

A Novel Unified Approach to Invariance in Control

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Abstract

In this paper, we propose a novel, unified, general approach to investigate sufficient and necessary conditions under which four types of convex sets, polyhedra, polyhedral cones, ellipsoids and Lorenz cones, are invariant sets for a linear continuous or discrete dynamical system. In proving invariance of ellipsoids and Lorenz cones for discrete systems, instead of the traditional Lyapunov method, our novel proofs are based on the S -lemma, which enables us to extend invariance conditions to any set represented by a quadratic inequality. Such sets include nonconvex and unbounded sets. Finally, according to the framework of our novel method, sufficient and necessary conditions for continuous systems are derived from the sufficient and necessary conditions for the corresponding discrete systems that are obtained by Euler methods.

Keywords:

Invariant Set, Dynamical System, Polyhedron, Lorenz Cone, Farkas Lemma, S -Lemma

1. Introduction

Positively invariant sets play a central role in the theory and applications of dynamical systems. Stability, control and preservation of constraints of dynamical systems can be formulated, somehow in a geometrical way, with

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the help of positively invariant sets. For a given dynamical system, both of continuous or discrete time, a subset of the state space is called positively invariant set for the dynamical system if containing the system state at a certain time then forward in time all the states remain within the positively invariant set. Geometrically, the trajectories cannot escape from a positively invariant set if the initial state belongs to the set. The dynamical system is often a controlled system of which the maximal (or minimal) positively invariant set is to be constructed.

It is well known, see e.g., Blanchini [6], Blanchini and Miani [8], and Polanski [21], that the Lyapunov stability theory is used as a powerful tool in obtaining many important results in control theory. The basic framework of the Lyapunov stability theory synthesizes the identification and computation of a Lyapunov function of a dynamical system. Usually positive definite quadratic functions serve as candidate Lyapunov functions. Sufficient and necessary conditions for positive invariance of a polyhedral set with respect to discrete dynamical systems were first proposed by Bitsoris [4, 5]. A novel positively invariant polyhedral cone was constructed by Horváth [16]. The Riccati equation was proved to be connected with ellipsoidal sets as invariant sets of linear dynamical systems, see e.g., Lin et al. [18] and Zhou et al. [30]. Some fundamental results about Lorenz cones as invariant sets are given by Birkhoff [3], Schneider and Vidyasagar [24], Stern and Wolkowicz [25], and Valcher and Farina [28]. Birkhoff [3] proposed a necessary condition for a linear transformation to be invariant on a convex cone. The concept of cross positive matrices was introduced by Schneider and Vidyasagar [24] in an attempt to prove positive invariance of a Lorenz cone. According to Nagumo's theorem [20] and the theory of cross positive matrices, Stern and Wolkowicz [25] presented sufficient and necessary conditions for a Lorenz cone to be positively invariant with respect to a linear continuous system. A novel proof of the spectral characterization of real matrices that leaves a polyhedral cone invariant was proposed by Valcher and Farina [28]. The spectral properties of the matrices, e.g., theorems of Perron-Frobenius type, were connected to set positive invariance by Vandergraft [24].

In this paper we deal with dynamical systems in finite dimensional spaces and introduce a novel and unified method for the determination of whether a set is a positively invariant set for a linear dynamical system. Here the sets are ellipsoids, polyhedral sets or - not necessarily convex - second order sets including Lorenz cones. In addition, we formulate optimization methods to check the resulting equivalent conditions.

The main tool in the continuous time case consists of the explicit computation of the tangent cones of the positively invariant sets and their application along the lines of the Nagumo theorem [20]. This theorem says that a set is positively invariant, under some conditions on solvability of the underlying differential equation, if and only if at each point of the set, the vector field of the differential equation points toward the tangent cone at that point. The resulting conditions are constructive in the sense that they can be checked by well established optimization methods. Our unified approach is based on optimization methodology. The analysis in the discrete case is based on the theorems of alternatives of optimization, namely on the Farkas lemma [23] and the S -lemma [22, 29]. Let us mention that the technique with the tangent cones in the continuous time case and the theorem of alternatives of optimization in the discrete case show common features.

First, in the paper, we consider various sets as candidates for positively invariant sets with respect to a discrete system. Sufficient and necessary conditions for the four types of sets are derived using the Farkas lemma [23] and the S -lemma [22, 29], respectively. The Farkas lemma and the S -lemma are frequently referred to as Theorems of the Alternatives in the optimization literature. Note that the approach based on the Farkas lemma is originally due to Bitsoris [4, 5]. Our approach, based on the S -lemma for ellipsoids and Lorenz cones, is not only simpler compared to the traditional Lyapunov theory based approach, but also highlights the strong relationship between control and optimization theories. It also enables us to extend invariance conditions to any set represented by a quadratic inequality. Such sets include nonconvex and unbounded sets. Positively invariant sets for continuous systems are linked to the ones for discrete systems by applying Euler method. The forward Euler method or backward Euler method is used to discretize a continuous system to a discrete system. According to [17], we have that both the continuous and discrete systems can share the same set as a positively invariant set when the discretization steplength is bounded by a certain value. In [17], we prove that there exists a uniform upper bound of the steplength for both the forward and backward Euler methods (only the backward Euler method) such that the discrete and continuous systems can share a polyhedron or a polyhedral cone (an ellipsoid or a Lorenz cone) as a positively invariant set. Some geometrical properties of the original trajectory, i.e., the solution of the continuous system, the forward Euler trajectory, and the backward Euler trajectory are discussed. Then, sufficient and necessary conditions under which the four types of convex sets are positively invariant

sets for the continuous systems are derived by using Euler methods and the corresponding sufficient and necessary conditions for the discrete systems.

The novelty of this paper is that we propose a simple, novel, unified approach, different from the traditional Lyapunov stability theory approach, to derive conditions for the four types of sets to be positively invariant sets with respect to discrete systems. Our approach is based on the S -lemma when ellipsoids and Lorenz cones are considered for the discrete system. We also establish a framework according to Euler methods to derive conditions for the four types of sets with respect to the continuous systems to be positively invariant.

Notation and Conventions. To avoid unnecessary repetitions, the following notations and conventions are used in this paper. A dynamical system, positively invariant, and sufficient and necessary condition for positive invariance are called a *system*, *invariant*, and *invariance condition*, respectively. The sets considered in this paper are non-empty, closed, and convex sets if not specified otherwise. The interior and the boundary of a set \mathcal{S} is denoted by $\text{int}(\mathcal{S})$ and $\partial\mathcal{S}$, respectively. A matrix Q to be positive definite, positive semidefinite, negative definite, and negative semidefinite matrix is denoted by $Q \succ 0$, $Q \succeq 0$, $Q \prec 0$, and $Q \preceq 0$, respectively. The i -th row of a matrix G is denoted by G_i^T . The eigenvalues of a real symmetric matrix Q , whose eigenvalues are always real, are ordered as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and the corresponding eigenvectors are denoted by u_1, u_2, \dots, u_n . The spectrum of Q is represented by $\lambda(Q) = \max\{|\lambda_i(Q)|\}$, and $\text{inertia}\{Q\} = \{\alpha, \beta, \gamma\}$ indicates that the number of positive, zero, and negative eigenvalues of Q are α, β , and γ , respectively. The index set $\{1, 2, \dots, n\}$ is denoted by $\mathcal{I}(n)$. The inner product of vectors $x, y \in \mathbb{R}^n$ is represented by $x^T y$.

This paper is organized as follows: in Section 2, the related basic concepts and theorems are introduced. Our main results are shown in Section 3, in which invariance conditions of polyhedral sets, ellipsoids, and Lorenz cones for continuous and discrete systems are presented. In Section 4, some numerical examples are given to illustrate the invariance conditions presented in Section 3. Finally, our conclusions are summarized in Section 5.

2. Basic Concepts and Theorems

In this section, the basic concepts and theorems related to invariant sets for dynamical systems are introduced.

2.1. Linear Dynamical System

In this paper, we consider discrete and continuous linear dynamical systems, respectively described by the following equations:

$$x_{k+1} = Ax_k, \tag{1}$$

$$\dot{x}(t) = Ax(t), \tag{2}$$

where $A \in \mathbb{R}^{n \times n}$ is a constant real matrix, $x_k, x(t) \in \mathbb{R}^n$ are the state variables, $t \in \mathbb{R}$, and $k \in \mathbb{N}$. We may assume, without loss of generality, that A is not the zero matrix. The study of invariant sets is the main subject of this paper, thus now we introduce invariant sets for both discrete and continuous linear systems.

Definition 2.1. *A set $\mathcal{S} \subseteq \mathbb{R}^n$ is an invariant set for the discrete system (1) if $x_k \in \mathcal{S}$ implies $x_{k+1} \in \mathcal{S}$, for all $k \in \mathbb{N}$.*

Definition 2.2. *A set $\mathcal{S} \subseteq \mathbb{R}^n$ is an invariant set for the continuous system (2) if $x(0) \in \mathcal{S}$ implies $x(t) \in \mathcal{S}$, for all $t \geq 0$.*

In fact, the sets given in Definition 2.1 and 2.2 conventionally refer to positively invariant sets. Considering that only positively invariant sets are studied in this paper, we simply call them invariant sets. One can prove the following properties: the operators A (or¹ e^{At}) leave \mathcal{S} invariant if \mathcal{S} is an invariant set for the discrete (or continuous) systems.

Proposition 2.3. [2, 9] *The set \mathcal{S} is an invariant set for the discrete system (1) if and only if $A\mathcal{S} \subseteq \mathcal{S}$. Similarly, the set \mathcal{S} is an invariant set for the continuous system (2) if and only if $e^{At}\mathcal{S} \subseteq \mathcal{S}$.*

2.2. Convex Sets

In this paper, we investigate invariance conditions for a family of convex sets, namely polyhedral sets, ellipsoids, and Lorenz cones.

A *polyhedron*, denoted by $\mathcal{P} \subseteq \mathbb{R}^n$, can be defined as the intersection of a finite number of half-spaces:

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Gx \leq b\}, \tag{3}$$

¹The exponential function with respect to a matrix is defined as $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (A^k t^k)$.

where $G \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, or equivalently, as the sum of the convex combination of a finite number of points and the conic combination of a finite number of vectors:

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{\ell_1} \theta_i x^i + \sum_{j=1}^{\ell_2} \hat{\theta}_j \hat{x}^j, \sum_{i=1}^{\ell_1} \theta_i = 1, \theta_i \geq 0, \hat{\theta}_j \geq 0 \right\}, \quad (4)$$

where $x^1, \dots, x^{\ell_1}, \hat{x}^1, \dots, \hat{x}^{\ell_2} \in \mathbb{R}^n$. The *vertices* of \mathcal{P} are a subset of $x^i, i \in \mathcal{I}(\ell_1)$, and the *extreme rays* of \mathcal{P} are represented as $x^i + \alpha \hat{x}^j, \alpha > 0$, for some $i \in \mathcal{I}(\ell_1)$ and $j \in \mathcal{I}(\ell_2)$. We highlight that a bounded polyhedron, i.e., $\ell_2 = 0$ in (4), is called a *polytope*.

