

The constant objective value property for combinatorial optimization problems

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Abstract

Given a combinatorial optimization problem, we aim at characterizing the set of all instances for which every feasible solution has the same objective value.

Our central result deals with multi-dimensional assignment problems. We show that for the axial and for the planar d -dimensional assignment problem instances with constant objective value property are characterized by sum-decomposable arrays. We provide a counterexample to show that the result does not carry over to general d -dimensional assignment problems.

Our result for the axial d -dimensional assignment problems can be shown to carry over to the axial d -dimensional transportation problem. Moreover, we obtain characterizations when the constant objective value property holds for the minimum spanning tree problem, the shortest path problem and the minimum weight maximum cardinality matching problem.

Keywords. Constant objective value; admissible transformation; multi-dimensional assignment problem; sum-decomposable array.

1 Introduction

In this paper we will deal with combinatorial optimization problems of the following form. We are given a ground set $F = \{1, \dots, n\}$, a real cost vector $c = (c(1), \dots, c(n))$ and a set of feasible solutions $\mathcal{F} \subseteq 2^{\{1, \dots, n\}}$.

The *objective value* of a feasible solution $F \in \mathcal{F}$ is given by the so-called *sum objective function*

$$c(F) := \sum_{i \in F} c(i).$$

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The goal is to find a feasible solution F^* such that $c(F^*)$ is minimal.

The traveling salesman problem, the linear assignment problems, the shortest path problem, Lawler's quadratic assignment problem and many other well-known combinatorial optimization problems fall into the class of combinatorial optimization problems described above.

Definition 1.1. *We say that an instance of a combinatorial optimization problem has the constant objective value property (COVP) if every feasible solution has the same objective value.*

For various classes of combinatorial optimization problems our goal is to characterize the set of instances with the COVP, or in other words the space of all cost vectors for which every feasible solution has the same objective value.

Related results. The constant objective value property is closely connected to the notion of *admissible transformations* introduced in 1971 by Vo-Khac [20].

Definition 1.2. *A transformation T of the cost vector C to the new cost vector $\tilde{C} = (\tilde{c}(1), \tilde{c}(2), \dots, \tilde{c}(n))$ is called admissible with index $z(T)$, if*

$$c(F) = \tilde{c}(F) + z(T) \quad \text{for all } F \in \mathcal{F}.$$

Note that admissible transformations preserve the relative order of the objective values of all feasible solutions. It is well known that admissible transformations can be used as optimality criterion and to obtain lower bounds which are useful for hard combinatorial optimization problems.

Consider the combinatorial optimization problem $\min_{F \in \mathcal{F}} c(F)$. Let T be an admissible transformation from the original cost vector c to the new cost vector \tilde{c} such that there exists a feasible solution F^* with the following properties:

- (i) $\tilde{c}(i) \geq 0$ for all $i \in \{1, \dots, n\}$,
- (ii) $\tilde{c}(F^*) = 0$.

Then F^* is an optimal solution with the objective value $z(T)$. If the condition (ii) is not satisfied or we cannot prove it to hold, then $z(T)$ gives a lower bound.

For the type of combinatorial optimization problem stated above there is a one-to-one correspondence between admissible transformations that transform the cost vector $(c(1), \dots, c(n))$ to $(\tilde{c}(1), \dots, \tilde{c}(n))$, and cost vectors $B = (b(1), b(2), \dots, b(n))$ that fulfill the COVP. The correspondence is obtained by $c(i) = \tilde{c}(i) + b(i)$, for all i . Then the index of the corresponding admissible transformation is $z(T) = \sum_{i \in F} b(i)$ for any $F \in \mathcal{F}$.

The correspondence between the COVP and admissible transformations provides a further source of motivation for investigating COVP type characterizations.

The notion of admissible transformations can be generalized to the algebraic setting and applied to a wider class of combinatorial optimization problems including the case of bottleneck objective functions, see [6]. Note, however, that for the bottleneck objective function which is given by $c(F) = \max_{i \in F} c(i)$ there is no one-to-one correspondence between the COVP and admissible transformations.

Berenguer [2] characterized the set of all admissible transformations for the travelling salesman problem and the multiple salesmen version. All admissible transformations for the TSP are obtained by adding real values to rows and columns of the distance matrix. In view of the correspondence mentioned above this result can be rephrased as a result on the COVP for the TSP as follows (this has been noted already by Gilmore, Lawler and Shmoys [11]).

