ &\leqslant|||\boldsymbol{x}_n-\mathbf{z}|||^2+\|\mathbf{R}_n\boldsymbol{x}_n-\mathbf{R}_n\mathbf{z}\|^2-\|\boldsymbol{x}_n-\mathbf{z}\|^2.\end{aligned} \tag{4.14}$$

Now set $\chi=\min\{1/\sup_{k\in\mathbb{N}}\alpha_k,1/\sup_{k\in\mathbb{N}}\beta_k\}-1$. Then $\chi\in\,]0,+\infty[$ and since, for every $n\in\mathbb{N}$, $\mathbf{R}_n$ is β_n-averaged, Proposition 3.7 and (4.14) yield

$$(\forall n\in\mathbb{N})\quad \mathsf{E}(|||\boldsymbol{t}_n-\mathbf{z}|||^2\,|\,\boldsymbol{\mathcal{X}}_n)+\theta_n(\mathbf{z})\leqslant|||\boldsymbol{x}_n-\mathbf{z}|||^2, \tag{4.15}$$

where

$$(\forall n \in \mathbb{N}) \quad \theta_n(\mathbf{z}) = \chi\big(\|\boldsymbol{r}_n - \mathbf{T}_n\boldsymbol{r}_n - \mathbf{R}_n\mathbf{z} + \mathbf{z}\|^2 + \|\boldsymbol{x}_n - \boldsymbol{r}_n - \mathbf{z} + \mathbf{R}_n\mathbf{z}\|^2\big) \tag{4.16}$$

$$\leqslant \frac{1-\alpha_n}{\alpha_n}\|\boldsymbol{r}_n - \mathbf{T}_n\boldsymbol{r}_n - \mathbf{R}_n\mathbf{z} + \mathbf{z}\|^2 + \frac{1-\beta_n}{\beta_n}\|\boldsymbol{x}_n - \boldsymbol{r}_n - \mathbf{z} + \mathbf{R}_n\mathbf{z}\|^2. \tag{4.17}$$

We have thus shown that properties (i)–(iii) of Theorem 2.5 hold with $(\forall n \in \mathbb{N})\ \mu_n = \nu_n = 0$. Next, let $\mathbf{Z}$ be a countable set which is dense in $\mathbf{F}$. Then (2.14) asserts that

$$(\forall \mathbf{z} \in \mathbf{Z})(\exists\, \Omega_{\mathbf{z}} \in \mathcal{F}) \quad \mathsf{P}(\Omega_{\mathbf{z}}) = 1 \quad \text{and} \quad (\forall \omega \in \Omega_{\mathbf{z}}) \quad \sum_{n\in\mathbb{N}} \lambda_n\theta_n(\mathbf{z},\omega) < +\infty. \tag{4.18}$$

Moreover, the event $\widetilde{\Omega} = \bigcap_{\mathbf{z}\in\mathbf{Z}} \Omega_{\mathbf{z}}$ is almost certain, i.e., $\mathsf{P}(\widetilde{\Omega}) = 1$. Now fix $\mathbf{z} \in \mathbf{F}$. By density, we can extract from $\mathbf{Z}$ a sequence $(\mathbf{z}_k)_{k\in\mathbb{N}}$ such that $\mathbf{z}_k \to \mathbf{z}$. In turn, since $\inf_{n\in\mathbb{N}} \lambda_n > 0$, we derive from (4.16) and (4.18) that

$$(\forall k \in \mathbb{N})(\forall \omega \in \widetilde{\Omega}) \quad \begin{cases} \boldsymbol{r}_n(\omega) - \mathbf{T}_n\boldsymbol{r}_n(\omega) - \mathbf{R}_n\mathbf{z}_k + \mathbf{z}_k \to \mathbf{0} \\ \boldsymbol{x}_n(\omega) - \boldsymbol{r}_n(\omega) - \mathbf{z}_k + \mathbf{R}_n\mathbf{z}_k \to \mathbf{0}. \end{cases} \tag{4.19}$$

Now set $\zeta = \sup_{n\in\mathbb{N}} \sqrt{\beta_n/(1-\beta_n)}$, and $(\forall n \in \mathbb{N})\ \mathbf{S}_n = \mathbf{Id} - \mathbf{R}_n$ and $\boldsymbol{p}_n = \boldsymbol{r}_n - \mathbf{T}_n\boldsymbol{r}_n$. Then it follows from Proposition 3.7 that the operators $(\mathbf{S}_n)_{n\in\mathbb{N}}$ are ζ-Lipschitzian. Consequently

$$\begin{aligned}(\forall k \in \mathbb{N})(\forall n \in \mathbb{N})(\forall \omega \in \widetilde{\Omega}) \quad & -\zeta\|\mathbf{z}_k - \mathbf{z}\| \leqslant -\|\mathbf{S}_n\mathbf{z}_k - \mathbf{S}_n\mathbf{z}\| \\ &\leqslant \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}\| - \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}_k\| \leqslant \|\mathbf{S}_n\mathbf{z}_k - \mathbf{S}_n\mathbf{z}\| \leqslant \zeta\|\mathbf{z}_k - \mathbf{z}\| \end{aligned} \tag{4.20}$$

and, therefore, (4.19) yields

$$\begin{aligned}(\forall k \in \mathbb{N}) \quad -\zeta\|\mathbf{z}_k - \mathbf{z}\| &\leqslant \varliminf_{n\to+\infty} \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}\| - \lim_{n\to+\infty} \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}_k\| \\ &= \varliminf_{n\to+\infty} \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}\| \\ &\leqslant \varlimsup_{n\to+\infty} \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}\| \\ &\leqslant \varlimsup_{n\to+\infty} \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}\| - \lim_{n\to+\infty} \|\boldsymbol{p}_n(\omega) + \mathbf{S}_n\mathbf{z}_k\| \\ &\leqslant \zeta\|\mathbf{z}_k - \mathbf{z}\|. \end{aligned} \tag{4.21}$$

Since $\|\mathbf{z}_k - \mathbf{z}\| \to 0$ and $\mathsf{P}(\widetilde{\Omega}) = 1$, we obtain $\boldsymbol{p}_n + \mathbf{S}_n\mathbf{z} \to \mathbf{0}$ P-a.s., which proves (4.2). Likewise, set $(\forall n \in \mathbb{N})\ \boldsymbol{q}_n = \boldsymbol{x}_n - \boldsymbol{r}_n$. Then, proceeding as in (4.21), (4.19) yields $\boldsymbol{q}_n + \mathbf{S}_n\mathbf{z} \to \mathbf{0}$, which establishes (4.3). Finally, the weak convergence claim follows from (v) and Theorem 2.5. □

Remark 4.2 Consider the special case when only one-block is present ($m = 1$) and when the error sequences $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, as well as $\boldsymbol{x}_0$, are deterministic. Then the setting of Theorem 4.1 is found in [19, Theorem 6.3]. Our framework therefore makes it possible to design block-coordinate versions of the algorithms which comply with the two-layer format of [19, Theorem 6.3], such as the forward-backward algorithm [19] or the algorithm of [50]. Theorem 4.1 will be applied to block-coordinate forward-backward splitting in Section 5.2.