A *polyhedral cone*, denoted by $\mathcal{C}_{\mathcal{P}} \subseteq \mathbb{R}^n$, can be also considered as a special class of polyhedra, and it can be defined as:

$$\mathcal{C}_{\mathcal{P}} = \{x \in \mathbb{R}^n \mid Gx \leq 0\}, \quad (5)$$

or equivalently,

$$\mathcal{C}_{\mathcal{P}} = \left\{ x \in \mathbb{R}^n \mid x = \sum_{j=1}^{\ell} \hat{\theta}_j \hat{x}^j, \hat{\theta}_j \geq 0 \right\}, \quad (6)$$

where $G \in \mathbb{R}^{m \times n}$, and $\hat{x}^1, \dots, \hat{x}^{\ell} \in \mathbb{R}^n$.

An *ellipsoid*, denoted by $\mathcal{E} \subseteq \mathbb{R}^n$, centered at the origin, is defined as:

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid x^T Q x \leq 1\}, \quad (7)$$

where $Q \in \mathbb{R}^{n \times n}$ and $Q \succ 0$. Any ellipsoid with nonzero center can be transformed to an ellipsoid centered at the origin.

A *Lorenz cone*², denoted by $\mathcal{C}_{\mathcal{L}} \subseteq \mathbb{R}^n$, with vertex at the origin, is defined as:

$$\mathcal{C}_{\mathcal{L}} = \{x \in \mathbb{R}^n \mid x^T Q x \leq 0, x^T Q u_n \leq 0\}, \quad (8)$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric nonsingular matrix with one negative eigenvalue λ_n , i.e., $\text{inertia}\{Q\} = \{n - 1, 0, 1\}$. Similar to ellipsoids, any Lorenz cone with nonzero vertex can be transformed to a Lorenz cone with vertex at the origin. For every Lorenz cone given as in (8), there exists an orthonormal

²A Lorenz cone is sometimes also called an ice cream cone, or a second order cone.

basis $U = [u_1, u_2, \dots, u_n]$, i.e., $u_i^T u_j = \delta_{ij}$, where u_i is the eigenvector corresponding to the eigenvalue, λ_i , of Q , and δ_{ij} is the Kronecker delta function, such that $Q = U\Lambda^{\frac{1}{2}}\tilde{I}\Lambda^{\frac{1}{2}}U^T$, where $\Lambda^{\frac{1}{2}} = \text{diag}\{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_{n-1}}, \sqrt{-\lambda_n}\}$ and $\tilde{I} = \text{diag}\{1, \dots, 1, -1\}$. In particular, the Lorenz cone with $Q = \tilde{I}$ is denoted by $\mathcal{C}_{\tilde{I}}^*$, then we have $\mathcal{C}_{\tilde{I}}^* = \{x \in \mathbb{R}^n \mid x^T \tilde{I} x \leq 0, x^T e_n \geq 0\}$, where $e_n = (0, \dots, 0, 1)^T$. And we call $\mathcal{C}_{\tilde{I}}^*$ the *standard Lorenz cone*.

2.3. Basic Theorems

The Farkas lemma [23] and the S -lemma [22, 29], both of which are also called the Theorem of Alternatives, are fundamental tools to derive the invariance conditions for discrete systems in our study. The S -lemma proved by Yakubovich [29] is somewhat analogous to a special case of the nonlinear Farkas lemma, see Pólik and Terlaky [22].

Theorem 2.4. (Farkas lemma [23]) *Let $P \in \mathbb{R}^{m \times n}$ and $d \in \mathbb{R}^m$. Then the following two statements are equivalent:*

1. *There exists a vector $z \in \mathbb{R}^n$, such that $Pz = d$, and $z \geq 0$;*
2. *For all $y \in \mathbb{R}^m$ with $P^T y \leq 0$, the inequality $d^T y \leq 0$ holds.*

Theorem 2.5. (S-lemma [22, 29]) *Let $g_1(y), g_2(y) : \mathbb{R}^n \rightarrow \mathbb{R}$ be quadratic functions, and suppose that there is a $\hat{y} \in \mathbb{R}^n$ such that $g_2(\hat{y}) < 0$. Then the following two statements are equivalent:*

1. *There exists no $y \in \mathbb{R}^n$, such that $g_1(y) < 0, g_2(y) \leq 0$.*
2. *There exists a scalar $\rho \geq 0$, such that $g_1(y) + \rho g_2(y) \geq 0$, for all $y \in \mathbb{R}^n$.*

The use of the Theorems of Alternatives 2.4 and 2.5 to derive the invariance conditions is due to the Proposition 2.3. According to Proposition 2.3, to prove that a set \mathcal{S} is an invariant set for a discrete system, we need to prove $A\mathcal{S} \subseteq \mathcal{S}$, which is equivalent to $(\mathbb{R}^n \setminus \mathcal{S}) \cap (A\mathcal{S}) = \emptyset$. Since we assume that \mathcal{S} is a closed set, we have that $\mathbb{R}^n \setminus \mathcal{S}$ is an open set. Thus we have that the inequality systems, which are combined by a strictly inequality corresponding to $\mathbb{R}^n \setminus \mathcal{S}$ and an inequality or some equalities corresponding to $A\mathcal{S}$, have no solution. This is one of the statements in the Theorems of Alternatives 2.4 or 2.5.

For invariance conditions for continuous systems, the concept of *tangent cone* plays an important role in our analysis.

Definition 2.6. Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a closed convex set, and $x \in \mathcal{S}$. The tangent cone of \mathcal{S} at x , denoted by $\mathcal{T}_{\mathcal{S}}(x)$, is given as

$$\mathcal{T}_{\mathcal{S}}(x) = \left\{ y \in \mathbb{R}^n \mid \liminf_{t \rightarrow 0^+} \frac{\text{dist}(x + ty, \mathcal{S})}{t} = 0 \right\}, \quad (9)$$

where $\text{dist}(x, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|x - s\|$.

The following classic result proposed by Nagumo [20] provides a general criterion to determine whether a closed convex set is an invariant set for a continuous system. This theorem, however, is not valid for discrete systems, for which one can find a counterexample in [7].

Theorem 2.7. (Nagumo [7, 20]) Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a closed convex set, and assume that the system $\dot{x}(t) = f(x(t))$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuous mapping, admits a globally unique solution for every initial point $x(0) \in \mathcal{S}$. Then \mathcal{S} is an invariant set for this system if and only if

$$f(x) \in \mathcal{T}_{\mathcal{S}}(x), \text{ for all } x \in \partial\mathcal{S}, \quad (10)$$

where $\mathcal{T}_{\mathcal{S}}(x)$ is the tangent cone of \mathcal{S} at x .

In fact, Nagumo's Theorem 2.7 has a geometrical interpretation as follows: for any trajectory that starts in \mathcal{S} , it has to go through $\partial\mathcal{S}$ if it will go out of \mathcal{S} . Then one needs only to consider the property of this trajectory on $\partial\mathcal{S}$. Note that $f(x)$ is the derivative of the trajectory, thus (10) ensures that the trajectory will point inside \mathcal{S} on the boundary, which means \mathcal{S} is an invariant set. The disadvantage of Theorem 2.7, however, is that it may be difficult to verify whether (10) holds for all points on the boundary of a given set. According to Nagumo's Theorem 2.7, the key is to derive the formula of the tangent cone on the boundary of the set.

3. Invariance Conditions

In this section, we present the invariance conditions, i.e., sufficient and necessary conditions under which polyhedral sets, ellipsoids, and Lorenz cones are invariant sets for discrete and continuous systems. For each convex set, the invariance conditions for discrete systems are first derived by using the Theorems of Alternatives, i.e., the Farkas lemma or the S -lemma. Then the invariance conditions for continuous systems are derived by using a discretization method to discretize the continuous system and applying the invariance conditions for the obtained discrete systems.

3.1. Polyhedral Sets

Since every polyhedral set has two different representations as shown in Section 2.2, we present the invariance conditions for both forms, respectively. *Nonnegative* and *off-diagonal nonnegative matrices* are used in the invariance conditions.

Definition 3.1. *A matrix H is called a nonnegative matrix, denoted by $H \geq 0$, if $H_{ij} \geq 0$ for all i, j . A matrix L is called an off-diagonal nonnegative matrix, denoted by $L \geq_o 0$, if $L_{ij} \geq 0$ for $i \neq j$.*

3.1.1. Invariance Conditions for Discrete Systems

The invariance condition of a polyhedral sets given as in (3) for a discrete system is presented by Bitsoris in [5].

Theorem 3.2. (Bitsoris [5]) *A polyhedron \mathcal{P} given as in (3) is an invariant set for the discrete system (1) if and only if there exists a nonnegative matrix $H \in \mathbb{R}^{m \times m}$, such that $HG = GA$ and $Hb \leq b$.*

Proof. The basic idea of the proof is applying Proposition 2.3 and the Farkas lemma. According to Proposition 2.3, it suffices to prove $\mathcal{P} \subseteq \mathcal{P}' = \{x \mid GAx \leq b\}$, which is equivalent to $(\mathbb{R}^n \setminus \mathcal{P}') \cap \mathcal{P} = \emptyset$, i.e., there is no x , such that $Gx \leq b$ and $(GA)_i^T x > b_i$, for all $i \in \mathcal{I}(n)$. According to the Farkas lemma 2.4, for each $i \in \mathcal{I}(n)$, there exists a vector $h_i \geq 0$, such that $G^T h_i = (GA)_i$, and $b^T h_i \leq b_i$. Theorem 3.2 is immediate if we write these equalities and inequalities in a matrix form. \square

We highlight that Castelan and Hennes [11] present an algebraic characterization of the matrix G satisfying the conditions in Theorem 3.2. They prove that the invariance condition implies that the kernel of G is an A -invariant subspace.

The invariance condition of a polyhedral set given as in (4) for discrete systems is provided in Theorem 3.3. Note that a similar result is presented in [7], which considers only the case when the set is a polytope.

