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A DC (DIFFERENCE OF CONVEX FUNCTIONS) APPROACH OF THE MPECS

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Abstract. This article deals with a study of the MPEC problem based on a reformulation to a DC problem (Difference of Convex functions). This reformulation is obtained by a partial penalization of the constraints. In this article we prove that a classical optimality condition for a DC program, if a constraint qualification is satisfied for MPEC, it is a necessary and sufficient condition for a feasible point of an MPEC to be a strongly stationary point. Moreover we have proposed an algorithm to solve the MPEC problem based on the DC reformulation, and have studied the stationarity properties of the limit of the sequences generated by this algorithm.

1. Introduction.

One of the motivations of the MPEC is the link with the bi-level programming. Consider the following problem

$$\begin{array}{ll} \min & g(x,y) \\ \text{such that} & x \in w \\ & y \in S(x), \end{array}$$

with,

$$S(x) := \arg\min_{C(x)} \varphi(x,\cdot) \text{ and } C(x) = \{y \ : \ G_e(x,y) = 0 \ , \ G(x,y) \leq 0\}.$$

That is a bi-level program. Writing the KKT system associated to the problem $\min\limits_{C(x)} \varphi(x,\cdot)$, we obtain the following MPEC/MPCC:

$$\begin{array}{ll} \min & g(x,y) \\ \text{such that} & x \in w \\ & \nabla_y \mathcal{L}(x,y,\lambda,\mu) = 0 \\ & G_e(x,y) = 0 \\ & 0 \leq G(x,y) \perp \lambda \geq 0, \end{array}$$

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with $\mathcal{L}$ the lagrangian function associated to the problem $\min\limits_{C(x)} \varphi(x, \cdot)$. The bi-level program and the MPEC associated are equivalent for global solutions if a Slater's constraint qualification is satisfied for the second level problem. For local solutions the equivalence can be ensure under Slater's constraint qualification and constant rank constraint qualification [4, Dempe, Dutta, 2012].

The term MPEC stands for Mathematical Programming with Equilibrium Constraints. The MPECs are very often used in the economy theory. For example a competitive market with one leader, like a liberalized electricity market, can be written as a bilevel program, and therefore as an MPEC. The authors Hobbs and Pang [9] study an electricity market without thermal losses on the transmission lines, and obtain an MPEC with linear complementary constaints. Other references about the electricity market and its interactions with the MPECs are given [1, 3, 10, 11].

One of the methods to solve a MPEC problem is a SQP method, which was studied in [7]. This method permits a numerical resolution of a problem with linear complementary constraints with quadratic rate of convergence under good assumption. The authors [12] proposed a penalization method in the case where the complementary contraints are not linear, in order to replace the nonlinear complementary contraints by linear contraints and apply SQP method. The authors [5] consider an interior point method, and obtain a super linear rate of convergence. A method of relaxation of the complementary constraint was considered in [16], and the paper studies the stationarity properties of the limit when the relaxation parameter tends to zero. The authors [13] have worked about the relaxation and penalization of the complementary contraint of the MPEC. They have examined the properties of distance between the solutions of the MPEC and of the relaxed problem, moreover they have studied the boundness of the Lagrange multipliers under assumption

The topic of this paper is to introduce a DC approach for the MPEC, DC means "Difference of Convex functions". The DC methods were introduced in 1985 by Pham Dhin Tao. The first part of the paper gives the classical definitions about the MPEC, the second part gives a first order optimality condition for the DC program following the path of Tao and Pham [18, 20]. The third part gives a reformulation of the MPEC to a DC program, and we give a necesary and sufficient condition for a feasible point for MPEC to be a strongly stationary point in terms of optimality conditions for the equivalent DC program. The last part presents an algorithm based on the DC method, with a regularization of the penalization term in order to ensure the differentiability of the objective function.

2. **Definitions and preliminary results.** We consider an MPEC formulated as a non linear program with complementary constraints:

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \leq 0\ ,\ h(x) = 0 \\ & 0 \leq x_1 \perp x_2 \geq 0 \end{array} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$, $x_1 \in \mathbb{R}^m$ and $x_2 \in \mathbb{R}^m$ such that $x := (x_0, x_1, x_2)$.

The functions g_i are supposed to be convex, and h is supposed to be affine. Therefore, the set

$$\Omega := \{x \in \mathbb{R}^n \ : \ g(x) \leq 0 \ , \ h(x) = 0 \ , \ x_1 \geq 0 \ , \ x_2 \geq 0\} \tag{2}$$

is a convex set. The problem MPEC can be written as

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & x \in \Omega \\ & \langle x_1, x_2 \rangle = 0. \end{array}$$

We say that the constraint set Ω is **qualificated** at a point $\bar{x} \in \Omega$ if it is satisfied a qualification condition ensuring that the following inclusion holds:

$$N_\Omega(\bar{x}) \subset \left\{ {}^t\nabla g(\bar{x})\lambda^g +^t \nabla h(\bar{x})\lambda^h - \begin{pmatrix} 0 \\ \nu_1 \\ \nu_2 \end{pmatrix} \middle| \begin{array}{l} \lambda^h \in \mathbb{R}^q \ , \ 0 \geq g(\bar{x}) \perp \lambda^g \geq 0 \\ 0 \leq \nu_1 \perp \bar{x}_1 \geq 0 \ , \ 0 \leq \nu_2 \perp \bar{x}_2 \geq 0. \end{array} \right\} \tag{3}$$

For example it can be LICQ or MFCQ or the calmness of the perturbated set-valued map

$$M(y) = \{x \in \mathbb{R}^{p+q+2m} \mid G(x) + y \in D\},$$

with $G(x) := (g(x), h(x), x_1, x_2)$ and $D := (\mathbb{R}_-)^p \times \{0\}^q \times (\mathbb{R}_+)^{2m}$.

The feasible set of MPEC (1) will be denoted by C. Given $\bar{x} \in C$, we define the following index sets of active and inactive constraints:

$$\begin{array}{rcl} I_g(\bar{x}) & := & \{i \in \{1, \cdots, p\} \mid g_i(\bar{x}) = 0\} \\ I_g^c(\bar{x}) & := & \{i \in \{1, \cdots, p\} \mid g_i(\bar{x}) < 0\} \\ I_1(\bar{x}) & := & \{i \in \{1, \cdots, m\} \mid \bar{x}_{1,i} = 0\} \\ I_2(\bar{x}) & := & \{i \in \{1, \cdots, m\} \mid \bar{x}_{2,i} = 0\} \\ I_{12}(\bar{x}) & := & I_1(\bar{x}) \cap I_2(\bar{x}). \end{array} \tag{4}$$

For this class of problem, there exist many notions of stationary point. Two notions of stationary point, the **weakly stationary point**, and the **strongly stationary point**, are defined below.

Definition 2.1. A feasible point $\bar{x}$ of the MPEC is said to be **weakly stationary** if there exists a vector of MPEC multipliers $(\lambda_g, \lambda_h, \hat{\nu}_1, \hat{\nu}_2)$ such that:

$$\begin{array}{rcl} \nabla f(\bar{x}) + \nabla g(\bar{x})^T \bar{\lambda}^g + \nabla h(\bar{x})^T \bar{\lambda}^h - (0, \hat{\nu}_1, \hat{\nu}_2) & = & 0 \\ h(\bar{x}) = 0, \ g(\bar{x}) \leq 0, \ \bar{\lambda}^g \geq 0, \ \langle \bar{\lambda}^g, g(\bar{x}) \rangle & = & 0 \\ \forall i \notin I_1(\bar{x}) \ : \qquad \bar{x}_{1,i} \geq 0, \ \hat{\nu}_{1,i} \geq 0, \ \hat{\nu}_{1,i}\bar{x}_{1,i} & = & 0 \\ \forall i \notin I_2(\bar{x}) \ : \qquad \bar{x}_{2,i} \geq 0, \ \hat{\nu}_{2,i} \geq 0, \ \hat{\nu}_{2,i}\bar{x}_{2,i} & = & 0 \end{array}$$

In addition, the feasible vector $\bar{x}$ is called a **strongly stationary** point if $\hat{\nu}_{1,i} \geq 0$, $\hat{\nu}_{2,i} \geq 0$, for all $i \in I_{12}(\bar{x})$.