An $n \times n$ real matrix $C = (c_{ij})$ is called *sum matrix* if there exist two real n -dimensional vectors $U = (u_i)$ and $V = (v_i)$ such that

$$c_{ij} = u_i + v_j \quad \text{for all } i, j \in \{1, \dots, n\}. \quad (1)$$

Theorem 1.1. (Berenguer [2], Gilmore et al. [11])

The TSP instance with the $n \times n$ cost matrix $C = (c_{ij})$ has the COVP if and only if C is a sum matrix.

For the TSP the diagonal entries of C do not play a role and can be ignored. Berenguer's proof works for the linear assignment problem as well, i.e. an instance of the linear assignment problem with cost matrix $C = (c_{ij})$ has the COVP if and only if C is a sum matrix.

Some classes of admissible transformations for different types of assignment problems are listed by Burkard [4]. However, no COVP type characterizations are provided.

Organization of the paper. The structure of the rest of the paper is as follows. In Section 2 we investigate the problem of characterizing the instances with the COVP for multi-dimensional assignment problems. We show that for the multi-dimensional axial and planar case the cost arrays with the COVP are precisely the class of sum-decomposable arrays which can be represented as sums of lower dimensional arrays of appropriate dimension (for the precise definition see Section 2). We furthermore provide a counterexample which shows that sum-decomposability is not necessarily required for the COVP to hold for general multi-dimensional assignment problems. In Section 3 the result for the axial d -dimensional assignment problem is carried over to the axial d -dimensional transportation problem. Finally, in Section 4 we deal with COVP characterizations for the minimum

spanning tree problem, the shortest path problem and the minimum weight maximum cardinality matching problem.

2 The COVP for d -dimensional assignment problems

Berenguer's result for the classical linear assignment problem motivated us to ask for COVP type characterizations for multi-dimensional assignment problems.

Multi-dimensional assignment problems are a generalization of the classical linear assignment problem to more than 2 dimensions.

2.1 General multi-dimensional assignment problems

Two classical ways of generalizing the notion of assignments to three dimensions are the so-called *axial 3-dimensional (or 3-index) assignment problem* and the *planar 3-dimensional (or 3-index) assignment problem*. A further generalization is obtained by the class of d -dimensional assignment problems defined in the sequel. For more on the topic of assignment problems see the book by Burkard et al. [5] and the references cited therein.

A general multi-dimensional assignment problem is specified by two parameters d and s , where d is the number of indices and s describes the number of fixed indices in the constraints. Informally speaking, we want to find a set of n^s elements of a d -dimensional $n \times n \times \dots \times n$ array C with minimum total sum, such that for every s fixed indices of C exactly one element is chosen.

Formally, the (d, s) assignment problem, (d, s) -AP for short, can be stated in the following way.

Definition 2.1. *Let d and s be integers with $0 < s < d$. The input of the (d, s) -AP consists of an integer $n \geq 1$ and a d -dimensional $n \times n \times \dots \times n$ cost array C which associates the cost $c(i_1, i_2, \dots, i_d)$ to the d -tuple $(i_1, i_2, \dots, i_d) \in \{1, \dots, n\}^d$. Let \mathcal{Q}_s be the set of all subsets of $K = \{1, \dots, d\}$ with cardinality s , i.e., $\mathcal{Q}_s = \{Q: Q \subset K, |Q| = s\}$. For any set $Q = \{q_1, q_2, \dots, q_s\} \in \mathcal{Q}_s$ of fixed indices and any s -tuple $t = (i_{q_1}, \dots, i_{q_s}) \in \{1, \dots, n\}^s$, let $T(Q, t)$ be the set of d -tuples t' such that t' and t coincide with respect to the fixed indices. The general (d, s)*

assignment problem (d, s) -AP can be stated as

$$\begin{aligned} & \min \sum_{i_1=1}^n \cdots \sum_{i_d=1}^n c(i_1, i_2, \dots, i_d) x(i_1, i_2, \dots, i_d) \\ & \sum_{(i_1, \dots, i_d) \in T(Q, (j_{q_1}, \dots, j_{q_s}))} x(i_1, i_2, \dots, i_d) = 1 \text{ for all } Q = \{q_1, \dots, q_s\} \in \mathcal{Q}_s \text{ and} \\ & \text{all } (j_{q_1}, \dots, j_{q_s}) \in \{1, \dots, n\}^s \\ & x(i_1, i_2, \dots, i_d) \in \{0, 1\} \text{ for all } (i_1, i_2, \dots, i_d) \in \{1, \dots, n\}^d. \end{aligned}$$

Note that in each of the equality constraints above the sum essentially extends over $d - s$ variables (corresponding to the free indices from the set $K \setminus Q$).