Theorem 3.3. *A polyhedron \mathcal{P} given as in (4) is an invariant set for the discrete system (1) if and only if there exists a nonnegative matrix $L \in \mathbb{R}^{(\ell_1 + \ell_2) \times (\ell_1 + \ell_2)}$, such that $XL = AX$ and $\tilde{1}L = \bar{1}$, where $X = [x^1, \dots, x^{\ell_1}, \hat{x}^1, \dots, \hat{x}^{\ell_2}]$, $\tilde{1} = (1_{\ell_1}, 0_{\ell_2})$, $\bar{1} = 1_{\ell_1 + \ell_2}$.*

Proof. Note that \mathcal{P} given as in (4) is an invariant set for discrete system if and only if $Ax^i \in \mathcal{P}$, for all $i \in \mathcal{I}(\ell_1)$, and $A\hat{x}^j \in \mathcal{P}$, for all $j \in \mathcal{I}(\ell_2)$. Then there exist $\theta_{p_1}^i, \theta_{p_1}^j \geq 0$, for all $p_1 \in \mathcal{I}(\ell_1)$, with $\sum_{p_1=1}^{\ell_1} \theta_{p_1}^i = 1, \sum_{p_1=1}^{\ell_1} \theta_{p_1}^j = 1$, and there exist $\hat{\theta}_{p_2}^i, \hat{\theta}_{p_2}^j \geq 0$, for all $p_2 \in \mathcal{I}(\ell_2)$, such that

$$Ax^i = \sum_{p_1=1}^{\ell_1} \theta_{p_1}^i x^{p_1} + \sum_{p_2=1}^{\ell_2} \hat{\theta}_{p_2}^i x^{p_2}, \quad A\hat{x}^j = \sum_{p_1=1}^{\ell_1} \theta_{p_1}^j x^{p_1} + \sum_{p_2=1}^{\ell_2} \hat{\theta}_{p_2}^j x^{p_2}. \quad (11)$$

Let $L = [\theta^1, \dots, \theta^{\ell_1}, \hat{\theta}^1, \dots, \hat{\theta}^{\ell_2}]$, then the theorem is immediate by (11). \square

A polyhedral cone is a special polyhedral set, thus we have the following invariance condition of a polyhedral cone for discrete systems.

Corollary 3.4. 1). A polyhedral cone $\mathcal{C}_{\mathcal{P}}$ given as in (5) is an invariant set for the discrete system (1) if and only if there exists a nonnegative matrix $H \in \mathbb{R}^{m \times m}$, such that $HG = GA$.

2). A polyhedral cone $\mathcal{C}_{\mathcal{P}}$ given as in (6) is an invariant set for the discrete system (1) if and only if there exists a nonnegative matrix $L \in \mathbb{R}^{\ell \times \ell}$, such that $XL = AX$, where $X = [\hat{x}^1, \dots, \hat{x}^{\ell}]$.

For a given polyhedral set and a discrete system, according to Theorem 3.2 (Theorem 3.3, or Corollary 3.4), to determine whether the set is an invariant set for the system is equivalent to verify the existence of a nonnegative matrix H (or L), which is actually a linear optimization problem. Rather than computing H (or L) directly, it is more efficient to sequentially solve some small subproblems. Let us choose polyhedron \mathcal{P} as given in (3) and Theorem 3.2 as an example to illustrate this idea. We can sequentially examine the feasibility of the subproblems. Find $h_i \in \mathbb{R}^n$, such that $h_i G = G_i A$, $h_i \geq 0$, and $h_i b \leq b_i$, for all $i \in \mathcal{I}(n)$. Clearly, these are linear feasibility problems which can be verified in polynomial time, e.g., by using interior point methods [23]. If all of these linear optimization problems are feasible, then their solutions forms such a nonnegative matrix H . Otherwise, we can conclude that the set is not an invariant set for the system, and the computation is terminated at the first infeasible subproblem.

3.1.2. Invariance Conditions for Continuous Systems

According to [17], we have that both the forward and backward Euler methods are invariance preserving for a polyhedral set. Blanchini [7] presents

a proof to derive the invariance condition for a polyhedron using the forward Euler method, thus we present a proof using the backward Euler method. We first present the following invariance condition which is obtained by using Nagumo's Theorem 2.7.

Lemma 3.5. *Let a polyhedron \mathcal{P} be given as in (3), and at the index set of the constraints which are active at x be $\mathcal{I}_x \neq \emptyset$. Then \mathcal{P} is an invariant set for the continuous system (2) if and only if for every $x \in \partial\mathcal{P}$, i.e., $G_i^T x = b_i$, for $i \in \mathcal{I}_x$, we have*

$$G_i^T Ax \leq 0, \quad i \in \mathcal{I}_x. \quad (12)$$

Proof. The tangent cone at x , where $G_i^T x = b_i$ for $i \in \mathcal{I}_x$, is (see Page 138 in [14]) $\mathcal{T}_{\mathcal{P}}(x) = \{y \mid G_i^T y \leq 0, i \in \mathcal{I}_x\}$. Then the lemma immediately follows from Nagumo's Theorem 2.7. \square

We now present another invariance condition of a polyhedron in the form of (3) for the continuous system (2).

Theorem 3.6. *A polyhedron \mathcal{P} given as in (3) is an invariant set for the continuous system (2) if and only if there exists an off-diagonal matrix $\tilde{H} \in \mathbb{R}^{m \times m}$, such that $\tilde{H}G = GA$ and $\tilde{H}b \leq 0$.*

Proof. For the “if” part, we denote by \mathcal{I}_x the index set of the constraints which are active at x . We consider an $x \in \partial\mathcal{P}$, i.e., $x \in \{x \mid G_i^T x = b_i, i \in \mathcal{I}_x\}$. Since $\tilde{H}G = GA$, we have $\tilde{H}_i^T Gx = G_i^T Ax$, for every $i \in \mathcal{I}_x$. Since $\tilde{H} \succeq_o 0$ and $x \in \partial\mathcal{P}$, we have \tilde{H}_j is free, $G_j^T x = b_j$ for $j = i$, and $\tilde{H}_j \geq 0, G_j^T x \leq b_j$ for $j \neq i$. Thus $\sum_{j=1}^m \tilde{H}_j(G_j^T x - b_j) \leq 0$, i.e., $\tilde{H}_i^T Gx \leq \tilde{H}_i^T b$. Since $\tilde{H}b \leq 0$, we have $\tilde{H}_i^T b \leq 0$. Then, we have $G_i^T Ax = \tilde{H}_i^T Gx \leq \tilde{H}_i^T b \leq 0$. According to Lemma 3.5, we have that \mathcal{P} is an invariant set for the continuous system.

For the “only if” part, according to [17], we have that there exists a $\hat{\tau} > 0$, such that \mathcal{P} is also an invariant set for the discrete system $x_{k+1} = (I - A\Delta t)^{-1}x_k$, for every $0 \leq \Delta t \leq \hat{\tau}$. Then, according to Theorem 3.2, there exists a matrix $H(\Delta t) \geq 0$, such that

$$H(\Delta t)G = G(I - A\Delta t)^{-1}, \quad \text{and} \quad H(\Delta t)b \leq b. \quad (13)$$

Clearly, we have $\lim_{\Delta t \rightarrow 0} H(\Delta t) = I$. By reformulating (13), we have

$$\frac{H(\Delta t) - I}{\Delta t}G = H(\Delta t)GA, \quad \text{and} \quad \frac{H(\Delta t) - I}{\Delta t}b \leq 0.$$

Note that $\lim_{\Delta t \rightarrow 0} H(\Delta t)GA = GA$, thus $\tilde{H} := \lim_{\Delta t \rightarrow 0} \frac{H(\Delta t) - I}{\Delta t}$ exists. Since $H(\Delta t) \geq 0$, we have $\tilde{H} \geq_o 0$. The ‘‘only if’’ part is immediate by letting Δt approach 0. \square

We consider the invariance condition of the polyhedron in the form of (4) for the continuous system (2). For an arbitrary convex set in \mathbb{R}^n , we have the following conclusion.

Lemma 3.7. *Let \mathcal{S} be a convex set in \mathbb{R}^n . For any $\ell \in \mathbb{N}$ and $x, y^1, y^2, \dots, y^\ell \in \mathcal{S}$ satisfying $x = \sum_{i=1}^\ell \beta_i y^i$, where $\beta_i > 0$ for every $i \in \mathcal{I}(\ell)$, we have $\mathcal{T}_{\mathcal{S}}(y^i) \subseteq \mathcal{T}_{\mathcal{S}}(x)$ for every $i \in \mathcal{I}(\ell)$.*

Proof. Let $i \in \mathcal{I}(\ell)$. For every $y \in \mathcal{T}_{\mathcal{S}}(x^i)$ and $t > 0$, we have

$$\text{dist}(y^i + ty, \mathcal{S}) = \inf_{s^i \in \mathcal{S}} \|y^i + ty - s^i\| = \lim_{k \rightarrow \infty} \|y^i + ty - s^{ik}\|, \quad (14)$$

where $s^{ik} \in \mathcal{S}$, $k \in \mathbb{N}$, is a sequence of points in \mathcal{S} that converge to the point on $\partial\mathcal{S}$ closest to $y^i + ty$. For every $k \in \mathbb{N}$, since $x = \sum_{i=1}^\ell \beta_i y^i$ and $\beta_i > 0$, we have

$$\|y^i + ty - s^{ik}\| = \left\| y^i + \sum_{j \neq i} \frac{\beta_j}{\beta_i} y^j + ty - \left(s^{ik} + \sum_{j \neq i} \frac{\beta_j}{\beta_i} y^j \right) \right\| = \frac{1}{\beta_i} \|x + t\beta_i y - \tilde{s}^{ik}\|, \quad (15)$$

where $\tilde{s}^{ik} = \beta_i s^{ik} + \sum_{j \neq i} \beta_j y^j$, i.e., \tilde{s}^{ik} is a convex combination of s^{ik} and $\{y^j\}_{j \neq i}$. Thus $\tilde{s}^{ik} \in \mathcal{S}$, which implies $\text{dist}(x + t\beta_i y, \mathcal{S}) \leq \|x + t\beta_i y - \tilde{s}^{ik}\|$. Further, according to (15) and (14), we have

$$\text{dist}(x + t\beta_i y, \mathcal{S}) \leq \beta_i \lim_{k \rightarrow \infty} \|x + t\beta_i y - \tilde{s}^{ik}\| = \beta_i \text{dist}(y^i + ty, \mathcal{S}), \quad (16)$$

which, according to $y \in \mathcal{T}_{\mathcal{S}}(x^i)$, implies $\beta_i y \in \mathcal{T}_{\mathcal{S}}(x)$. Finally, note that $\mathcal{T}_{\mathcal{S}}(x)$ is a cone, thus we have $y \in \mathcal{T}_{\mathcal{S}}(x)$. \square

For the polyhedron \mathcal{P} given as in (4), a vertex of \mathcal{P} is given as x^i , for some $i \in \mathcal{I}(\ell_1)$, and an extreme ray of \mathcal{P} is represented as $x^i + \alpha \hat{x}^j$, $\alpha > 0$, for some $i \in \mathcal{I}(\ell_1)$ and $j \in \mathcal{I}(\ell_2)$. Applying Lemma 3.7 to \mathcal{P} , we have the following Corollary 3.8 about the relationship between tangent cones at a vector and the vertices and extreme rays of \mathcal{P} . Note that $\mathcal{T}_{\mathcal{P}}(x) = \mathbb{R}^n$ for every $x \in \text{int}(\mathcal{S})$, thus Corollary 3.8 is only nontrivial for $x \in \partial\mathcal{P}$.