Associated with any given feasible vector $\bar{x}$ of MPEC (1), there is a nonlinear program called the *tightened* NLP (TNLP($\bar{x}$)):

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \leq 0\ ,\ h(x) = 0 \\ & x_{1,i} = 0,\ \forall i \in I_1(\bar{x}) \\ & x_{1,i} \geq 0,\ \forall i \notin I_1(\bar{x}) \\ & x_{2,i} = 0,\ \forall i \in I_2(\bar{x}) \\ & x_{2,i} \geq 0,\ \forall i \notin I_2(\bar{x}) \end{array} \tag{5}$$

Note that a feasible point of MPEC (1) $\bar{x}$ is weakly stationary if and only if there exists a vector MPEC multipliers $\lambda = (\lambda^g, \lambda^h, \hat{\nu}_1, \hat{\nu}_2)$ such that $(\bar{x}, \lambda)$ is a KKT stationary point of the TNLP (5).

We also define the *relaxed* NLP (RNLP($\bar{x}$)) as follows:

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \leq 0\ ,\ h(x) = 0 \\ & x_{1,i} = 0,\ \forall i \notin I_2(\bar{x}) \\ & x_{1,i} \geq 0,\ \forall i \in I_2(\bar{x}) \\ & x_{2,i} = 0,\ \forall i \notin I_1(\bar{x}) \\ & x_{2,i} \geq 0,\ \forall i \in I_1(\bar{x}) \end{array} \tag{6}$$

Definition 2.2. A vector $\bar{x}$ satisfies the MPEC-LICQ (resp. MPEC-MFCQ) at a feasible point $\bar{x}$ if the corresponding RNLP($\bar{x}$) (6) satisfies LICQ (MFCQ) at $\bar{x}$.

A very important link between the solutions of the MPEC (1) and the strongly stationary points:

Theorem 2.3 (Fukushima, Pang, 1999)**.** *If the MPEC-LICQ holds at a local minimizer $\bar{x}$ of the MPEC, then $\bar{x}$ is a strongly stationary point of MPEC.*

We now define the B(ouligand)-stationarity for MPECs.

Definition 2.4. A feasible point $\bar{x}$ is said to be a B-stationary point if $d = 0$ solves the following *linear program with equilibrium constraints*, with the vector $d \in \mathbb{R}^n$ being the decision variable:

$$\begin{array}{ll} \min & \langle \nabla f(\bar{x}), d \rangle \\ \text{subject to} & g(\bar{x}) + \nabla g(\bar{x}) d \leq 0\ ,\ h(\bar{x}) + \nabla h(\bar{x}) d = 0 \\ & 0 \leq \bar{x}_1 + d_1 \perp \bar{x}_2 + d_2 \geq 0 \end{array} \tag{7}$$

The B-stationarity for MPECs are related to the strongly stationary points:

Theorem 2.5. [15] *If a feasible point for MPEC is a strong stationary point for the MPEC, then it is a B-stationary point. Conversely, if $\bar{x}$ is a B-stationary point of the MPEC, and MPEC-LICQ holds at $\bar{x}$, then $\bar{x}$ is a strongly stationary point for MPEC.*

3. Generalities about the minimization of the difference of convex functions.

During this section we give some results for the minimization of the diffence of convex functions. Let X a Banach space, $u : X \to \mathbb{R} \cup \{+\infty\}$ and $v : X \to \mathbb{R} \cup \{+\infty\}$ two convex, proper and lsc functions. We consider the following problem

$$\min_{x \in X} u(x) - v(x) \tag{8}$$

which is a DC (Difference of Convex functions) problem. By convention $+\infty - (+\infty) = +\infty$. This section gives a necesary and sufficient condition for local solution of (8). For more information, see [18, 20].

In the following proposition, we recall a necesary condition for $\bar{x}$ to be a local solution of the problem (8), under assumption $\text{dom}(u) \subset \text{dom}(v)$, which ensures that for all $x \in \mathbb{R}^n$, we have $u(x) - v(x) > -\infty$.

Proposition 1. *We suppose that $dom(u) \subset dom(v)$. A necessary condition for $\bar{x}$ to be a solution of problem (8) is*

$$\partial v(\bar{x}) \subset \partial u(\bar{x}).$$

Moreover, if v is a polyhedral convex function, the above inclusion is a sufficient condition for $\bar{x}$ to be a solution of (8).

We recall that a function $g : X \to \mathbb{R} \cup \{+\infty\}$ is a polyhedral convex function if there exist an integer $p \in \mathbb{N}$, elements $a_1, \cdots, a_p$ of X^*, some reals $b_1, \cdots, b_p$, a polyhedral convex set $S \subset X$ such that

$$\forall x \in X \ , \ g(x) = \max_{i=1,\cdots,p} (\langle a_i, x \rangle + b_i) + \delta_S(x).$$

By Rockafellar [14], the conjugate of a polyhedral convex function is polyhedral convex and the sum of polyhedral convex functions is polyhedral convex too.

4. **Reformulation of the MPEC to DC program.** We come back to the MPEC problem. The main idea is to penalize the constraint of complementarity and to formulate the MPEC into a DC program. Since the optimality conditions for the DC programs provide necesary and sufficient conditions for an element x to be a local solution, we need to penalize and to ensure that the local solutions are the same for the initial and penalized problems.

For the rest of this section, we set

$$\Omega := \{x \in \mathbb{R}^n \ : \ h(x) = 0 \ , \ g(x) \leq 0 \ , \ x_1 \geq 0 \ , \ x_2 \geq 0\} \tag{9}$$

and

$$\Delta := \{x \in \mathbb{R}^n \ : \ \langle x_1, x_2 \rangle \leq 0\} \tag{10}$$

with $x := (x_0, x_1, x_2)$. Note that the feasible set of the problem (1) is $\Omega \cap \Delta$. We recall that Ω is a convex set because g_i are convex functions and h is an affine function.

4.1. **Partial penalization of MPEC.** We define the constant $\alpha \geq 0$ by:

$$\alpha = \inf \left\{ \frac{\text{dist}(x, \Delta)}{\text{dist}(x, \Omega \cap \Delta)} \ : \ x \in \Omega \setminus \Delta \right\}. \tag{11}$$

We can easily verify that the constant $\alpha \in [0, 1]$ and:

Proposition 2. *If $\alpha > 0$ then for all Lipschitz continuous functions $f : \Omega \to \mathbb{R}$, for all $\mu > \frac{L}{\alpha}$ (with L being the constant of Lipschitz of f on Ω), the optimization problem*

$$\begin{array}{ll} \min & f(x) + \mu dist(x, \Delta) \\ \textit{subject to} & g(x) \leq 0 \ , \ h(x) = 0 \\ & x_1 \geq 0 \ , \ x_2 \geq 0 \end{array}$$

admits the same local and global solutions than the initial MPEC (1).

The next remark will be useful for the proof of the Theorem 4.2.

Actually, it is possible to prove that $\alpha > 0$ if and only if for all Lipschitz-continuous function f, there exists a real $\mu > 0$ such that the above optimization problem admits the same solution than the initiatial MPEC (1).

