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Two-Term Disjunctions on the Second-Order Cone

Fatma Kılınç-Karzan · Sercan Yıldız

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Abstract Balas introduced disjunctive cuts in the 1970s for mixed-integer linear programs. Several recent papers have attempted to extend this work to mixed-integer conic programs. In this paper we study the structure of the convex hull of a two-term disjunction applied to the second-order cone, and develop a methodology to derive closed-form expressions for convex inequalities describing the resulting convex hull. Our approach is based on first characterizing the structure of undominated valid linear inequalities for the disjunction and then using conic duality to derive a family of convex, possibly nonlinear, valid inequalities that correspond to these linear inequalities. We identify and study the cases where these valid inequalities can equivalently be expressed in conic quadratic form and where a single inequality from this family is sufficient to describe the convex hull. In particular, our results on two-term disjunctions on the second-order cone generalize related results on split cuts by Modaresi, Kılınç, and Vielma, and by Andersen and Jensen.

Keywords Mixed-integer conic programming, second-order cone programming, cutting planes, disjunctive cuts

Mathematics Subject Classification (2010) 90C11, 90C26

1 Introduction

1.1 Motivation and Related Work

A mixed-integer conic program is a problem of the form

$$\sup\{d^\top x : \ Ax = b, \ x \in \mathbb{K}, \ x_j \in \mathbb{Z} \ \forall j \in J\}$$

where $\mathbb{K} \subset \mathbb{R}^n$ is a regular (full-dimensional, closed, convex, and pointed) cone, A is an $m \times n$ real matrix, d and b are real vectors of appropriate dimensions, and

Tepper School of Business
Carnegie Mellon University, Pittsburgh, PA
E-mail: {fkilinc,syildiz}@andrew.cmu.edu

$J \subseteq \{1,\ldots,n\}$. Mixed-integer conic programming (MICP) models arise naturally as robust versions of mixed-integer linear programming (MILP) models in finance, management, and engineering [11,15]. MILP is the special case of MICP where $\mathbb{K}$ is the nonnegative orthant, and it has itself numerous applications. A successful approach to solving MILP problems has been to first solve the natural continuous relaxation obtained by dropping the integrality conditions, then add cuts to eliminate regions that do not contain any feasible integer points, and finally perform branch-and-bound using this strengthened formulation. A powerful way of generating such cuts is to impose a valid disjunction on the continuous relaxation, and derive tight convex inequalities for the resulting disjunctive set. Such inequalities are known as *disjunctive cuts*. Specifically, the integrality conditions on the variables x_j, $j \in J$, imply *split disjunctions* of the form $\pi^\top x \le \pi_0 \vee \pi^\top x \ge \pi_0 + 1$ where $\pi_0 \in \mathbb{Z}$, $\pi_j \in \mathbb{Z}$, $j \in J$, and $\pi_j = 0$, $j \notin J$. Following this approach, the feasible region for MICP problems can be relaxed to $\{x \in \mathbb{K} :\ Ax = b, \pi^\top x \le \pi_0 \vee \pi^\top x \ge \pi_0 + 1\}$. More general two-term disjunctions arise in complementarity [24,34] and other non-convex optimization [8,16] problems. Therefore, it is interesting to study relaxations of MICP problems of the form

$$\begin{aligned} &\sup\{d^\top x :\ x \in C_1 \cup C_2\} \quad \text{where} \\ C_i := \{x \in \mathbb{K} :\ &Ax = b,\ c_i^\top x \ge c_{i,0}\} \quad \text{for} \quad i \in \{1,2\}. \end{aligned} \tag{1}$$

Note that the feasible region $C_1 \cup C_2$ of the problem above is still possibly non-convex, but using its particular disjunctive structure, one can build convex relaxations by deriving valid inequalities. The tightest convex relaxations of $C_1 \cup C_2$ that one can obtain through this procedure are its convex hull $\text{conv}(C_1 \cup C_2)$ and closed convex hull $\overline{\text{conv}}(C_1 \cup C_2)$. When $\mathbb{K}$ is the nonnegative orthant, Bonami et al. [14] characterize $\overline{\text{conv}}(C_1 \cup C_2)$ by a finite set of linear inequalities. The purpose of this paper is to study the structure of $\overline{\text{conv}}(C_1 \cup C_2)$ for other cones such as the second-order (Lorentz) cone $\mathbb{L}^n := \{x \in \mathbb{R}^n :\ \|(x_1;\ldots;x_{n-1})\|_2 \le x_n\}$ and provide the explicit description of $\overline{\text{conv}}(C_1 \cup C_2)$ with convex inequalities in the space of the original variables. We first review related results from the literature.

Disjunctive cuts were introduced by Balas [4] for MILP in the early 1970s. Since then, disjunctive cuts have been studied extensively in mixed integer linear and nonlinear optimization [5,32,7,19,31,17,25,16]. *Chvátal-Gomory*, *lift-and-project*, *mixed-integer rounding (MIR)*, and *split cuts* are all special types of disjunctive cuts. Recent efforts on extending the cutting plane theory for MILP to the MICP setting include the work of Çezik and Iyengar [18] for Chvatal-Gomory cuts, Stubbs and Mehrotra [33], Drewes [21], Drewes and Pokutta [22], and Bonami [13] for lift-and-project cuts, and Atamtürk and Narayanan [2,3] for MIR cuts. Kılınç-Karzan [26] analyzed properties of minimal valid linear inequalities for general conic sets with a disjunctive structure and showed that these are sufficient to describe their closed convex hulls. Such general sets from [26] include two-term disjunctions on the cone $\mathbb{K}$ considered in this paper. Bienstock and Michalka [12] studied the characterization and separation of valid linear inequalities that convexify the epigraph of a convex, differentiable function restricted to a non-convex domain. While the second-order cone can be viewed naturally as the epigraph of the function $\|\cdot\|_2$, we note here that this function is not differentiable and a two-term disjunction on the domain of $\|\cdot\|_2$ would require $c_{1,n} = c_{2,n} = 0$ in our setting. In the last few years, there has been growing interest in developing closed-form expressions for convex inequalities that fully describe the convex hull of a disjunctive conic set. Dadush et al. [20] and Andersen and Jensen [1] derived split cuts for ellipsoids and the second-order cone, respectively. Modaresi et al. [28] extended this work on split disjunctions to essentially all cross-sections of the second-order cone, and studied their theoretical and computational relations with extended formulations and conic MIR inequalities in [29]. Belotti et al. [10] studied the families of quadratic surfaces that have fixed intersections with two given hyperplanes and showed that these families can be described by a single parameter. Building on this, in [9] they identified a procedure for constructing two-term disjunctive cuts under the assumptions that $C_1 \cap C_2 = \emptyset$ and the sets $\{x \in \mathbb{K} : Ax = b, c_1^\top x = c_{1,0}\}$ and $\{x \in \mathbb{K} : Ax = b, c_2^\top x = c_{2,0}\}$ are bounded.

In this paper we study general two-term disjunctions $c_1^\top x \geq c_{1,0} \vee c_2^\top x \geq c_{2,0}$ on conic sets and give closed-form expressions for the disjunctive cuts that can be obtained from these disjunctions in a large class of instances. We focus on the case where C_1 and C_2 in (1) above have an empty set of equations $Ax = b$. That is to say, we consider

$$C_1 := \{x \in \mathbb{K} : c_1^\top x \geq c_{1,0}\} \quad \text{and} \quad C_2 := \{x \in \mathbb{K} : c_2^\top x \geq c_{2,0}\}. \tag{2}$$

Our results can, however, easily be extended to two-term disjunctions on sets $\{x \in \mathbb{R}^n : Ax - b \in \mathbb{K}\}$ where A has full row rank through the affine transformation discussed in [1]. Our main contribution in this paper is to give an explicit outer description of $\overline{\text{conv}}(C_1 \cup C_2)$ when $\mathbb{K}$ is the second-order cone. Similar results have previously appeared in [1], [28], and [9]. Nevertheless, our work is set apart from [1] and [28] by the fact that we study two-term disjunctions $c_1^\top x \geq c_{1,0} \vee c_2^\top x \geq c_{2,0}$ on the cone $\mathbb{K}$ in full generality and do not restrict our attention to split disjunctions, which are defined by *parallel* hyperplanes. Furthermore, unlike [9], we do not assume that $C_1 \cap C_2 = \emptyset$ and the sets $\{x \in \mathbb{K} : c_1^\top x = c_{1,0}\}$ and $\{x \in \mathbb{K} : c_2^\top x = c_{2,0}\}$ are bounded. Our analysis shows that the resulting convex hulls can turn out to be significantly more complex in the absence of such assumptions. We also stress that our proof techniques originate from a conic duality perspective and are completely different from what is employed in the aforementioned papers; in particular, we believe that they are intuitive in terms of their derivation and transparent in explaining the structure of the resulting convex hulls. Therefore, we hope that they have the potential to be instrumental in extending several important results in this growing area of research.

A preliminary version of this paper has appeared in [27]. Yıldız and Cornuéjols [35] have recently extended our convex hull characterizations to two-term disjunctions on cross-sections of the second-order cone. They have shown that a single valid inequality of the type derived in this paper can be used to describe $\overline{\text{conv}}(C_1 \cup C_2)$ explicitly even when the definition of C_1 and C_2 in (1) includes a nonempty set of equations $Ax = b$.

1.2 Outline of the Paper

We would like to give a brief outline of our results before we proceed. In Section 2 we introduce the tools that are useful in our analysis. In Section 2.1 we set out our notation and identify the cases that are of main interest to us with Conditions 1 and 2. These conditions are indeed mild requirements and mainly introduced to avoid trivial pathological cases. We discuss the pathologies that arise in the absence of these conditions in Section 2.3. It is a well-known fact from convex analysis that $\overline{\text{conv}}(C_1 \cup C_2)$ can be described by linear inequalities alone. However, the set of linear inequalities that are valid for $\overline{\text{conv}}(C_1 \cup C_2)$ will typically be very large, and only a small subset of these will be needed besides the cone constraint $x \in \mathbb{K}$ in a minimal description of $\overline{\text{conv}}(C_1 \cup C_2)$. In Section 2.2, for a general regular cone $\mathbb{K}$, we identify and characterize the structure of a subset of strong valid linear inequalities which, along with the cone constraint $x \in \mathbb{K}$, are sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$. These inequalities are tight on $\overline{\text{conv}}(C_1 \cup C_2)$ and $\mathbb{K}$-minimal in the sense defined in [26]. We term such linear inequalities "undominated."

In Section 3 we focus on the case where $\mathbb{K}$ is the second-order cone, $\mathbb{L}^n$. In Section 3.1 we prove our main result, Theorem 3, which we state below in a slightly modified form. The proof of Theorem 3 uses conic duality, along with the characterization of undominated valid linear inequalities from Section 2.2, to derive a family of convex, possibly linear or conic, valid inequalities (3) for $\overline{\text{conv}}(C_1 \cup C_2)$.

Theorem 1 *Let C_1 and C_2 be defined as in* (2) *where $\mathbb{K} = \mathbb{L}^n$ and $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$. Suppose Conditions 1 and 2 are satisfied. For any $\beta_1, \beta_2 > 0$ such that $\beta_1 c_1 - \beta_2 c_2 \notin \pm \operatorname{int} \mathbb{L}^n$, let $\mathcal{N}(\beta_1, \beta_2) := \|\beta_1 \tilde{c_1} - \beta_2 \tilde{c_2}\|_2^2 - (\beta_1 c_{1,n} - \beta_2 c_{2,n})^2$. Then the inequality*

$$2 \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\} - (\beta_1 c_1 + \beta_2 c_2)^\top x \leq \sqrt{\left((\beta_1 c_1 - \beta_2 c_2)^\top x\right)^2 + \mathcal{N}(\beta_1, \beta_2)\left(x_n^2 - \|\tilde{x}\|_2^2\right)} \quad (3)$$

is valid for $\overline{\text{conv}}(C_1 \cup C_2)$. Furthermore, inequalities (3) *are sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$ along with the cone constraint $x \in \mathbb{L}^n$.*

In Section 3.2 we explore when (3) can equivalently be expressed in conic quadratic form (Propositions 3 and 4).

In Section 4 we identify and study the cases where only one inequality of the form (3) is sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$ (Theorem 4). In the special case where $C_1 \cup C_2$ is defined by a split disjunction on the second-order cone, the convex hull is closed and our results show that a single additional valid inequality in conic quadratic form is always sufficient to describe $\text{conv}(C_1 \cup C_2)$. We formulate this conclusion into Theorem 2, which we state below. This recovers the related results of [1] and [28] on split disjunctions on the second-order cone.

Theorem 2 *Let C_1 and C_2 be defined by a split disjunction in* (2) *where $\mathbb{K} = \mathbb{L}^n$ and $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$. Suppose Condition 1 is satisfied. If $c_{1,0} = c_{2,0} = 1$, then*

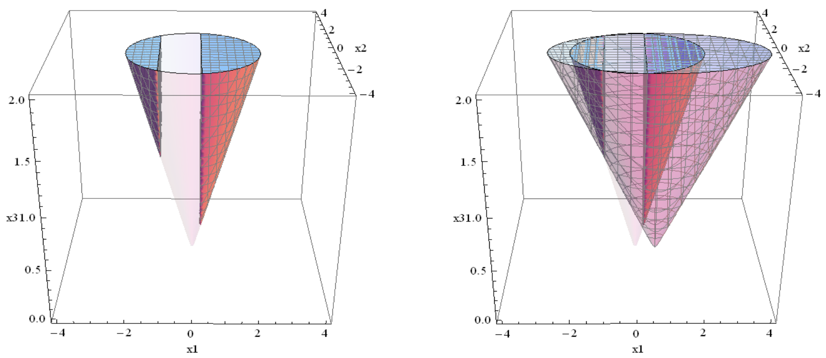
$$\text{conv}(C_1 \cup C_2) = \left\{ x \in \mathbb{L}^n : \mathcal{N} x - 2(c_1^\top x - 1) \begin{pmatrix} \tilde{c}_1 - \tilde{c}_2 \\ -c_{1,n} + c_{2,n} \end{pmatrix} \in \mathbb{L}^n \right\}$$

where $\mathcal{N} := \|\tilde{c_1} - \tilde{c_2}\|_2^2 - (c_{1,n} - c_{2,n})^2$; otherwise, $\text{conv}(C_1 \cup C_2) = \mathbb{L}^n$.