5 Applications to operator splitting

Let $\mathsf{A}\colon \mathsf{H} \to 2^{\mathsf{H}}$ be a set-valued operator and let A^{-1} be its inverse, i.e., $(\forall(\mathsf{x},\mathsf{u}) \in \mathsf{H}^2)$ $\mathsf{x} \in \mathsf{A}^{-1}\mathsf{u} \Leftrightarrow \mathsf{u} \in \mathsf{A}\mathsf{x}$. The resolvent of A is $\mathsf{J}_{\mathsf{A}} = (\mathrm{Id} + \mathsf{A})^{-1}$. If A is monotone, then J_{A} is single-valued and nonexpansive and, furthermore, if A is maximally monotone, then $\operatorname{dom} \mathsf{J}_{\mathsf{A}} = \mathsf{H}$. We denote by $\Gamma_0(\mathsf{H})$ the class of lower semicontinuous convex functions $f\colon \mathsf{H} \to \left]-\infty,+\infty\right]$ such that $\mathsf{f} \not\equiv +\infty$. The subdifferential of $\mathsf{f} \in \Gamma_0(\mathsf{H})$ is the maximally monotone operator

$$\partial \mathsf{f}\colon \mathsf{H} \to 2^{\mathsf{H}}\colon \mathsf{x} \mapsto \big\{\mathsf{u} \in \mathsf{H} \;\big|\; (\forall \mathsf{y} \in \mathsf{H})\;\; \langle \mathsf{y} - \mathsf{x} \mid \mathsf{u}\rangle + \mathsf{f}(\mathsf{x}) \leqslant \mathsf{f}(\mathsf{y})\big\}. \tag{5.1}$$

For every $\mathsf{x} \in \mathsf{H}$, $\mathsf{f} + \|\mathsf{x} - \cdot\|^2/2$ has a unique minimizer, which is denoted by $\mathrm{prox}_{\mathsf{f}}\mathsf{x}$ [42]. We have

$$\mathrm{prox}_{\mathsf{f}} = \mathsf{J}_{\partial \mathsf{f}}. \tag{5.2}$$

For background on convex analysis and monotone operator theory, see [7]. We continue to use the standing Notation 3.1.

5.1 Random block-coordinate Douglas-Rachford splitting

We propose a random sweeping, block-coordinate version of the Douglas-Rachford algorithm with stochastic errors. The purpose of this algorithm is to construct iteratively a zero of the sum of two maximally monotone operators and it has found applications in numerous areas; see, e.g., [7, 9, 11, 21, 24, 29, 35, 39, 46, 47, 48].

Proposition 5.1 *Set* $\mathsf{D} = \{0,1\}^m \smallsetminus \{\mathbf{0}\}$ *and, for every* $i \in \{1,\ldots,m\}$, *let* $\mathsf{A}_i\colon \mathsf{H}_i \to 2^{\mathsf{H}_i}$ *be maximally monotone and let* $\mathsf{B}_i\colon \mathbf{H} \to 2^{\mathsf{H}_i}$. *Suppose that* $\mathbf{B}\colon \mathbf{H} \to 2^{\mathbf{H}}\colon \mathbf{x} \mapsto \times_{i=1}^m \mathsf{B}_i\mathbf{x}$ *is maximally monotone and that the set* $\mathbf{F}$ *of solutions to the problem*

$$\text{find}\;\; \mathsf{x}_1 \in \mathsf{H}_1,\ldots,\mathsf{x}_m \in \mathsf{H}_m \;\;\text{such that}\;\; (\forall i \in \{1,\ldots,m\}) \quad 0 \in \mathsf{A}_i\mathsf{x}_i + \mathsf{B}_i(\mathsf{x}_1,\ldots,\mathsf{x}_m) \tag{5.3}$$

is nonempty. Set $\mathbf{B}^{-1}\colon \mathbf{u} \mapsto \times_{i=1}^m \mathsf{C}_i\mathbf{u}$ *where, for every* $i \in \{1,\ldots,m\}$, $\mathsf{C}_i\colon \mathsf{H} \to 2^{\mathsf{H}_i}$. *We also consider the set* $\mathbf{F}^*$ *of solutions to the dual problem*

$$\text{find}\;\; \mathsf{u}_1 \in \mathsf{H}_1,\ldots,\mathsf{u}_m \in \mathsf{H}_m \;\;\text{such that}\;\; (\forall i \in \{1,\ldots,m\})\; 0 \in -\mathsf{A}_i^{-1}(-\mathsf{u}_i) + \mathsf{C}_i(\mathsf{u}_1,\ldots,\mathsf{u}_m). \tag{5.4}$$

Let $\gamma \in \left]0,+\infty\right[$, *let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence in* $\left]0,2\right[$ *such that* $\inf_{n\in\mathbb{N}} \mu_n > 0$ *and* $\sup_{n\in\mathbb{N}} \mu_n < 2$, *let* $\boldsymbol{x}_0$, $\boldsymbol{z}_0$, $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, *and* $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ *be* $\mathbf{H}$*-valued random variables, and let* $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ *be identically distributed* D*-valued random variables. Set* $\mathbf{J}_{\gamma\mathbf{B}}\colon \mathbf{x} \mapsto (\mathsf{Q}_i\mathbf{x})_{1\leqslant i\leqslant m}$ *where, for every* $i \in \{1,\ldots,m\}$, $\mathsf{Q}_i\colon \mathbf{H} \to \mathsf{H}_i$, *iterate*

$$\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor \begin{array}{l}
\text{for } i = 1,\ldots,m\\
\left\lfloor \begin{array}{l}
z_{i,n+1} = z_{i,n} + \varepsilon_{i,n}\big(\mathsf{Q}_i(x_{1,n},\ldots,x_{m,n}) + b_{i,n} - z_{i,n}\big)\\
x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\mu_n\big(\mathsf{J}_{\gamma\mathsf{A}_i}(2z_{i,n+1} - x_{i,n}) + a_{i,n} - z_{i,n+1}\big),
\end{array}\right.
\end{array}\right.
\end{array} \tag{5.5}$$

and set $(\forall n \in \mathbb{N})$ $\boldsymbol{\mathcal{E}}_n = \sigma(\boldsymbol{\varepsilon}_n)$. *Assume that the following hold:*

(i) $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2 \,|\, \boldsymbol{\mathcal{X}}_n)} < +\infty$ *and* $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2 \,|\, \boldsymbol{\mathcal{X}}_n)} < +\infty$.

(ii) *For every* $n \in \mathbb{N}$*,* $\boldsymbol{\mathcal{E}}_n$ *and* $\boldsymbol{\mathcal{X}}_n$ *are independent.*

(iii) $(\forall i \in \{1,\ldots,m\}) \displaystyle\sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=1} \mathsf{P}[\boldsymbol{\varepsilon}_0 = \boldsymbol{\epsilon}] > 0.$

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to a* $\mathbf{H}$*-valued random variable* $\boldsymbol{x}$ *such that* $\boldsymbol{z} = \mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}$ *is an* $\mathbf{F}$*-valued random variable and* $\boldsymbol{u} = \gamma^{-1}(\boldsymbol{x}-\boldsymbol{z})$ *is an* $\mathbf{F}^*$*-valued random variable. Furthermore, suppose that:*

(iv) $J_{\gamma\mathbf{B}}$ *is weakly sequentially continuous and* $\boldsymbol{b}_n \rightharpoonup \mathbf{0}$ P-a.s.

Then $\boldsymbol{z}_n \rightharpoonup \boldsymbol{z}$ P-a.s. *and* $\gamma^{-1}(\boldsymbol{x}_n - \boldsymbol{z}_n) \rightharpoonup \boldsymbol{u}$ P-a.s.