Let $X = (x(i_1, i_2, \dots, i_d))$ be a feasible solution of the integer program stated above. Then the set $F = \{(i_1, i_2, \dots, i_d) : x(i_1, i_2, \dots, i_d) = 1\}$ is a feasible solution of the (d, s) -AP and the value $\sum_{(i_1, \dots, i_d) \in F} c(i_1, \dots, i_d)$ is the cost (objective value) of a feasible solution F . Hence the (d, s) -AP fits into the setting of combinatorial optimization problems introduced in Section 1.

Using the (d, s) -AP notation, the classical linear assignment problem is the $(2, 1)$ -AP, while the axial and the planar 3-dimensional assignment problems correspond to the $(3, 1)$ -AP and the $(3, 2)$ -AP, respectively. More generally, we refer to the $(d, 1)$ -AP as *axial d -dimensional assignment problem*, and to the $(d, d - 1)$ -AP as *planar d -dimensional assignment problem*.

Let us remark that for $d \geq 4$, there is no consensus in the literature which problem version is referred to as planar d -dimensional assignment problem. Our decision to refer to the $(d, d - 1)$ -AP in which all constraints involve single sums as planar problem is in accordance with the axial/planar nomenclature which has been introduced in the original paper by Schell [18] and is used by Spieksma [19], Frieze and Sorkin [9] and others. A group of authors around Appa, see e.g. [1] refer to the $(d, 2)$ -AP as planar problem.

The axial and the planar 3-dimensional assignment problem are known to be NP-hard [10, 12]. As a consequence thereof both the axial and the planar d -dimensional assignment problems are NP-hard for all $d \geq 3$.

The following observation collects a few well known facts about the structure of the set of feasible solutions of the (d, s) -AP. These will turn out to be helpful in later parts of the paper.

Observation 2.1.

- (i) *The feasible solutions of the $(d, 1)$ -AP of size n can be represented by a set of $d - 1$ permutations over $\{1, \dots, n\}$.*
- (ii) *A set of d -tuples F is a feasible solution of the $(d, d - 1)$ -AP of size n if and only if F “contains” n pairwise disjoint feasible solutions of the $(d - 1, d - 2)$ -AP.*

(iii) *The feasible solutions of the $(d, 2)$ -AP for $d \geq 3$ correspond to sets of $d - 2$ mutually orthogonal Latin squares.*

Proof. Ad (i): Note that the set of feasible solutions contains n elements and can be written as $F = \{(i, \phi_1(i), \dots, \phi_{d-1}(i))\}$ where each ϕ_k is a permutation over $\{1, \dots, n\}$.

Ad (ii): Define n sets of $(d - 1)$ -tuples obtained from F by fixing one index, for example the first one: $F_i = \{(a_2, a_3, \dots, a_d) : (i, a_2, a_3, \dots, a_d) \in F\}$ for $i = 1, 2, \dots, n$. Then every F_i is a feasible solution of the $(d - 1, d - 2)$ -AP because fixing $d - 2$ indices in F_i corresponds to fixing $d - 1$ indices in F . Also, if there are F_i and F_j that are not disjoint, then there would be two elements in F that coincide on $d - 1$ indices which is a contradiction. The same construction works in the other direction too.