Corollary 3.8. *Let a polyhedron \mathcal{P} be given as in (4), and $x \in \mathcal{P}$, and x be represented as in (4). Let $\mathcal{I}_1 = \{i \in \mathcal{I}(\ell_1) \mid \theta_i > 0\}$ and $\mathcal{I}_2 = \{j \in \mathcal{I}(\ell_2) \mid \hat{\theta}_j > 0\}$. Then $\mathcal{T}_{\mathcal{P}}(x^i) \subseteq \mathcal{T}_{\mathcal{P}}(x)$ and $\mathcal{T}_{\mathcal{P}}(x^i + \alpha \hat{x}^j) = \mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j) \subseteq \mathcal{T}_{\mathcal{P}}(x)$ for $i \in \mathcal{I}_1, j \in \mathcal{I}_2$, and $\alpha > 0$, where $x^i + \alpha \hat{x}^j$ is an extreme ray of \mathcal{P} .*

The exact representations of the tangent cones at vertices or extreme rays of \mathcal{P} given as in (4) are presented in Lemma 3.9 below.

Lemma 3.9. *Let a polyhedron \mathcal{P} be given as in (4), and $\mathcal{I}'_1 = \{i \in \mathcal{I}(\ell_1) \mid \text{for any } j \in \mathcal{I}(\ell_2), x^i + \hat{x}^j \text{ is not an extreme ray.}\}$, $\mathcal{I}''_1 = \mathcal{I}(\ell_1) \setminus \mathcal{I}'_1$, then*

1). *For every $i \in \mathcal{I}'_1$, we have $\mathcal{T}_{\mathcal{P}}(x^i) = \{y \in \mathbb{R}^n \mid y = \sum_{p=1}^{\ell_1} \alpha_p x^p, \alpha_p \geq 0, p \neq i, \sum_{p=1}^{\ell_1} \alpha_p = 0\}$.*

2). *For every $i \in \mathcal{I}''_1$, we have $\mathcal{T}_{\mathcal{P}}(x^i) = \{y \in \mathbb{R}^n \mid y = \sum_{p=1}^{\ell_1} \alpha_p x^p + \sum_{q=1}^{\ell_2} \hat{\alpha}_q \hat{x}^q, \alpha_p, \hat{\alpha}_q \geq 0, p \neq i, \sum_{p=1}^{\ell_1} \alpha_p = 0\}$.*

3). *For every $i \in \mathcal{I}''_1$ and $j \in \mathcal{I}(\ell_2)$ such that $x^i + \hat{x}^j$ is an extreme ray, we have $\mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j) = \{y \in \mathbb{R}^n \mid y = \sum_{q=1}^{\ell_2} \hat{\alpha}_q \hat{x}^q, \hat{\alpha}_q \geq 0, j \neq q\}$.*

Proof. For $i \in \mathcal{I}'_1$, we have $\mathcal{T}_{\mathcal{P}}(x^i) = \{y \mid y = \sum_{p=1, p \neq i}^{\ell_1} \alpha_p (x^p - x^i), \alpha_p \geq 0\}$, which yields 1) by introducing α_i . The other two cases are easy to verify. \square

Lemma 3.10. *Let \mathcal{C} be a closed convex cone. If $x + \alpha y \in \mathcal{C}$ for $\alpha > 0$, then $x, y \in \mathcal{C}$.*

The following lemma presents an invariance condition for a polyhedron in the form of (4) for the continuous system (2).

Lemma 3.11. *Let a polyhedron \mathcal{P} be given as in (4). Then \mathcal{P} is an invariant set for the continuous system (2) if and only if $Ax^i \in \mathcal{T}_{\mathcal{P}}(x^i)$ and $A\hat{x}^j \in \mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j)$ for $i \in \mathcal{I}(\ell_1)$ and $j \in \mathcal{I}(\ell_2)$.*

Proof. For $i \in \mathcal{I}(\ell_1)$ and $j \in \mathcal{I}(\ell_2)$ when $x^i + \alpha \hat{x}^j$ for $\alpha > 0$ is an extreme ray, according to Nagumo's Theorem 2.7, we have $Ax^i \in \mathcal{T}_{\mathcal{P}}(x^i)$ and $A(x^i + \alpha \hat{x}^j) \in \mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j)$, which implies, by Lemma 3.10, $A\hat{x}^j \in \mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j)$. Note that $\mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j) = \mathbb{R}^n$ when $x^i + \alpha \hat{x}^j$ is not an extreme ray, the "only if" part follows.

For the "if" part, we choose $x \in \mathcal{P}$. We represent x as $x = \sum_{i \in \mathcal{I}_1} \theta_i x^i + \sum_{j \in \mathcal{I}_2} \hat{\theta}_j \hat{x}^j$, where $\mathcal{I}_1 = \{i \in \mathcal{I}(\ell_1) \mid \theta_i > 0\}$ and $\mathcal{I}_2 = \{j \in \mathcal{I}(\ell_2) \mid \hat{\theta}_j > 0\}$. Then according to Corollary 3.8, we have $Ax = \sum_{i \in \mathcal{I}_1} \theta_i Ax^i + \sum_{j \in \mathcal{I}_2} \hat{\theta}_j A\hat{x}^j \in (\cup_{i \in \mathcal{I}_1} \mathcal{T}_{\mathcal{P}}(x^i)) \cup (\cup_{j \in \mathcal{I}_2} \mathcal{T}_{\mathcal{P}}(x^i + \hat{x}^j)) \subseteq \mathcal{T}_{\mathcal{P}}(x)$. Finally, the "if" part follows by Nagumo's Theorem 2.7. \square

Note that Lemma 3.11 is only nontrivial when $x^i + \hat{x}^j$ is an extreme ray of \mathcal{P} . By Lemma 3.9 and Lemma 3.11, the following corollary is immediate.

Corollary 3.12. *Let a polyhedron \mathcal{P} be given as in (4). Then \mathcal{P} is an invariant set for the continuous system (2) if and only if for x^i , $i \in \mathcal{I}(\ell_1)$, there exist $\alpha_p^i, \hat{\alpha}_q^i \geq 0$ for $p \neq i$, such that*

$$Ax^i = \sum_{p=1}^{\ell_1} \alpha_p^i x^p + \sum_{q=1}^{\ell_2} \hat{\alpha}_q^i \hat{x}^q, \text{ and } \sum_{p=1}^{\ell_1} \alpha_p = 0, \quad (17)$$

for \hat{x}^j , $j \in \mathcal{I}(\ell_2)$, there exist $\hat{\alpha}_q^j \geq 0$ for $q \neq j$, such that $A\hat{x}^j = \sum_{q=1}^{\ell_2} \hat{\alpha}_q^j \hat{x}^q$.

Theorem 3.13. *A polyhedron \mathcal{P} given as in (4) is an invariant set for the continuous system (2) if and only if there exists a matrix $L \in \mathbb{R}^{(\ell_1+\ell_2) \times (\ell_1+\ell_2)}$ and $\tilde{L} \geq_o 0$, such that $X\tilde{L} = AX$ and $\bar{1}\tilde{L} = \bar{0}$, where $X = [x^1, \dots, x^{\ell_1}, \hat{x}^1, \dots, \hat{x}^{\ell_2}]$, $\bar{1} = [1_{\ell_1}, 0_{\ell_2}]$.*

Proof. This proof is similar to the one given in Theorem 3.6. We denote the i -th column of \tilde{L} by $(l_{1,i}, \dots, l_{\ell_1+\ell_2,i})^T$.

For the “if” part, we consider x^i with $i \in \mathcal{I}(\ell_1)$. Since $\tilde{L} \geq_o 0$, $X\tilde{L} = AX$, and $\bar{1}\tilde{L} = \bar{0}$, we have $Ax^i = \sum_{p=1}^{\ell_1} l_{p,i} x^p + \sum_{q=1}^{\ell_2} l_{\ell_1+q,i} \hat{x}^q$, with $\sum_{p=1}^{\ell_1} l_{p,i} = 0$, and $l_{p,i} \geq 0$, for $p \neq i$. The argument for \hat{x}^j with $j \in \mathcal{I}(\ell_2)$ is similar. Then, according to Corollary 3.12, we have that \mathcal{P} is an invariant set for the continuous system.

For the “only if” part, according to Theorem 3.3, we know that there exists a matrix $L(\Delta t)$ and a scalar $\hat{\tau} > 0$, such that

$$XL(\Delta t) = (I - \Delta t A)^{-1} X, \quad \bar{1}L(\Delta t) = \bar{1}, \text{ for } 0 \leq \Delta t \leq \hat{\tau}. \quad (18)$$

Clearly, we have $\lim_{\Delta t \rightarrow 0} L(\Delta t) = I$. By reformulating (18), we have

$$X \frac{L(\Delta t) - I}{\Delta t} = AXL(\Delta t), \quad \bar{1} \frac{L(\Delta t) - I}{\Delta t} = \bar{0}.$$

Note that $\lim_{\Delta t \rightarrow 0} AXL(\Delta t) = AX$, thus $\tilde{L} := \lim_{\Delta t \rightarrow 0} \frac{L(\Delta t) - I}{\Delta t}$ exists. Since $L(\Delta t) \geq 0$, we have $\tilde{L} \geq_o 0$. The “only if” part is immediate by letting Δt approach 0. \square

Since the invariance conditions for a polyhedral cone given in the two different forms can be obtained by similar discussions as above. Here we only present these invariance conditions without providing the proofs.

Corollary 3.14. 1). A polyhedral cone $\mathcal{C}_{\mathcal{P}}$ given as in (5) is an invariant set for the continuous system (2) if and only if there exists an off-diagonal matrix $\tilde{H} \in \mathbb{R}^{m \times m}$, such that $\tilde{H}G = GA$.

2). A polyhedral cone $\mathcal{C}_{\mathcal{P}}$ given as in (6) is an invariant set for the continuous system (2) if and only if there exists an off-diagonal matrix $\tilde{L} \in \mathbb{R}^{\ell \times \ell}$, such that $X\tilde{L} = AX$, where $X = [\hat{x}^1, \dots, \hat{x}^{\ell}]$.