Proof. Set $\mathbf{A}\colon \mathbf{H} \to 2^{\mathbf{H}}\colon \mathbf{x} \mapsto \times_{i=1}^m \mathsf{A}_i \mathsf{x}_i$ and $(\forall i \in \{1,\ldots,m\})$ $\mathsf{T}_i = (2\mathsf{J}_{\gamma\mathsf{A}_i} - \mathrm{Id}\,) \circ (2\mathsf{Q}_i - \mathrm{Id}\,)$. Then $\mathbf{T} = (2\mathbf{J}_{\gamma\mathbf{A}} - \mathbf{Id}\,) \circ (2\mathbf{J}_{\gamma\mathbf{B}} - \mathbf{Id}\,)$ is nonexpansive as the composition of two nonexpansive operators [7, Corollary 23.10(ii)]. Furthermore $\operatorname{Fix}\mathbf{T} \neq \varnothing$ since [19, Lemma 2.6(iii)]

$$\mathbf{J}_{\gamma\mathbf{B}}(\operatorname{Fix}\mathbf{T}) = \operatorname{zer}(\mathbf{A}+\mathbf{B}) = \mathbf{F} \neq \varnothing. \tag{5.6}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \lambda_n = \mu_n/2 \quad \text{and} \quad \boldsymbol{e}_n = 2\big(\mathbf{J}_{\gamma\mathbf{A}}(2\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n + 2\boldsymbol{b}_n - \boldsymbol{x}_n) - \mathbf{J}_{\gamma\mathbf{A}}(2\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n - \boldsymbol{x}_n) + \boldsymbol{a}_n - \boldsymbol{b}_n\big). \tag{5.7}$$

Then we derive from (5.5) that

$$\begin{aligned}(\forall n \in \mathbb{N})(\forall i \in \{1,\ldots,m\}) \quad x_{i,n+1} &= x_{i,n} + \varepsilon_{i,n}\mu_n\big(\mathsf{J}_{\gamma\mathsf{A}_i}\big(2\mathsf{Q}_i\boldsymbol{x}_n + 2b_{i,n} - x_{i,n}\big) + a_{i,n} - z_{i,n+1}\big)\\ &= x_{i,n} + \varepsilon_{i,n}\lambda_n\big(2\mathsf{J}_{\gamma\mathsf{A}_i}\big(2\mathsf{Q}_i\boldsymbol{x}_n - x_{i,n}\big) + e_{i,n} - 2\mathsf{Q}_i\boldsymbol{x}_n\big)\\ &= x_{i,n} + \varepsilon_{i,n}\lambda_n\big(\mathsf{T}_i\boldsymbol{x}_n + e_{i,n} - x_{i,n}\big),\end{aligned} \tag{5.8}$$

which is precisely the iteration process (3.13). Furthermore, we infer from (5.7) and the nonexpansiveness of $\mathbf{J}_{\gamma\mathbf{A}}$ [7, Corollary 23.10(i)] that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|\boldsymbol{e}_n\|^2 &\leqslant 4\|\mathbf{J}_{\gamma\mathbf{A}}(2\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n + 2\boldsymbol{b}_n - \boldsymbol{x}_n) - \mathbf{J}_{\gamma\mathbf{A}}(2\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n - \boldsymbol{x}_n) + \boldsymbol{a}_n - \boldsymbol{b}_n\|^2\\ &\leqslant 12\big(\|\mathbf{J}_{\gamma\mathbf{A}}(2\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n + 2\boldsymbol{b}_n - \boldsymbol{x}_n) - \mathbf{J}_{\gamma\mathbf{A}}(2\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n - \boldsymbol{x}_n)\|^2 + \|\boldsymbol{a}_n\|^2 + \|\boldsymbol{b}_n\|^2\big)\\ &\leqslant 12\big(\|\boldsymbol{a}_n\|^2 + 5\|\boldsymbol{b}_n\|^2\big)\end{aligned} \tag{5.9}$$

and therefore that

$$(\forall n \in \mathbb{N}) \quad \sqrt{\mathsf{E}(\|\boldsymbol{e}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n)} \leqslant 2\sqrt{3}\Big(\sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n)} + \sqrt{5}\sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n)}\Big). \tag{5.10}$$

Thus, we deduce from (i) that $\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{e}_n\|^2\,|\,\boldsymbol{\mathcal{X}}_n)} < +\infty$. Altogether, the almost sure weak convergence of $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ to a $(\operatorname{Fix}\mathbf{T})$-valued random variable $\boldsymbol{x}$ follows from Corollary 3.4. In turn, (5.6) asserts that $\boldsymbol{z} = \mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x} \in \mathbf{F}$ P-a.s. Now set $\boldsymbol{u} = \gamma^{-1}(\boldsymbol{x}-\boldsymbol{z})$. Then, P-a.s.,

$$\boldsymbol{z} = \mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x} \;\Leftrightarrow\; \boldsymbol{x} - \boldsymbol{z} \in \gamma\mathbf{B}\boldsymbol{z} \;\Leftrightarrow\; \boldsymbol{z} \in \mathbf{B}^{-1}\boldsymbol{u} \tag{5.11}$$

and

$$\begin{aligned}\boldsymbol{x} \in \operatorname{Fix}\mathbf{T} \quad &\Leftrightarrow \quad \boldsymbol{x} = (2\mathbf{J}_{\gamma\mathbf{A}} - \mathbf{Id}\,)(2\boldsymbol{z} - \boldsymbol{x})\\ &\Leftrightarrow \quad \boldsymbol{z} = \mathbf{J}_{\gamma\mathbf{A}}(2\boldsymbol{z} - \boldsymbol{x})\\ &\Leftrightarrow \quad \boldsymbol{z} - \boldsymbol{x} \in \gamma\mathbf{A}\boldsymbol{z}\\ &\Leftrightarrow \quad -\boldsymbol{z} \in -\mathbf{A}^{-1}(-\boldsymbol{u}).\end{aligned} \tag{5.12}$$

These imply that $\mathbf{0} \in -\mathbf{A}^{-1}(-\boldsymbol{u}) + \mathbf{B}^{-1}\boldsymbol{u}$ P-a.s., i.e., that $\boldsymbol{u} \in \mathbf{F}^*$ P-a.s. Finally, assume that (iv) holds. Then there exists $\widetilde{\Omega} \in \mathcal{F}$ such that $\mathsf{P}(\widetilde{\Omega}) = 1$ and $(\forall \omega \in \widetilde{\Omega})$ $\mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}_n(\omega) \rightharpoonup \mathbf{J}_{\gamma\mathbf{B}}\boldsymbol{x}(\omega) = \boldsymbol{z}(\omega)$. Now let $i \in \{1, \ldots, m\}$, $\omega \in \widetilde{\Omega}$, and $\mathbf{v} \in \mathbf{H}$. Then $\langle \mathsf{Q}_i\boldsymbol{x}_n(\omega) \mid \mathsf{v}_i\rangle \to \langle z_i(\omega) \mid \mathsf{v}_i\rangle$ and (5.5) yields

$$(\forall n \in \mathbb{N}) \quad \langle z_{i,n+1}(\omega) \mid \mathsf{v}_i\rangle = \langle z_{i,n}(\omega) \mid \mathsf{v}_i\rangle + \varepsilon_{i,n}(\omega)\big(\langle \mathsf{Q}_i\boldsymbol{x}_n(\omega) \mid \mathsf{v}_i\rangle + \langle b_{i,n}(\omega) \mid \mathsf{v}_i\rangle - \langle z_{i,n}(\omega) \mid \mathsf{v}_i\rangle\big). \tag{5.13}$$

However, according to (iii), at the expense of possibly taking ω in a smaller almost sure event, $\varepsilon_{i,n}(\omega) = 1$ infinitely often. Hence, there exists a monotone sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_n \to +\infty$ and, for $n \in \mathbb{N}$ sufficiently large,

$$\langle z_{i,n+1}(\omega) \mid \mathsf{v}_i\rangle = \langle \mathsf{Q}_i\boldsymbol{x}_{k_n}(\omega) \mid \mathsf{v}_i\rangle + \langle b_{i,k_n}(\omega) \mid \mathsf{v}_i\rangle. \tag{5.14}$$

Thus, since $\langle \mathsf{Q}_i\boldsymbol{x}_{k_n}(\omega) \mid \mathsf{v}_i\rangle \to \langle z_i(\omega) \mid \mathsf{v}_i\rangle$ and $\langle b_{i,k_n}(\omega) \mid \mathsf{v}_i\rangle \to 0$, $\langle z_{i,n+1}(\omega) - z_i(\omega) \mid \mathsf{v}_i\rangle \to 0$. Hence, $\langle \boldsymbol{z}_{n+1}(\omega) - \boldsymbol{z}(\omega) \mid \mathbf{v}\rangle = \sum_{i=1}^m \langle z_{i,n+1}(\omega) - z_i(\omega) \mid \mathsf{v}_i\rangle \to 0$. This shows that $\boldsymbol{z}_n \rightharpoonup \boldsymbol{z}$ P-a.s., which allows us to conclude that $\gamma^{-1}(\boldsymbol{x}_n - \boldsymbol{z}_n) \rightharpoonup \boldsymbol{u}$ P-a.s. □

Remark 5.2 Let us make some connections between Proposition 5.1 and existing results.