Ad (iii): Assume that the set $F = \{(i_1, i_2, \dots, i_d)\}$ of d -tuples is a feasible solution of the $(d, 2)$ -AP. We can represent F as an $n \times n$ table T with $(d - 2)$ -tuples as entries in the following way: The $(d - 2)$ -tuple (i_3, \dots, i_d) is the entry in row i_1 and column i_2 of T if and only if (i_1, \dots, i_d) is an element of F . Since F is a feasible solution of the $(d, 2)$ -AP, each row and each column of T contain every integer from 1 to n exactly once on the k -position for all $k = 1, \dots, d - 2$. Moreover, each $(d - 2)$ -tuple of pairwise distinct integers from $\{1, \dots, n\}$ appears exactly once in T . Hence T can be interpreted as a set of $d - 2$ mutually orthogonal Latin squares (the k -th component of the entries of T yields the k -th Latin square). \square

What distinguishes the general (d, s) -AP from the special cases with $s = 1$ (axial problem) and with $s = d - 1$ (planar problem) is that there does not need to exist feasible solutions for every value of n and that not much is known on the structure of the set of feasible solutions for the (d, s) -AP for general n , cf. Appa et al. [1].

Infeasible instances and instances with very few feasible solutions provide obstacles to our intended COVP characterization as we aim for conditions that are sufficient and necessary for the property that all feasible solutions have the same objective value. For this reason the feasibility topic for the (d, s) -AP plays a role for us and we briefly review a few basic results from the literature.

The question for which values n the general (d, s) -AP has feasible solutions is a very difficult problem which is still open for many combinations of (d, s) and is related to a number of difficult problems in combinatorics. More specifically, the general (d, s) -AP of size n has a feasible solution if there exists an s -transversal design with d groups of size n , or equivalently there exists an orthogonal array $\text{OA}(n, d, s)$ of index 1, strength s and order n [7].

The existence of pairs of orthogonal Latin squares (also known as Graeco-Latin squares) has been a famous open problem for a long time going back

to Euler until it finally has been proved by Bose et al. [3] that Graeco-Latin squares and thus feasible solutions of the $(4, 2)$ -AP exist for every $n \geq 3$ except for $n = 6$.

The number of mutually orthogonal Latin squares of size n (note the connection to the feasibility of the $(d, 2)$ -AP) is unknown for general n . It is known however that this number is at most $n - 1$ and that the upper bound is achieved if n is a prime power. It is also known that there exist $n - 1$ mutually orthogonal Latin squares if and only if there exists a projective plane of order n [7].

2.2 Sum-decomposable arrays

In this subsection we investigate the vector spaces of *sum-decomposable arrays*. These will occur as solutions of various COVP characterization problems. Sum-decomposable arrays generalize the concept of sum matrices to higher dimensions.

Informally, a d -dimensional $n \times n \times \cdots \times n$ real array C is sum-decomposable with parameters d and s (and size n) if C can be obtained as sum of $\binom{d}{s}$ s -dimensional arrays, one for each subset of $\{1, \dots, d\}$ of size s . For example, in the case $d = 3$ and $s = 2$, $C = (c_{ijk})$ is sum-decomposable if there exist three two-dimensional real arrays $A = (a_{ij})$, $B = (b_{ij})$ and $D = (d_{ij})$ such that $c_{ijk} = a_{ij} + b_{ik} + d_{jk}$. A formal definition follows.

Definition 2.2. *Let n, d and s be integers such that $d > s > 0$ and $n > 1$. Let again $\mathcal{Q}_s = \{Q: Q \subset \{1, \dots, d\}, |Q| = s\}$. Then the d -dimensional $n \times n \times \cdots \times n$ real array C is called sum-decomposable with parameters d and s and size n if there exist $\binom{d}{s}$ s -dimensional $n \times n \times \cdots \times n$ real arrays, one for each $Q \in \mathcal{Q}_s$ such that*

$$c(i_1, i_2, \dots, i_d) = \sum_{Q \in \mathcal{Q}_s} a^Q(h_Q(i_1, i_2, \dots, i_d))$$

where A^Q denotes the array associated with Q and $h_Q(i_1, i_2, \dots, i_d)$ denotes the s -tuple associated with Q , i.e., $h_Q(i_1, i_2, \dots, i_d) = (i_{q_1}, \dots, i_{q_s})$ for $Q = \{q_1, \dots, q_s\}$, $q_1 < q_2 < \cdots < q_s$.

We denote the vector space of all sum-decomposable real arrays of size n with parameters d and s by $\text{SAVS}(d, s, n)$.