Remark 1. We suppose that f is Lipschitz continuous on Ω (with L its constant of Lipschitz). Let $\mu > L/\alpha$, and let $\bar{x} \in \Omega$. Since $\|\nabla f(\bar{x})\| \leq L$, the problem

$$\begin{array}{ll} \min & \langle \nabla f(\bar{x}), x \rangle + \mu \text{dist}(x, \Delta) \\ \text{subject to} & g(x) \leq 0 \ , \ h(x) = 0 \\ & x_1 \geq 0 \ , \ x_2 \geq 0 \end{array}$$

admits the same solution than

$$\begin{array}{ll} \min & \langle \nabla f(\bar{x}), x \rangle \\ \text{subject to} & g(x) \leq 0 \ , \ h(x) = 0 \\ & x_1 \geq 0 \ , \ x_2 \geq 0 \\ & \langle x_1, x_2 \rangle = 0. \end{array}$$

We now give a sufficient and necesary condition for $\alpha > 0$, in the case where Ω is bounded and closed.

Proposition 3. *We suppose that Ω is bounded and closed. Then $\alpha = 0$ if and only if there exists an element $\bar{x} \in \Omega \cap \Delta$, a sequence $(x^k) \in (\Omega \setminus \Delta)^{\mathbb{N}}$ satisfying $\lim x^k = \bar{x}$ and*

$$\lim_{n \to +\infty} \frac{dist(x^k, \Delta)}{dist(x^k, \Omega \cap \Delta)} = 0.$$

Proof. By definition of α, if $\alpha = 0$ then there exists a sequence $(x^k) \in (\Omega \setminus \Delta)^{\mathbb{N}}$ satisfying

$$\lim_{n \to +\infty} \frac{\text{dist}(x^k, \Delta)}{\text{dist}(x^k, \Omega \cap \Delta)} = 0.$$

Since the set Ω is compact, there exists a subsequence of $(x^k)_k$ converging to an element $\bar{x} \in \Omega$.
Without losing generality, we can suppose that the whole sequence (x^k) converges. Since the sequence (x^k) is bounded, the sequence $(\text{dist}(x^k, \Delta \cap \Omega))_n$ is also bounded, thus $\lim_{n \to \infty} \text{dist}(x^k, \Delta) = 0$. Therefore, by continuity of $\text{dist}(\cdot, \Delta)$, we obtain $\text{dist}(\bar{x}, \Delta) = 0$, thus $\bar{x} \in \Delta$, and then $\bar{x} \in \Omega \cap \Delta$.

The converse is clear by the definition of α. □

The following proposition gives a case where $\alpha > 0$.

Remark 2. If Ω is bounded, and h and g are affine functions, it can be shown that $\alpha > 0$.

4.2. **Reformulation of the MPEC into DC program.** We suppose during all this section that $\alpha > 0$ (α is defined by (11)) and f is Lipschitz-continuous on Ω with constant of Lipschitz L. By the Proposition 2, for all $\mu > L/\alpha$, the MPEC (1) is equivalent to the problem

$$\begin{array}{ll} \min & f(x) + \mu \text{dist}(x, \Delta) \\ \text{subject to} & g(x) \leq 0 \ , \ h(x) = 0 \\ & x_1 \geq 0 \ , \ x_2 \geq 0 \end{array}$$

In this subsection we reformulate the MPEC into a DC program, and show that the optimality condition for DC programs is equivalent, under qualification condition, to the strongly stationarity. We set a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties:

- $(\mathcal{H}_1)$ For all $x \in \mathbb{R}^n$ with $x_1 \geq 0$, $x_2 \geq 0$, $\Phi(x) = 0 \Longleftrightarrow \langle x_1, x_2 \rangle = 0$
- $(\mathcal{H}_2)$ There exists a constant $c > 0$ such that for all $x \in \mathbb{R}^n$ verifying $x_1 \geq 0$ and $x_2 \geq 0$, one has $\Phi(x) \geq c\text{dist}(x, \Delta)$.
- $(\mathcal{H}_3)$ The function Φ is concave on $\mathbb{R}^n$.
- $(\mathcal{H}_4)$ The subdifferential $\partial(-\Phi)$ is uniformly bounded on the set $\{x \in \mathbb{R}^n \ : \ x_1 \geq 0 \text{ and } x_2 \geq 0\}$.

Example 4.0.1. The functions

$$\Phi_0(x) = \sum_{i=1}^{m} \min\{x_{1,i}, x_{2,i}\} \tag{12}$$

and

$$\Phi_1(x) = \sum_{i=1}^{m} \left(x_{1,i} + x_{2,i} - \sqrt{x_{1,i}^2 + x_{2,i}^2} \right) \tag{13}$$

satisfy all the above hypotheties.

A function Φ satisfying the above assumptions gives an error bound for the distance between x and Δ, and thus by the Proposition 2, for all $\mu > \frac{L}{c\alpha}$, the optimization problem

$$\begin{array}{ll} \min & f(x) + \mu \Phi(x) \\ \text{subject to} & g(x) \leq 0 \ , \ h(x) = 0 \\ & x_1 \geq 0 \ , \ x_2 \geq 0 \end{array} \tag{14}$$

admits the same local and global solutions as the MPEC problem (1), with c the constant of the hypothety $\mathcal{H}_2$.

We now suppose that f can be written as a difference of convex function on Ω. Writing $f = f_1 - f_2$, with f_1 and f_2 two convex functions. By the concavity of Φ, the MPEC problem (1) can be written as a DC program:

$$\min_{x \in \mathbb{R}^n} (f_1 + \delta_\Omega)(x) - (f_2 - \mu \Phi)(x). \tag{15}$$

If $\bar{x}$ is a local solution of (1) and MPEC-LICQ holds at this point, then $\bar{x}$ is a strongly stationary point by Theorem 2.3, but moreover $\bar{x}$ solves (19), therefore by the Proposition 1, the point $\bar{x}$ satisfies $\partial(f_2 - \mu\Phi)(\bar{x}) \subset \partial(f_1 + \delta_\Omega)(\bar{x})$. If the function f is differentiable, by the equality $f = f_1 - f_2$, it is equivalent to $\partial(-\mu\Phi)(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x})$. This observation asks us if the notion of strongly stationarity is linked to the inclusion $\partial(-\mu\Phi)(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x})$ with Φ satisfying the hypotheties $\mathcal{H}_1$-$\mathcal{H}_4$. The following proposition gives an answer to this question. Before this we need the tecnical lemma:

Lemma 4.1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfying the hypotheties $\mathcal{H}_1$-$\mathcal{H}_4$, and $\bar{x}$ a feasible point for MPEC (1). There exist two elements $x^* \in \partial(-\Phi)(\bar{x})$ and $\bar{x}^* \in \partial(-\Phi)(\bar{x})$ satisfying:*

$$\begin{cases} x^*_{1,i} = \bar{x}^*_{1,i} \leq -c \text{ and } x^*_{2,i} = \bar{x}^*_{2,i} = 0 \text{ if } i \in I_1(\bar{x}) \setminus I_2(\bar{x}) \\ x^*_{1,i} = \bar{x}^*_{1,i} = 0 \text{ and } x^*_{2,i} = \bar{x}^*_{2,i} \leq -c \text{ if } i \in I_2(\bar{x}) \setminus I_1(\bar{x}) \\ x^*_{1,i} \leq -c \text{ , } x^*_{2,i} = 0 \text{ , } \bar{x}^*_{1,i} = 0, \; \bar{x}^*_{2,i} \leq -c \text{ if } i \in I_1(\bar{x}) \cap I_2(\bar{x}), \end{cases}$$

with c the constant defined by $\mathcal{H}_2$.