(i) In the standard case of a single block ($m = 1$) and when all the variables are deterministic, the above primal convergence result goes back to [29] and to [39] in the unrelaxed case.

(ii) In minimization problems, the alternating direction method of multipliers (ADMM) is strongly related to an application of the Douglas-Rachford algorithm to the dual problem [35]. This connection can be used to construct a random block-coordinate ADMM algorithm. Let us note that such an algorithm was recently proposed in [37] in a finite-dimensional setting, where single-block, unrelaxed, and error-free iterations were used.

Next, we apply Proposition 5.1 to devise a primal-dual block-coordinate algorithm for solving a class of structured inclusion problems investigated in [22].

Corollary 5.3 *Set* $\mathsf{D} = \{0,1\}^{m+p} \smallsetminus \{\mathbf{0}\}$, *let* $(\mathsf{G}_k)_{1\leqslant k\leqslant p}$ *be separable real Hilbert spaces, and set* $\mathbf{G} = \mathsf{G}_1 \oplus \cdots \oplus \mathsf{G}_p$. *For every* $i \in \{1, \ldots, m\}$, *let* $\mathsf{A}_i \colon \mathsf{H}_i \to 2^{\mathsf{H}_i}$ *be maximally monotone and, for every* $k \in \{1, \ldots, p\}$, *let* $\mathsf{B}_k \colon \mathsf{G}_k \to 2^{\mathsf{G}_k}$ *be maximally monotone, and let* $\mathsf{L}_{ki} \colon \mathsf{H}_i \to \mathsf{G}_k$ *be linear and bounded. It is assumed that the set* $\mathbf{F}$ *of solutions to the problem*

$$\textit{find}\ \ \mathsf{x}_1 \in \mathsf{H}_1, \ldots, \mathsf{x}_m \in \mathsf{H}_m\ \ \textit{such that}\ \ (\forall i \in \{1, \ldots, m\})\ 0 \in \mathsf{A}_i\mathsf{x}_i + \sum_{k=1}^p \mathsf{L}_{ki}^*\mathsf{B}_k\Bigg(\sum_{j=1}^m \mathsf{L}_{kj}\mathsf{x}_j\Bigg) \tag{5.15}$$

is nonempty. We also consider the set $\mathbf{F}^*$ *of solutions to the dual problem*

$$\textit{find}\ \ \mathsf{v}_1 \in \mathsf{G}_1, \ldots, \mathsf{v}_p \in \mathsf{G}_p\ \ \textit{such that}\ \ (\forall k \in \{1, \ldots, p\})\ 0 \in -\sum_{i=1}^m \mathsf{L}_{ki}\mathsf{A}_i^{-1}\Bigg(-\sum_{l=1}^p \mathsf{L}_{li}^*\mathsf{v}_l\Bigg) + \mathsf{B}_k^{-1}\mathsf{v}_k. \tag{5.16}$$

Let $\gamma \in \left]0, +\infty\right[$, *let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence in* $\left]0, 2\right[$ *such that* $\inf_{n\in\mathbb{N}} \mu_n > 0$ *and* $\sup_{n\in\mathbb{N}} \mu_n < 2$, *let* $\boldsymbol{x}_0$, $\boldsymbol{z}_0$, $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, *and* $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ *be* $\mathbf{H}$*-valued random variables, let* $\boldsymbol{y}_0$, $\boldsymbol{w}_0$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, *and* $(\boldsymbol{d}_n)_{n\in\mathbb{N}}$ *be* $\mathbf{G}$*-valued random variables, and let* $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ *be identically distributed* D*-valued random variables. Set*

$$\mathbf{V} = \left\{ (\mathsf{x}_1, \ldots, \mathsf{x}_m, \mathsf{y}_1, \ldots, \mathsf{y}_p) \in \mathbf{H} \oplus \mathbf{G} \;\middle|\; (\forall k \in \{1, \ldots, p\})\ \mathsf{y}_k = \sum_{i=1}^m \mathsf{L}_{ki}\mathsf{x}_i \right\} \tag{5.17}$$

and $\mathbf{P_V}\colon \mathbf{x} \mapsto (\mathsf{Q}_j\mathbf{x})_{1\leqslant j\leqslant m+p}$ *where* $(\forall i \in \{1,\ldots,m\})$ $\mathsf{Q}_i\colon \mathbf{H}\oplus\mathbf{G} \to \mathsf{H}_i$ *and* $(\forall k \in \{1,\ldots,p\})$ $\mathsf{Q}_{m+k}\colon \mathbf{H}\oplus\mathbf{G} \to \mathsf{G}_k$*, iterate*

$$
\begin{array}{l}
\textit{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\textit{for } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
z_{i,n+1} = z_{i,n} + \varepsilon_{i,n}\big(\mathsf{Q}_i(x_{1,n},\ldots,x_{m,n},y_{1,n},\ldots,y_{p,n}) + c_{i,n} - z_{i,n}\big)\\
x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\mu_n\big(\mathsf{J}_{\gamma\mathsf{A}_i}(2z_{i,n+1} - x_{i,n}) + a_{i,n} - z_{i,n+1}\big)
\end{array}
\right.\\
\textit{for } k = 1,\ldots,p\\
\left\lfloor
\begin{array}{l}
w_{k,n+1} = w_{k,n} + \varepsilon_{m+k,n}\big(\mathsf{Q}_{m+k}(x_{1,n},\ldots,x_{m,n},y_{1,n},\ldots,y_{p,n}) + d_{k,n} - w_{k,n}\big)\\
y_{k,n+1} = y_{k,n} + \varepsilon_{m+k,n}\mu_n\big(\mathsf{J}_{\gamma\mathsf{B}_k}(2w_{k,n+1} - y_{k,n}) + b_{k,n} - w_{k,n+1}\big),
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{5.18}
$$

and set $(\forall n\in\mathbb{N})$ $\mathcal{Y}_n = \sigma(\boldsymbol{x}_j,\boldsymbol{y}_j)_{0\leqslant j\leqslant n}$ *and* $\mathcal{E}_n = \sigma(\boldsymbol{\varepsilon}_n)$*. In addition, assume that the following hold:*

(i) $\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2\,|\,\mathcal{Y}_n)} < +\infty$, $\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2\,|\,\mathcal{Y}_n)} < +\infty$, $\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{c}_n\|^2\,|\,\mathcal{Y}_n)} < +\infty$, $\sum_{n\in\mathbb{N}}\sqrt{\mathsf{E}(\|\boldsymbol{d}_n\|^2\,|\,\mathcal{Y}_n)} < +\infty$, $\boldsymbol{b}_n \rightharpoonup \mathbf{0}$ P-a.s.*, and* $\boldsymbol{d}_n \rightharpoonup \mathbf{0}$ P-a.s.