For $Q = \{j_1, j_2, \dots, j_s\} \in \mathcal{Q}_s$ let V_Q denote the vector space of all d -dimensional $n \times n \times \cdots \times n$ arrays $C = (c(i_1, i_2, \dots, i_d))$ for which there exists a mapping $f: \{1, 2, \dots, n\}^s \mapsto \mathbb{R}$ with $c(i_1, i_2, \dots, i_d) = f(i_{j_1}, i_{j_2}, \dots, i_{j_s})$. In other words, the value $c(i_1, i_2, \dots, i_d)$ does only depend on the s indices from the set Q and not on all d indices. Let $\mathcal{Q}_s = \{Q_1, \dots, Q_{\binom{d}{s}}\}$. Note that

$$\text{SAVS}(d, s, n) = V_{Q_1} + V_{Q_2} + \cdots + V_{Q_{\binom{d}{s}}}. \quad (2)$$

Proposition 2.2. *Let $\text{SAVS}(d, s, n)$ be expressed as in (2). Then we have*

$$(i) \dim(V_Q) = n^s \text{ for all } Q \in \mathcal{Q}_s.$$

$$(ii) \dim\left(\bigcap_{i=1}^k V_{Q_i}\right) = n^{|\bigcap_{i=1}^k Q_i|}.$$

$$(iii) \dim(\text{SAVS}(d, s, n)) = \dim\left(\sum_{i=1}^{\binom{d}{s}} V_{Q_i}\right) = \sum_{k=1}^{\binom{d}{s}} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \dots < i_k \leq \binom{d}{s}} \dim(V_{Q_{i_1}} \cap \dots \cap V_{Q_{i_k}}) \right).$$

$$(iv) \dim(\text{SAVS}(d, d-1, n)) = n^d - (n-1)^d$$

$$\dim(\text{SAVS}(d, 1, n)) = dn - d + 1.$$

Proof. Ad (i): Follows directly from the definition of V_Q .

Ad (ii): Let $k = 2$ and consider $J = \{j_1, j_2, \dots, j_s\}$, $K = \{k_1, k_2, \dots, k_s\} \in \mathcal{Q}_s$. Further, let C, E be two arrays from V_J and V_K , respectively. Hence, $c(i_1, i_2, \dots, i_d) = f(i_{j_1}, i_{j_2}, \dots, i_{j_s})$ and $e(i_1, i_2, \dots, i_d) = g(i_{k_1}, i_{k_2}, \dots, i_{k_s})$ for some $f, g: \{1, 2, \dots, n\}^s \mapsto \mathbb{R}$. Then for every $A = (a(i_1, \dots, i_d)) \in V_J \cap V_K$, we have that $a(i_1, i_2, \dots, i_d) = t(i_{q_1}, \dots, i_{q_{|J \cap K|}})$, $q_i \in J \cap K$, for some $t: \{1, 2, \dots, n\}^{|J \cap K|} \mapsto \mathbb{R}$. Hence, $\dim(V_J \cap V_K) = n^{|J \cap K|}$. The case $k \geq 3$ follows from an inductive argument which settles (ii).

Ad (iii): The case $k = 2$ follows from the fact that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) \quad (3)$$

holds for any two subspaces U_1 and U_2 of a vector space.

Next we deal with the case $k = 3$. The general case then follows by induction. Let $Q_i, Q_j, Q_k \in \mathcal{Q}_s$. Note that (3) implies that

$$\begin{aligned} \dim(V_{Q_i} + V_{Q_j} + V_{Q_k}) &= \\ & \dim(V_{Q_i}) + \dim(V_{Q_j} + V_{Q_k}) - \dim(V_{Q_i} \cap (V_{Q_j} + V_{Q_k})) \end{aligned} \quad (4)$$

To proceed further it suffices to prove that the subspaces V_Q are distributive with respect to sum and intersection, that is,

$$V_{Q_i} \cap (V_{Q_j} + V_{Q_k}) = (V_{Q_i} \cap V_{Q_j}) + (V_{Q_i} \cap V_{Q_k}). \quad (5)$$

It is easy to check that using (5) and (3) in (4) above leads to the claim (iii) for $k = 3$.

So it only remains to be shown that the distributivity property (5) which does not hold in general is true in our case. To that end, we construct bases

for the vector spaces V_Q for $Q \in \mathcal{Q}_s$ and for the vector space V of all d -dimensional $n \times \dots \times n$ real arrays. The distributivity then follows from [15, Prop. 7.1 of Chap. 1, p. 15].