According to Theorem 3.13 and Corollary 3.14, verifying if a polyhedron given as in (4) or polyhedral cone given as in (6) is an invariant set for the continuous system (2) can be done by solving a series of linear optimization problems.

3.2. Ellipsoids

In this section, we consider the invariance condition for ellipsoids which are represented by a quadratic inequality.

3.2.1. Invariance Conditions for Discrete Systems

The S -lemma and Proposition 2.3 are our main tools to obtain the invariance condition of an ellipsoid for a discrete system. First, we present a technical lemma.

Lemma 3.15. For all $x \in \mathbb{R}^n$, $x^T Q x \geq \alpha$ holds if and only if $Q \succeq 0$, and $\alpha \leq 0$.

Proof. The “if” part is trivial, we only prove the “only if” part. Letting $x = 0$ yields $\alpha \leq 0$. Assume $Q \not\succeq 0$, then there exists an $\hat{x} \neq 0$, such that $\hat{x}^T Q \hat{x} = \beta < 0$. Let $x = \gamma \hat{x}$, $\gamma > \sqrt{\alpha/\beta}$, then $x^T Q x = \gamma^2 \beta < \alpha$, which contradicts to $x^T Q x \geq \alpha$. \square

Theorem 3.16. An ellipsoid \mathcal{E} given as in (7) is an invariant set for the discrete system (1) if and only if

$$\exists \mu \in [0, 1], \text{ such that } A^T Q A - \mu Q \preceq 0. \quad (19)$$

Proof. According to Proposition 2.3, to prove this theorem is equivalent to prove $\mathcal{E} \subseteq \mathcal{E}'$, where $\mathcal{E} = \{x \mid x^T Q x \leq 1\}$ and $\mathcal{E}' = \{x \mid x^T A^T Q A x \leq 1\}$. Clearly, $\mathcal{E} \subseteq \mathcal{E}'$ holds if and only if the following inequality system has no solution:

$$-x^T A^T Q A x + 1 < 0, \quad x^T Q x - 1 \leq 0, \quad \text{for all } x \in \mathbb{R}^n. \quad (20)$$

Note that the left sides of the two inequalities in (20) are both quadratic functions, thus, according to the S -lemma, we have that (20) has no solution is equivalent to that there exists $\mu \geq 0$, such that $-x^T A^T Q A x + 1 + \mu(x^T Q x - 1) \geq 0$, or equivalently,

$$x^T(\mu Q - A^T Q A)x \geq \mu - 1. \quad (21)$$

The theorem follows by applying Lemma 3.15 to (21). \square

We can also consider an ellipsoid as an invariant set for a system in the following perspective. Invariance of a bounded set for a system is possible only if the system is non-expansive, which means that for discrete system (1), all eigenvalues of A are in a closed unit disc of the complex plane. Then it becomes clear that (19) has a solution only if (1) is non-expansive. One can conclude from this that there is an invariant ellipsoid for (1) if and only if (19) has a solution for a positive definite Q . Moreover, the smallest μ solving (19) is the largest eigenvalue of $W A^T Q A W$, where W is the symmetric positive definite square root of Q^{-1} , i.e., $W^2 = Q^{-1}$.

In fact, the interval of the scalar μ in Theorem 3.16 can be explained in the following way. Assume $\mu \leq 0$, then $A^T Q A - \mu Q$ is always a positive definite matrix. On the other hand, assume $\mu > 1$, consider the discrete system $x_{k+1} = -x_k$, one can prove that $\{x \mid x^T Q x \leq 1\}$ is an invariant set for this system. However, in this case, we have $A^T Q A - \mu Q = (1 - \mu)Q$, which is always a positive definite matrix. Thus μ should be in the internal $[0, 1]$.

Apart from the simplicity, another advantage of the approach given in the proof of Theorem 3.16 is that it obtains a sufficient and necessary condition. Also, this approach highlights the close relationship between the theory of invariant sets and the Theorem of Alternatives, which is a fundamental result in optimization society.

Corollary 3.17. *Condition (19) holds if and only if*

$$\exists \nu \in [0, 1], \text{ such that } \tilde{Q} = \begin{pmatrix} Q^{-1} & A \\ A^T & \nu Q \end{pmatrix} \succeq 0. \quad (22)$$

Proof. First, $Q \succ 0$ yields $Q^{-1} \succ 0$. By Schur's lemma [10], $\tilde{Q} \succeq 0$ if and only if its Schur complement $\nu Q - A^T(Q^{-1})^{-1}A = \nu Q - A^T Q A \succeq 0$, i.e., (19) holds. \square

Corollary 3.18. *Condition (19) holds if and only if*

$$A^TQA - Q \preceq 0. \quad (23)$$

Proof. The “if” part is immediate by letting $\mu = 1$ in (19). For the “only if” part, we let $\nu = 1 - \mu$, which, by reformulating (19), yields $A^TQA - Q \preceq -\nu Q \preceq 0$, for $\nu \in [0, 1]$, where the second “ \preceq ” holds due to the fact that $\nu \geq 0$ and $Q \succ 0$. \square

The left side of (23) is called the Lyapunov operator [9] in discrete form in dynamical system. Corollary 3.18 is consistent with the invariance condition of an ellipsoid for discrete system given in [7, 9]. The invariance condition presented in [7] is the same as (23) without the equality. This is since contractivity rather than invariance of a set for a system is analyzed in [7]. Lyapunov method is used to derive condition (23) in [9]. Apparently, condition (23) is easier to apply than condition (19), since the former one involves only about the ellipsoid and the system.

The attentive reader may observe that the positive definiteness assumption for matrix Q is never used in the proof of Theorem 3.16. That assumption was only needed to ensure that the set \mathcal{S} is convex. Recall that the quadratic functions in the S -lemma are not necessarily convex, thus we can extend Theorem 3.16 to general sets which are represented by a quadratic inequality.

Theorem 3.19. *A set $\mathcal{S} = \{x \in \mathbb{R}^n \mid x^T Q x \leq 1\}$, where $Q \in \mathbb{R}^{n \times n}$, is an invariant set for the discrete system (1) if and only if*

$$\exists \mu \in [0, 1], \text{ such that } A^TQA - \mu Q \preceq 0. \quad (24)$$

The proof of Theorem 3.19 is the same as that of Theorem 3.16, so we do not duplicate that proof here. A trivial example that satisfy the condition in is given by choosing Q to be any indefinite matrix, $A = I$, and we choose $\mu = 1$. It is easy to see that for this choice condition (24) holds. Further exploring the implications of possibly using nonconvex and unbounded invariant sets is far from the main focus of our paper, so this topic remains the subject of further research.

3.2.2. Invariance Condition for Continuous System

We first present an interesting result about the solution of continuous system.

Proposition 3.20. *The solution of the continuous system (2) is on the boundary of the ellipsoid \mathcal{E} given as in (7) (or the Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given as in (8)) if and only if*

$$\sum_{i=0}^{k-1} \frac{1}{(k-1)!} \binom{k-1}{i} (A^i)^T Q A^{k-i-1} = 0, \text{ for } k = 2, 3, \dots \quad (25)$$

Proof. We consider only ellipsoids, and the proof for Lorenz cones is similar. The solution of (2) is given as $x(t) = e^{At}x_0$, thus $x(t) \in \partial\mathcal{E}$ if and only if $x_0^T (e^{At})^T Q e^{At} x_0 = 1$, which can be expanded, by substituting $e^{At} = \sum_{i=0}^{\infty} \frac{A^i}{i!} t^i$, as

$$\sum_{k=1}^{\infty} t^{k-1} x_0^T \tilde{Q}_{k-1} x_0 = 1, \text{ where } \tilde{Q}_{k-1} = \sum_{i=0}^{k-1} \frac{1}{(i)!(k-i-1)!} (A^i)^T Q A^{k-i-1},$$

for any $x_0^T Q x_0 = 1$ and $t \geq 0$. Thus, $\tilde{Q}_{k-1} = 0$, for $k \geq 2$. Also, note that $\frac{1}{(k-1)!} \binom{k-1}{i} = \frac{1}{(i)!(k-i-1)!}$, condition (25) is immediate. \square

In particular, when $k = 2$, condition (25) yields $A^T Q + Q A = 0$. The left hand side of this equation is called Lyapunov operator in continuous form. We now present invariance condition of an ellipsoid for the continuous system.

Theorem 3.21. *An ellipsoid \mathcal{E} given as in (7) is an invariant set for the continuous system (2) if and only if*

$$A^T Q + Q A \preceq 0. \quad (26)$$

Proof. According to Lemma 3.32, we have that condition (36) holds. Then $x^T (A^T Q + Q A) x \leq 0$, for every $x \in \partial\mathcal{E}$. For an arbitrary $y \in \mathbb{R}^n$, there exists an $x \in \partial\mathcal{E}$, such that $y = \gamma x$, for some $\gamma \in \mathbb{R}$. Then $y^T (A^T Q + Q A) y = \frac{1}{\gamma^2} x^T (A^T Q + Q A) x \leq 0$, which yields condition (26). \square

The presented method in the proof of Theorem 3.21 is simpler than the traditional Lyapunov method to derive the invariance condition. However, the approach in the proof cannot be used for Lorenz cones, since the origin is not in the interior of Lorenz cones.

3.3. Lorenz Cones

A Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given as in (8) also has a quadratic form, but the way to obtain the invariance condition of a Lorenz cone for discrete system is much more complicated than an ellipsoid. The difficulty is mainly due to the existence of the second constraint in (8).

3.3.1. Invariance Condition for Discrete System

The representation of the nonconvex set $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}}) = \{x \mid x^T Q x \leq 0\}$ involves only the quadratic form, which is almost the same as an ellipsoid, we can first derive the invariance condition of this set for discrete system. Recall that the S -lemma does not require that the quadratic functions have to be convex, thus the S -lemma is still valid for the nonconvex set.