(ii) *For every* $n\in\mathbb{N}$*,* $\mathcal{E}_n$ *and* $\mathcal{Y}_n$ *are independent.*

(iii) $(\forall j\in\{1,\ldots,m+p\})$ $\displaystyle\sum_{\epsilon\in\mathsf{D},\epsilon_j=1}\mathsf{P}[\boldsymbol{\varepsilon}_0 = \epsilon] > 0.$

Then $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* $\mathbf{F}$*-valued random variable, and* $(\gamma^{-1}(\boldsymbol{w}_n - \boldsymbol{y}_n))_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* $\mathbf{F}^*$*-valued random variable.*

Proof. Set $\mathbf{A}\colon \mathbf{H} \to 2^{\mathbf{H}}\colon \mathbf{x} \mapsto \times_{i=1}^m \mathsf{A}_i\mathsf{x}_i$, $\mathbf{B}\colon \mathbf{G} \to 2^{\mathbf{G}}\colon \mathbf{y} \mapsto \times_{k=1}^p \mathsf{B}_k\mathsf{y}_k$, and $\mathbf{L}\colon \mathbf{H} \to \mathbf{G}\colon \mathbf{x} \mapsto \big(\sum_{i=1}^m \mathsf{L}_{ki}\mathsf{x}_i\big)_{1\leqslant k\leqslant p}$. Furthermore, let us introduce

$$
\mathbf{K} = \mathbf{H}\oplus\mathbf{G},\quad \mathbf{C}\colon \mathbf{K}\to 2^{\mathbf{K}}\colon (\mathbf{x},\mathbf{y}) \mapsto \mathbf{A}\mathbf{x}\times\mathbf{B}\mathbf{y},\quad\text{and}\quad \mathbf{V} = \big\{(\mathbf{x},\mathbf{y})\in\mathbf{K} \;\big|\; \mathbf{L}\mathbf{x} = \mathbf{y}\big\}. \tag{5.19}
$$

Then the primal-dual problem (5.15)–(5.16) can be rewritten as

$$
\text{find } (\mathbf{x},\mathbf{v})\in\mathbf{K} \text{ such that } \begin{cases}\mathbf{0}\in\mathbf{A}\mathbf{x} + \mathbf{L}^*\mathbf{B}\mathbf{L}\mathbf{x}\\ \mathbf{0}\in -\mathbf{L}\mathbf{A}^{-1}(-\mathbf{L}^*\mathbf{v}) + \mathbf{B}^{-1}\mathbf{v}.\end{cases} \tag{5.20}
$$

The normal cone operator to $\mathbf{V}$ is [7, Example 6.42]

$$
\mathbf{N_V}\colon \mathbf{K}\to 2^{\mathbf{K}}\colon (\mathbf{x},\mathbf{y}) \mapsto \begin{cases}\mathbf{V}^\perp, & \text{if } \mathbf{L}\mathbf{x} = \mathbf{y};\\ \varnothing, & \text{if } \mathbf{L}\mathbf{x} \neq \mathbf{y},\end{cases} \quad\text{where}\quad \mathbf{V}^\perp = \big\{(\mathbf{u},\mathbf{v})\in\mathbf{K} \;\big|\; \mathbf{u} = -\mathbf{L}^*\mathbf{v}\big\}. \tag{5.21}
$$

Now let $(\mathbf{x},\mathbf{y})\in\mathbf{K}$. Then

$$
\begin{aligned}
(\mathbf{0},\mathbf{0})\in\mathbf{C}(\mathbf{x},\mathbf{y}) + \mathbf{N_V}(\mathbf{x},\mathbf{y}) \quad&\Leftrightarrow\quad \begin{cases}(\mathbf{x},\mathbf{y})\in\mathbf{V}\\ (\mathbf{0},\mathbf{0})\in(\mathbf{A}\mathbf{x}\times\mathbf{B}\mathbf{y}) + \mathbf{V}^\perp\end{cases}\\
&\Leftrightarrow\quad \begin{cases}\mathbf{L}\mathbf{x} = \mathbf{y}\\ (\exists\,\mathbf{u}\in\mathbf{A}\mathbf{x})(\exists\,\mathbf{v}\in\mathbf{B}\mathbf{y})\ \mathbf{u} = -\mathbf{L}^*\mathbf{v}\end{cases}\\
&\Rightarrow\quad (\exists\,\mathbf{v}\in\mathbf{B}(\mathbf{L}\mathbf{x}))\ -\mathbf{L}^*\mathbf{v}\in\mathbf{A}\mathbf{x}\\
&\Rightarrow\quad (\exists\,\mathbf{v}\in\mathbf{G})\ \mathbf{L}^*\mathbf{v}\in\mathbf{L}^*\mathbf{B}\mathbf{L}\mathbf{x} \text{ and } -\mathbf{L}^*\mathbf{v}\in\mathbf{A}\mathbf{x}\\
&\Leftrightarrow\quad \mathbf{x} \text{ solves (5.15).}
\end{aligned} \tag{5.22}
$$

Since $\mathbf{C}$ and $\mathbf{N_V}$ are maximally monotone, it follows from [7, Proposition 23.16] that the iteration process (5.18) is an instance of (5.5) for finding a zero of $\mathbf{C}+\mathbf{N_V}$ in $\mathbf{K}$. The associated dual problem consists of finding a zero of $-\mathbf{C}^{-1}(-\cdot)+\mathbf{N_V}^{-1}$. Let $(\mathbf{u},\mathbf{v})\in\mathbf{K}$. Then (5.21) yields

$$\begin{aligned}(\mathbf{0},\mathbf{0})\in -\mathbf{C}^{-1}(-\mathbf{u},-\mathbf{v})+\mathbf{N_V}^{-1}(\mathbf{u},\mathbf{v}) \quad&\Leftrightarrow\quad (\mathbf{0},\mathbf{0})\in -\mathbf{C}^{-1}(-\mathbf{u},-\mathbf{v})+\mathbf{N}_{\mathbf{V}^\perp}(\mathbf{u},\mathbf{v})\\ &\Leftrightarrow\quad \begin{cases}(\mathbf{u},\mathbf{v})\in\ \mathbf{V}^\perp\\ (\mathbf{0},\mathbf{0})\in\big(-\mathbf{A}^{-1}(-\mathbf{u})\times-\mathbf{B}^{-1}(-\mathbf{v})\big)+\mathbf{V}\end{cases}\\ &\Leftrightarrow\quad \begin{cases}\mathbf{u}=-\mathbf{L}^*\mathbf{v}\\ (\exists\,\mathbf{x}\in-\mathbf{A}^{-1}(-\mathbf{u}))(\exists\,\mathbf{y}\in-\mathbf{B}^{-1}(-\mathbf{v}))\ \mathbf{L}\mathbf{x}=\mathbf{y}\end{cases}\\ &\Rightarrow\quad (\exists\,\mathbf{x}\in-\mathbf{A}^{-1}(\mathbf{L}^*\mathbf{v}))\ \mathbf{L}\mathbf{x}\in-\mathbf{B}^{-1}(-\mathbf{v})\\ &\Rightarrow\quad (\exists\,\mathbf{x}\in\mathbf{H})\ \ \mathbf{L}\mathbf{x}\in-\mathbf{L}\mathbf{A}^{-1}(\mathbf{L}^*\mathbf{v}))\text{ and }-\mathbf{L}\mathbf{x}\in\mathbf{B}^{-1}(-\mathbf{v})\\ &\Leftrightarrow\quad -\mathbf{v}\text{ solves (5.16).}\end{aligned}\tag{5.23}$$

The convergence result therefore follows from Proposition 5.1 using (5.22), (5.23), and the weak continuity of $\mathbf{P_V}=\mathbf{J}_{\gamma\mathbf{N_V}}$ [7, Proposition 28.11(i)]. □