Example 1 As an application of Theorem 2, consider the split disjunction $4x_1 \geq 1 \vee -x_1 \geq 1$ on the second-order cone $\mathbb{L}^3$. Let $e^i \in \mathbb{R}^n$ be the i^{th} standard unit vector, $i \in \{1, \ldots, n\}$. Theorem 2 states that in this case $\text{conv}(C_1 \cup C_2)$ is the set of points $x \in \mathbb{L}^3$ that satisfy the conic quadratic inequality

$$5x - 2(4x_1 - 1)e^1 \in \mathbb{L}^3.$$

Figures 1(a) and (b) show the disjunctive set $C_1 \cup C_2$ and the conic quadratic inequality which is introduced to convexify $C_1 \cup C_2$, respectively.



(a) $C_1 \cup C_2$

(b) Conic quadratic inequality describing $\text{conv}(C_1 \cup C_2)$

Fig. 1: Sets associated with the split disjunction $4x_1 \geq 1 \vee -x_1 \geq 1$ on $\mathbb{L}^3$.

In spite of these good news, there are cases where it is not possible to obtain $\overline{\text{conv}}(C_1 \cup C_2)$ with a single inequality of the form (3). In Section 5 we study these cases and outline a technique to characterize $\overline{\text{conv}}(C_1 \cup C_2)$ with closed-form formulas in general.

2 Preliminaries

The main purpose of this section is to characterize the structure of undominated valid linear inequalities for $\overline{\text{conv}}(C_1 \cup C_2)$ when $\mathbb{K}$ is a regular cone and C_1 and C_2 are defined as in (2). First, we present our notation and assumptions.

2.1 Notation and Assumptions

Given a set $S \subseteq \mathbb{R}^n$, we let $\text{span}\, S$, $\text{int}\, S$, and $\text{bd}\, S$ denote the linear span, interior, and boundary of S, respectively. We use $\text{rec}\, S$ to refer to the recession cone of a convex set S. The *dual cone* of $\mathbb{K} \subseteq \mathbb{R}^n$ is $\mathbb{K}^* := \{y \in \mathbb{R}^n : y^\top x \geq 0 \, \forall x \in \mathbb{K}\}$. Recall that the dual cone $\mathbb{K}^*$ of a regular cone $\mathbb{K}$ is also regular and the dual of $\mathbb{K}^*$ is $\mathbb{K}$ itself.

We can always scale the inequalities $c_1^\top x \geq c_{1,0}$ and $c_2^\top x \geq c_{2,0}$ defining the disjunction so that their right-hand sides are 0 or ± 1. Therefore, from now on we assume that $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ for notational convenience.

When $C_1 \subseteq C_2$, we have $\overline{\text{conv}}(C_1 \cup C_2) = C_2$. Similarly, when $C_1 \supseteq C_2$, we have $\overline{\text{conv}}(C_1 \cup C_2) = C_1$. In the remainder we focus on the case where $C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.

Condition 1 *$C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.*

In particular, Condition 1 implies $C_1, C_2 \neq \emptyset$ and $C_1, C_2 \neq \mathbb{K}$. Hence, $c_i \notin -\mathbb{K}^*$ when $c_{i,0} = +1$ and $c_i \notin \mathbb{K}^*$ when $c_{i,0} = -1$. We also need the following technical condition in our analysis.

Condition 2 *C_1 and C_2 are strictly feasible sets. That is, $C_1 \cap \operatorname{int} \mathbb{K} \neq \emptyset$ and $C_2 \cap \operatorname{int} \mathbb{K} \neq \emptyset$.*

Under Condition 2, the sets C_1 and C_2 always have nonempty interior. Note that the set C_i is always strictly feasible when it is nonempty and $c_{i,0} \in \{\pm 1\}$. Therefore, we need Condition 2 to supplement Condition 1 only when $c_{1,0} = 0$ or $c_{2,0} = 0$. We note that Condition 2 is essentially used in proving the sufficiency results, that is, explicit closed convex hull characterizations, and even in the absence of Condition 2, our techniques yield convex valid inequalities. We evaluate the necessity of Condition 2 for our sufficiency results with an example in Section 2.3.

Throughout the paper, we consider sets C_1 and C_2 which are defined as in (2) with $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and which satisfy Conditions 1 and 2. We say that such sets C_1 and C_2 satisfy the *basic disjunctive setup*.

Conditions 1 and 2 have several simple implications, which we state next. The first lemma extends ideas from Balas [6] to disjunctions on more general convex sets. Its proof is left to the appendix.

Lemma 1 *Let $S \subset \mathbb{R}^n$ be a closed, convex, pointed set, $S_1 := \{x \in S : c_1^\top x \geq c_{1,0}\}$, and $S_2 := \{x \in S : c_2^\top x \geq c_{2,0}\}$ for $c_1, c_2 \in \mathbb{R}^n$ and $c_{1,0}, c_{2,0} \in \mathbb{R}$. Suppose $S_1 \not\subseteq S_2$ and $S_1 \not\supseteq S_2$. Then*

(i) $S_1 \cup S_2$ is not convex unless $S_1 \cup S_2 = S$,
(ii) $\overline{\operatorname{conv}}(S_1 \cup S_2) = \operatorname{conv}(S_1^+ \cup S_2^+)$ where $S_1^+ := S_1 + \operatorname{rec} S_2$ and $S_2^+ := S_2 + \operatorname{rec} S_1$.

Clearly, when $\overline{\operatorname{conv}}(C_1 \cup C_2) = \mathbb{K}$, we do not need to derive any new inequalities to get a description of the closed convex hull. The next lemma obtains a natural consequence of Condition 1 through conic duality.

Lemma 2 *Consider C_1, C_2 defined as in (2). Suppose Condition 1 holds. Then the following system of inequalities in the variable β_1 is inconsistent:*

$$\beta_1 \geq 0, \quad \beta_1 c_{1,0} \geq c_{2,0}, \quad c_2 - \beta_1 c_1 \in \mathbb{K}^*. \tag{4}$$

Similarly, the following system of inequalities in the variable β_2 is inconsistent:

$$\beta_2 \geq 0, \quad \beta_2 c_{2,0} \geq c_{1,0}, \quad c_1 - \beta_2 c_2 \in \mathbb{K}^*. \tag{5}$$

Proof Suppose there exists β_1^* satisfying (4). For all $x \in \mathbb{K}$, this implies $(c_2 - \beta_1^* c_1)^\top x \geq 0 \geq c_{2,0} - \beta_1^* c_{1,0}$. Then any point $x \in C_1$ satisfies $\beta_1^* c_1^\top x \geq \beta_1^* c_{1,0}$ and therefore, $c_2^\top x \geq c_{2,0}$. Hence, $C_1 \subseteq C_2$ which contradicts Condition 1. The proof for the inconsistency of (5) is similar. □

2.2 Properties of Undominated Valid Linear Inequalities

It is well-known that the closed convex hull of any set can be described by valid linear inequalities alone (see, e.g., [23, Theorem 4.2.3]). In this section, using the special structure of the set $C_1 \cup C_2$, we show that a subset of strong valid linear inequalities is sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$. Besides being smaller in size, this subset of linear inequalities also has a particular structure which is instrumental in the derivation of the nonlinear valid inequality of Theorem 3 in Section 3.

A valid linear inequality $\mu^\top x \geq \mu_0$ for a feasible set $S \subseteq \mathbb{K}$ is said to be *tight* if $\inf_x\{\mu^\top x : x \in S\} = \mu_0$ and *strongly tight* if there exists $x^* \in S$ such that $\mu^\top x^* = \mu_0$. A valid linear inequality $\nu^\top x \geq \nu_0$ for a strictly feasible set $S \subseteq \mathbb{K}$ is said to *dominate* another valid linear inequality $\mu^\top x \geq \mu_0$ if it is not a positive multiple of $\mu^\top x \geq \mu_0$ and implies $\mu^\top x \geq \mu_0$ together with the cone constraint $x \in \mathbb{K}$. Furthermore, a valid linear inequality $\mu^\top x \geq \mu_0$ is said to be *undominated* if there does not exist another valid linear inequality $\nu^\top x \geq \nu_0$ such that $(\mu - \nu, \mu_0 - \nu_0) \in \mathbb{K}^* \times -\mathbb{R}_+ \setminus \{(0,0)\}$. This notion of domination is closely tied with the $\mathbb{K}$-minimality definition of [26] which says that a valid linear inequality $\mu^\top x \geq \mu_0$ is $\mathbb{K}$-*minimal* if there does not exist another valid linear inequality $\nu^\top x \geq \nu_0$ such that $(\mu - \nu, \mu_0 - \nu_0) \in (\mathbb{K}^* \setminus \{0\}) \times -\mathbb{R}_+$. In particular, a valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ is undominated in the sense considered here if and only if it is $\mathbb{K}$-minimal and tight on $\overline{\text{conv}}(C_1 \cup C_2)$.

In [26], Kılınç-Karzan defines and studies $\mathbb{K}$-minimal inequalities for disjunctive conic sets of the form

$$\left\{x \in \mathbb{R}^n : Ax \in H, \, x \in \mathbb{K}\right\},$$

where H is an arbitrary set and $\mathbb{K}$ is a regular cone. Our set $C_1 \cup C_2$ can be represented in this form as

$$\left\{x \in \mathbb{R}^n : \begin{pmatrix} c_1^\top \\ c_2^\top \end{pmatrix} x = \begin{pmatrix} \{c_{1,0}\} + \mathbb{R}_+ \\ \mathbb{R} \end{pmatrix} \bigcup \begin{pmatrix} \mathbb{R} \\ \{c_{2,0}\} + \mathbb{R}_+ \end{pmatrix}, \, x \in \mathbb{K}\right\}.$$

Because $C_1 \cup C_2$ is full-dimensional under Condition 2, we can use Proposition 3.2 of [26] to conclude that the extreme rays of the convex cone of valid linear inequalities

$$\left\{(\mu, \mu_0) \in \mathbb{R}^n \times \mathbb{R} : \, \mu^\top x \geq \mu_0 \; \forall x \in \overline{\text{conv}}(C_1 \cup C_2)\right\}$$

are either tight, $\mathbb{K}$-minimal inequalities or implied by the cone constraint $x \in \mathbb{K}$. Hence, one needs to add only undominated valid linear inequalities to the cone constraint $x \in \mathbb{K}$ to obtain an outer description of $\overline{\text{conv}}(C_1 \cup C_2)$.

Because C_1 and C_2 are strictly feasible sets by Condition 2, conic duality implies that a linear inequality $\mu^\top x \geq \mu_0$ is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ if and only if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies

$$\begin{aligned} \mu &= \alpha_1 + \beta_1 c_1, \\ \mu &= \alpha_2 + \beta_2 c_2, \\ \beta_1 c_{1,0} \geq \mu_0, &\quad \beta_2 c_{2,0} \geq \mu_0, \\ \alpha_1, \alpha_2 \in \mathbb{K}^*, &\quad \beta_1, \beta_2 \in \mathbb{R}_+. \end{aligned} \tag{6}$$

This system can be strengthened significantly when we consider *undominated* valid linear inequalities.

Proposition 1 *Let C_1, C_2 satisfy the basic disjunctive setup. Then, up to positive scaling, any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ has the form $\mu^\top x \geq \mu_0$ with $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying*

$$\begin{aligned} \mu &= \alpha_1 + \beta_1 c_1, \\ \mu &= \alpha_2 + \beta_2 c_2, \\ \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\} &= \mu_0, \\ \alpha_1, \alpha_2 \in \text{bd}\, \mathbb{K}^*, &\quad \beta_1, \beta_2 \in \mathbb{R}_+ \setminus \{0\}. \end{aligned} \tag{7}$$

Proof Let $\nu^\top x \geq \nu_0$ be a valid inequality for $\overline{\text{conv}}(C_1 \cup C_2)$. Then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $(\nu, \nu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies (6). If $\beta_1 = 0$ or $\beta_2 = 0$, then $\nu^\top x \geq \nu_0$ is implied by the cone constraint $x \in \mathbb{K}$. If $\min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\} > \nu_0$, then $\nu^\top x \geq \nu_0$ is implied by the valid inequality $\nu^\top x \geq \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$. Hence, we can assume without any loss of generality that any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ has the form $\nu^\top x \geq \nu_0$ with $(\nu, \nu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying

$$\begin{aligned} \nu &= \alpha_1 + \beta_1 c_1, \\ \nu &= \alpha_2 + \beta_2 c_2, \\ \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\} &= \nu_0, \\ \alpha_1, \alpha_2 \in \mathbb{K}^*, &\ \beta_1, \beta_2 \in \mathbb{R}_+ \setminus \{0\}. \end{aligned}$$

We are now going to show that when $\alpha_1 \in \text{int}\, \mathbb{K}^*$ or $\alpha_2 \in \text{int}\, \mathbb{K}^*$, any such inequality is either dominated or equivalent to a valid inequality $\mu^\top x \geq \nu_0$ that satisfies (7). Assume without any loss of generality that $\alpha_2 \in \text{int}\, \mathbb{K}^*$. There are two cases that we need to consider: $\alpha_1 = 0$ and $\alpha_1 \neq 0$.