Let the function $\tilde{\Phi} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{\Phi}(x) = \sum_{i \in I_1(\bar{x}) \setminus I_2(\bar{x})} -x^*_{1,i} \min(x_{1,i}, x_{2,i}) + \sum_{i \in I_2(\bar{x}) \setminus I_1(\bar{x})} -x^*_{2,i} \min(x_{1,i}, x_{2,i}) \quad (16)$$

$$+ \sum_{i \in I_1(\bar{x}) \cap I_2(\bar{x})} \min(-x^*_{1,i} x_{1,i}, -\bar{x}^*_{2,i} x_{2,i}). \quad (17)$$

Then $\tilde{\Phi}$ satisfies the hypotheties $\mathcal{H}_1$-$\mathcal{H}_4$, the constant c in $\mathcal{H}_2$ for $\tilde{\Phi}$ is the same than for Φ, and moreover one has

$$\partial(-\tilde{\Phi})(\bar{x}) \subset \partial(-\Phi)(\bar{x}).$$

Proof. Remember that for $x \in \mathbb{R}^n$, we write $x = (x_0, x_1, x_2)$, where x_0, x_1 and x_2 are three vectors. The notation $x_{j,i}$ denotes the $i-th$ component of the vector x_j, with $j \in \{0, 1, 2\}$. Actually, this notation will be useful only for $j \in \{1, 2\}$ during this proof. For simplicity, we denote by $\bar{x}_{-(j,i)}$ the vector $\bar{x}$ without the component (j, i), with $j \in \{1, 2\}$ and $i \in \{1, \cdots, m\}$. Therefore, the notation $(x_{j,i}, \bar{x}_{-(j,i)})$ denotes the vector in which component (j, i) is equal to $x_{j,i}$, and the other components are the same as $\bar{x}$.

Let $i \in I_1(\bar{x}) \setminus I_2(\bar{x})$. Since $\langle \bar{x}_1, \bar{x}_2 \rangle = 0$ and $\bar{x}_{2,i} > 0$, there exists a neighborhood of $\bar{x}_{2,i}$ such that for all $x_{2,i}$ in this neighborhood, one has $\langle x_1, x_2 \rangle = 0$ with $x = (x_{2,i}, \bar{x}_{-(2,i)})$, which implies that $\Phi(x) = 0$. Therefore, $\partial_{x_{2,i}}(-\Phi)(\bar{x}) = \{0\}$.

Now let $x^*_{1,i} \in \partial_{x_{1,i}}(-\Phi)(x)$. Since $\bar{x}_{2,i} > 0$, if $x_{1,i} > 0$, then $\langle x_1, x_2 \rangle > 0$ with $x = (x_{1,i}, \bar{x}_{-(1,i)})$, and $\text{dist}(x, \Delta) = x_{1,i}$. Therefore by the hypothety $\mathcal{H}_2$, one has $-\Phi(x) \leq -c x_{1,i}$, thus $x^*_{1,i} \leq -c$. Setting $\bar{x}^*_{2,i} = x^*_{2,i} = 0$ and $\bar{x}^*_{1,i} = x^*_{1,i}$, we obtain that $(x^*_{1,i}, x^*_{2,i}) = (\bar{x}^*_{1,i}, \bar{x}^*_{2,i}) \in \partial_{(x_{1,i}, x_{2,i})}(-\Phi)(\bar{x})$.

In the same way we treat the case where $i \in I_2(\bar{x}) \setminus I_1(\bar{x})$.

Now we consider $i \in I_1(\bar{x}) \cap I_2(\bar{x})$. Let the sequence $(x^n_{1,i})_n$ defined by $x^n_{1,i} = \frac{1}{n}$. By previous case there exists, for all $n \in \mathbb{N}$, $(y^n_{1,i}, 0) \in \partial_{x_{1,i}, x_{2,i}}(-\Phi)(x^n)$, with $x^n = (x^n_{1,i}, \bar{x}_{-(1,i)})$. By the hypothety $\mathcal{H}_4$, the sequence $(y^n_{1,i})_n$ is bounded, without losing generality we suppose that the sequence $(y^n_{1,i})_n$ converges to an element denoted by $x^*_{1,i}$. Since the subdifferential of a convex function has a closed graph, we deduce that $(x^*_{1,i}, 0) \in \partial_{x_{1,i}, x_{2,i}}(-\Phi)(\bar{x})$ since $x^n \to \bar{x}$. Moreover by previous case, one has $y^n_{1,i} \leq -c$, for all $n \in \mathbb{N}$, thus $x^*_{1,i} \leq -c$. Setting $x^*_{2,i} = 0$, we obtain $(x^*_{1,i}, x^*_{2,i}) \in \partial_{(x_{1,i}, x_{2,i})}(-\Phi)(\bar{x})$.

In the same way, there exists a real $\bar{x}^*_{2,i} \leq -c$ such that $(\bar{x}^*_{1,i}, \bar{x}^*_{2,i}) \in \partial_{(x_{1,i}, x_{2,i})}(-\Phi)(\bar{x})$ with $\bar{x}^*_{1,i} = 0$. Thus we complete by $x^*_0 := 0$ and $\bar{x}^*_0 := 0$ and obtain two elements

$x^* := (x_0^*, x_1^*, x_2^*)$ and $\bar{x}^* := (\bar{x}_0^*, \bar{x}_1^*, \bar{x}_2^*)$ of $\partial(-\Phi)(\bar{x})$ satisfying the above conditions.

We now consider the function $\tilde{\Phi}$ defined in (16). We can easily verify that for all $x \in \mathbb{R}^n$, if $x_1 \geq 0$ and $x_2 \geq 0$, then $\tilde{\Phi}(x) \geq c \sum_{i=1}^m \min(x_{1,i}, x_{2,i}) \geq 0$, and $\tilde{\Phi}(x) = 0$ if and only if $\langle x_1, x_2 \rangle = 0$. This ensures that the hypothety $\mathcal{H}_1$ is satisfied.

Since for all nonegative reals $\alpha_1, \cdots, \alpha_m$, one has $\sum_{i=1}^m \alpha_i \geq \left(\sum_{i=1}^m \alpha_i^2\right)^{\frac{1}{2}}$, we deduce that for all $x_1 \geq 0$, $x_2 \geq 0$, we have

$$\begin{aligned} \tilde{\Phi}(x) &\geq c \sum_{i=1}^m \min(x_{1,i}, x_{2,i}) \\ &\geq c \left(\sum_{i=1}^m \min(x_{1,i}, x_{2,i})^2 \right)^{\frac{1}{2}} \\ &= c\mathrm{dist}(x, \Delta). \end{aligned}$$

Therefore hypothety $\mathcal{H}_2$ is satisfied by $\tilde{\Phi}$ with the constant c. The hypothety $\mathcal{H}_3$ follows from the concavity of the function $(x, y) \to \min(x, y)$.

The hypothety $\mathcal{H}_4$ follows from the fact that $-\Phi$ is a polyhedral convex function, thus its subdifferential is uniformely bounded.

In order to prove the inclusion $\partial(-\tilde{\Phi})(\bar{x}) \subset \partial(-\Phi)(\bar{x})$, we compute the subdifferential $\partial(-\tilde{\Phi})(\bar{x})$, and obtain that $\tilde{x}^* \in \partial(-\Phi)(\bar{x})$ if and only if

$$\begin{cases} (\tilde{x}_{1,i}^*, \tilde{x}_{2,i}^*) = (x_{1,i}^*, 0) \text{ if } i \in I_1(\bar{x}) \setminus I_2(\bar{x}) \\ (\tilde{x}_{1,i}^*, \tilde{x}_{2,i}^*) = (0, x_{2,i}^*) \text{ if } i \in I_2(\bar{x}) \setminus I_1(\bar{x}) \\ (\tilde{x}_{1,i}^*, \tilde{x}_{2,i}^*) \in \mathrm{conv}\{(x_{1,i}^*, 0), (0, \bar{x}_{2,i}^*)\} \text{ if } i \in I_1(\bar{x}) \cap I_2(\bar{x}), \end{cases}$$

Finally the set $\partial(-\tilde{\Phi})(\bar{x})$ is the convex hull of some elements of $\partial(-\Phi)(\bar{x})$, the set $\partial(-\Phi)(\bar{x})$ is convex, then we have $\partial(-\tilde{\Phi})(\bar{x}) \subset \partial(-\Phi)(\bar{x})$.

□

The above lemma permits us to prove the theorem below.