Remark 5.4 The parametrization (5.19) made it possible to reduce the structured primal-dual problem (5.15)–(5.16) to a basic two-operator inclusion, to which the block-coordinate Douglas-Rachford algorithm (5.5) could be applied. A similar parametrization was used in [1] in a different context. We also note that, at each iteration of Algorithm (5.18), components of the projector $\mathbf{P_V}$ need to be activated. This operator is expressed as

$$\big(\forall(\mathbf{x},\mathbf{y})\in\mathbf{H}\oplus\mathbf{G}\big)\quad \mathbf{P_V}\colon(\mathbf{x},\mathbf{y})\mapsto(\mathbf{t},\mathbf{L}\mathbf{t})=(\mathbf{x}-\mathbf{L}^*\mathbf{s},\mathbf{y}+\mathbf{s})\tag{5.24}$$

where $\mathbf{t}=(\mathbf{Id}+\mathbf{L}^*\mathbf{L})^{-1}(\mathbf{x}+\mathbf{L}^*\mathbf{y})$ and $\mathbf{s}=(\mathbf{Id}+\mathbf{L}\mathbf{L}^*)^{-1}(\mathbf{L}\mathbf{x}-\mathbf{y})$ [1, Lemma 3.1]. This formula allows us to compute the components of $\mathbf{P_V}$, which is especially simple when $\mathbf{Id}+\mathbf{L}^*\mathbf{L}$ or $\mathbf{Id}+\mathbf{L}\mathbf{L}^*$ is easily inverted.

The previous result leads to a random block-coordinate primal-dual proximal algorithm for solving a wide range of structured convex optimization problems.

Corollary 5.5 *Set* $\mathsf{D}=\{0,1\}^{m+p}\smallsetminus\{\mathbf{0}\}$, *let* $(\mathsf{G}_k)_{1\leqslant k\leqslant p}$ *be separable real Hilbert spaces, and set* $\mathbf{G}=\mathsf{G}_1\oplus\cdots\oplus\mathsf{G}_p$. *For every* $i\in\{1,\dots,m\}$, *let* $\mathsf{f}_i\in\Gamma_0(\mathsf{H}_i)$ *and, for every* $k\in\{1,\dots,p\}$, *let* $\mathsf{g}_k\in\Gamma_0(\mathsf{G}_k)$, *and let* $\mathsf{L}_{ki}\colon\mathsf{H}_i\to\mathsf{G}_k$ *be linear and bounded. It is assumed that there exists* $(\mathsf{x}_1,\dots,\mathsf{x}_m)\in\mathbf{H}$ *such that*

$$(\forall i\in\{1,\dots,m\})\quad 0\in\partial\mathsf{f}_i(\mathsf{x}_i)+\sum_{k=1}^{p}\mathsf{L}_{ki}^*\partial\mathsf{g}_k\Bigg(\sum_{j=1}^{m}\mathsf{L}_{kj}\mathsf{x}_j\Bigg).\tag{5.25}$$

Let $\mathbf{F}$ *be the set of solutions to the problem*

$$\underset{\mathsf{x}_1\in\mathsf{H}_1,\dots,\mathsf{x}_m\in\mathsf{H}_m}{\text{minimize}}\quad \sum_{i=1}^{m}\mathsf{f}_i(\mathsf{x}_i)+\sum_{k=1}^{p}\mathsf{g}_k\Bigg(\sum_{i=1}^{m}\mathsf{L}_{ki}\mathsf{x}_i\Bigg)\tag{5.26}$$

and let $\mathbf{F}^*$ *be the set of solutions to the dual problem*

$$\underset{\mathsf{v}_1\in\mathsf{G}_1,\dots,\mathsf{v}_p\in\mathsf{G}_p}{\text{minimize}}\quad \sum_{i=1}^{m}\mathsf{f}_i^*\Bigg(-\sum_{k=1}^{p}\mathsf{L}_{ki}^*\mathsf{v}_k\Bigg)+\sum_{k=1}^{p}\mathsf{g}_k^*(\mathsf{v}_k).\tag{5.27}$$

Let $\gamma \in \left]0,+\infty\right[$, let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\left]0,2\right[$ such that $\inf_{n\in\mathbb{N}}\mu_n > 0$ and $\sup_{n\in\mathbb{N}}\mu_n < 2$, let $\boldsymbol{x}_0$, $\boldsymbol{z}_0$, $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ be H-valued random variables, let $\boldsymbol{y}_0$, $\boldsymbol{w}_0$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{d}_n)_{n\in\mathbb{N}}$ be G-valued random variables, and let $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ be identically distributed D-valued random variables. Define V as in (5.17) and set $\mathsf{P_V}\colon \mathsf{x} \mapsto (\mathsf{Q}_j\mathsf{x})_{1\leqslant j\leqslant m+p}$ where $(\forall i \in \{1,\ldots,m\})$ $\mathsf{Q}_i\colon \mathsf{H}\oplus\mathsf{G} \to \mathsf{H}_i$ and $(\forall k \in \{1,\ldots,p\})$ $\mathsf{Q}_{m+k}\colon \mathsf{H}\oplus\mathsf{G}\to\mathsf{G}_k$, and iterate

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
z_{i,n+1} = z_{i,n} + \varepsilon_{i,n}\big(\mathsf{Q}_i(x_{1,n},\ldots,x_{m,n},y_{1,n},\ldots,y_{p,n}) + c_{i,n} - z_{i,n}\big)\\
x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\mu_n\big(\operatorname{prox}_{\gamma \mathsf{f}_i}(2z_{i,n+1} - x_{i,n}) + a_{i,n} - z_{i,n+1}\big)
\end{array}
\right.\\
\text{for } k = 1,\ldots,p\\
\left\lfloor
\begin{array}{l}
w_{k,n+1} = w_{k,n} + \varepsilon_{m+k,n}\big(\mathsf{Q}_{m+k}(x_{1,n},\ldots,x_{m,n},y_{1,n},\ldots,y_{p,n}) + d_{k,n} - w_{k,n}\big)\\
y_{k,n+1} = y_{k,n} + \varepsilon_{m+k,n}\mu_n\big(\operatorname{prox}_{\gamma \mathsf{g}_k}(2w_{k,n+1} - y_{k,n}) + b_{k,n} - w_{k,n+1}\big).
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{5.28}
$$

In addition, assume that conditions (i)–(iii) of Corollary 5.3 are satisfied. Then $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ converges weakly P-a.s. *to an* F*-valued random variable, and $(\gamma^{-1}(\boldsymbol{w}_n - \boldsymbol{y}_n))_{n\in\mathbb{N}}$ converges weakly* P-a.s. *to an* F^**-valued random variable.*

Proof. Using the same arguments as in [22, Proposition 5.4] one sees that this is an application of Corollary 5.3 with, for every $i \in \{1,\ldots,m\}$, $\mathsf{A}_i = \partial \mathsf{f}_i$ and, for every $k \in \{1,\ldots,p\}$, $\mathsf{B}_k = \partial \mathsf{g}_k$. ☐

Remark 5.6 Sufficient conditions for (5.25) to hold are provided in [22, Proposition 5.3].

5.2 Random block-coordinate forward-backward splitting

The forward-backward algorithm addresses the problem of finding a zero of the sum of two maximally monotone operators, one of which has a strongly monotone inverse (see [2, 19] for historical background). It has been applied to a wide variety of problems among which mechanics, partial differential equations, best approximation, evolution inclusions, signal and image processing, convex optimization, learning theory, inverse problems, statistics, and game theory [2, 7, 13, 14, 19, 24, 26, 28, 36, 41, 55, 56, 57]. In this section we design a block-coordinate version of this algorithm with random sweeping and stochastic errors.