First suppose $\alpha_1 = 0$. We have $\alpha_2 = \beta_1 c_1 - \beta_2 c_2 \in \text{int}\, \mathbb{K}^*$. By Lemma 2 and taking $\beta_1, \beta_2 > 0$ into account, we conclude $\beta_2 c_{2,0} < \beta_1 c_{1,0}$. Hence, $\nu_0 = \beta_2 c_{2,0}$. If $\nu_0 > 0$, let $0 < \epsilon' < \beta_1$ be such that $\alpha_2' := \alpha_2 - \epsilon' c_1 \in \mathbb{K}^*$ and $\beta_2 c_{2,0} \leq \beta_1 c_{1,0} - \epsilon' c_{1,0}$ and define $\beta_1' := \beta_1 - \epsilon'$ and $\mu := \nu - \epsilon' c_1$. If $\nu_0 \leq 0$, let $\epsilon' > 0$ be such that $\alpha_2' := \alpha_2 + \epsilon' c_1 \in \mathbb{K}^*$ and $\beta_2 c_{2,0} \leq \beta_1 c_{1,0} + \epsilon' c_{1,0}$ and define $\beta_1' := \beta_1 + \epsilon'$ and $\mu := \nu + \epsilon' c_1$. In either case, the inequality $\mu^\top x \geq \nu_0$ is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ because $(\mu, \nu_0, \alpha_1, \alpha_2', \beta_1', \beta_2)$ satisfies (6). Furthermore, it dominates (or, in the case of $\nu_0 = 0$, is equivalent to) $\nu^\top x \geq \nu_0$ because $\mu = \frac{\beta_1'}{\beta_1} \nu$ and $\beta_1' < \beta_1$ when $\nu_0 > 0$ and $\beta_1' > \beta_1$ when $\nu_0 \leq 0$.

Now suppose $\alpha_1 \neq 0$. Let $0 < \epsilon'' \leq 1$ be such that $\alpha_2'' := \alpha_2 - \epsilon'' \alpha_1 \in \mathbb{K}^*$, and define $\alpha_1'' := (1 - \epsilon'') \alpha_1$ and $\mu := \nu - \epsilon'' \alpha_1$. The inequality $\mu^\top x \geq \nu_0$ is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ because $(\mu, \nu_0, \alpha_1'', \alpha_2'', \beta_1, \beta_2)$ satisfies (6). Furthermore, $\mu^\top x \geq \nu_0$ dominates $\nu^\top x \geq \nu_0$ since $\nu - \mu = \epsilon'' \alpha_1 \in \mathbb{K}^* \setminus \{0\}$. □

The system (7) is homogeneous in the tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$. Therefore, in an undominated valid linear inequality $\mu^\top x \geq \mu_0$, we can assume without any loss of generality that the whole tuple has been scaled by a positive real number so that $\beta_1 = 1$ or $\beta_2 = 1$.

Proposition 2 *Let C_1, C_2 satisfy the basic disjunctive setup. Then, up to positive scaling, any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ has the form $\mu^\top x \geq$*

μ_0 *with* $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ *satisfying one of the following systems:*

$$(i)\quad \begin{aligned} &\mu = \alpha_1 + \beta_1 c_1,\\ &\mu = \alpha_2 + c_2,\\ &\beta_1 c_{1,0} \geq c_{2,0} = \mu_0,\\ &\alpha_1, \alpha_2 \in \operatorname{bd}\mathbb{K}^*,\\ &\beta_1 \in \mathbb{R}_+ \setminus \{0\}, \beta_2 = 1, \end{aligned} \qquad (ii)\quad \begin{aligned} &\mu = \alpha_1 + c_1,\\ &\mu = \alpha_2 + \beta_2 c_2,\\ &\beta_2 c_{2,0} \geq c_{1,0} = \mu_0,\\ &\alpha_1, \alpha_2 \in \operatorname{bd}\mathbb{K}^*,\\ &\beta_2 \in \mathbb{R}_+ \setminus \{0\}, \beta_1 = 1. \end{aligned} \tag{8}$$

Keeping $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ in mind, observe that the first of the two systems in (8) is infeasible when $c_{2,0} > c_{1,0}$ and the second is infeasible when $c_{1,0} > c_{2,0}$. Therefore, in these cases it suffices to consider only one of these systems. When $c_{1,0} = c_{2,0}$ however, one may need valid linear inequalities that are associated with either of the two systems in (8) to be able to describe $\overline{\text{conv}}(C_1 \cup C_2)$. Still, for this case Proposition 2 implies that any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ can be written in the form $\mu^\top x \geq \mu_0$ with $\mu_0 = c_{1,0} = c_{2,0}$.

Note that in any tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying the first system in (8), we must have $c_2 - \beta_1 c_1 \notin \pm \operatorname{int}\mathbb{K}^*$ since having $c_2 - \beta_1 c_1 \in \pm \operatorname{int}\mathbb{K}^*$ would contradict either $\alpha_1 = \alpha_2 + (c_2 - \beta_1 c_1) \in \operatorname{bd}\mathbb{K}^*$ or $\alpha_2 = \alpha_1 - (c_2 - \beta_1 c_1) \in \operatorname{bd}\mathbb{K}^*$. Similarly, in any tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying the second system in (8), we must have $c_1 - \beta_2 c_2 \notin \pm \operatorname{int}\mathbb{K}^*$. Proposition 2 and the discussion above lead us to Corollary 1. For ease of exposition, we define the sets

$$\begin{aligned} B_1^{C_1,C_2} &:= \{\beta_1 > 0 :\ \beta_1 c_{1,0} \geq c_{2,0},\ c_2 - \beta_1 c_1 \notin \pm \operatorname{int}\mathbb{K}^*\},\\ B_2^{C_1,C_2} &:= \{\beta_2 > 0 :\ \beta_2 c_{2,0} \geq c_{1,0},\ c_1 - \beta_2 c_2 \notin \pm \operatorname{int}\mathbb{K}^*\},\\ M^{C_1,C_2}(\beta_1, \beta_2) &:= \{\mu \in \mathbb{R}^n :\ \exists \alpha_1, \alpha_2 \in \operatorname{bd}\mathbb{K}^*,\ \mu = \alpha_1 + \beta_1 c_1 = \alpha_2 + \beta_2 c_2\}. \end{aligned}$$

Corollary 1 *Let* C_1, C_2 *satisfy the basic disjunctive setup. The closed convex hull of* $C_1 \cup C_2$ *is given by*

$$\overline{\text{conv}}(C_1 \cup C_2) = \left\{ x \in \mathbb{R}^n :\ \begin{array}{l} \mu^\top x \geq c_{2,0},\ \forall \mu \in M^{C_1,C_2}(\beta_1, 1), \beta_1 \in B_1^{C_1,C_2}, \\ \mu^\top x \geq c_{1,0},\ \forall \mu \in M^{C_1,C_2}(1, \beta_2), \beta_2 \in B_2^{C_1,C_2} \end{array} \right\}.$$

2.3 Revisiting Condition 2

When $C_i = \{x \in \mathbb{K} :\ c_i^\top x \geq c_{i,0}\}$ is nonempty and $c_{i,0} \in \{\pm 1\}$, it is not difficult to show that C_i has to be strictly feasible. Therefore, Condition 2 is not needed when, for instance, C_1 and C_2 are nonempty sets defined by a split disjunction which excludes the origin. Indeed, the only situation where Condition 2 may be needed in addition to Condition 1 occurs when $c_{1,0} = 0$ or $c_{2,0} = 0$. Note that in such a case, linear inequalities that satisfy system (6) (or (7)) are still valid for $\overline{\text{conv}}(C_1 \cup C_2)$; they may just not be sufficient to define it completely. We next give an example which shows that Condition 2 is necessary to establish the sufficiency of the linear inequalities that satisfy (6) (or (7)) when $c_{1,0} = c_{2,0} = 0$.

Consider the second-order cone $\mathbb{L}^3$ and the disjunction $x_1 - x_3 \geq 0 \vee -x_1 - x_3 \geq 0$ ($c_1 := e^1 - e^3$, $c_2 := -e^1 - e^3$, $c_{1,0} = c_{2,0} = 0$). Note that $c_1, c_2 \in -\operatorname{bd}\mathbb{L}^3$, and C_1 and C_2 are the rays generated by $e^1 + e^3$ and $-e^1 + e^3$, respectively. Therefore, $\overline{\text{conv}}(C_1 \cup C_2) = \{x \in \mathbb{L}^3 :\ x_2 = 0\}$ and $x_2 \geq 0$ is a valid inequality for $\overline{\text{conv}}(C_1 \cup C_2)$. However, letting $\mu = e^2$ in (6), we see that any α_1 which satisfies $\mu = \alpha_1 + \beta_1 c_1$ for some $\beta_1 \in \mathbb{R}$ cannot be in $\mathbb{L}^3$ because $\alpha_1 = -\beta_1 e^1 + e^2 + \beta_1 e^3 \notin \mathbb{L}^3$.

3 Deriving the Disjunctive Cut

In this section we focus on the case where $\mathbb{K}$ is the second-order cone $\mathbb{L}^n = \{x \in \mathbb{R}^n : \|\tilde{x}\|_2 \leq x_n\}$ and $\tilde{x} := (x_1; \ldots; x_{n-1})$. Recall that the dual cone of $\mathbb{L}^n$ is again $\mathbb{L}^n$. As in the previous section, we consider sets C_1 and C_2 which satisfy the basic disjunctive setup. When in addition $\mathbb{K} = \mathbb{L}^n$, we say that C_1 and C_2 satisfy the *second-order cone disjunctive setup.*

3.1 A Convex Valid Inequality

Proposition 1 gives a nice characterization of the form of undominated linear inequalities valid for $\overline{\text{conv}}(C_1 \cup C_2)$. In the following we use this characterization and show that, for a given pair (β_1, β_2) satisfying the conditions of Proposition 1, one can group all of the corresponding linear inequalities $\mu^\top x \geq \mu_0$, where $\mu \in M^{C_1,C_2}(\beta_1, \beta_2)$ and $\mu_0 = \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$, into a single convex, possibly nonlinear, inequality (3) valid for $\overline{\text{conv}}(C_1 \cup C_2)$. Then Theorem 1 immediately follows from the fact that the inequalities (3) that are associated with all such pairs (β_1, β_2) yields $\overline{\text{conv}}(C_1 \cup C_2)$. By Corollary 1, without any loss of generality, we focus on the case $c_{1,0} \geq c_{2,0}$ and the linear inequalities $\mu^\top x \geq c_{2,0}$ where $\mu \in M^{C_1,C_2}(\beta_1, 1)$ and $\beta_1 \in B_1^{C_1,C_2}$. For any fixed $\beta_1 \in B_1^{C_1,C_2}$, $\beta_1 c_1 - c_2 \notin \pm\, \text{int}\, \mathbb{L}^n$. Furthermore, $\beta_1 c_1 - c_2 \notin -\mathbb{L}^n$ by Lemma 2. This leaves us two distinct cases to consider: $\beta_1 c_1 - c_2 \in \text{bd}\, \mathbb{L}^n$ and $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$.

Remark 1 Let C_1, C_2 satisfy the second-order cone disjunctive setup. For any $\beta_1 \in B_1^{C_1,C_2}$ such that $\beta_1 c_1 - c_2 \in \text{bd}\, \mathbb{L}^n$, the inequality

$$\beta_1 c_1^\top x \geq c_{2,0} \tag{9}$$

is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ and dominates all valid linear inequalities $\mu^\top x \geq c_{2,0}$ such that $\mu \in M^{C_1,C_2}(\beta_1, 1)$.

Proof Since $\beta_1 \in B_1^{C_1,C_2}$, we have $\beta_1 c_{1,0} \geq c_{2,0}$. The validity of (9) follows easily from $\beta_1 c_{1,0} \geq c_{2,0}$ for C_1 and $\beta_1 c_1 - c_2 \in \mathbb{L}^n$ for C_2. Let $\mu \in M^{C_1,C_2}(\beta_1, 1)$. Then $\mu - \beta_1 c_1 =: \alpha_1 \in \mathbb{L}^n$, and since $\beta_1 c_1^\top x \geq c_{2,0}$ is valid as well, we have that $\mu^\top x \geq c_{2,0}$ is dominated unless $\alpha_1 = 0$. □

We note that Remark 1 remains true in the general case where $\mathbb{K}$ is an arbitrary regular cone.

Theorem 3 *Let C_1, C_2 satisfy the second-order cone disjunctive setup. For any $\beta_1 \in B_1^{C_1,C_2}$ such that $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$, the inequality*

$$2c_{2,0} - (\beta_1 c_1 + c_2)^\top x \leq \sqrt{((\beta_1 c_1 - c_2)^\top x)^2 + \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right)} \tag{10}$$

with

$$\mathcal{N}_1(\beta_1) := \|\beta_1 \tilde{c_1} - \tilde{c_2}\|_2^2 - (\beta_1 c_{1,n} - c_{2,n})^2 \tag{11}$$

is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ and implies all valid linear inequalities $\mu^\top x \geq c_{2,0}$ such that $\mu \in M^{C_1,C_2}(\beta_1, 1)$.