Theorem 4.2. *We suppose f differentiable on Ω. Let a function Φ satisfying the hypotheties $\mathcal{H}_1$-$\mathcal{H}_4$, and c a constant satisfying the assumption $\mathcal{H}_2$. Let $\mu > \frac{L}{c\alpha}$ large enough such that the problem*

$$\min_{x \in \Omega} f(x) + \mu\Phi(x)$$

admits the same local and global solutions than (1). Let $\bar{x}$ a feasible point for MPEC (1). If $\bar{x}$ is a strongly stationary point for MPEC, then the following inclusion holds:

$$\partial(-\mu\Phi)(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x}).$$

If moreover MPEC-LICQ holds at $\bar{x}$, then the converse holds true.

Proof. We first suppose that $\bar{x}$ is a strongly stationary point. By Theorem 2.5, 0 is solution of the problem

$$\begin{array}{ll} \min & \langle \nabla f(\bar{x}), d \rangle \\ \text{subject to} & g(\bar{x}) + \nabla g(\bar{x})d \leq 0 \;,\; h(\bar{x}) + \nabla h(\bar{x})d = 0 \\ & 0 \leq \bar{x}_1 + d_1 \perp \bar{x}_2 + d_2 \geq 0, \end{array}$$

By convexity of the functions g_i, one has $\{d \in \mathbb{R}^n : g(\bar{x}+d) \leq 0\} \subset \{d \in \mathbb{R}^n : g(\bar{x}) + \nabla g(\bar{x})d \leq 0\}$. Moreover, since h is an affine function and $h(\bar{x}) = 0$, one has $h(\bar{x}+d) = \nabla h(\bar{x})d$. Therefore $d = 0$ solves the problem

$$\begin{array}{ll} \min & \langle \nabla f(\bar{x}), d\rangle \\ \text{subject to} & g(\bar{x}+d) \leq 0\ ,\ h(\bar{x}+d) = 0 \\ & 0 \leq \bar{x}_1 + d_1 \perp \bar{x}_2 + d_2 \geq 0. \end{array}$$

By the Remark 1, $d = 0$ is solution of the following problem

$$\min_{x\in\mathbb{R}^n} \langle \nabla f(\bar{x}+d), d\rangle + \delta_\Omega(\bar{x}+d) - (-\mu\Phi)(\bar{x}+d).$$

Since the set Ω is convex, the function $\langle \nabla f(\bar{x}), \cdot\rangle + \delta_\Omega(\bar{x}+\cdot)$ is a convex function. Moreover, Φ is concave by assumption ($\mathcal{H}_3$), thus the above program is a DC program, which prove that $\partial(-\mu\Phi)(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x})$ by Proposition 1.

Conversely, we suppose that $\partial(-\mu\Phi)(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x})$. Let $\tilde{\Phi}$ defined like in the Lemma 4.1. By the inclusion $\partial(-\tilde{\Phi})(\bar{x}) \subset \partial(-\Phi)(\bar{x})$ we have

$$\partial(-\mu\tilde{\Phi})(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x}).$$

Since $-\mu\tilde{\Phi}$ is a convex polyhedral function, by Proposition 1 the point $\bar{x}$ is a local solution of the DC program

$$\min_{x\in\mathbb{R}^n} \langle \nabla f(\bar{x}), x\rangle + \delta_\Omega(x) - (-\mu\tilde{\Phi})(x).$$

Since $\tilde{\Phi}$ satisfies the hypotheties $\mathcal{H}_1$-$\mathcal{H}_4$, and the hypothety $\mathcal{H}_2$ with the same constant c than for the function Φ, $\bar{x}$ is also a local solution of the MPEC

$$\begin{array}{ll} \min & \langle \nabla f(\bar{x}), x\rangle \\ \text{subject to} & g(x) \leq 0\ ,\ h(x) = 0 \\ & 0 \leq x_1 \perp x_2 \geq 0, \end{array}$$

which proves by Theorem 2.3 that $\bar{x}$ is a strongly stationary point for MPEC since MPEC-LICQ holds at $\bar{x}$. □

We can observe that the inclusion $\partial(-\mu\Phi)(\bar{x}) \subset \nabla f(\bar{x}) + N_\Omega(\bar{x})$ uses only convex analysis tools because $-\mu\Phi$ is a convex function and Ω is a convex set, though the constraint sets of the MPECs are not convex in general.

Another notion of stationary point for the DC program $\min_{x\in X} u(x) - v(x)$, with u and v convex function, is that $\partial u(\bar{x}) \cap \partial v(\bar{x}) \neq \emptyset$ (see Definition 5.1). This notion, applied to the DC reformulation of MPEC, is

$$\partial(-\mu\Phi)(\bar{x}) \cap (\nabla f(\bar{x}) + N_\Omega(\bar{x})) \neq \emptyset.$$

The following proposition shows shows that it is linked to the weakly stationarity for $\bar{x}$. Before we need a lemma:

Lemma 4.3. *Let x a feasible point for MPEC, and $x^* \in \partial(-\Phi)(x)$. Then for all $i \notin I_1(x)$, one has $x^*_{1,i} = 0$ and for all $i \notin I_2(x)$, one has $x^*_{2,i} = 0$.*

Proof. Let $i \notin I_1(x)$ and $x^* \in \partial(-\Phi)(x)$. For all $x'_{1,i}$ in a neighborhood of $x_{1,i}$, for all $x' := (x_0, (x_{1,1}, \cdots, x_{1,i-1}, x'_{1,i}, x_{1,i+1}, \cdots, x_{1,m}), x_2)$, one has $x'_1 \geq 0$, $x'_2 \geq 0$ and $\langle x'_1, x'_2\rangle = 0$ since $x_{2,i} = 0$ and $x_{1,i} > 0$. Therefore, one has $\Phi(x') = 0$, for all $x'_{1,i}$ in a neighborhood of $x_{1,i}$, which ensure that $x^*_{1,i} = 0$. By the same way, if $i \notin I_2(x)$, one has $x^*_{2,i} = 0$. □

Proposition 4. *We suppose f differentiable on Ω. Let a function Φ satisfying the hypotheties $\mathcal{H}_1$-$\mathcal{H}_4$, and $\mu > \frac{L}{c\alpha}$ such that the problem*

$$\min_{x\in\Omega} f(x) + \mu\Phi(x)$$

admits the same local and global solutions than (1). Let $\bar{x}$ a feasible point for MPEC (1). If $\bar{x}$ satisfies the following property

$$\partial(-\mu\Phi)(\bar{x}) \cap (\nabla f(\bar{x}) + N_\Omega(\bar{x})) \neq \emptyset,$$

and the constraint set Ω is qualificated at $\bar{x}$, then $\bar{x}$ is a weakly stationary point for MPEC.

Proof. Since $\partial(-\mu\Phi)(\bar{x}) \cap (\nabla f(\bar{x}) + N_\Omega(\bar{x})) \neq \emptyset$, there exists an element $-\bar{x}^* \in \partial(-\Phi)(\bar{x})$ such that $-\mu\bar{x}^* \in \nabla f(\bar{x}) + N_\Omega(\bar{x})$. Therefore one has

$$\nabla f(\bar{x}) + \mu\bar{x}^* \in -N_\Omega(\bar{x}).$$

Therefore, since the constraint set Ω is qualificated at $\bar{x}$, by (3), there exist Lagrange multipliers λ^g, λ^h, ν_1, ν_2 such that

$$\nabla f(\bar{x}) +^t \nabla g(\bar{x})\lambda^g +^t \nabla h(\bar{x})\lambda^h - \begin{pmatrix} 0 \\ \nu_1 - \mu\bar{x}_1^* \\ \nu_2 - \mu\bar{x}_2^* \end{pmatrix} = 0$$

and

$$0 \geq g(\bar{x}) \perp \lambda^g \geq 0\ ,\ 0 \leq \nu_1 \perp \bar{x}_1 \geq 0\ ,\ 0 \leq \nu_2 \perp \bar{x}_2 \geq 0.$$