Proposition 5.7 *Set $\mathsf{D} = \{0,1\}^m \smallsetminus \{\mathbf{0}\}$ and, for every $i \in \{1,\ldots,m\}$, let $\mathsf{A}_i\colon \mathsf{H}_i \to 2^{\mathsf{H}_i}$ be maximally monotone and let $\mathsf{B}_i\colon \mathsf{H} \to \mathsf{H}_i$. Suppose that*

$$(\exists\,\vartheta \in \left]0,+\infty\right[)(\forall \mathsf{x} \in \mathsf{H})(\forall \mathsf{y} \in \mathsf{H}) \quad \sum_{i=1}^{m} \langle \mathsf{x}_i - \mathsf{y}_i \mid \mathsf{B}_i\mathsf{x} - \mathsf{B}_i\mathsf{y}\rangle \geqslant \vartheta \sum_{i=1}^{m} \big\|\mathsf{B}_i\mathsf{x} - \mathsf{B}_i\mathsf{y}\big\|^2, \tag{5.29}$$

and that the set F of solutions to the problem

$$\text{find } \mathsf{x}_1 \in \mathsf{H}_1,\ldots,\mathsf{x}_m \in \mathsf{H}_m \text{ such that } (\forall i \in \{1,\ldots,m\}) \quad 0 \in \mathsf{A}_i\mathsf{x}_i + \mathsf{B}_i(\mathsf{x}_1,\ldots,\mathsf{x}_m) \tag{5.30}$$

is nonempty. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $\left]0,2\vartheta\right[$ such that $\inf_{n\in\mathbb{N}}\gamma_n > 0$ and $\sup_{n\in\mathbb{N}}\gamma_n < 2\vartheta$, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\left]0,1\right]$ such that $\inf_{n\in\mathbb{N}}\lambda_n > 0$. Let $\boldsymbol{x}_0$, $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ be H-valued random variables, and let $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ be identically distributed D-valued random variables. Iterate

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for } i = 1,\ldots,m\\
\left\lfloor\; x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\lambda_n\big(\mathsf{J}_{\gamma_n\mathsf{A}_i}\big(x_{i,n} - \gamma_n(\mathsf{B}_i(x_{1,n},\ldots,x_{m,n}) + c_{i,n})\big) + a_{i,n} - x_{i,n}\big),\right.
\end{array}
\right.
\end{array}
\tag{5.31}
$$

and set $(\forall n \in \mathbb{N})$ $\mathcal{E}_n = \sigma(\varepsilon_n)$. *In addition, assume that the following hold:*

(i) $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{a}_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$ *and* $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{c}_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$.

(ii) *For every* $n \in \mathbb{N}$, $\mathcal{E}_n$ *and* $\mathcal{X}_n$ *are independent.*

(iii) $(\forall i \in \{1, \ldots, m\}) \displaystyle\sum_{\boldsymbol{\epsilon}\in\mathsf{D},\epsilon_i=1} \mathsf{P}[\boldsymbol{\varepsilon}_0 = \boldsymbol{\epsilon}] > 0.$

Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* $\mathbf{F}$*-valued random variable.*

Proof. Set $\mathbf{A}\colon \mathbf{H} \to 2^{\mathbf{H}}\colon \mathbf{x} \mapsto \times_{i=1}^{m} \mathsf{A}_i \mathsf{x}_i$, $\mathbf{B}\colon \mathbf{H} \to \mathbf{H}\colon \mathbf{x} \mapsto (\mathsf{B}_i\mathbf{x})_{1\leqslant i\leqslant m}$, and, for every $n \in \mathbb{N}$, $\alpha_n = 1/2$, $\beta_n = \gamma_n/(2\vartheta)$, $\mathbf{T}_n = \mathbf{J}_{\gamma_n\mathbf{A}}$, $\mathbf{R}_n = \mathbf{Id} - \gamma_n\mathbf{B}$, and $\boldsymbol{b}_n = -\gamma_n\boldsymbol{c}_n$. Then, $\mathbf{F} = \operatorname{zer}(\mathbf{A} + \mathbf{B})$ and, for every $n \in \mathbb{N}$, $\mathbf{T}_n$ is α_n-averaged [7, Corollary 23.8], $\mathbf{T}_n\colon \mathbf{x} \mapsto (\mathsf{J}_{\gamma_n A_i}\mathsf{x}_i)_{1\leqslant i\leqslant m}$ [7, Proposition 23.16], $\mathbf{R}_n$ is β_n-averaged [7, Proposition 4.33], and $\operatorname{Fix}(\mathbf{T}_n \circ \mathbf{R}_n) = \mathbf{F}$ [7, Proposition 25.1(iv)]. Moreover, $\sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{b}_n\|^2 \,|\, \mathcal{X}_n)} \leqslant 2\vartheta \sum_{n\in\mathbb{N}} \sqrt{\mathsf{E}(\|\boldsymbol{c}_n\|^2 \,|\, \mathcal{X}_n)} < +\infty$ and (5.31) is a special case of (4.1). To show that the result is an application of Theorem 4.1, it remains to establish condition (v) of that theorem. To this end, in view of (4.2) and (4.3), it is enough to fix $\mathbf{x} \in \mathbf{H}$, $\mathbf{z} \in \mathbf{F}$, a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, and $\omega \in \Omega$ such that

$$\begin{cases} \boldsymbol{x}_{k_n}(\omega) \rightharpoonup \mathbf{x} \\ \mathbf{T}_n(\mathbf{R}_n\boldsymbol{x}_n(\omega)) - \mathbf{R}_n\boldsymbol{x}_n(\omega) + \mathbf{R}_n\mathbf{z} \to \mathbf{z} \\ \mathbf{R}_n\boldsymbol{x}_n(\omega) - \boldsymbol{x}_n(\omega) - \mathbf{R}_n\mathbf{z} \to -\mathbf{z}, \end{cases} \tag{5.32}$$

and to show that $\mathbf{x} \in \mathbf{F}$. Note that, since $\inf_{n\in\mathbb{N}} \gamma_n > 0$, (5.32) implies that

$$\begin{cases} \boldsymbol{x}_{k_n}(\omega) \rightharpoonup \mathbf{x} \\ \mathbf{J}_{\gamma_n\mathbf{A}}\big(\boldsymbol{x}_n(\omega) - \gamma_n\mathbf{B}\boldsymbol{x}_n(\omega)\big) - \boldsymbol{x}_n(\omega) = \mathbf{T}_n\big(\mathbf{R}_n\boldsymbol{x}_n(\omega)\big) - \boldsymbol{x}_n(\omega) \to \mathbf{0} \\ \mathbf{B}\boldsymbol{x}_n(\omega) \to \mathbf{B}\mathbf{z}. \end{cases} \tag{5.33}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{y}_n = \mathbf{J}_{\gamma_n\mathbf{A}}(\boldsymbol{x}_n - \gamma_n\mathbf{B}\mathbf{x}_n) \quad \text{and} \quad \boldsymbol{u}_n = \gamma_n^{-1}(\boldsymbol{x}_n - \boldsymbol{y}_n) - \mathbf{B}\boldsymbol{x}_n. \tag{5.34}$$

Then, it follows from (5.33) that $\boldsymbol{x}_{k_n}(\omega) \rightharpoonup \mathbf{x}$, $\boldsymbol{y}_n(\omega) - \boldsymbol{x}_n(\omega) \to \mathbf{0}$, and $\mathbf{B}\boldsymbol{x}_n(\omega) \to \mathbf{B}\mathbf{z}$. Thus, $\boldsymbol{x}_{k_n}(\omega) \rightharpoonup \mathbf{x}$, $\mathbf{B}\boldsymbol{x}_{k_n}(\omega) \to \mathbf{B}\mathbf{z}$ and, since [7, Example 20.28] asserts that $\mathbf{B}$ is maximally monotone, we deduce from [7, Proposition 20.33(ii)] that $\mathbf{B}\mathbf{x} = \mathbf{B}\mathbf{z}$. We also derive from (5.33) that $\boldsymbol{y}_{k_n}(\omega) \rightharpoonup \mathbf{x}$ and $\boldsymbol{u}_{k_n}(\omega) \to -\mathbf{B}\mathbf{z} = -\mathbf{B}\mathbf{x}$. Since (5.34) implies that $(\boldsymbol{y}_{k_n}(\omega), \boldsymbol{u}_{k_n}(\omega))_{n\in\mathbb{N}}$ lies in the graph of $\mathbf{A}$, it follows from [7, Proposition 20.33(ii)] that $-\mathbf{B}\mathbf{x} \in \mathbf{A}\mathbf{x}$, i.e., $\mathbf{x} \in \mathbf{F}$. ☐

Remark 5.8 Here are a few remarks regarding Proposition 5.7.