We call an array from V *elementary* if a single entry is 1 and all other entries are 0. It is easy to see that the set of n^d d -dimensional $n \times n \times \dots \times n$ elementary arrays forms a basis for V .

Next we construct a basis for the subspace V_Q where $Q = \{j_1, j_2, \dots, j_s\}$. Let A_{k_1, \dots, k_s} be the 0-1 d -dimensional array such that its entry at position (i_1, \dots, i_d) is 1 if $i_{j_1} = k_1, \dots, i_{j_s} = k_s$ and 0 otherwise. Then the set of arrays $\{A_{k_1, \dots, k_s} : (k_1, \dots, k_s) \in \{1, \dots, n\}^s\}$ forms a basis for V_Q . Note that every element of the basis for V_Q can be written as linear combination of elementary arrays.

Ad (iv): Note that any intersection of k distinct subsets of $\{1, 2, \dots, d\}$ has cardinality $d - k$. Hence, from (ii) and (iii) it follows that

$$\dim(\text{SAVS}(d, d-1, n)) = \sum_{i=1}^d (-1)^{i+1} \binom{d}{i} n^{d-i}$$

which is equal to $n^d - (n-1)^d$ by the binomial theorem.

Since the intersection of any distinct 1-element sets is empty it follows that $\dim(\text{SAVS}(d, 1, n)) = dn - d + 1$. \square

2.3 The COVP for the axial case: $(d, 1)$ -AP

Now we turn to the problem of characterizing the instances of the axial d -dimensional assignment problem with the COVP.

Theorem 2.3. *An instance of the $(d, 1)$ -AP with d -dimensional $n \times n \times \dots \times n$ cost array C has the constant objective value property (COVP) if and only if C is a sum-decomposable array with parameters d and 1.*

Proof. Note that one direction follows immediately, i.e. if C is the sum of d vectors, then every feasible solution has the same objective value.

Conversely, assume that every feasible solution has the same objective value. For integers $i_1, i_2, \dots, i_d \in \{2, 3, \dots, n\}$ consider the following d pairs of d -tuples:

$$\begin{aligned} &(1, 1, \dots, 1), \quad (i_1, i_2, \dots, i_d) \\ &(i_1, 1, \dots, 1), \quad (1, i_2, \dots, i_d) \\ &(1, i_2, 1, \dots, 1), \quad (i_1, 1, i_3, \dots, i_d) \\ &\quad \vdots \\ &(1, \dots, 1, i_d), \quad (i_1, \dots, i_{d-1}, 1). \end{aligned}$$

There exists a set of $n-2$ d -tuples which completes each of these pairs to a feasible solution; for example the set $\{(k_1^j, k_2^j, \dots, k_d^j) : j = 2, \dots, n-1, k_l^j =$

j if $j < i_l$ and $k_l^j = j + 1$ otherwise, $l = 1, \dots, d$. By assumption we get

$$c(i_1, i_2, \dots, i_d) = c(i_1, 1, \dots, 1) + c(1, i_2, \dots, i_d) - c(1, \dots, 1) \quad (6)$$

$$= c(1, i_2, 1, \dots, 1) + c(i_1, 1, i_3, \dots, i_d) - c(1, \dots, 1) \quad (7)$$

\vdots

$$= c(1, \dots, 1, i_d) + c(i_1, \dots, i_{d-1}, 1) - c(1, \dots, 1).$$

Due to (6) there exist a vector $V_1 = (v_1(i))$ and an $(d-1)$ -dimensional array $G_1 = (g_1(i_1, \dots, i_{d-1}))$ such that $c(i_1, i_2, \dots, i_d) = v_1(i_1) + g_1(i_2, \dots, i_d)$. Analogously, from (7) it follows that there exist a vector V_2 and an $(d-1)$ -dimensional array G_2 such that $c(i_1, i_2, \dots, i_d) = v_2(i_2) + g_2(i_1, i_3, \dots, i_d)$. Hence, it follows that $c(i_1, i_2, \dots, i_d) = v_1(i_1) + v_2(i_2) + g_{1,2}(i_3, \dots, i_d)$ for some $(d-2)$ -dimensional array $G_{1,2} = (g_{1,2}(i_1, \dots, i_{d-2}))$. Using the remaining equations in an analogous manner we finally obtain that C is the sum of d vectors, i.e.,