Theorem 3.22. *The nonconvex set $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ is an invariant set for the discrete system (1) if and only if*

$$\exists \mu \geq 0, \text{ such that } A^T Q A - \mu Q \preceq 0. \quad (27)$$

Proof. The proof is closely following the ideas in the proof of Theorem 3.16. The only difference is that the right side in (21) is 0 rather than $1 - \mu$, which is why the condition $\mu \leq 1$ is absent in this case. \square

The invariance condition for $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ shown in (27) is similar to the one proposed by Loewy and Schneider in [19]. They proved by contradiction using the properties of copositive matrices that when the rank of A is greater than 1, $A\mathcal{C}_{\mathcal{L}} \subseteq \mathcal{C}_{\mathcal{L}}$ or $-A\mathcal{C}_{\mathcal{L}} \subseteq \mathcal{C}_{\mathcal{L}}$ if and only if (27) holds. They also concluded, see Lemma 3.1 in [19], that $A\mathcal{C}_{\mathcal{L}} \subseteq \mathcal{C}_{\mathcal{L}}$ if and only if there exist two vectors $x, y \in \mathcal{C}_{\mathcal{L}}$, such that $A = xy^T$, when the rank of A is 1. In fact, substituting such A into (27) and letting $\mu = 0$, we have $A^T Q A - \mu Q = (xy^T)^T Q (xy^T) = y(x^T Q x)y^T = (x^T Q x)yy^T$. It is clear that $x^T Q x \leq 0$ due to $x \in \mathcal{C}_{\mathcal{L}}$. Note that the matrix yy^T is a ranked one and positive semidefinite matrix with eigenvalues 0 and $y^T y$, thus we also have $A^T Q A - \mu Q \preceq 0$. Thus condition (27) is also satisfied when the rank of A is 1.

In fact, the interval of the scalar μ in condition (27) can also be considered in the following way. Assume $\mu < 0$, we choose $A = Q = \text{diag}\{-1, \dots, -1, 1\}$. Then the Lorenz cone is an invariant set for the system, since such Lorenz cone is a self-dual cone³. The left hand side in (27) is, however, now simplified

³A self-dual cone is a cone that coincides with its dual cone, where the dual cone for a cone \mathcal{C} is defined as $\{y \mid x^T y \geq 0, \forall x \in \mathcal{C}\}$.

to $(1 - \mu)Q$ that cannot be negative semidefinite because $\text{inertia}\{Q\} = \{n - 1, 0, 1\}$. Thus the scalar μ in (27) should be nonnegative.

It is hard to apply directly Schur lemma, which is different from an ellipsoid, to the case of a Lorenz cone, since the matrix Q for a Lorenz cone is neither positive nor negative semidefinite. However, in some special cases, e.g., $Q = \text{diag}\{-1, \dots, -1, 1\}$, where there exists lower dimension sub-matrix which is negative definite, the Schur complement of this sub-matrix is possible to compute such that a similar semidefinite problem as (22) can be obtained. In fact, the scalar μ is hard to eliminate, since a Lorenz cone is unbounded.

To find the scalar μ in (27) is essentially a semidefinite optimization (SDO) problem. Various celebrated SDO solvers, e.g., SeDuMi [26], CVX [13], and SDPT3 [27] have been shown robust performance in solving a SDO problems numerically.

Corollary 3.23. *If $\lambda_1(A^TQA) \leq 0$, then the Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given as in (8) is an invariant set for the discrete system (1).*

Corollary 3.23 gives a simple sufficient condition such that a Lorenz cone is an invariant set, but it is only valid when the A is a singular matrix. In fact, if A is nonsingular, by Sylvester's law of inertia [15], we have that $\lambda_1(A^TQA) > 0$.

The interval of the scalar μ in (27) can be tightened by incorporating the eigenvalues and eigenvectors of Q . Such a tighter condition is presented in Corollary 3.24.

Corollary 3.24. *If condition (27) holds, then*

$$\max \left\{ 0, \max_{1 \leq i \leq n-1} \left\{ \frac{u_i^T A^T Q A u_i}{\lambda_i} \right\} \right\} \leq \mu \leq \frac{u_n^T A^T Q A u_n}{\lambda_n}. \quad (28)$$

Proof. Multiplying condition (27) by u_i^T from the left and u_i from the right, we have $u_i^T A^T Q A u_i - \mu u_i^T Q u_i \leq 0$. Since $u_i^T Q u_i = \lambda_i u_i^T u_i = \lambda_i > 0$, for $i \in \mathcal{I}(n - 1)$, and $u_n^T Q u_n = \lambda_n < 0$, condition (28) follows immediately. \square

Corollary 3.24 presents tighter bounds for the scalar μ in (28) in terms of an algebraic form. The existence of a scalar μ implies that the upper bound should be no less than the lower bound in (28). However, this is not always true. We now present a geometrical interpretation of the interval of the scalar μ , that can be directly derived from Corollary 3.24.

Corollary 3.25. *The relationship between the vector Au_i , and the scalars $u_i^T A^T Q Au_i$, and μ are as follows:*

- *If $Au_n \notin \mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$, then μ does not exist.*
- *If $Au_i \in \mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ for $i \in \mathcal{I}(n-1)$, then*
 - *if $Au_n \in \partial\mathcal{C}_{\mathcal{L}} \cup (-\partial\mathcal{C}_{\mathcal{L}})$, then $\mu = 0$.*
 - *if $Au_n \in \text{int } \mathcal{C}_{\mathcal{L}} \cup (-\text{int } \mathcal{C}_{\mathcal{L}})$, then $\mu \in \left[0, \frac{u_n^T A^T Q Au_n}{\lambda_n}\right]$.*
- *Let $\mathcal{I} = \{i \mid Au_i \notin \mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})\}$. If the set $\mathcal{I} \subseteq \mathcal{I}(n-1)$ is nonempty, then*
 - *if $Au_n \in \partial\mathcal{C}_{\mathcal{L}} \cup (-\partial\mathcal{C}_{\mathcal{L}})$, then μ does not exist.*
 - *if $Au_n \in \text{int } (\mathcal{C}_{\mathcal{L}}) \cup (-\text{int } (\mathcal{C}_{\mathcal{L}}))$, then*
 - * *if there exist $i^* \in \mathcal{I}$, such that $\frac{u_{i^*}^T A^T Q Au_{i^*}}{\lambda_{i^*}} > \frac{u_n^T A^T Q Au_n}{\lambda_n}$, then μ does not exist.*
 - * *otherwise, $\mu \in \left[\max_{i \in \mathcal{I}} \left\{ \frac{u_i^T A^T Q Au_i}{\lambda_i} \right\}, \frac{u_n^T A^T Q Au_n}{\lambda_n}\right]$*

We now consider the invariance condition of a Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given as in (8), which is a convex set and can handle expansive systems, for

Lemma 3.26. [25] *A Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given as in (8) can be written as a linear transformation of the standard Lorenz cone $\mathcal{C}_{\mathcal{L}}^*$ as $\mathcal{C}_{\mathcal{L}} = T\mathcal{C}_{\mathcal{L}}^*$, where T is nonsingular*

$$T = \left[\frac{u_1}{\sqrt{\lambda_1}}, \dots, \frac{u_{n-1}}{\sqrt{\lambda_{n-1}}}, \frac{u_n}{\sqrt{-\lambda_n}} \right]. \quad (29)$$

Lemma 3.27. *A Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given as in (8) is an invariant set for the discrete system (1) if and only if the standard Lorenz cone $\mathcal{C}_{\mathcal{L}}^*$ is an invariant set for the following discrete system*

$$x_{k+1} = T^{-1} A T x_k, \quad (30)$$

where T is defined as (29).

Proof. The Lorenz cone $\mathcal{C}_{\mathcal{L}}$ is an invariant set for (1) if and only if $A\mathcal{C}_{\mathcal{L}} \subseteq \mathcal{C}_{\mathcal{L}}$. This holds if and only if $A T \mathcal{C}_{\mathcal{L}}^* \subseteq T \mathcal{C}_{\mathcal{L}}^*$, which is equivalent to $T^{-1} A T \mathcal{C}_{\mathcal{L}}^* \subseteq \mathcal{C}_{\mathcal{L}}^*$. \square

The invariance condition of a Lorenz cone for discrete systems is presented in Theorem 3.28. Although we have developed such invariance condition independently, it was brought to our attention recently that the invariance condition is the same as the one proposed by Aliluiko and Mazko in [1]. But our proof is more straightforward.

Theorem 3.28. *A Lorenz cone $\mathcal{C}_{\mathcal{L}}$ (or $-\mathcal{C}_{\mathcal{L}}$) given as in (8) is an invariant set for the discrete system (1) if and only if*

$$\exists \mu \geq 0, \text{ such that } A^TQA - \mu Q \preceq 0, \quad u_n^T Au_n \geq 0, \quad u_n^T AQ^{-1}A^T u_n \leq 0, \quad (31)$$

where u_n is the eigenvector corresponding to the unique negative eigenvalue λ_n of Q .

Proof. Since the proof for the Lorenz cone $-\mathcal{C}_{\mathcal{L}}$ can be derived by choosing u_n to be $-u_n$, we only present the proof for $\mathcal{C}_{\mathcal{L}}$. For an arbitrary $x \in \mathcal{C}_{\mathcal{L}}$, by Theorem 3.22, we have that $Ax \in \mathcal{C}_{\mathcal{L}}$ or $Ax \in -\mathcal{C}_{\mathcal{L}}$ if and only if condition (27) is satisfied. To ensure that only $Ax \in \mathcal{C}_{\mathcal{L}}$ holds, some additional conditions should be added.

According to Lemma 3.27, we consider $\mathcal{C}_{\mathcal{L}}^*$ and the discrete system (30), where the coefficient matrix, denoted by \tilde{A} , can be explicitly written as

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} u_1^T Au_1 & \cdots & \sqrt{-\frac{\lambda_1}{\lambda_n}} u_1^T Au_n \\ \vdots & \ddots & \vdots \\ \sqrt{-\frac{\lambda_n}{\lambda_1}} u_n^T Au_1 & \cdots & u_n^T Au_n \end{bmatrix}.$$

Then, according to Theorem 3.22, condition (27) implies

$$\exists \mu \geq 0, \text{ such that } (T^{-1}AT)^T \tilde{I} T^{-1}AT - \mu \tilde{I} \preceq 0, \quad (32)$$

where $\tilde{I} = \text{diag}\{1, \dots, 1, -1\}$. Note that $T^TQT = \tilde{I}$, condition (32) is equivalent to

$$\exists \mu \geq 0, \text{ such that } A^TQA - \mu Q \preceq 0.$$

Recall that we denote the i -th row of a matrix M by M_i^T . Also, the second constraint in the formulae of $\mathcal{C}_{\mathcal{L}}^*$ requires that every $x_k \in \mathcal{C}_{\mathcal{L}}^*$ implies that the last coordinate in x_{k+1} is nonnegative, i.e., $\tilde{A}_n^T x_k \geq 0$, for all $x_k \in \mathcal{C}_{\mathcal{L}}^*$. Note

that $\mathcal{C}_{\mathcal{L}}^*$ is a self-dual cone, we have $\tilde{A}_n^T x \geq 0$, for all $x \in \mathcal{C}_{\mathcal{L}}^*$ if and only if $\tilde{A}_n \in \mathcal{C}_{\mathcal{L}}^*$. Thus, we have

$$\tilde{A}_n^T = \sqrt{-\lambda_n} \left(\frac{1}{\sqrt{\lambda_1}} u_n^T A u_1, \frac{1}{\sqrt{\lambda_2}} u_n^T A u_2, \dots, \frac{1}{\sqrt{-\lambda_n}} u_n^T A u_n \right) = \sqrt{-\lambda_n} u_n^T A T. \quad (33)$$