(i) Proposition 5.7 generalizes [19, Corollary 6.5], which does not allow for block-processing and uses deterministic variables.

(ii) Problem (5.30) was considered in [2], where it was shown to capture formulations encountered in areas such as evolution equations, game theory, optimization, best approximation, and network flows. It also models domain decomposition problems in partial differential equations [10].

(iii) Proposition 5.7 generalizes [2, Theorem 2.9], which uses a fully parallel deterministic algorithm in which all the blocks are used at each iteration, i.e., $(\forall n \in \mathbb{N})(\forall i \in \{1,\ldots,m\})$ $\varepsilon_{i,n} = 1$.

(iv) As shown in [23, 25], strongly monotone composite inclusion problems can be solved by applying the forward-backward algorithm to the dual problem. Using Proposition 5.7 we can obtain a block-coordinate version of this primal-dual framework. Likewise, it was shown in [25, 27, 58] that suitably renormed versions of the forward-backward algorithm applied in the primal-dual space yielded a variety of methods for solving composite inclusions in duality. Block-coordinate versions of these methods can be devised via Proposition 5.7.

Next, we present an application of Proposition 5.7 to block-coordinate convex minimization.

Corollary 5.9 *Set* $\mathsf{D} = \{0,1\}^m \smallsetminus \{\mathbf{0}\}$ *and let* $(\mathsf{G}_k)_{1\leqslant k\leqslant p}$ *be separable real Hilbert spaces. For every* $i \in \{1,\ldots,m\}$*, let* $\mathsf{f}_i \in \Gamma_0(\mathsf{H}_i)$ *and, for every* $k \in \{1,\ldots,p\}$*, let* $\tau_k \in \left]0,+\infty\right[$*, let* $\mathsf{g}_k\colon \mathsf{G}_k \to \mathbb{R}$ *be a differentiable convex function with a* τ_k*-Lipschitz-continuous gradient, and let* $\mathsf{L}_{ki}\colon \mathsf{H}_i \to \mathsf{G}_k$ *be linear and bounded. It is assumed that* $\min_{1\leqslant k\leqslant p}\sum_{i=1}^m \|\mathsf{L}_{ki}\|^2 > 0$ *and that the set* $\mathbf{F}$ *of solutions to the problem*

$$\underset{\mathsf{x}_1\in\mathsf{H}_1,\ldots,\mathsf{x}_m\in\mathsf{H}_m}{\text{minimize}} \quad \sum_{i=1}^m \mathsf{f}_i(\mathsf{x}_i) + \sum_{k=1}^p \mathsf{g}_k\Bigg(\sum_{i=1}^m \mathsf{L}_{ki}\mathsf{x}_i\Bigg) \tag{5.35}$$

is nonempty. Set

$$\vartheta = \frac{1}{\displaystyle\sum_{k=1}^p \tau_k \sum_{i=1}^m \|\mathsf{L}_{ki}\|^2}, \tag{5.36}$$

let $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $\left]0,2\vartheta\right[$ *such that* $\inf_{n\in\mathbb{N}}\gamma_n > 0$ *and* $\sup_{n\in\mathbb{N}}\gamma_n < 2\vartheta$*, and let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $\left]0,1\right]$ *such that* $\inf_{n\in\mathbb{N}}\lambda_n > 0$*. Let* $\boldsymbol{x}_0$*,* $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$*, and* $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ *be* $\mathbf{H}$*-valued random variables, and let* $(\boldsymbol{\varepsilon}_n)_{n\in\mathbb{N}}$ *be identically distributed* D*-valued random variables. Iterate*

$$\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for } i = 1,\ldots,m\\
\left\lfloor
\begin{array}{l}
r_{i,n+1} = \varepsilon_{i,n}\big(x_{i,n} - \gamma_n\big(\sum_{k=1}^p \mathsf{L}_{ki}^*\nabla\mathsf{g}_k\big(\sum_{j=1}^m \mathsf{L}_{kj}x_{j,n}\big) + c_{i,n}\big)\big)\\
x_{i,n+1} = x_{i,n} + \varepsilon_{i,n}\lambda_n\big(\operatorname{prox}_{\gamma_n\mathsf{f}_i} r_{i,n} + a_{i,n} - x_{i,n}\big).
\end{array}
\right.
\end{array}
\right.
\end{array} \tag{5.37}$$

In addition, assume that conditions (i)–(iii) in Proposition 5.7 are satisfied. Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ *converges weakly* P-a.s. *to an* $\mathbf{F}$*-valued random variable.*

Proof. As shown in [2, Section 4], (5.35) is a special case of (5.30) with

$$\mathsf{A}_i = \partial\mathsf{f}_i \quad\text{and}\quad \mathsf{B}_i\colon (\mathsf{x}_j)_{1\leqslant j\leqslant m} \mapsto \sum_{k=1}^p \mathsf{L}_{ki}^*\nabla\mathsf{g}_k\Bigg(\sum_{j=1}^m \mathsf{L}_{kj}\mathsf{x}_j\Bigg). \tag{5.38}$$

Now set $\mathbf{h}\colon \mathbf{H} \to \mathbb{R}\colon \mathbf{x} \mapsto \sum_{k=1}^p \mathsf{g}_k\big(\sum_{i=1}^m \mathsf{L}_{ki}\mathsf{x}_i\big)$. Then $\mathbf{h}$ is a Fréchet-differentiable convex function and $\mathbf{B} = \nabla\mathbf{h}$ is Lipschitz-continuous with constant $1/\vartheta$, where ϑ is given in (5.36). It therefore follows from the Baillon-Haddad theorem [7, Theorem 18.15] that (5.29) holds with this constant. Since, in view of (5.2), (5.31) specializes to (5.37), the claim follows from Proposition 5.7. □

Remark 5.10 Let us make a few observations about Corollary 5.9.

(i) If more assumptions are available about the problem, the Lipschitz constant ϑ of (5.36) can be improved. Some examples are given in [12].

(ii) A deterministic block-coordinate forward-backward algorithm was proposed in [8, Section 3.6] in the special case of (5.35) when

$$\mathsf{H} \text{ is a Euclidean space}, \quad p = 1, \quad \text{and} \quad (\forall i \in \{1, \ldots, m\}) \quad \mathsf{L}_{1i} = \mathrm{Id} \tag{5.39}$$

for convex functions satisfying the Kurdyka-Łojasiewicz inequality. In that method, the sweeping proceeds by activating only one block at each iteration according to a periodic schedule. Moreover, errors and relaxations are not allowed. This approach was extended in [17] to an error-tolerant form with a cyclic sweeping rule whereby each block is used at least once within a preset number of consecutive iterations.

(iii) A block-coordinate forward-backward method with random seeping was proposed in [52] in the special case of (5.39). That method uses only one block at each iteration, no relaxation, and no error terms. The asymptotic analysis of [52] provides a lower bound on the probability that $(f + g_1)(x_n)$ be close to $\inf(f + g_1)(\mathsf{H})$, with no result on the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$. Related work is presented in [43], where the proposed algorithm allows for several blocks to be activated simultaneously.

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