Proof Consider the set $M^{C_1,C_2}(\beta_1,1)$. Because $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$, Moreau's decomposition theorem implies that there exist $\mu^*, \alpha_1^* \neq 0, \alpha_2^* \neq 0$ such that $\alpha_1^* \perp \alpha_2^*$, $\alpha_1^*, \alpha_2^* \in \operatorname{bd}\mathbb{L}^n$, and $\mu^* = \beta_1 c_1 + \alpha_1^* = c_2 + \alpha_2^*$. Hence, $M^{C_1,C_2}(\beta_1,1)$ is indeed nonempty. We can write

$$\begin{aligned} M^{C_1,C_2}(\beta_1,1) &= \left\{\mu \in \mathbb{R}^n : \|\tilde{\mu} - \beta_1\tilde{c}_1\|_2 = \mu_n - \beta_1 c_{1,n},\ \|\tilde{\mu} - \tilde{c}_2\|_2 = \mu_n - c_{2,n}\right\} \\ &= \left\{\mu \in \mathbb{R}^n : \begin{array}{c} \|\tilde{\mu} - \tilde{c}_2\|_2 = \|\tilde{\mu} - \beta_1\tilde{c}_1\|_2 + \beta_1 c_{1,n} - c_{2,n}, \\ \|\tilde{\mu} - \beta_1\tilde{c}_1\|_2 = \mu_n - \beta_1 c_{1,n} \end{array}\right\}. \end{aligned}$$

After taking the square of both sides of the first equation in $M^{C_1,C_2}(\beta_1,1)$, noting $\beta_1 c_1 - c_2 \notin -\mathbb{L}^n$, and replacing the term $\|\tilde{\mu} - \beta_1\tilde{c}_1\|_2$ with $\mu_n - \beta_1 c_{1,n}$, we arrive at

$$M^{C_1,C_2}(\beta_1,1) = \left\{\mu \in \mathbb{R}^n : \begin{array}{c} \tilde{\mu}^\top(\beta_1\tilde{c}_1 - \tilde{c}_2) - \mu_n(\beta_1 c_{1,n} - c_{2,n}) = \frac{\mathcal{M}}{2}, \\ \|\tilde{\mu} - \beta_1\tilde{c}_1\|_2 = \mu_n - \beta_1 c_{1,n} \end{array}\right\}$$

where $\mathcal{M} := \beta_1^2(\|\tilde{c}_1\|_2^2 - c_{1,n}^2) - (\|\tilde{c}_2\|_2^2 - c_{2,n}^2)$.

Note that, by Corollary 1, $x \in \overline{\operatorname{conv}}(C_1 \cup C_2)$ and $\beta_1 \in B_1^{C_1,C_2}$ (thus $\beta_1 c_{1,0} \geq c_{2,0}$) imply

$$\begin{aligned} &\Rightarrow x \in \mathbb{L}^n \text{ and } \mu^\top x \geq c_{2,0} \quad \forall \mu \in M^{C_1,C_2}(\beta_1,1). \\ &\Leftrightarrow x \in \mathbb{L}^n \text{ and } \inf_{\mu}\left\{\mu^\top x :\ \mu \in M^{C_1,C_2}(\beta_1,1)\right\} \geq c_{2,0}. \end{aligned}$$

Unfortunately, the optimization problem stated above is non-convex due to the second equality constraint in the description of $M^{C_1,C_2}(\beta_1,1)$. We show below that the natural convex relaxation for this problem is tight. Indeed, consider the relaxation

$$\inf_{\mu}\left\{\mu^\top x : \begin{array}{c} \tilde{\mu}^\top(\beta_1\tilde{c}_1 - \tilde{c}_2) - \mu_n(\beta_1 c_{1,n} - c_{2,n}) = \frac{\mathcal{M}}{2}, \\ \|\tilde{\mu} - \beta_1\tilde{c}_1\|_2 \leq \mu_n - \beta_1 c_{1,n} \end{array}\right\}$$

The feasible region of this relaxation is the intersection of a hyperplane with a closed, convex cone shifted by the vector $\beta_1 c_1$. Any solution which is feasible to the relaxation but not the original problem can be expressed as a convex combination of solutions feasible to the original problem. Because we are optimizing a linear function, this shows that the relaxation is equivalent to the original problem. Thus, we have

$$\begin{aligned} &x \in \overline{\operatorname{conv}}(C_1 \cup C_2) \Rightarrow \\ &\qquad x \in \mathbb{L}^n \text{ and } \inf_{\mu}\left\{\mu^\top x : \begin{array}{c} \tilde{\mu}^\top(\beta_1\tilde{c}_1 - \tilde{c}_2) - \mu_n(\beta_1 c_{1,n} - c_{2,n}) = \frac{\mathcal{M}}{2}, \\ \|\tilde{\mu} - \beta_1\tilde{c}_1\|_2 \leq \mu_n - \beta_1 c_{1,n} \end{array}\right\} \geq c_{2,0} \end{aligned}$$

which is exactly the same as

$$\begin{aligned} &x \in \overline{\operatorname{conv}}(C_1 \cup C_2) \Rightarrow \\ &x \in \mathbb{L}^n \text{ and } \inf_{\mu}\left\{\mu^\top x : \begin{array}{c} \tilde{\mu}^\top(\beta_1\tilde{c}_1 - \tilde{c}_2) - \mu_n(\beta_1 c_{1,n} - c_{2,n}) = \frac{\mathcal{M}}{2}, \\ \mu - \beta_1 c_1 \in \mathbb{L}^n \end{array}\right\} \geq c_{2,0}. \end{aligned} \tag{12}$$

The minimization problem in the last line above is feasible since μ^*, defined at the beginning of the proof, is a feasible solution. Indeed, it is strictly feasible since

$\alpha_1^* + \alpha_2^*$ is a recession direction of the feasible region and belongs to $\operatorname{int} \mathbb{L}^n$. Hence, its dual problem is solvable whenever it is feasible, strong duality applies, and we can replace the problem in the last line with its dual without any loss of generality.

Considering the definition of $\mathcal{N}_1(\beta_1) = \|\beta_1 \tilde{c_1} - \tilde{c_2}\|_2^2 - (\beta_1 c_{1,n} - c_{2,n})^2$ and the hypothesis that $\beta_1 c_1 - c_2 \notin \pm \mathbb{L}^n$, we get $\mathcal{N}_1(\beta_1) > 0$. Then

$$
\begin{aligned}
& x \in \overline{\operatorname{conv}}(C_1 \cup C_2) \\
& \Rightarrow x \in \mathbb{L}^n \text{ and } \max_{\rho,\tau} \left\{ \beta_1 c_1^\top \rho + \frac{\mathcal{M}}{2}\tau : \begin{array}{c} \rho + \tau \begin{pmatrix} \beta_1 \tilde{c}_1 - \tilde{c}_2 \\ -\beta_1 c_{1,n} + c_{2,n} \end{pmatrix} = x, \\ \rho \in \mathbb{L}^n \end{array} \right\} \ge c_{2,0}. \\
& \Leftrightarrow x \in \mathbb{L}^n \text{ and } \max_{\tau} \left\{ \beta_1 c_1^\top x - \frac{\mathcal{N}_1(\beta_1)}{2}\tau : x + \tau \begin{pmatrix} -\beta_1 \tilde{c}_1 + \tilde{c}_2 \\ \beta_1 c_{1,n} - c_{2,n} \end{pmatrix} \in \mathbb{L}^n \right\} \ge c_{2,0},
\end{aligned}
$$

and since the optimum solution will be on the boundary of feasible region,

$$
\begin{aligned}
& \Leftrightarrow x \in \mathbb{L}^n \text{ and } \min\{\tau_-, \tau_+\} \le \frac{2(\beta_1 c_1^\top x - c_{2,0})}{\mathcal{N}_1(\beta_1)} \\
& \qquad \text{where } \tau_\pm := \frac{(\beta_1 c_1 - c_2)^\top x \pm \sqrt{((\beta_1 c_1 - c_2)^\top x)^2 + \mathcal{N}_1(\beta_1)(x_n^2 - \|\tilde{x}\|_2^2)}}{\mathcal{N}_1(\beta_1)}. \\
& \Leftrightarrow x \in \mathbb{L}^n \text{ and } \tau_- \le \frac{2(\beta_1 c_1^\top x - c_{2,0})}{\mathcal{N}_1(\beta_1)}. \\
& \Leftrightarrow x \in \mathbb{L}^n \text{ and } \mathcal{N}_1(\beta_1)\tau_- \le 2(\beta_1 c_1^\top x - c_{2,0}).
\end{aligned}
$$

Rearranging the terms of the inequality in the last expression above yields (10). □

The next two observations follow directly from the proof of Theorem 3.

Remark 2 Under the hypotheses of Theorem 3, the set of points that satisfy (10) in $\mathbb{L}^n$ is convex.

Proof The inequality (10) is equivalent to (12) by construction. The left-hand side of (12) is a concave function of x written as the pointwise-infimum of linear functions, while the right-hand side is a constant. □

We note here that one can convert (10) into an inequality that is convex on all of $\mathbb{R}^n$ by rewriting it as $g(x) \le 0$ where $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$
g(x) := \begin{cases} 2c_{2,0} - (\beta_1 c_1 + c_2)^\top x - \sqrt{((\beta_1 c_1 - c_2)^\top x)^2 + \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right)}, & x \in \mathbb{L}^n, \\ +\infty, & x \notin \mathbb{L}^n. \end{cases}
$$

However, we avoid using this construction to keep the notation simple in what follows.

Remark 3 Inequality (10) reduces to the linear inequality (9) in $\mathbb{L}^n$ when $\beta_1 c_1 - c_2 \in \operatorname{bd} \mathbb{L}^n$.

Proof When $\beta_1 c_1 - c_2 \in \operatorname{bd} \mathbb{L}^n$, $\mathcal{N}_1(\beta_1) = 0$. Together with $x \in \mathbb{L}^n$, this also implies $(\beta_1 c_1 - c_2)^\top x \ge 0$, and hence, (10) of Theorem 3 becomes $2c_{2,0} - (\beta_1 c_1 + c_2)^\top x \le (\beta_1 c_1 - c_2)^\top x$. This is equivalent to (9). □

Some comments about Theorem 3 are in order.

Remark 4 For $\beta_1 \in B_1^{C_1,C_2}$, the inequality (10) can in fact be considered to be derived for the relaxed disjunction $\beta_1 c_1^\top x \geq c_{2,0} \vee c_2^\top x \geq c_{2,0}$. Indeed, under the hypothesis $\beta_1 \in B_1^{C_1,C_2}$ of Theorem 3, we have $\beta_1 c_1^\top x \geq \beta_1 c_{1,0} \geq c_{2,0}$ for all $x \in C_1$. However, somewhat contrary to intuition, inequalities (10) obtained from such weaker disjunctions are sometimes necessary for a complete description of $\overline{\text{conv}}(C_1 \cup C_2)$. We will study these cases in more detail in Section 5.

Remark 5 Inequality (10) has a simple geometrical meaning when the two sides of the relaxed disjunction $\beta_1 c_1^\top x \geq c_{2,0} \vee c_2^\top x \geq c_{2,0}$ on $\mathbb{L}^n$ do not intersect, except possibly at their boundary. Consider a point $x \in \mathbb{R}^n$ which is on the hyperplane defined by $\beta_1 c_1^\top x = c_{2,0}$. Then disjointness of the two sides of the disjunction implies $c_2^\top x \leq c_{2,0}$. Replacing $\beta_1 c_1^\top x$ with $c_{2,0}$ on both sides of (10), we can see that when $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$ ($\mathcal{N}_1(\beta_1) > 0$), such a point x satisfies (10) if and only if $x \in \pm\mathbb{L}^n$. Similarly, a point x which is on the hyperplane defined by $c_2^\top x = c_{2,0}$ satisfies (10) if and only if $x \in \pm\mathbb{L}^n$. Thus, the region defined by (10) has the same cross-section as $\pm\mathbb{L}^n$ at the hyperplanes defined by the equalities $\beta_1 c_1^\top x = c_{2,0}$ and $c_2^\top x = c_{2,0}$.

Suppose C_1 and C_2 have been chosen so that $c_{1,0} \geq c_{2,0}$. When $c_{1,0} > c_{2,0}$, $B_2^{C_1,C_2} = \emptyset$, and by Corollary 1, the family of inequalities given in Remark 1 and Theorem 3 is sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$. On the other hand, when $c_{1,0} = c_{2,0}$, we also need to consider valid linear inequalities $\mu^\top x \geq c_{1,0}$ such that $\mu \in M^{C_1,C_2}(1, \beta_2)$ and $\beta_2 \in B_2^{C_1,C_2}$. Following Remark 1 and Theorem 3, such linear inequalities can be condensed into a single inequality for every fixed $\beta_2 \in B_2^{C_1,C_2}$. In particular, letting $\mathcal{N}_2(\beta_2) := \|\tilde{c_1} - \beta_2 \tilde{c_2}\|_2^2 - (c_{1,n} - \beta_2 c_{2,n})^2$, one can derive the valid inequalities

$$\beta_2 c_2^\top x \geq c_{1,0}, \text{ and} \tag{13}$$

$$2c_{1,0} - (c_1 + \beta_2 c_2)^\top x \leq \sqrt{\left((c_1 - \beta_2 c_2)^\top x\right)^2 + \mathcal{N}_2(\beta_2)\left(x_n^2 - \|\tilde{x}\|_2^2\right)} \tag{14}$$

for $\beta_2 \in B_2^{C_1,C_2}$ such that $\beta_2 c_2 - c_1 \in \text{bd}\,\mathbb{L}^n$ and $\beta_2 c_2 - c_1 \notin \pm\mathbb{L}^n$, respectively. Using these inequalities and recalling Remark 3, $\overline{\text{conv}}(C_1 \cup C_2)$ can be written as

$$\overline{\text{conv}}(C_1 \cup C_2) = \left\{x \in \mathbb{R}^n : \, x \text{ satisfies } (10)\ \forall \beta_1 \in B_1^{C_1,C_2} \text{ and } (14)\ \forall \beta_2 \in B_2^{C_1,C_2}\right\}.$$

Note that for any $\beta_1 \in B_1^{C_1,C_2}$ and $\beta_2 \in B_2^{C_1,C_2}$, we have $\mathcal{N}_1(\beta_1) \geq 0$ and $\mathcal{N}_2(\beta_2) \geq 0$; hence, the right hand sides of the inequalities (10) and (14) above are well-defined for any $x \in \mathbb{L}^n$. Thus, this proves Theorem 1 of Section 1.2. In the remainder of this section, we continue to focus on the case where $\beta_2 = 1$ and $\beta_1 \in B_1^{C_1,C_2}$ with the understanding that our results are also applicable to the symmetric situation.

3.2 A Conic Quadratic Form

While having a convex valid inequality is nice in general, there are certain cases where (10) can be expressed in conic quadratic form. In the following, we first identify a symmetry condition that guarantees the existence of an equivalent conic quadratic form for (10) and then show that such a symmetry condition holds under a disjointness condition.