Let the vector $\lambda = (\lambda^g, \lambda^h, \hat{\nu}_1, \hat{\nu}_2)$ defined by $(\hat{\nu}_{1,i}, \hat{\nu_{2,i}}) = (\nu_1 - \mu\bar{x}^*_{1,i}, \nu_2 - \mu\bar{x}^*_{2,i})$. Let $i \notin I_1(\bar{x})$, since $\bar{x} \in \Delta$, we have $\bar{x}_{2,i} = 0$, moreover since $\bar{x}_{1,i} > 0$, one has $\bar{x}^*_{1,i} = 0$ by Lemma (4.3). Moreover, $\nu_{1,i} = 0$ by complementarity, therefore $\hat{\nu}_{1,i} = 0$. By the same way, if $x_{2,i} > 0$, then $\hat{\nu}_{2,i} = 0$. That implies that $(\bar{x}, \lambda)$ is a KKT stationary point of the TNLP (5), thus $\bar{x}$ is a weakly stationary point for MPEC. □

5. **Resolution of MPEC problem with the reformulation into DC program.** - During this section we introduce the DC method, following the path of Pham Dinh Tao and Le Thi [17, 18, 19, 20], and after we apply it to the MPEC.

5.1. **The DC Method.** The DC algorithm is based on the following observation. Let u and v lsc, convex and proper functions. Then we have the following equalities:

$$\begin{aligned}
\inf_{x\in X}\{u(x) - v(x)\} &= \inf_{x\in X}\{u(x) - v^{**}(x)\} \text{ since } v \text{ is a convex function} \\
&= \inf_{x\in X}\{u(x) - \sup_{y\in X^*}\{\langle y, x\rangle - v^*(y)\}\} \\
&= \inf_{x\in X}\inf_{y\in X^*}\{u(x) - [\langle y, x\rangle - v^*(y)]\} \\
&= \inf_{y\in X^*}\{v^*(y) - u^*(y)\}
\end{aligned}$$

with

$$g^*(y) = \sup_{x\in X}(\langle y, x\rangle - g(x))$$

the adjoint function of g.

The perfect symmetry between the primal program and the dual program is referred to as the d.c. duality.

Based on the above d.c. duality, Pham Dinh and Le Thi [18, 20] proposed the DC Algorithm, abbreviated as DCA, which consists of starting from an initial point $x_0 \in \mathrm{dom}(u)$, and constructing two sequences $(x^k)_k$ and $(y^k)_k$ satisfying $y^k \in \partial v(x^k)$ and $x^{k+1} \in \partial u^*(y^k)$.

It has been proved in [20, Lemma 3.6] that the sequences $(x^k)_k$ and $(y^k)_k$ are well defined if and only if $\mathrm{dom}\partial u \subset \mathrm{dom}\partial v$ and $\mathrm{dom}\partial v^* \subset \mathrm{dom}\partial u^*$.

We can observe that $\partial v(x^k) = \arg\min_{X^*} v^* - \langle \cdot, x^k\rangle$ and $\partial u^*(y^k) = \arg\min_X u - \langle y^k, \cdot\rangle$. Therefore the iterates y^k and x^{k+1} are obtained solving the two following optimization problems:

$$y^k \text{ solves } \min_{y\in X^*} v^*(y) - \langle y, x^k\rangle \qquad (D_n),$$

$$x^{k+1} \text{ solves } \min_{x\in X} u(x) - \langle y^k, x\rangle \qquad (P_n).$$

Suppose v differentiable. Then the problem (P_n) is reduced to

$$\min_{x\in X} u(x) - \langle \nabla v(x^k), x - x^k\rangle,$$

which is a linearization of the concave part of the objective function.

We need to define a notion of stationary points adapted to the DC program.

Definition 5.1. We say that the point x is a stationary point for the DC program

$$\min(u(x) - v(x))$$

if one has $\partial u(x) \cap \partial v(x) \neq \emptyset$.

If u and v are differentiable, it corresponds to $\nabla(u - v)(x) = 0$, which is the classical first order optimality condition.

The following theorem give us very important informations about the behaviour of the sequences (x^k) and (y^k) generated by DCA.

Theorem 5.2. *Assume u, $v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, lsc and convex, and suppose that $domu \subset domv$ and $domv^* \subset domu^*$. Then:*

1. *The sequences (x^k) and (y^k) are well-defined.*
2. *The sequence $(u(x^k) - v(x^k))_k$ is decreasing.*
3. *Every limit point x^∞ of (x^k) is a critical point of $u-v$, i.e. $\partial u(x^\infty) \cap \partial v(x^\infty) \neq \emptyset$.*

The DCA Algorithm does not ensure that a limit point x^∞ of the sequence (x^k) satisfies $\partial v(x^\infty) \subset \partial u(x^\infty)$. There exists a complete form of DCA which ensures that all limit point x^∞ satisfies the inclusion $\partial v(x^\infty) \subset \partial u(x^\infty)$. In the complete form of DCA, we impose the following choice of y^k and x^{k+1}:

$$x^{k+1} \in \arg\min\{u(x) - v(x) \ : \ x \in \partial u^*(y^k)\}$$

and

$$y^k \in \arg\min\{v^*(y) - u^*(y) \ : \ y \in \partial v(x^k)\}.$$

The fact that all limit of a sequence generated by the complete form of DCA satisfies $\partial v(x^\infty) \subset \partial u(x^\infty)$ follows from the following theorem.

Theorem 5.3. [17, 19]*Let $\bar{x} \in X$. One has $\partial v(\bar{x}) \subset \partial u(\bar{x})$ if and only if there there exists an element $\bar{y} \in X^*$ such that $\bar{x} \in \partial v^*(\bar{y})$ with*

$$\bar{y} \in \arg\min \{v^*(y) - u^*(y) \ : \ y \in X^*\}.$$

5.2. **Application for solving MPEC.** We choose the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ defined by $\Phi(x) := \sum_{i=1}^m \left(x_{1,i} + x_{2,i} - \sqrt{x_{1,i}^2 + x_{2,i}^2}\right)$. The objective function f is written as

$$f = f_1 - f_2, \tag{18}$$

with f_1 and f_2 two convex functions of Ω. That is, under good assumptions (see section 4), the MPEC is equivalent to the DC program

$$\min_{x \in \mathbb{R}^n} (f_1 + \delta_\Omega)(x) - (f_2 - \mu\Phi)(x). \tag{19}$$

The DC algorithm, starting from an initial point $x_0 \in \mathrm{dom} u$, constructs two sequences $(x^k)_k$ and $(y^k)_k$ by

$$y^k \in \partial(f_2 - \mu\Phi)(x^k),$$

and

$$x^{k+1} \in \partial(f_1 + \delta_C)^*(y^k).$$

Remark 3. By the Theorem 5.2 all limit x^∞ of the sequence (x^k) satisfies $\partial(f_1 + \delta_\Omega)(x^\infty) \cap \partial(f_2 - \mu\Phi)(x^\infty) \neq \emptyset$, which is equivalent, under assumption of differentiability of f_1 and f_2, to

$$\partial(-\mu\Phi)(x^\infty) \cap (\nabla f(x^\infty) + N_\Omega(x^\infty)) \neq \emptyset.$$

Therefore, by the Proposition 4, all limit x^∞ of (x^k) is a weakly stationary point for MPEC under Constraints Qualification at x^∞.

We can see that the function Φ is differentiable at a point x if for all $i \in \{1, \cdots, m\}$, $x_{1,i} \neq 0$ or $x_{2,i} \neq 0$. If x satisfies $x_1 \geq 0$ and $x_2 \geq 0$, thus Φ is differentiable at x if $x_1 + x_2 > 0$ (which ensures that $x_{1,i} > 0$ or $x_{2,i} > 0$, for all $i \in \{1, \cdots, m\}$).