Substituting (33) into the first inequality in the formulae of $\mathcal{C}_{\mathcal{L}}^*$, we have

$$-\lambda_n (T^T A^T u_n)^T \tilde{I} (T^T A^T u_n) \leq 0. \quad (34)$$

Since $\lambda_n < 0$ and $T \tilde{I} T^T = \sum_{i=1}^n \frac{u_i u_i^T}{\lambda_i} = Q^{-1}$, where the second equality is due to the spectral decomposition of Q^{-1} , we have that (34) is equivalent to $u_n^T A Q^{-1} A^T u_n \leq 0$. Also, substituting (33) into the second inequality in the formulae of $\mathcal{C}_{\mathcal{L}}^*$ yields $u_n^T A u_n \geq 0$. The proof is complete. \square

Corollary 3.29. *The inequality $u_n^T A Q^{-1} A^T u_n \leq 0$ holds if and only if $u_n^T A x \geq 0$, for all $x \in \mathcal{C}_{\mathcal{L}}$.*

Proof. Since $x^T Q x \leq 0$ can be written as $x^T U \Lambda^{\frac{1}{2}} \tilde{I} \Lambda^{\frac{1}{2}} U^T x \leq 0$, we have $x \in \mathcal{C}_{\mathcal{L}}$ if and only if $\Lambda^{\frac{1}{2}} U^T x \in \mathcal{C}_{\mathcal{L}}^*$. Similarly, since $Q^{-1} = U \Lambda^{-\frac{1}{2}} \tilde{I} \Lambda^{-\frac{1}{2}} U^T$, then $u_n^T A Q^{-1} A^T u_n \leq 0$ can be written as $u_n^T A U \Lambda^{-\frac{1}{2}} \tilde{I} \Lambda^{-\frac{1}{2}} U^T A^T u_n \leq 0$, which yields $\Lambda^{-\frac{1}{2}} U^T A^T u_n \in \mathcal{C}_{\mathcal{L}}^*$. Since the set $\mathcal{C}_{\mathcal{L}}^*$ is a self-dual cone, we have $(\Lambda^{-\frac{1}{2}} U^T A^T u)^T (\Lambda^{\frac{1}{2}} U^T x) \geq 0$, which can be simplified to $u_n^T A x \geq 0$, for all $x \in \mathcal{C}_{\mathcal{L}}$. \square

The normal plane of the eigenvector u_n that contains the origin separates \mathbb{R}^n into two half spaces. Corollary 3.29 presents a geometrical interpretation that A transforms the Lorenz cone $\mathcal{C}_{\mathcal{L}}$ to the half space that contains eigenvector u_n , i.e., $A \mathcal{C}_{\mathcal{L}} \subseteq \{y \mid u_n^T y \geq 0\}$. Moreover, note that $u_n^T A x = (A^T u_n)^T x$, which shows that the vector $A^T u_n$ is in the dual cone of $\mathcal{C}_{\mathcal{L}}$.

Corollary 3.30. *If condition (31) holds, then*

$$0 \leq \mu \leq \frac{u_n^T A^T Q A u_n}{\lambda_n}. \quad (35)$$

Proof. The proof is analogous to the one given in the proof of Corollary 3.24. \square

The interval for the scalar μ in condition (35) is wider but simpler than the one presented in Corollary 3.24. Analogous to Corollary 3.25, we present an intuitive geometrical interpretation of μ for Lorenz cones.

Corollary 3.31. *The relationship between the vector Au_n , and the scalars $u_n^T A^T Q Au_n$, and μ are as follows.*

- *If $Au_n \notin \mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$, then μ does not exist.*
- *If $Au_n \in \partial\mathcal{C}_{\mathcal{L}} \cup (-\partial\mathcal{C}_{\mathcal{L}})$, then $\mu = 0$.*
- *If $Au_n \in \text{int}(\mathcal{C}_{\mathcal{L}}) \cup (-\text{int}(\mathcal{C}_{\mathcal{L}}))$, then $\mu \in \left[0, \frac{u_n^T A^T Q Au_n}{\lambda_n}\right]$.*

3.3.2. Invariance Condition for Continuous System

Now we consider the invariance condition of Lorenz cones for the continuous system. The following invariance conditions is first given by Stern and Wolkowicz [25], where they consider only Lorenz cones and their proof is using the concept of the cross-positivity. Here we present a simple proof.

Lemma 3.32. [25] *An ellipsoid \mathcal{E} given in the form of (7) (or a Lorenz cone $\mathcal{C}_{\mathcal{L}}$ given in the form of (8)) is an invariant set for the continuous system (2) if and only if*

$$(Ax)^T Qx \leq 0, \text{ for all } x \in \partial\mathcal{E} \text{ (or } x \in \partial\mathcal{C}_{\mathcal{L}}). \quad (36)$$

Proof. We consider only ellipsoids, and it is analogous for Lorenz cones. Note that $\partial\mathcal{E} = \{x \mid x^T Qx = 1\}$, thus the outer normal vector of \mathcal{E} at $x \in \partial\mathcal{E}$ is Qx . Then we have $\mathcal{T}_{\mathcal{E}}(x) = \{y \mid y^T Qx \leq 0\}$, thus this theorem follows by Theorem 2.7. \square

We also need to analyze the eigenvalue of a sum of two symmetric matrices for the invariance conditions for continuous systems. And the following lemma is a useful tool in our analysis.

Lemma 3.33. (Weyl [15]) *Let $M, N \in \mathbb{R}^{n \times n}$ be two symmetric matrices. Then*

$$\lambda_{i+j-1}(M+N) \leq \lambda_i(M) + \lambda_j(N) \leq \lambda_{i+j-n}(M+N), \text{ for } 1 \leq i, j, i+j-1 \leq n. \quad (37)$$

The left and right inequalities in (37) are called the Weyl inequality and the dual Weyl inequality, respectively. According to Lemma 3.33, we immediately have the following lemma that shows a fact that the spectrum of a matrix is stable under a small perturbation by another matrix.

Lemma 3.34. *Let M and N be two symmetric matrices. Then*

- *if there exists a $\hat{\tau} > 0$, such that $M + \tau N \preceq 0$, for $0 < \tau \leq \hat{\tau}$, then $M \preceq 0$.*
- *if $M \prec 0$, then there exists a $\hat{\tau} > 0$, such that $M + \tau N \preceq 0$, for $0 < \tau \leq \hat{\tau}$.*

Proof. We consider only the first one, and it is analogous for the second one. If $\lambda_1(M) \leq 0$, then we are done. If $\lambda_n(N) \geq 0$, we have $M \preceq -\tau N \preceq 0$, then we are also done. Then for $\lambda_n(N) = \gamma < 0$ and $\lambda_1(M) = \delta > 0$, we can find a sufficiently small τ , e.g., $0 < \tau < \min\{\hat{\tau}, -\frac{\delta}{\gamma}\}$, such that $\delta + \tau\gamma > 0$, i.e., $\lambda_1(M) + \lambda_n(\tau N) > 0$. By the dual Weyl inequality in (37) we have $\lambda_1(M + \tau N) \geq \lambda_1(M) + \lambda_n(\tau N) > 0$, which contradicts to $M + \tau N \preceq 0$. \square

Similar to the case for discrete system, we first consider the invariance condition of the nonconvex set $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ for the continuous system.

Theorem 3.35. *The nonconvex set $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ is an invariant set for the continuous system (2) if and only if*

$$\exists \eta \in \mathbb{R}, \text{ such that } A^T Q + Q A - \eta Q \preceq 0. \quad (38)$$

Proof. For the “if” part, i.e., condition (38) holds, then for every $x \in \partial\mathcal{C}_{\mathcal{L}} \cup (-\partial\mathcal{C}_{\mathcal{L}})$, we have $(Ax)^T Q x = (Ax)^T Q x - \frac{\eta}{2} x^T Q x = \frac{1}{2} x^T (A^T Q + Q A - \eta Q) x \leq 0$. Thus, by Lemma 3.32, the set $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ is an invariant set for continuous system.

Next, we prove the “only if” part. For $x_k \in \mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$, there exists a $\hat{\tau} > 0$, such that $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ is also an invariant set for $x_{k+1} = (I - A\Delta t)^{-1} x_k$, for every $0 \leq \Delta t \leq \hat{\tau}$. By Theorem 3.22 and $(I - A\Delta t)^{-1} = I + A\Delta t + A^2\Delta t^2 + \dots$, we have

$$\exists \mu(\Delta t) \geq 0, \text{ such that } \frac{1 - \mu(\Delta t)}{\Delta t} Q + (A^T Q + Q A) + \Delta t K(\Delta t) \preceq 0,$$

where $K(\Delta t) = (A^T Q A + A^{2T} Q + Q A^2) + \Delta t (A^{2T} Q A + A^T Q A^2 + A^{3T} Q + Q A^3) + \dots$. Since Q and A are constant matrices, and applying the fact that $\|M + N\| \leq \|M\| + \|N\|$ and $\|MN\| \leq \|M\| \|N\|$, we have

$$\begin{aligned} \|K(\Delta t)\| &\leq \sum_{i=3}^{\infty} i \|Q\| \|A\|^{i-1} (\Delta t)^{i-3} = \|Q\| \|A\|^2 \sum_{i=0}^{\infty} (i+3) (\Delta t \|A\|)^i \\ &= \|Q\| \|A\|^2 \frac{3 - 2\Delta t \|A\|}{(1 - \Delta t \|A\|)^2} \leq 8 \|Q\| \|A\|^2, \end{aligned}$$

where $\Delta t \leq 0.5\|A\|^{-1}$ such that $(3 - 2\Delta t\|A\|)/(1 - \Delta t\|A\|)^2 \leq 8$. Also, applying the relationship between spectral radius $\rho(A)$ and its induced norm, $\rho(A) \leq \|A\|$ (see [12]), to $K(\Delta t)$, we have

$$|\lambda_i(K(\Delta t))| \leq \rho(K(\Delta t)) \leq \|K(\Delta t)\| \leq 8\|Q\|\|A\|^2, \text{ for } i \in \mathcal{I}(n),$$

i.e., the eigenvalues of $K(\Delta t)$ are bounded. Then according to Lemma 3.34, we have

$$\exists \mu(\Delta t) \geq 0, \text{ such that } \frac{1 - \mu(\Delta t)}{\Delta t}Q + A^TQ + QA \preceq 0,$$

which, by letting $\eta = \frac{\mu(\Delta t)-1}{\Delta t}$, is the same as (38). The proof is complete. \square

The approach in the proof of Theorem 3.35 can be also used to prove Theorem 3.21. The only remaining invariance condition is the one of a Lorenz cone for continuous system.