Proposition 3 *Let C_1, C_2 satisfy the second-order cone disjunctive setup, and let $\beta_1 \in B_1^{C_1,C_2}$ be such that $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$. Let $x \in \mathbb{L}^n$ be a point for which*

$$-2c_{2,0} + (\beta_1 c_1 + c_2)^\top x \le \sqrt{((\beta_1 c_1 - c_2)^\top x)^2 + \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right)} \tag{15}$$

holds with $\mathcal{N}_1(\beta_1)$ defined as in (11). Then x satisfies (10) if and only if it satisfies the conic quadratic inequality

$$\mathcal{N}_1(\beta_1)x - 2(\beta_1 c_1^\top x - c_{2,0}) \begin{pmatrix} \beta_1 \tilde{c}_1 - \tilde{c}_2 \\ -\beta_1 c_{1,n} + c_{2,n} \end{pmatrix} \in \mathbb{L}^n. \tag{16}$$

Furthermore, if (15) holds for all $x \in \overline{\text{conv}}(C_1 \cup C_2)$, then (16) is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ and implies (10).

Proof Let $x \in \mathbb{L}^n$ be a point for which (15) holds. Then x satisfies (10) if and only if it satisfies

$$|2c_{2,0} - (\beta_1 c_1 + c_2)^\top x| \le \sqrt{((\beta_1 c_1 - c_2)^\top x)^2 + \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right)}.$$

We can take the square of both sides without any loss of generality and rewrite this inequality as

$$\begin{aligned} &\left(2c_{2,0} - (\beta_1 c_1 + c_2)^\top x\right)^2 \le \left((\beta_1 c_1 - c_2)^\top x\right)^2 + \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right). \\ \Leftrightarrow \quad & 4(\beta_1 c_1^\top x - c_{2,0})(c_2^\top x - c_{2,0}) \le \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right). \end{aligned}$$

Because $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$, we have $\mathcal{N}_1(\beta_1) > 0$, and the above inequality is equivalent to

$$0 \le \mathcal{N}_1(\beta_1)^2\left(x_n^2 - \|\tilde{x}\|_2^2\right) - 4\mathcal{N}_1(\beta_1)(\beta_1 c_1^\top x - c_{2,0})(c_2^\top x - c_{2,0}).$$

The right-hand side of this inequality is identical to the following quadratic form which has a single negative eigenvalue:

$$\left(\mathcal{N}_1(\beta_1)x_n + 2(\beta_1 c_1^\top x - c_{2,0})(\beta_1 c_{1,n} - c_{2,n})\right)^2 - \left\|\mathcal{N}_1(\beta_1)\tilde{x} - 2(\beta_1 c_1^\top x - c_{2,0})(\beta_1 \tilde{c}_1 - \tilde{c}_2)\right\|_2^2.$$

Therefore, we arrive at

$$\left\|\mathcal{N}_1(\beta_1)\tilde{x} - 2(\beta_1 c_1^\top x - c_{2,0})(\beta_1 \tilde{c}_1 - \tilde{c}_2)\right\|_2^2 \le \left(\mathcal{N}_1(\beta_1)x_n + 2(\beta_1 c_1^\top x - c_{2,0})(\beta_1 c_{1,n} - c_{2,n})\right)^2.$$

Let

$$\begin{aligned} \mathcal{A}(x) &:= \left\|\mathcal{N}_1(\beta_1)\tilde{x} - 2(\beta_1 c_1^\top x - c_{2,0})(\beta_1 \tilde{c}_1 - \tilde{c}_2)\right\|_2 \text{ and} \\ \mathcal{B}(x) &:= \mathcal{N}_1(\beta_1)x_n + 2(\beta_1 c_1^\top x - c_{2,0})(\beta_1 c_{1,n} - c_{2,n}). \end{aligned}$$

We have just proved that x satisfies (10) if and only if it satisfies $\mathcal{A}(x)^2 \le \mathcal{B}(x)^2$. In order to finish the proof, all we need to show is that $\mathcal{A}(u)^2 \le \mathcal{B}(u)^2$ is equivalent to

$\mathcal{A}(u) \leq \mathcal{B}(u)$ for all $u \in \mathbb{L}^n$. It will be enough to show that either $\mathcal{A}(u)+\mathcal{B}(u) > 0$ or $\mathcal{A}(u) = \mathcal{B}(u) = 0$ holds for all $u \in \mathbb{L}^n$. Suppose $\mathcal{A}(u)+\mathcal{B}(u) \leq 0$ for some $u \in \mathbb{L}^n$. Using the triangle inequality, we can write

$$\begin{aligned}
0 &\geq \mathcal{A}(u) + \mathcal{B}(u) \\
&= \left\| \mathcal{N}_1(\beta_1)\tilde{u} - 2(\beta_1 c_1^\top u - c_{2,0})(\beta_1\tilde{c}_1 - \tilde{c}_2) \right\|_2 \\
&\qquad + \mathcal{N}_1(\beta_1)u_n + 2(\beta_1 c_1^\top u - c_{2,0})(\beta_1 c_{1,n} - c_{2,n}) \\
&\geq -\mathcal{N}_1(\beta_1)\,\|\tilde{u}\|_2 + 2|\beta_1 c_1^\top u - c_{2,0}|\,\|\beta_1\tilde{c}_1 - \tilde{c}_2\|_2 \\
&\qquad + \mathcal{N}_1(\beta_1)u_n - 2|\beta_1 c_1^\top u - c_{2,0}||\beta_1 c_{1,n} - c_{2,n}| \\
&= \mathcal{N}_1(\beta_1)(u_n - \|\tilde{u}\|_2) + 2|\beta_1 c_1^\top u - c_{2,0}|(\|\beta_1\tilde{c}_1 - \tilde{c}_2\|_2 - |\beta_1 c_{1,n} - c_{2,n}|).
\end{aligned}$$

Because $u \in \mathbb{L}^n$ and $\beta_1 c_1 - c_2 \notin \pm\mathbb{L}^n$, we have $u_n - \|\tilde{u}\|_2 \geq 0$ and $\|\beta_1\tilde{c}_1 - \tilde{c}_2\|_2 - |\beta_1 c_{1,n} - c_{2,n}| > 0$. Hence, $\beta_1 c_1^\top u = c_{2,0}$. This implies $\mathcal{A}(u) + \mathcal{B}(u) = \mathcal{N}_1(\beta_1)(u_n + \|\tilde{u}\|_2)$ which is strictly positive unless $u = 0$, but then $\mathcal{A}(u) = \mathcal{B}(u) = 0$.

The second claim of the proposition follows immediately from the first under the hypothesis that (15) holds for all $x \in \overline{\text{conv}}(C_1 \cup C_2)$. □

Note that by changing the roles of $\beta_1 c_1$ and c_2, the proof of Proposition 3 can be repeated to show that a point $x \in \mathbb{L}^n$ for which (15) holds satisfies (10) if and only if it satisfies

$$\mathcal{N}_1(\beta_1)x + 2(c_2^\top x - c_{2,0}) \begin{pmatrix} \beta_1\tilde{c}_1 - \tilde{c}_2 \\ -\beta_1 c_{1,n} + c_{2,n} \end{pmatrix} \in \mathbb{L}^n.$$

We next give a sufficient condition, based on a disjointness property of the intersection of C_1 and C_2, under which (15) is satisfied by every point in $\mathbb{L}^n$. Note that this condition thus allows our convex inequality (10) to be represented in an equivalent conic quadratic form (16).

Proposition 4 *Let C_1, C_2 satisfy the second-order cone disjunctive setup. Let $\beta_1 \in B_1^{C_1,C_2}$ be such that $\beta c_1 - c_2 \notin \pm\mathbb{L}^n$. Then* (15) *holds for all $x \in \mathbb{L}^n$ that satisfy $\beta_1 c_1^\top x \leq c_{2,0}$ or $c_2^\top x \leq c_{2,0}$. Furthermore, if*

$$\{x \in \mathbb{L}^n : \; \beta_1 c_1^\top x > c_{2,0}, \; c_2^\top x > c_{2,0}\} = \emptyset, \tag{17}$$

then (15) *holds for all $x \in \mathbb{L}^n$ and* (16) *is equivalent to* (10) *on $\mathbb{L}^n$.*

Proof Let $x \in \mathbb{L}^n$ satisfy $\beta_1 c_1^\top x \leq c_{2,0}$ or $c_2^\top x \leq c_{2,0}$. Using Theorem 3 on the disjunction $-\beta_1 c_1^\top u \geq -c_{2,0}$ or $-c_2^\top u \geq -c_{2,0}$ shows that x satisfies (15). The second claim of the proposition now follows immediately from Proposition 3 and (17). □

Condition (17) of Proposition 4 says that the two sides of the relaxed disjunction $\beta_1 c_1^\top x \geq c_{2,0} \vee c_2^\top x \geq c_{2,0}$ on the cone $\mathbb{L}^n$ have to be almost disjoint. This condition, together with the results of Proposition 3 and Theorem 3, identifies some cases in which (10) can be expressed in an equivalent conic quadratic form. For instance, in a split disjunction on the cone $\mathbb{L}^n$, it is easy to see using Lemma 1 that C_1 and C_2 are both nonempty and $\overline{\text{conv}}(C_1 \cup C_2) \neq \mathbb{L}^n$ if and only if $c_1, c_2 \notin \pm\mathbb{L}^n$ and $c_{1,0} = c_{2,0} = 1$. Furthermore, $C_1 \cap C_2 = \emptyset$; hence, for a proper two-sided split disjunction on $\mathbb{L}^n$, (17) is trivially satisfied with $\beta_1 = 1$. Moreover, in the next section, we will see that $\overline{\text{conv}}(C_1 \cup C_2)$ can be described completely with linear inequalities $\mu^\top x \geq 1$ such that $\mu \in M^{C_1,C_2}(1,1)$ ($\beta_1 = \beta_2 = 1$ in (7)) when $C_1 \cup C_2$ is defined by such a split disjunction.

4 When does a Single Inequality Suffice?

In this section we give two conditions under which a single convex inequality of the type derived in Theorem 3, together with the cone constraint $x \in \mathbb{L}^n$, describes $\overline{\text{conv}}(C_1 \cup C_2)$ completely. The main result of this section is Theorem 4 which we state below.

Theorem 4 *Let C_1, C_2 satisfy the second-order cone disjunctive setup with $c_1 - c_2 \notin \pm\mathbb{L}^n$ and $c_{1,0} \geq c_{2,0}$. Then the inequality*

$$2c_{2,0} - (c_1 + c_2)^\top x \leq \sqrt{\left((c_1 - c_2)^\top x\right)^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|_2^2\right)} \tag{18}$$

is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ with $\mathcal{N} = \|\tilde{c_1} - \tilde{c_2}\|_2^2 - (c_{1,n} - c_{2,n})^2$. Furthermore,

$$\overline{\text{conv}}(C_1 \cup C_2) = \{x \in \mathbb{L}^n : \ x \textit{ satisfies } (18)\},$$

when, in addition,

(i) $c_1 \in \mathbb{L}^n$, *or* $c_2 \in \mathbb{L}^n$, *or*
(ii) $c_{1,0} = c_{2,0} \in \{\pm 1\}$ *and undominated valid linear inequalities that are tight on both C_1 and C_2 are sufficient to describe $\overline{\text{conv}}(C_1 \cup C_2)$.*

Theorem 4 shows that, under certain conditions, $\overline{\text{conv}}(C_1 \cup C_2)$ is completely described by the cone constraint $x \in \mathbb{L}^n$ and the inequality obtained by setting $\beta_1 = 1$ in (10). A recent result of Yıldız and Cornuéjols [35] complements this sufficiency result by showing that a similar statement is true when $c_{1,0} = c_{2,0} = 0$.

Theorem 5 *[35, Theorem 2] Let C_1 and C_2 satisfy the second-order cone disjunctive setup with $c_{1,0} = c_{2,0} = 0$. Then*

$$\text{conv}(C_1 \cup C_2) = \left\{ x \in \mathbb{L}^n : -(\beta_1 c_1 + c_2)^\top x \leq \sqrt{((\beta_1 c_1 - c_2)^\top x)^2 + \mathcal{N}(\beta_1)\left(x_n^2 - \|\tilde{x}\|^2\right)} \right\}$$

where $\beta_1 = \sqrt{\|\tilde{c}_2\|_2^2 - c_{2,n}^2} \Big/ \sqrt{\|\tilde{c}_1\|_2^2 - c_{1,n}^2}\,.$

The proof of Theorem 4 will require additional results on the structure of undominated valid linear inequalities. These are the subject of the next section.

4.1 Further Properties of Undominated Valid Linear Inequalities

In this section we continue studying the disjunction $c_1^\top x \geq c_{1,0} \vee c_2^\top x \geq c_{2,0}$ on a regular cone $\mathbb{K}$ and refine the results of Section 2.2 on the structure of undominated valid linear inequalities. The results that we are going to present in this section hold for any regular cone $\mathbb{K}$.

The lemma below shows that the statement of Proposition 2 can be strengthened substantially when $c_1 \in \mathbb{K}^*$ or $c_2 \in \mathbb{K}^*$.

Lemma 3 *Let C_1, C_2 satisfy the basic disjunctive setup with $c_{1,0} \geq c_{2,0}$. Suppose $c_1 \in \mathbb{K}^*$ or $c_2 \in \mathbb{K}^*$. Then, up to positive scaling, any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ has the form $\mu^\top x \geq c_{2,0}$ where $\mu \in M^{C_1,C_2}(1,1)$.*

Proof First note that having $c_i \in \mathbb{K}^*$ implies $\text{rec}\, C_i = \mathbb{K}$. Therefore, when $c_{2,0} \leq 0$, we can use Lemma 1 to conclude $\overline{\text{conv}}(C_1 \cup C_2) = \mathbb{K}$. In this case all valid inequalities for $\overline{\text{conv}}(C_1 \cup C_2) = \mathbb{K}$ are implied by the cone constraint $x \in \mathbb{K}$, and the claim holds trivially because there are no undominated valid inequalities. Thus, we only need to consider the situation in which $c_{1,0} = c_{2,0} = 1$.