Therefore, if $x_1^k + x_2^k > 0$, then y^k is uniquely determined by

$$y^k = \nabla f_2(x^k) - \mu \nabla\Phi(x^k).$$

We can give a first result of convergence of DCA for MPEC.

Proposition 5. *We construct the sequence (x^k) by the DCA with*

$$\Phi(x) = \sum_{i=1}^m \left(x_{1,i} + x_{2,i} - \sqrt{x_{1,i}^2 + x_{2,i}^2}\right)$$

and suppose that f_1 defined in (18) is strongly convex on Ω. Suppose that for all k, $x_1^k + x_2^k > 0$. Let a limit x^∞ of x^k. If $\langle x_1^\infty, x_2^\infty \rangle = 0$ and MPEC-LICQ holds, then x^∞ is a strongly stationary point for MPEC.

Proof. We show that the sequence generated by DCA is a sequence generated by the complete form of DCA. Actually, all sequence generated by the complete form of DCA is also generated by DCA. Therefore it is sufficient to show that given an starting point x_0, at each step, the DCA generates unique iterates y^k and x^{k+1}.

Since the function $f_2 - \mu\Phi$ is differentiable, y^k is uniquely determined by $y^k = \nabla f_2(x^k) - \mu\nabla\Phi(x^k)$. Since f_1 is strongly convex, the set-valued map $\nabla f_1 + N_\Omega$ is maximal monotone, thus $(\nabla f_1 + N_\Omega)^{-1}(y^k)$ is a singleton. Since $\partial(f_1 + 1_\Omega)^*(y^k) = (\nabla f_1 + N_\Omega)^{-1}(y^k)$ and $x^{k+1} \in \partial(f_1 + 1_\Omega)^*(y^k)$, we deduce that x^{k+1} is uniquely determined by $x^{k+1} = (\nabla f_1 + N_\Omega)^{-1}(y^k)$.

Therefore the sequences (x^k) and (y^k) generated by DCA are the same that the sequences generated by the complete form of DCA, thus we have, for all limit x^∞ of (x^k),

$$\partial(-\mu\Phi(x^\infty)) \subset \nabla f(x^\infty) + N_\Omega(x^\infty).$$

Since MPEC-LICQ holds, by Theorem 4.2, x^∞ is a strongly stationary point of MPEC. □

5.3. **A regularized scheme of DCA.** We now regularize the function Φ in order to ensure its differentiability, by the following way:

$$\Phi_\rho(x) = \sum_{i=1}^{m}(x_{1,i} + x_{2,i} + \rho - \sqrt{x_{1,i}^2 + x_{2,i}^2 + \rho^2}). \tag{20}$$

The DC Algorithm regularized scheme consists in linearizing the concave part of the objective function: given the vector x^k, we choose a real $\rho_k > 0$ and we compute the next iterate x^{k+1} solving

$$\begin{array}{ll} \min & f_1(x) - \langle \nabla f_2(x^k) - \mu\nabla_x\Phi_{\rho_k}(x^k), x\rangle \\ \text{subject to} & g(x) \leq 0 \\ & h(x) = 0 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{array} \tag{21}$$

We give some properties about the function Φ_ρ defined above (20), first we introduce the following function $\theta : \mathbb{R}^3 \to \mathbb{R}$.

Proposition 6. *Let $\theta : \mathbb{R}^3 \to \mathbb{R}$ defined by $\theta(a, b, \rho) = a + b + \rho - \sqrt{a^2 + b^2 + \rho^2}$. The function θ satisfies the following properties:*

1. *For all $a, b \in \mathbb{R}_+$, $\theta(a, b, 0) \geq \frac{2}{2+\sqrt{2}}\min\{a, b\}$.*
2. *The function $\theta(\cdot,\cdot,0)$ is differentiable for all $(a, b) \in (\mathbb{R}_+)^2 \setminus \{(0, 0)\}$, and $\partial_{(a,b)}(-\theta(\cdot,\cdot,0))(0, 0) = B((-1,-1), 1)$.*
3. *The function θ is Lipschitz-continuous on $\mathbb{R}^3$.*

Proof. 1. Let $a, b \in \mathbb{R}$ such that $0 < a \leq b$. We have:

$$\begin{aligned} \theta(a, b, 0) &= a + b - \sqrt{a^2 + b^2} \\ &= \frac{2ab}{a + b + \sqrt{a^2 + b^2}} \\ &\geq \frac{2ab}{2b + \sqrt{2b^2}} \\ &= \frac{2}{2 + \sqrt{2}}a \\ &= \frac{2}{2 + \sqrt{2}}\min\{a, b\}. \end{aligned}$$

If $0 < b \leq a$ we obtain the result by the same calculus. If $a = 0$ and $b \geq 0$ or $b = 0$ and $a \geq 0$, then $\theta(a, b, 0) = 0 = \min\{a, b\}$, that proves the first result of the Proposition.

2. It is clear that $\theta(\cdot,\cdot,0)$ is differentiable over $(\mathbb{R}_+)^2 \setminus \{(0,0)\}$. The equality $\partial_{(a,b)}(-\theta(0,0,0)) = B((-1,-1),1)$ is deduced by the equality $\partial\|\cdot\|(0) = B(0,1)$, with $\|\cdot\|$ an euclidian norm.
3. It is easy to verify that for all $(a,b,\rho) \in \mathbb{R}^3$, for all $x^* \in \partial\theta(a,b,\rho)$, one has $\|x^*\| \leq 3\sqrt{2}$, which ensures that θ is Lipschitz-continuous on $\mathbb{R}^3$.

□

Since we have $\Phi_\rho(x) = \displaystyle\sum_{i=1}^{m} \theta(x_{1,i}, x_{2,i}, \rho)$, we can deduce the following properties about the function $(\rho, x) \to \Phi_\rho(x)$

Proposition 7. *The function Φ satisfies the following properties:*

1. *There exists a constant $c > 0$ such that for all $x \in \Omega$, $\Phi_0(x) \geq c dist(x, \Delta)$.*
2. *For all $x \in \Omega$, $x^* \in \partial(-\Phi_0)(x)$ if and only if $(x^*_{1,i}, x^*_{2,i}) = -\nabla_{x_{1,i},x_{2,i}}\theta(x_{1,i}, x_{2,i}, 0)$ if $(x_{1,i}, x_{2,i}) \neq (0,0)$ and $(x^*_{1,i}, x^*_{2,i}) \in B((-1,-1),1)$ if $(x_{1,i}, x_{2,i}) = (0,0)$.*
3. *The function $(\rho, x) \to \Phi_\rho(x)$ is Lipschitz-continuous on $\mathbb{R}^n$.*

Proof. It is a direct consequence of the Proposition 6. □

The following proposition gives a stationary property of the limit of the sequence $(x^k)_k$.

Proposition 8. *We choose Φ_ρ defined as in (20), and at each step, we choose ρ_k such that $\rho_k^2 = o(\min\{x_{1,i,n}^2 + x_{2,i,n}^2 \ : \ i = 1, \cdots, m\})$. All limit $\bar{x}$ of the sequence (x^k) such that $\langle \bar{x}_1, \bar{x}_2\rangle = 0$ and MPEC-LICQ holds is a weakly stationary point for MPEC.*

Proof. Let $\bar{x}$ a limit of the sequence (x^k). Since the gradient $\nabla\Phi_{\rho_k}(x_n)$ is bounded by a constant M which does not depend on n by Proposition 7, there exists a subsequence $(x_{n_k})_k$ such that $\nabla\Phi_{\rho_{n_k}}(x_{n_k}) \longrightarrow \bar{x}^*$. We show that $-\bar{x}^* \in \partial(-\Phi_0)(\bar{x})$.