Theorem 3.36. *A Lorenz cone $\mathcal{C}_{\mathcal{L}}$ (or $-\mathcal{C}_{\mathcal{L}}$) is an invariant set for the continuous system (2) if and only if there exists $\eta \in \mathbb{R}$, such that*

$$\exists \eta \in \mathbb{R}, \text{ such that } A^TQ + QA - \eta Q \preceq 0. \quad (39)$$

Proof. Consider the continuous system with $x_0 \in \mathcal{C}_{\mathcal{L}}$, according to Theorem 3.35, the trajectory $x(t)$ will stay in $\mathcal{C}_{\mathcal{L}} \cup (-\mathcal{C}_{\mathcal{L}})$ if condition (38) is satisfied. If $x(t)$ would move over to $-\mathcal{C}_{\mathcal{L}}$, then $x(t)$ must go through the origin, i.e., $x(t^*) = 0$ for some $t^* \geq 0$. Note that $x(t) = e^{A(t-t^*)}x(t^*) = 0$ for any $t > t^*$, i.e., the origin is an equilibrium point, which means $\mathcal{C}_{\mathcal{L}}$ is an invariant set for the continuous system. Thus the theorem is immediate. \square

In fact, a direct proof of Theorem 3.36 can be given as follows: one can also prove that the second and third conditions in (31) by choosing sufficiently small Δt . To be specific, for the second condition in (31), we have

$$u_n^T(I - \Delta tA)^{-1}u_n \geq 0, \text{ if and only if } \|u_n\|^2 + \sum_{i=1}^{\infty}(\Delta t)^i u_n^T A^i u_n \geq 0, \quad (40)$$

where the second term, when $\Delta t < \|A\|^{-1}$, can be bounded as follows: $|\sum_{i=1}^{\infty}(\Delta t)^i u_n^T A^i u_n| \leq \|u_n\|^2 \frac{\Delta t\|A\|}{(1-\Delta t\|A\|)}$. Thus, we can choose $\Delta t < 0.5\|A\|^{-1}$,

such that condition (40) holds. Similarly, the third condition in (31) can be transformed to

$$u_n^T(I - \Delta t A)^{-1} Q^{-1} (I - \Delta t A)^{-T} u_n \leq 0, \text{ if and only if } \frac{1}{\lambda_n} \|u_n\|^2 + K(\Delta t) \leq 0, \quad (41)$$

where we use the fact that u_n is the eigenvector corresponding to the eigenvalue λ_n^{-1} of Q^{-1} , and $K(\Delta t) = \Delta t u_n^T (A Q^{-1} + Q^{-1} A^T) u_n + (\Delta t)^2 u_n^T (A Q^{-1} A + A^2 Q^{-1} + Q^{-1} A^2 T) u_n + \dots$. We note that $\text{inertia}\{Q\} = \{n - 1, 0, 1\}$ implies $\text{inertia}\{Q^{-1}\} = \{n - 1, 0, 1\}$, then we have that Q^{-1} exists, which yields the following: $|K(\Delta t)| \leq \|u\|^2 (2\Delta t \|A\| \|Q^{-1}\| + 3\Delta t^2 \|A\|^2 \|Q^{-1}\| + \dots) = \|u\|^2 \|Q^{-1}\| \frac{2\Delta t \|A\| - (\Delta t \|A\|)^2}{(1 - \Delta t \|A\|)^2}$. We can choose $\Delta t \leq \min\{0.5 \|A\|^{-1}, (\|A\| (1 - 4\lambda_n \|Q^{-1}\|)^{-1})\}$, such that (41) holds. In fact,

$$\begin{aligned} \frac{1}{\lambda_n} \|u_n\|^2 + K(\Delta t) &\leq \|u_k\|^2 \left(\frac{1}{\lambda_n} + \|Q^{-1}\| \frac{2\Delta t \|A\| - (\Delta t \|A\|)^2}{(1 - \Delta t \|A\|)^2} \right) \\ &\leq \|u\|^2 \left(\frac{1}{\lambda_n} + \|Q^{-1}\| \frac{4\Delta t \|A\|}{1 - \Delta t \|A\|} \right) \leq 0. \end{aligned}$$

Condition (39) is the same as the one presented in [25], whose proof is much more complicated than ours. Finding the value of η in Theorem 3.35 and 3.36 is essentially a semidefinite optimization problem. For example, we can use the following semidefinite optimization problem:

$$\begin{aligned} \max \quad &\eta \\ \text{s.t.} \quad &A^T Q + Q A - \eta Q \preceq 0. \end{aligned} \quad (42)$$

When the optimal solution η^* is nonnegative, we can claim that the Lorenz cone is an invariant set for the continuous system. Various celebrated SDO solvers, e.g., SeDuMi, CVX, and SDPT3, can be used to solve SDO problem (42).

Corollary 3.37. *If condition (38) and (39) hold, then*

$$\max_{1 \leq i \leq n-1} \{u_i^T (A^T + A) u_i\} \leq \eta \leq u_n^T (A^T + A) u_n. \quad (43)$$

Proof. The proof is similar to the one presented in the proof of Corollary 3.24 by noting that $u_i^T (A^T Q + Q A) u_i = 2(A u_i)^T Q u_i$, and $Q u_i = \lambda_i u_i$. \square

4. Examples

In this section, we present some simple examples to illustrate the invariance conditions presented in Section 3. Since it is straightforward for discrete systems, we only present examples for continuous systems. The following two examples consider polyhedral sets for continuous systems.

Example 4.1. Consider the polyhedron $\mathcal{P} = \{(\xi, \eta) \mid \xi + \eta \leq 1, -\xi + \eta \leq 1, \xi - \eta \leq 1, -\xi - \eta \leq 1\}$, and the continuous system $\dot{\xi} = -\xi, \dot{\eta} = -\eta$.

The solution of the system is $\xi(t) = \xi_0 e^{-t}, \eta(t) = \eta_0 e^{-t}$, so $(\xi(t), \eta(t)) \in \mathcal{P}$ for all $t \geq 0$, i.e., the polyhedron is an invariant set for the continuous system. This can also be verified by Theorem 3.6. We have

$$H = -I_4, G = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, A = -I_2,$$

which satisfy $HG = GA$ and $Hb \leq 0$. Thus Theorem 3.6 yields that \mathcal{P} is an invariant set for this continuous system.

Example 4.2. Consider the polyhedral cone $\mathcal{C}_{\mathcal{P}}$ generated by the extreme rays $x^1 = (1, 1, 1)^T, x^2 = (-1, 1, 1)^T, x^3 = (1, -1, 1)^T$, and $x^4 = (-1, -1, 1)^T$, and the continuous system $\dot{\xi} = \xi, \dot{\eta} = \eta, \dot{\zeta} = \zeta$.

The solution of the system is $\xi(t) = \xi_0 e^t, \eta(t) = \eta_0 e^t, \zeta(t) = \zeta_0 e^t$, thus one can easily verify that the polyhedral cone is an invariant set for this continuous system. This can also be verified by Corollary 3.14. We have

$$X = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tilde{L} = I_4, A = I_3,$$

which satisfy that $X\tilde{L} = AX$. Thus Corollary 3.14 yields that $\mathcal{C}_{\mathcal{P}}$ is an invariance set for this continuous system.

The following two examples consider ellipsoids and Lorenz cones for continuous systems.

Example 4.3. Consider the ellipsoid $\mathcal{E} = \{(\xi, \eta) \mid \xi^2 + \eta^2 \leq 1\}$, and the system $\dot{\xi} = -\eta, \dot{\eta} = \xi$.

The solution of the system is $\xi(t) = \alpha \cos t + \beta \sin t$ and $\eta(t) = \alpha \sin t - \beta \cos t$, where α, β are two parameters depending on the initial condition. The solution trajectory is a circle, thus the system is invariant on this ellipsoid. Also, we have

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Q = I_2, \quad A^T Q + Q A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \preceq 0,$$

which shows that, according to Theorem 3.21, the ellipsoid is an invariant set for this continuous system.

Example 4.4. Consider the Lorenz cone $\mathcal{C}_{\mathcal{L}} = \{(\xi, \eta, \zeta) \mid \xi^2 + \eta^2 \leq \zeta^2, \zeta \geq 0\}$, and the system $\dot{\xi} = \xi - \eta, \dot{\eta} = \xi + \eta, \dot{\zeta} = \zeta$.

The solution is $\xi(t) = e^t(\alpha \cos t + \beta \sin t)$, $\eta(t) = e^t(\alpha \sin t - \beta \cos t)$ and $\zeta(t) = \gamma e^t$, where α, β, γ are three parameters depending on the initial condition. It is easy to verify that this Lorenz cone is an invariant set for the continuous system. Also, by letting $\eta \leq -2$, we have

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = I_3, \quad A^T Q + Q A + \eta Q = \begin{bmatrix} \eta + 2 & 0 & 0 \\ 0 & \eta + 2 & 0 \\ 0 & 0 & \eta + 2 \end{bmatrix} \preceq 0,$$

which shows that, according to Theorem 3.36, the Lorenz cone is an invariant set for this continuous system.

5. Conclusions

The study of invariant sets for dynamical systems is an important topic both for the theory and computational practice in control theory and numerical analysis. In this paper, we consider four convex sets as invariant set for both linear continuous and discrete systems, which have significant implementations in the real world problems. Specifically, four convex sets are polyhedra, polyhedral cones, ellipsoids, and Lorenz cones, all of which are extensively studied in optimization theory. Sufficient and necessary conditions under which a discrete or continuous system is invariant on each of these convex sets are presented in a general, and unified approach. We first consider discrete systems, followed by continuous systems. We develop the connection between discrete systems and continuous systems by using the forward or backward Euler methods. This paper not only presents the invariance

conditions of the four convex sets for continuous and discrete systems by using simple proofs, but also establishes a framework, which may be used for other convex sets as invariant sets, to derive the invariance conditions for both continuous and discrete systems.

We also show that by applying the S-lemma one can extend invariance conditions to any set represented by a quadratic inequality. Such sets include nonconvex and unbounded sets. The future research interests mainly focus on two directions. The first one is extending the results in our paper to nonlinear dynamical systems and other sets. The second one is exploring the applications of the results in our paper in control and other fields.

Acknowledgments

This research is supported by a Start-up grant of Lehigh University and by TAMOP-4.2.2.A-11/1KONV-2012-0012: Basic research for the development of hybrid and electric vehicles. The TAMOP Project is supported by the European Union and co-financed by the European Regional Development Fund.

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