Assume without any loss of generality that $c_2 \in \mathbb{K}^*$. Let $\nu^\top x \geq \nu_0$ be a valid inequality of the form given in Proposition 2. Then $\nu_0 = c_{1,0} = c_{2,0} = 1$, and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $(\nu, 1, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies one of the two systems in (8). In particular, $\nu = \alpha_1 + \beta_1 c_1 = \alpha_2 + \beta_2 c_2 \in \mathbb{K}^*$ and $\min\{\beta_1, \beta_2\} = 1$. We are going to show that either $\nu \in M^{C_1,C_2}(1,1)$ or $\nu^\top x \geq 1$ is dominated. If $\beta_1 = \beta_2 = 1$, $\nu \in M^{C_1,C_2}(1,1)$ and we are done. We divide the rest of the proof into the following two cases: $\beta_1 > \beta_2$ and $\beta_1 < \beta_2$.

First suppose $\beta_1 > \beta_2$. Then $\beta_2 = 1$ and $\nu = \alpha_1 + \beta_1 c_1 = \alpha_2 + c_2$. Having $\alpha_2 = 0$ contradicts Condition 1 through Lemma 2; therefore, $\alpha_2 \neq 0$. Let ϵ' be such that $0 < \epsilon' \leq \frac{\beta_1 - 1}{\beta_1}$, and define $\alpha_1' := (1-\epsilon')\alpha_1 + \epsilon' c_2$, $\beta_1' := (1-\epsilon')\beta_1$, $\alpha_2' := (1-\epsilon')\alpha_2$ and $\mu := \nu - \epsilon'\alpha_2$. The inequality $\mu^\top x \geq 1$ is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ because $(\mu, 1, \alpha_1', \alpha_2', \beta_1', 1)$ satisfies (6). Furthermore, $\mu^\top x \geq 1$ dominates $\nu^\top x \geq 1$ since $\nu - \mu = \epsilon'\alpha_2 \in \mathbb{K}^* \setminus \{0\}$.

Now suppose $\beta_2 > \beta_1 = 1$. Observe that $(\nu, 1, \alpha_1, \alpha_2 + (\beta_2 - 1)c_2, 1, 1)$ is also a solution satisfying (6). Having $\alpha_1 = 0$ contradicts Condition 1 through Lemma 2; therefore, $\alpha_1 \neq 0$. If $\alpha_2 + (\beta_2 - 1)c_2 \in \text{int}\, \mathbb{K}^*$, we can find a valid inequality that dominates $\nu^\top x \geq 1$ by subtracting a positive multiple of α_1 from μ as in the proof of Proposition 1. Otherwise, $\alpha_2 + (\beta_2 - 1)c_2 \in \text{bd}\, \mathbb{K}^*$ and $\nu \in M^{C_1,C_2}(1,1)$ since $\nu = \alpha_1 + c_1 = (\alpha_2 + (\beta_2 - 1)c_2) + c_2$. □

When $c_{1,0} = c_{2,0} \in \{\pm 1\}$, a similar result holds for undominated valid linear inequalities that are tight on both C_1 and C_2.

Lemma 4 *Let C_1, C_2 satisfy the basic disjunctive setup with $c_{1,0} = c_{2,0} \in \{\pm 1\}$. Then, up to positive scaling, any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ that is tight on both C_1 and C_2 has the form $\mu^\top x \geq c_{2,0}$ where $\mu \in M^{C_1,C_2}(1,1)$.*

Proof Let $\mu^\top x \geq \mu_0$ be an undominated valid inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ that is tight on both C_1 and C_2. Using Proposition 2, we can assume that $\mu_0 = c_{1,0} = c_{2,0}$ and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies one of the two systems in (8). In particular, either $\beta_1 = 1$ and $\beta_2\mu_0 \geq \mu_0$, or $\beta_2 = 1$ and $\beta_1\mu_0 \geq \mu_0$. In any case, $\min\{\beta_1\mu_0, \beta_2\mu_0\} = \mu_0$. We are going to show $\beta_1 = \beta_2 = 1$.

Consider the following pair of minimization problems

$$\inf_x \{\mu^\top x : x \in C_1\} \quad \text{and} \quad \inf_x \{\mu^\top x : x \in C_2\},$$

and their duals

$$\sup_{\delta,\gamma} \{\delta\mu_0 : \mu = \gamma + \delta c_1, \gamma \in \mathbb{K}^*, \delta \geq 0\} \text{ and}$$
$$\sup_{\delta,\gamma} \{\delta\mu_0 : \mu = \gamma + \delta c_2, \gamma \in \mathbb{K}^*, \delta \geq 0\}.$$

The pairs (α_1, β_1) and (α_2, β_2) are feasible solutions to the first and second dual problems, respectively. Because $\mu^\top x \geq \mu_0$ is tight on both C_1 and C_2, the optimal values of both minimization problems are μ_0 and we must have $\beta_1\mu_0 \leq \mu_0$ and $\beta_2\mu_0 \leq \mu_0$ by duality. This implies $\beta_1\mu_0 = \beta_2\mu_0 = \mu_0$ and $\beta_1 = \beta_2 = 1$. □

Proof of Theorem 4 The validity of (18) follows from Theorem 3 by setting $\beta_1 = 1$. Lemmas 3 and 4 show that we can limit ourselves to valid linear inequalities $\mu^\top x \geq c_{2,0}$ where $\mu \in M^{C_1,C_2}(1,1)$ in Corollary 1. When this is the case, the implication in (12) in the proof of Theorem 3 is actually an equivalence. □

4.2 A Topological Condition: Closedness of the Convex Hull

Next, we identify an important case where the family of tight inequalities specified in Lemma 4 is rich enough to describe $\overline{\text{conv}}(C_1 \cup C_2)$ completely. The key ingredient is the closedness of $\text{conv}(C_1 \cup C_2)$.

Proposition 5 *Let C_1, C_2 satisfy the basic disjunctive setup. Suppose* $\text{conv}(C_1 \cup C_2)$ *is closed. Then undominated valid linear inequalities that are strongly tight on both C_1 and C_2 are sufficient to describe* $\text{conv}(C_1 \cup C_2)$*, together with the cone constraint $x \in \mathbb{K}$.*

Proof Suppose $\text{conv}(C_1 \cup C_2)$ is closed. When $\text{conv}(C_1 \cup C_2) = \mathbb{K}$, no new inequalities are needed for a description of $\text{conv}(C_1 \cup C_2)$ and the claim holds trivially. Therefore, assume $\text{conv}(C_1 \cup C_2) \subsetneq \mathbb{K}$. We prove that given $u \in \mathbb{K} \setminus \text{conv}(C_1 \cup C_2)$, there exists an undominated valid inequality that separates u from $\text{conv}(C_1 \cup C_2)$ and is strongly tight on both C_1 and C_2.

Let $v \in \text{int}(\text{conv}(C_1 \cup C_2)) \setminus (C_1 \cup C_2)$. Note that such a point exists since otherwise, we have $\text{int}\,\text{conv}(C_1 \cup C_2) \subseteq C_1 \cup C_2$ which implies $\text{conv}(C_1 \cup C_2) \subseteq C_1 \cup C_2$ through the closedness of $C_1 \cup C_2$. By Lemma 1, this is possible only if $C_1 \cup C_2 = \mathbb{K}$ which we have already ruled out. Let $0 < \lambda < 1$ be such that $w := (1-\lambda)u + \lambda v \in \text{bd}(\text{conv}(C_1 \cup C_2))$. Then $w \in \mathbb{K} \setminus (C_1 \cup C_2)$ by the convexity of $\mathbb{K} \setminus (C_1 \cup C_2) = \{x \in \mathbb{K} : c_1^\top x < c_{1,0}, c_2^\top x < c_{2,0}\}$. Because $w \in \text{conv}(C_1 \cup C_2)$, there exist $x_1 \in C_1$, $x_2 \in C_2$, and $0 < \kappa < 1$ such that $w = \kappa x_1 + (1-\kappa)x_2$. Furthermore, by Corollary 1, the fact that $w \in \text{bd}(\text{conv}(C_1 \cup C_2))$ implies that there exists an undominated valid inequality $\mu^\top x \geq \mu_0$ for $\text{conv}(C_1 \cup C_2)$ such that $\mu^\top w = \mu_0$. Because $\mu^\top w = \kappa\mu^\top x_1 + (1-\kappa)\mu^\top x_2 = \mu_0$, $\mu^\top x_1 \geq \mu_0$, and $\mu^\top x_2 \geq \mu_0$, it must be the case that $\mu^\top x_1 = \mu^\top x_2 = \mu_0$. Thus, the inequality $\mu^\top x \geq \mu_0$ is strongly tight on both C_1 and C_2. The only thing that remains is to show that $\mu^\top x \geq \mu_0$

separates u from $\operatorname{conv}(C_1 \cup C_2)$. To see this, observe that $u = \frac{1}{1-\lambda}(w - \lambda v)$ and that $\mu^\top v > \mu_0$ since $v \in \operatorname{int}\operatorname{conv}(C_1 \cup C_2)$. Hence, we conclude

$$\mu^\top u = \frac{1}{1-\lambda}(\mu^\top w - \lambda \mu^\top v) < \mu_0.$$

□

Proposition 5 demonstrates the close relationship between the closedness of $\operatorname{conv}(C_1 \cup C_2)$ and the sufficiency of valid linear inequalities that are tight on both C_1 and C_2. This motivates us to investigate the cases where $\operatorname{conv}(C_1 \cup C_2)$ is closed.

The set $\operatorname{conv}(C_1 \cup C_2)$ is always closed when $c_{1,0} = c_{2,0} = 0$ (see, e.g., Rockafellar [30, Corollary 9.1.3]) or when C_1 and C_2 are defined by a split disjunction (see Dadush et al. [20, Lemma 2.3]). In Proposition 6 below, we generalize the result of Dadush et al.: We give a sufficient condition for $\operatorname{conv}(C_1 \cup C_2)$ to be closed and show that this condition is almost necessary. In Corollary 2, we show that the sufficient condition of Proposition 6 can be rewritten in a more specialized form using conic duality when the base set is the regular cone $\mathbb{K}$. The proofs of these results are left to the appendix.

Proposition 6 *Let $S \subset \mathbb{R}^n$ be a closed, convex, pointed set, $S_1 := \{x \in S : c_1^\top x \geq c_{1,0}\}$, and $S_2 := \{x \in S : c_2^\top x \geq c_{2,0}\}$ for $c_1, c_2 \in \mathbb{R}^n$ and $c_{1,0}, c_{2,0} \in \mathbb{R}$. Suppose $S_1 \not\subseteq S_2$ and $S_1 \not\supseteq S_2$. If*

$$\begin{aligned}
&\{r \in \operatorname{rec} S :\ c_2^\top r = 0\} \subseteq \{r \in \operatorname{rec} S :\ c_1^\top r \geq 0\} \text{ and} \\
&\{r \in \operatorname{rec} S :\ c_1^\top r = 0\} \subseteq \{r \in \operatorname{rec} S :\ c_2^\top r \geq 0\},
\end{aligned} \tag{19}$$

then $\operatorname{conv}(S_1 \cup S_2)$ is closed. Conversely, if

(i) there exists $r^* \in \operatorname{rec} S$ such that $c_1^\top r^* < 0 = c_2^\top r^*$ and the problem $\inf_x\{c_2^\top x : x \in S_1\}$ is solvable, or
(ii) there exists $r^* \in \operatorname{rec} S$ such that $c_2^\top r^* < 0 = c_1^\top r^*$ and the problem $\inf_x\{c_1^\top x : x \in S_2\}$ is solvable,

then $\operatorname{conv}(S_1 \cup S_2)$ is not closed.

Corollary 2 *Let C_1, C_2 satisfy the basic disjunctive setup. If there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $c_1 - \beta_2 c_2 \in \mathbb{K}^*$ and $c_2 - \beta_1 c_1 \in \mathbb{K}^*$, then $\operatorname{conv}(C_1 \cup C_2)$ is closed.*

Theorem 4, Proposition 5, and Corollary 2 imply that (18) is sufficient to describe $\operatorname{conv}(C_1 \cup C_2)$ when there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $c_1 - \beta_2 c_2 \in \mathbb{L}^n$ and $c_2 - \beta_1 c_1 \in \mathbb{L}^n$ and $c_{1,0} = c_{2,0} \in \{\pm 1\}$. Nevertheless, it is easy to construct instances where these hypotheses are not satisfied. We explore these cases further in Section 5.

Consider the case of $c_{1,0} = c_{2,0} \in \{0, \pm 1\}$. Then by Lemma 2, $c_1 - c_2 \notin \mathbb{L}^n$. Suppose also that

(a) condition (i) or (ii) of Theorem 4 is satisfied, and
(b) $\{x \in \mathbb{L}^n :\ c_1^\top x > c_{1,0},\ c_2^\top x > c_{2,0}\} = \emptyset$.

We note that statement (a) holds, for instance, when the sets C_1 and C_2 are defined by a split disjunction which excludes the origin because in this case $c_{1,0} = c_{2,0} = 1$ and $\mathrm{conv}(C_1 \cup C_2)$ is closed by Corollary 2. Moreover, statement (b) simply means that the two sets C_1 and C_2 defined by the disjunction do not meet, except possibly at their boundaries. This also holds for split disjunctions. Then by Theorem 4, $\overline{\mathrm{conv}}(C_1 \cup C_2)$ is completely described by (18) together with the cone constraint $x \in \mathbb{L}^n$. Furthermore, by Proposition 4, (15) is satisfied by every point in $\mathbb{L}^n$ with $\beta_1 = 1$, and by statement (b), we have that (18) can be expressed in an equivalent conic quadratic form (16). Thus, we conclude

$$\overline{\mathrm{conv}}(C_1 \cup C_2) = \left\{ x \in \mathbb{L}^n : \mathcal{N}x - 2(c_1^\top x - c_{1,0}) \begin{pmatrix} \tilde{c}_1 - \tilde{c}_2 \\ -c_{1,n} + c_{2,n} \end{pmatrix} \in \mathbb{L}^n \right\}$$

where $\mathcal{N} = \|\tilde{c_1} - \tilde{c_2}\|_2^2 - (c_{1,n} - c_{2,n})^2$.