Let $i \in \{1, \cdots, m\}$ such that $\bar{x}_{1,i}^2 + \bar{x}_{2,i}^2 > 0$. One has

$$\begin{aligned}
\frac{\partial\Phi_{\rho_{n_k}}}{\partial x_{1,i}}(x_{n_k}) &= 1 - \frac{x_{1,i,n_k}}{\sqrt{x_{1,i,n_k}^2 + x_{2,i,n_k}^2 + \rho_{n_k}^2}} \\
&\to 1 - \frac{\bar{x}_{1,i}}{\sqrt{\bar{x}_{1,i}^2 + \bar{x}_{2,i}^2}} \\
&= \frac{\partial\Phi_0}{\partial x_{1,i}}(\bar{x}).
\end{aligned}$$

In the same way, $\dfrac{\partial\Phi_{\rho_{n_k}}}{\partial x_{2,i}}(x_{n_k}) \longrightarrow \dfrac{\partial\Phi_0}{\partial x_{2,i}}(\bar{x})$. We suppose that $\bar{x}_{1,i}^2 + \bar{x}_{2,i}^2 = 0$. Then one has:

$$\begin{aligned}
(1 - \bar{x}^*_{1,i})^2 + (1 - \bar{x}^*_{2,i})^2 &= \lim_{k\to+\infty} \left(1 - \frac{\partial\Phi}{\partial x_{1,i}}(x_{n_k}, \rho_{n_k})\right)^2 + \left(1 - \frac{\partial\Phi}{\partial x_{2,i}}(x_{n_k}, \rho_{n_k})\right)^2 \\
&= \lim_{k\to+\infty} \frac{x_{1,i,n_k}^2 + x_{2,i,n_k}^2}{x_{1,i,n_k}^2 + x_{2,i,n_k}^2 + \rho_{n_k}^2} \\
&= 1 - \lim_{k\to+\infty} \frac{\rho_{n_k}^2}{x_{1,i,n_k}^2 + x_{2,i,n_k}^2 + \rho_{n_k}^2} \\
&= 1 \text{ because } \rho_{n_k}^2 = o(x_{1,i,n_k}^2 + x_{2,i,n_k}^2).
\end{aligned}$$

Therefore, by the Proposition 7, one has $-\bar{x}^* \in \partial_x(-\Phi_0)(\bar{x})$.

Since for all $n \in \mathbb{N}$, x^{k+1} solves the optimization problem (21), one has $\nabla f_1(x^{k+1}) - \nabla f_2(x^k) + \mu \nabla \Phi_{\rho_k}(x^k) \in -N_\Omega(x^k)$. Taking a sub-sequence, one has

$$\nabla f(\bar{x}) + \mu \bar{x}^* \in -N_\Omega(\bar{x}).$$

Therefore, one has $-\mu\bar{x}^* \in \partial(-\Phi)(\bar{x}) \cap (\nabla f(\bar{x}) + N_\Omega(\bar{x}))$, since MPEC-LICQ holds at the point $\bar{x}$, by the Proposition (4), $\bar{x}$ is a weakly stationary point for MPEC. □

We can now give a convergence result of this algorithm.

Proposition 9. *We suppose that f_1 or f_2 is μ-strongly convex, with $f = f_1 - f_2$, f_1 and f_2 convex, proper and differentiable on Ω. We suppose that the set Ω is bounded. Moreover, we chose the sequence of parameters $(\rho_k)_k$ such that the serie $\sum |\rho_{k+1} - \rho_k|$ converges and $\rho_k^2 = o(\min\{x_{1,i,n}^2 + x_{2,i,n}^2 \ : \ i = 1, \cdots, m\})$.*
If the set of weakly stationary points for MPEC is finite, then for all $x_0 \in \Omega$, the sequence (x^k) converges to a weakly stationary point for MPEC.

Proof. If there exist $k \in \mathbb{N}$ such that $x^k = x^{k+1}$ then the convergence of the sequence is clear. We suppose that for all $k \in \mathbb{N}$, one has $x^k \neq x^{k+1}$.

If f_1 is μ-strongly convex, then one has

$$f_1(x^{k+1}) \leq f_1(x^k) + \langle \nabla f_1(x^{k+1}), x^{k+1} - x^k \rangle - \frac{\mu}{2}\|x^{k+1} - x^k\|^2,$$

or else by convexity of f_1, one has

$$f_1(x^{k+1}) \leq f_1(x^k) + \langle \nabla f_1(x^{k+1}), x^{k+1} - x^k \rangle.$$

If f_1 is not μ-strongly convex, then f_2 is μ-strongly convex, and since $-\mu\Phi_{\rho_k}$ is convex, we obtain

$$\begin{aligned} f_2(x^{k+1}) - \mu\Phi_{\rho_k}(x^{k+1}) &\geq f_2(x^k) - \mu\Phi_{\rho_k}(x^k) + \frac{\mu}{2}\|x^{k+1} - x^k\|^2 \\ &+ \langle \nabla f_2(x^k) - \mu\nabla\Phi_{\rho_k}(x^k), x^{k+1} - x^k \rangle. \end{aligned}$$

If f_2 is not μ-strongly convex, it is only convex (and f_1 is μ-strongly convex), thus one has

$$f_2(x^{k+1}) - \mu\Phi_{\rho_k}(x^{k+1}) \geq f_2(x^k) - \mu\Phi_{\rho_k}(x^k) + \langle \nabla f_2(x^k) - \mu\nabla_x\Phi_{\rho_k}(x^k), x^{k+1} - x^k \rangle.$$

In both cases, the following inequalities hold:

$$\begin{aligned} f_1(x^{k+1}) - (f_2(x^{k+1}) - \mu\Phi_{\rho_k}(x^{k+1})) &\leq f_1(x^k) - (f_2(x^k) - \mu\Phi_{\rho_k}(x^k)) \\ &- \frac{\mu}{2}\|x^{k+1} - x^k\|^2 + \langle \nabla f_1(x^{k+1}) \\ &- (\nabla f_2(x^k) - \mu\nabla_x\Phi_{\rho_k}(x^k)), x^{k+1} - x^k \rangle \\ &\leq f_1(x^k) - (f_2(x^k) - \mu\Phi_{\rho_k}(x^k)) \\ &- \frac{\mu}{2}\|x^{k+1} - x^k\|^2. \end{aligned}$$

The last inequality holds true because x^{k+1} solves the problem (21) thus $\nabla f_1(x^{k+1}) - (\nabla f_2(x^k) - \mu\nabla\Phi_{\rho_k}(x^k)) \in -N_\Omega(x^{k+1})$.

Finally, using the Lipschitz-continuity of $(\rho, x) \to \Phi_\rho(x)$ (which it is ensured by Proposition 7), one has (with L a constant of Lipschitz-continuity of $(\rho, x) \to \Phi_\rho(x)$):

$$\begin{aligned}
\frac{\rho_2}{2}\sum_{k=0}^{p-1}\|x^{k+1}-x^k\|^2 &\leq f_1(x_0) - f_2(x_0) - f_1(x_p) + f_2(x_p) + \Phi_{\rho_p}(x_p) \\
&- \Phi_{\rho_0}(x_0) + L\sum_{k=0}^{p-1}|\rho_{k+1}-\rho_k| \\
&= f(x_0) - f(x_p) + \Phi_{\rho_0}(x_0) - \Phi_{\rho_0}(x_p) + L\sum_{k=0}^{p-1}|\rho_{k+1}-\rho_k|.
\end{aligned}$$

Since the function f and Φ are bounded from below and the serie $\sum |\rho_{k+1} - \rho_k|$ is convergent, we deduce that the serie $\sum \|x^{k+1} - x^k\|^2$ is also convergent, thus the sequence $(x^{k+1} - x^k)_k$ tends to zero. Therefore the set of limits of the sequence $(x^k)_k$ is connex.

Each limit $\bar{x}$ of $(x^k)_k$ is a weakly stationnary point for MPEC by Proposition 8. Thus the set of limits of (x^k) is finite and connex, thus it is a singleton, that's proves the convergence of the sequence (x^k).

□

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