Proof of Theorem 2 By Lemma 1, $\overline{\mathrm{conv}}(C_1 \cup C_2) = \mathbb{L}^n$ unless $c_{1,0} = c_{2,0} = 1$ because $0 \in C_1 \cup C_2$ and $\mathrm{rec}(\mathrm{conv}(C_1 \cup C_2)) = \mathbb{L}^n$. Therefore, suppose $c_{1,0} = c_{2,0} = 1$. Because C_1 and C_2 are both nonempty by hypothesis and c_2 is a negative multiple of c_1, we have $c_1, c_2 \notin \pm\mathbb{L}^n$ and $c_1 - c_2 \notin \pm\mathbb{L}^n$. Moreover, C_1 and C_2 are also both strictly feasible in this case; so Condition 2 is satisfied as well. After verifying statements (a) and (b) as above, Theorem 2 follows from Theorem 4 and Propositions 3 and 4. □

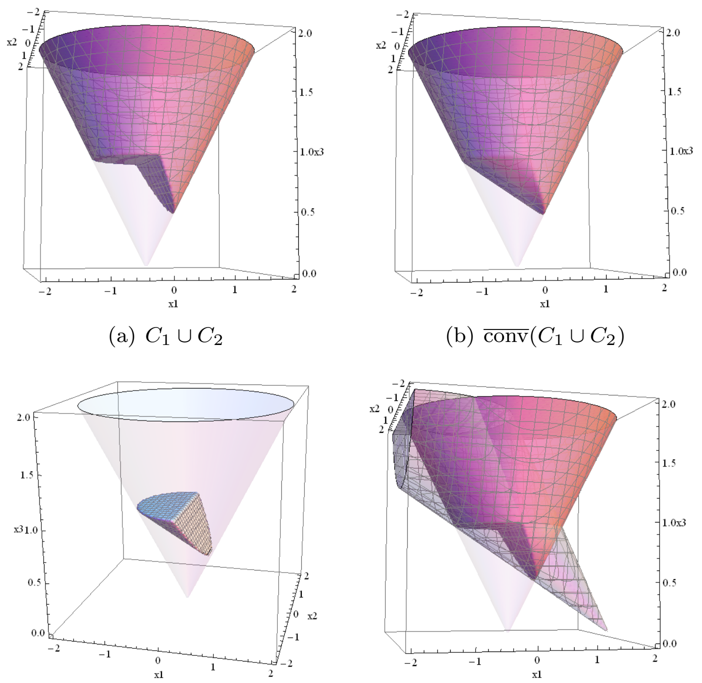
Theorem 4 and Proposition 3, together with Proposition 4, recover the results of [28] and [1] for split disjunctions on the cone $\mathbb{L}^n$ and extend them significantly to more general two-term disjunctions.

4.3 Example where a Single Inequality Suffices

Consider the cone $\mathbb{L}^3$ and the disjunction $x_3 \geq 1 \vee x_1 + x_3 \geq 1$ ($c_1 := e^3$, $c_2 := e^1 + e^3$, $c_{1,0} = c_{2,0} = 1$). Note that $c_1, c_2 \in \mathbb{L}^3$ in this example. Hence, we can use Theorem 4 to characterize the closed convex hull:

$$\overline{\mathrm{conv}}(C_1 \cup C_2) = \left\{ x \in \mathbb{L}^3 : 2 - (x_1 + 2x_3) \leq \sqrt{x_3^2 - x_2^2} \right\}.$$

Figures 2(a) and (b) depict the disjunctive set $C_1 \cup C_2$ and the associated closed convex hull, respectively. In order to give a better sense of the convexification operation, we plot the points added to $C_1 \cup C_2$ to generate the closed convex hull in Figure 2(c). We note that in this example the condition on the disjointness of the interiors of C_1 and C_2 that was required in Proposition 4 is violated. Nevertheless, the inequality that we provide is still intrinsically related to the conic quadratic inequality (16) of Proposition 3: The sets described by the two inequalities coincide in the region $\overline{\mathrm{conv}}(C_1 \cup C_2) \setminus (C_1 \cap C_2)$ as a consequence of Proposition 4. We display the corresponding cone for this example in Figure 2(d). Note that the resulting conic quadratic inequality is in fact not valid for some points in $\overline{\mathrm{conv}}(C_1 \cup C_2)$.



(a) $C_1 \cup C_2$

(b) $\overline{\text{conv}}(C_1 \cup C_2)$

(c) $\overline{\text{conv}}(C_1 \cup C_2) \setminus C_1 \cup C_2$

(d) Underlying cone generating the convex inequality

Fig. 2: Sets associated with the disjunction $x_3 \geq 1 \vee x_1 + x_3 \geq 1$ on $\mathbb{L}^3$.

5 When are Multiple Convex Inequalities Needed?

Lemma 4 allows us to simplify the characterization (7) of undominated valid linear inequalities which are tight on both C_1 and C_2 in the case $c_{1,0} = c_{2,0} \in \{\pm 1\}$. The next proposition shows the necessity the condition $c_{1,0} = c_{2,0}$ in the statement of this lemma. Unfortunately, when $c_{1,0} \neq c_{2,0}$, undominated valid linear inequalities are tight on exactly one of the two sets C_1 and C_2. The proof of this result is left to the appendix.

Proposition 7 *Let C_1, C_2 satisfy the basic disjunctive setup. If $c_{1,0} > c_{2,0}$, then any undominated valid linear inequality for $\overline{\text{conv}}(C_1 \cup C_2)$ is tight on C_2 but not on C_1.*

This result, when combined with Proposition 5, yields the following corollary.

Corollary 3 *Let C_1, C_2 satisfy the basic disjunctive setup with $c_{1,0} > c_{2,0}$. If $\overline{\text{conv}}(C_1 \cup C_2) \neq \mathbb{K}$, then $\text{conv}(C_1 \cup C_2)$ is not closed.*

Proof Suppose $\text{conv}(C_1 \cup C_2)$ is closed, and let $x \in \mathbb{K} \setminus \text{conv}(C_1 \cup C_2)$. By Proposition 5, there exists an undominated valid linear inequality which cuts off x from $\text{conv}(C_1 \cup C_2)$ and is tight on both C_1 and C_2. This contradicts Proposition 7. □

5.1 Describing the Closed Convex Hull

As Proposition 7 hints, there are cases where linear inequalities $\mu^\top x \geq \mu_0$ such that $\mu \in M^{C_1,C_2}(1,1)$ and $\mu_0 = \min\{c_{1,0}, c_{2,0}\}$ ($\beta_1 = \beta_2 = 1$ in (7)) may not be sufficient to define $\overline{\text{conv}}(C_1 \cup C_2)$. In this section, we study these cases when $\mathbb{K} = \mathbb{L}^n$ and outline a procedure to find closed-form expressions describing $\overline{\text{conv}}(C_1 \cup C_2)$.

Recall that, by Theorem 3 and Remark 3,

$$\overline{\text{conv}}(C_1 \cup C_2) = \{x \in \mathbb{L}^n : x \text{ satisfies (10) } \forall \beta \in B_1^{C_1,C_2} \text{ and (14) } \forall \beta \in B_2^{C_1,C_2}\}.$$

For ease of notation, let

$$\begin{aligned}
\mathcal{R} &:= \mathcal{R}(c_1, c_2, x) = (c_1^\top x)^2 + (\|\tilde{c}_1\|_2^2 - c_{1,n}^2)(x_n^2 - \|\tilde{x}\|_2^2),\\
\mathcal{P} &:= \mathcal{P}(c_1, c_2, x) = (c_1^\top x)(c_2^\top x) + (\tilde{c}_1^\top \tilde{c}_2 - c_{1,n}c_{2,n})(x_n^2 - \|\tilde{x}\|_2^2),\\
\mathcal{Q} &:= \mathcal{Q}(c_1, c_2, x) = (c_2^\top x)^2 + (\|\tilde{c}_2\|_2^2 - c_{2,n}^2)(x_n^2 - \|\tilde{x}\|_2^2),
\end{aligned}$$

and $f_1^{c_1,c_2,x}(\beta_1) := \beta_1 c_1^\top x + \sqrt{\mathcal{R}\beta_1^2 - 2\mathcal{P}\beta_1 + \mathcal{Q}}$. Then

$$\mathcal{R}\beta_1^2 - 2\mathcal{P}\beta_1 + \mathcal{Q} = \left((\beta_1 c_1 - c_2)^\top x\right)^2 + \mathcal{N}_1(\beta_1)\left(x_n^2 - \|\tilde{x}\|_2^2\right).$$

Similarly, define $f_2^{c_1,c_2,x}(\beta_2) := \beta_2 c_2^\top x + \sqrt{\mathcal{Q}\beta_2^2 - 2\mathcal{P}\beta_2 + \mathcal{R}}$ and note

$$\mathcal{Q}\beta_2^2 - 2\mathcal{P}\beta_2 + \mathcal{R} = \left((c_1 - \beta_2 c_2)^\top x\right)^2 + \mathcal{N}_2(\beta_2)\left(x_n^2 - \|\tilde{x}\|_2^2\right).$$

Through these definitions, we reach

$$\begin{aligned}
\overline{\text{conv}}(C_1 \cup C_2) &= \left\{x \in \mathbb{L}^n : \begin{array}{c} 2c_{2,0} - c_2^\top x \leq f_1^{c_1,c_2,x}(\beta_1) \;\; \forall \beta_1 \in B_1^{C_1,C_2}, \\ 2c_{1,0} - c_1^\top x \leq f_2^{c_1,c_2,x}(\beta_2) \;\; \forall \beta_2 \in B_2^{C_1,C_2} \end{array}\right\}\\
&= \left\{x \in \mathbb{L}^n : \begin{array}{c} 2c_{2,0} - c_2^\top x \leq \inf_{\beta_1 \in B_1^{C_1,C_2}} f_1^{c_1,c_2,x}(\beta_1), \\ 2c_{1,0} - c_1^\top x \leq \inf_{\beta_2 \in B_2^{C_1,C_2}} f_2^{c_1,c_2,x}(\beta_2) \end{array}\right\}. \qquad (20)
\end{aligned}$$

It follows that, for any given $x \in \mathbb{L}^n$, we can check whether $x \in \overline{\text{conv}}(C_1 \cup C_2)$ by calculating the optimal values of the problems on the right-hand side of the inequalities in (20). Furthermore, whenever the minimizer $\beta_1^* := \beta_1^*(c_1, c_2, x)$ of $\inf_{\beta_1 \in B_1^{C_1,C_2}} f_1^{c_1,c_2,x}(\beta_1)$ exists and can be expressed parametrically in terms of c_1, c_2, and x, one can replace the inequality $2c_{2,0} - c_2^\top x \leq \inf_{\beta_1 \in B_1^{C_1,C_2}} f_1^{c_1,c_2,x}(\beta_1)$ in (20) above with $2c_{2,0} - c_2^\top x \leq f_1^{c_1,c_2,x}(\beta_1^*)$ for all points $x \in \mathbb{L}^n$ such that $\beta_1^*(c_1, c_2, x) \in B_1^{C_1,C_2}$. Similarly, one can define $\beta_2^* := \beta_2^*(c_1, c_2, x)$ and replace $2c_{1,0} - c_1^\top x \leq \inf_{\beta_2 \in B_2^{C_1,C_2}} f_2^{c_1,c_2,x}(\beta_2)$ with $2c_{1,0} - c_1^\top x \leq f_2^{c_1,c_2,x}(\beta_2^*)$ for all points $x \in \mathbb{L}^n$ such that $\beta_2^*(c_1, c_2, x) \in B_2^{C_1,C_2}$. We illustrate this procedure on an example in the next section.

5.2 Example where Multiple Inequalities are Needed

Consider the cone $\mathbb{L}^3$ and the disjunction $-x_2 \geq 0 \vee -x_3 \geq -1$ ($c_1 := -e^2$, $c_{1,0} = 0$, $c_2 := -e^3$, $c_{2,0} = -1$). Since $c_{1,0} > c_{2,0}$, by Proposition 7, we know that any undominated valid linear inequality for $\overline{\mathrm{conv}}(C_1 \cup C_2)$ will be tight on C_2 but not on C_1. Therefore, we follow the approach outlined in Section 5.1. By noting that $c_2 - \beta_1 c_1 \in -\mathrm{int}\,\mathbb{L}^3$ for $0 \leq \beta_1 < 1$ and $c_2 - \beta_1 c_1 \notin \pm\mathrm{int}\,\mathbb{L}^3$ for $\beta_1 \geq 1$, we obtain $B_1^{C_1,C_2} = [1, \infty)$. Furthermore, for $\beta_1 = 1$, using (9) in Remark 1, we obtain $x_2 \leq 1$ as a valid linear inequality for all $x \in \overline{\mathrm{conv}}(C_1 \cup C_2)$. It is also clear that $B_2^{C_1,C_2} = \emptyset$.

Since we are interested in cutting off only points $x \in \mathbb{L}^3$ such that $x_2 \leq 1$ and $x \notin \overline{\mathrm{conv}}(C_1 \cup C_2)$, consider $x \in \mathbb{L}^3$ such that $0 < x_2 \leq 1$ and $x_3 > 1$. Note that $x \in \mathbb{L}^3$ and $x_2 > 0$ imply $x_3 - |x_1| > 0$. In this setup we have

$$\begin{aligned}\mathcal{R}(c_1, c_2, x) &= x_3^2 - x_1^2,\\ \mathcal{P}(c_1, c_2, x) &= x_2 x_3,\\ \mathcal{Q}(c_1, c_2, x) &= x_1^2 + x_2^2.\end{aligned}$$

The resulting $f_1^{c_1,c_2,x}$ is a convex function of β_1 and has a critical point at

$$\begin{aligned}\hat{\beta}_1 := \hat{\beta}_1(c_1, c_2, x) &= \frac{\mathcal{P}}{\mathcal{R}} - \frac{c_1^\top x}{\mathcal{R}}\sqrt{\frac{\mathcal{P}^2 - \mathcal{Q}\mathcal{R}}{(c_1^\top x)^2 - \mathcal{R}}}\\ &= \frac{x_2 x_3}{x_3^2 - x_1^2} + \frac{x_2}{x_3^2 - x_1^2}\sqrt{\frac{x_2^2 x_3^2 - (x_1^2 + x_2^2)(x_3^2 - x_1^2)}{(-1)(x_3^2 - x_1^2 - x_2^2)}}\\ &= \frac{x_2 x_3 + |x_1| x_2}{x_3^2 - x_1^2} = \frac{x_2}{x_3 - |x_1|}\end{aligned}$$

where in the last equation we used the fact that $x \in \mathbb{L}^3$ and thus $x_3 > 1$.

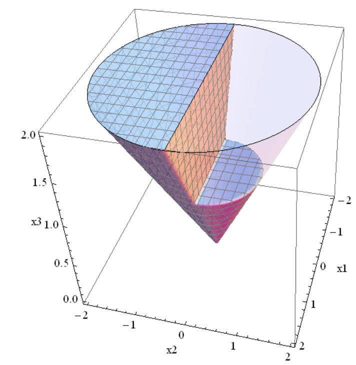
For any $x \in \mathbb{L}^3$ such that $x_2 \leq x_3 - |x_1|$, we have $\hat{\beta}_1 \leq 1$. By the convexity of $f_1^{c_1,c_2,x}$, the minimum of $f_1^{c_1,c_2,x}$ occurs at $\beta_1^* = \max\left\{\hat{\beta}_1, 1\right\} = 1$. Moreover, for any $x \in \mathbb{L}^3$ such that $x_2 \geq x_3 - |x_1|$, we have $\hat{\beta}_1 \geq 1$. For such points, $\beta_1^* = \hat{\beta}_1$ and $f_1^{c_1,c_2,x}(\beta_1^*) = |x_1| - \frac{x_2^2(x_3 + |x_1|)}{x_3^2 - x_1^2} = |x_1| - \frac{x_2^2}{x_3 - |x_1|}$. Therefore, for all $x \in \mathbb{L}^3$ such that $0 < x_2 \leq 1$, $x_3 > 1$, and $x_2 \geq x_3 - |x_1|$, we can enforce $2c_{2,0} - c_2^\top x \leq f_1^{c_1,c_2,x}(\hat{\beta}_1)$ which translates to $-2 + x_3 \leq |x_1| - \frac{x_2^2}{x_3 - |x_1|}$ in this example. Using $0 < x_2 \leq 1$ and $x_3 - |x_1| > 0$, we can rewrite this inequality as $1 + |x_1| - x_3 \leq \sqrt{1 - \max\{0, x_2\}^2}$. Putting this together with $x_2 \leq 1$, we arrive at

$$\begin{aligned}\overline{\mathrm{conv}}(C_1 \cup C_2) &= \left\{x \in \mathbb{L}^3 : -2 + x_3 \leq f_1^{c_1,c_2,x}(\beta_1) \quad \forall \beta_1 \in [1, \infty)\right\}\\ &= \left\{x \in \mathbb{L}^3 :\ x_2 \leq 1,\ 1 + |x_1| - x_3 \leq \sqrt{1 - \max\{0, x_2\}^2}\right\},\end{aligned}$$

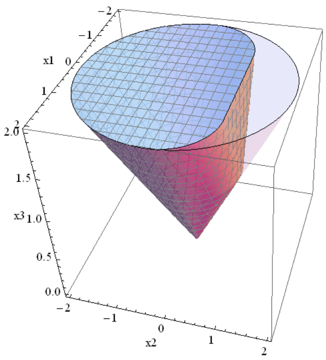
where both inequalities are convex on $\mathbb{R}^3$. In fact, both inequalities are conic quadratic representable in a lifted space as expected.

In Figures 3(a) and (b), we plot the disjunctive set $C_1 \cup C_2$ and the resulting $\overline{\mathrm{conv}}(C_1 \cup C_2)$, respectively. The closed convex hull is obtained by imposing various convex inequalities (10), each corresponding to a different value of $\beta_1 \in B_1^{C_1,C_2}$, on

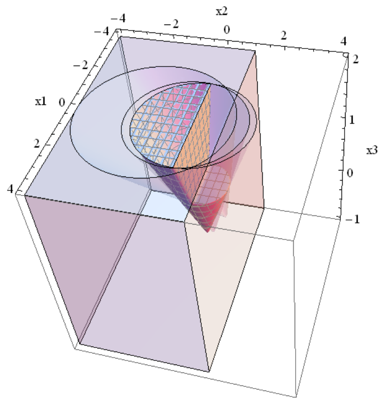
$\mathbb{L}^3$. In Figure 3(c) we show the conic quadratic counterparts (16) of the inequalities. Note that these inequalities are not necessarily valid for all points in $C_1 \cup C_2$ because Condition (17) is not satisfied, but they describe how the boundary of $\overline{\text{conv}}(C_1 \cup C_2)$ is formed outside $C_1 \cup C_2$. In Figure 3(d) we show the cross-section of $C_1 \cup C_2$ and the regions defined by the conic quadratic inequalities (16) at $x_3 = 4$.



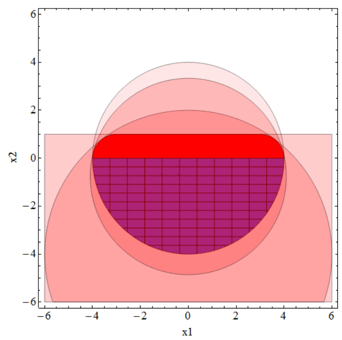
(a) $C_1 \cup C_2$



(b) $\overline{\text{conv}}(C_1 \cup C_2)$



(c) Underlying cones generating the convex inequalities



(d) Cross-section at $x_3 = 4$

Fig. 3: Sets associated with the disjunction $-x_2 \geq 0 \vee -x_3 \geq -1$ on $\mathbb{L}^3$.

6 Appendix

Proof of Lemma 1 To prove the first claim, suppose $S_1 \cup S_2 \subsetneq S$ and pick $x_0 \in S \setminus (S_1 \cup S_2)$. Also, pick $x_1 \in S_1 \setminus S_2$ and $x_2 \in S_2 \setminus S_1$. Let x' be the point on the line segment between x_0 and x_1 such that $c_1^\top x' = c_{1,0}$. Similarly, let x'' be the point between x_0 and x_2 such that $c_2^\top x'' = c_{2,0}$. Note that $x' \notin S_2$ and $x'' \notin S_1$ by the convexity of $S \setminus S_1$ and $S \setminus S_2$. Then a point that is a strict convex combination of x' and x'' is in $\text{conv}(S_1 \cup S_2)$ but not in $S_1 \cup S_2$.

Corollary 9.1.2 in [30] implies S_1^+ and S_2^+ are closed and $\text{rec}\, S_1^+ = \text{rec}\, S_2^+ = \text{rec}\, S_1 + \text{rec}\, S_2$ because S is pointed. The inclusions $S_1 \subseteq S_1^+$ and $S_2 \subseteq S_2^+$ imply that $\text{conv}(S_1 \cup S_2) \subseteq \text{conv}(S_1^+ \cup S_2^+)$. Furthermore, $\text{conv}(S_1^+ \cup S_2^+)$ is closed by Corollary 9.8.1 in [30] since S_1^+ and S_2^+ have the same recession cone. Hence, $\overline{\text{conv}}(S_1 \cup S_2) \subseteq \text{conv}(S_1^+ \cup S_2^+)$. We claim $\overline{\text{conv}}(S_1 \cup S_2) = \text{conv}(S_1^+ \cup S_2^+)$. Let $x^+ \in \text{conv}(S_1^+ \cup S_2^+)$. Then there exist $u_1 \in S_1$, $v_2 \in \text{rec}\, S_2$, $u_2 \in S_2$, and $v_1 \in \text{rec}\, S_1$ such that $x^+ \in \text{conv}\{u_1 + v_2, u_2 + v_1\}$. To prove the claim, it is enough to show that $u_1 + v_2, u_2 + v_1 \in \overline{\text{conv}}(S_1 \cup S_2)$. Consider the point $u_1 + v_2$ and the sequence

$$\left\{ \left(1 - \frac{1}{k}\right) u_1 + \frac{1}{k}\left(u_2 + k v_2\right) \right\}_{k \in \mathbb{N}}.$$

For any $k \in \mathbb{N}$, we have $u_1 \in S_1$ and $u_2 + k v_2 \in S_2$. Therefore, this sequence is in $\text{conv}(S_1 \cup S_2)$. Furthermore, it converges to $u_1 + v_2$ as $k \to \infty$ which implies $u_1 + v_2 \in \overline{\text{conv}}(S_1 \cup S_2)$. A similar argument shows $u_2 + v_1 \in \overline{\text{conv}}(S_1 \cup S_2)$ and proves the claim. □

Proof of Proposition 6 Let $S_1^+ := S_1 + \text{rec}\, S_2$ and $S_2^+ := S_2 + \text{rec}\, S_1$. We have $\text{conv}(S_1 \cup S_2) \subseteq \overline{\text{conv}}(S_1 \cup S_2) = \text{conv}(S_1^+ \cup S_2^+)$ by Lemma 1. We are going to show $\text{conv}(S_1^+ \cup S_2^+) \subseteq \text{conv}(S_1 \cup S_2)$ to prove that $\text{conv}(S_1 \cup S_2)$ is closed when (19) is satisfied. Let $x^+ \in S_1^+$. Then there exist $u_1 \in S_1$ and $v_2 \in \text{rec}(S_2)$ such that $x^+ = u_1 + v_2$. If $c_2^\top v_2 > 0$, then there exists $\epsilon \geq 1$ such that $x^+ + \epsilon v_2 \in S_2$ and we have $x^+ \in \text{conv}(S_1 \cup S_2)$. Otherwise, $c_2^\top v_2 = 0$, and by the hypothesis, $c_1^\top v_2 \geq 0$. This implies $x^+ \in S_1$, and thus $S_1^+ \subseteq \text{conv}(S_1 \cup S_2)$. Through a similar argument, one can show $S_2^+ \subseteq \text{conv}(S_1 \cup S_2)$. Hence, $S_1^+ \cup S_2^+ \subseteq \text{conv}(S_1 \cup S_2)$. Taking the convex hull of both sides yields $\text{conv}(S_1^+ \cup S_2^+) \subseteq \text{conv}(S_1 \cup S_2)$.

For the converse, suppose condition (i) holds, and let $x^* \in S_1$ be such that $c_2^\top x^* \leq c_2^\top x$ for all $x \in S_1$. Note that $c_2^\top x^* < c_{2,0}$ since otherwise, $S_1 \subseteq S_2$. Pick $\delta > 0$ such that $x' := x^* + \delta r^* \notin S_1$. Then $x' \notin S_2$ too because $c_2^\top x' = c_2^\top x^* < c_{2,0}$. For any $0 < \lambda < 1$, $x_1 \in S_1$, and $x_2 \in S_2$, we can write $c_2^\top(\lambda x_1 + (1 - \lambda) x_2) \geq \lambda c_2^\top x^* + (1 - \lambda) c_{2,0} > c_2^\top x'$. Hence, $x' \notin \text{conv}(S_1 \cup S_2)$. On the other hand, $x' \in S_1^+ \subseteq \text{conv}(S_1^+ \cup S_2^+) = \overline{\text{conv}}(S_1 \cup S_2)$ where the last equality follows from Lemma 1. □

Proof of Corollary 2 Suppose there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $c_1 - \beta_2 c_2 \in \mathbb{K}^*$ and $c_2 - \beta_1 c_1 \in \mathbb{K}^*$. Consider the following minimization problem

$$\inf_u \{c_1^\top u : \; c_2^\top u = 0, u \in \mathbb{K}\}$$

and its dual

$$\sup_\delta \{0 : \; c_1 - \delta c_2 \in \mathbb{K}^*\}.$$

Because β_2 is a feasible solution to the dual problem, we have $c_1^\top u \geq 0$ for all $u \in \mathbb{K}$ such that $c_2^\top u = 0$. Similarly, one can use the existence of β_1 to show that the second part of (19) holds too. Then by Proposition 6, $\text{conv}(C_1 \cup C_2)$ is closed. □

Proof of Proposition 7 Every undominated valid inequality has to be tight on either C_1 or C_2; otherwise, we can just increase the right-hand side to obtain a dominating valid inequality. By Proposition 2, undominated valid inequalities are of the form $\mu^\top x \geq \mu_0$ where $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies the first system in (8). In particular, we have $\beta_1 > 0$, $\beta_1 c_{1,0} \geq c_{2,0}$, and $\mu_0 = c_{2,0}$. Now consider the following minimization problem

$$\inf_u \{\mu^\top u : \ u \in C_1\}$$

and its dual

$$\sup_\delta \{\delta c_{1,0} : \ \mu - \delta c_1 \in \mathbb{K}^*, \ \delta \geq 0\}.$$

Note that β_1 is a feasible solution to the dual problem. The set C_1 is strictly feasible by Condition 2, so strong duality applies to this pair of conic optimization problems. The dual problem admits an optimal solution δ^* which satisfies $\delta^* \geq \beta_1 > 0$ because $c_{1,0} \geq 0$. Then

$$\text{sign}\{\delta^* c_{1,0}\} = \text{sign}\{c_{1,0}\} = c_{1,0} > c_{2,0} = \mu_0.$$

Hence, the inequality $\mu^\top x \geq \mu_0$ cannot be tight on C_1. □

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