$$c(i_1, i_2, \dots, i_d) = v_1(i_1) + v_2(i_2) + \dots + v_d(i_d),$$

where the vectors V_k can be chosen as follows:

$$v_1(i) = c(i, 1, \dots, 1) - \frac{d-1}{d}c(1, 1, \dots, 1),$$

\vdots

$$v_d(i) = c(1, \dots, 1, i) - \frac{d-1}{d}c(1, 1, \dots, 1).$$

□

2.4 The COVP for the planar case: $(d, d-1)$ -AP

We now turn to the planar case. Note that there are exactly two feasible solutions of the $(d, d-1)$ -AP when $n = 2$.

Definition 2.3. *We say that an instance of the $(d, d-1)$ -AP with cost array C has property P_2 if for every $2 \times 2 \times \dots \times 2$ sub-array of C which is obtained by restricting the index sets to $\{1, i_1\} \times \{1, i_2\} \times \{1, i_3\}$ the two feasible solutions on the resulting subproblem of size 2 have the same objective function value.*

Property P_2 and sum-decomposable cost arrays for the $(d, d-1)$ -AP are related as follows.

Lemma 2.4. *Let an instance of the $(d, d-1)$ -AP with cost array C be given. If the instance has property P_2 , then the cost array C is sum-decomposable with parameters d and $d-1$.*

Proof. Consider the $2 \times 2 \times \dots \times 2$ subarray D_2 of C obtained by restricting attention to the index set $\{1, i_1\} \times \{1, i_2\} \times \dots \times \{1, i_d\}$ with $i_j \in \{2, \dots, n\}$ for $j = 1, \dots, d$.

Exploiting the fact that the two feasible solutions for the subarray D_2 have the same objective function value we get

$$c(i_1, i_2, \dots, i_d) = \sum_{x \in I_1} c(x) - \sum_{x \in I_2} c(x) + \dots + (-1)^{d+1} \sum_{x \in I_d} c(x), \quad (8)$$

where I_i is the set of all d -tuples from $\{1, i_1\} \times \{1, i_2\} \times \dots \times \{1, i_d\}$ with exactly i ones. Then from (8) it follows that C can be expressed as the sum of d $(d-1)$ -dimensional arrays $A_j = (a_j(i_1, \dots, i_{d-1}))$, $j = 1, \dots, d$, such that:

$$\begin{aligned} a_1(i_2, i_3, \dots, i_d) &= \sum_{x \in I_1^1} c(x) - \frac{1}{2} \sum_{x \in I_2^1} c(x) + \dots + (-1)^{d+1} \frac{1}{d} \sum_{x \in I_d^1} c(x) \\ a_2(i_1, i_3, \dots, i_d) &= \sum_{x \in I_1^2} c(x) - \frac{1}{2} \sum_{x \in I_2^2} c(x) + \dots + (-1)^{d+1} \frac{1}{d} \sum_{x \in I_d^2} c(x) \\ &\vdots \\ a_d(i_1, i_2, \dots, i_{d-1}) &= \sum_{x \in I_1^d} c(x) - \frac{1}{2} \sum_{x \in I_2^d} c(x) + \dots + (-1)^{d+1} \frac{1}{d} \sum_{x \in I_d^d} c(x), \end{aligned}$$

where I_i^k is the set of all d -tuples from $\{1, i_1\} \times \{1, i_2\} \times \dots \times \{1, i_d\}$ with exactly i ones, one of which is on the k -th coordinate. \square

The following result relates property P_2 and the constant objective function value property COVP.

Proposition 2.5. *Let an instance of the $(d, d-1)$ -AP with cost array C be given that has the constant objective value property (COVP) and let $n \neq 3$. Then the instance fulfills property P_2 .*

Proof. For the ease of notation let us assume that $i_1 = i_2 = \dots = i_d = 2$. The general case is proved analogously. We will prove that both feasible solutions of the $(d, d-1)$ -AP on the sub-array of C with indices $\{1, 2\} \times \{1, 2\} \times \dots \times \{1, 2\}$ have the same objective value. When $n = 2$, this is trivially true.

So assume $n \geq 4$. We will build two different feasible solutions for the $(d, d-1)$ -AP, which we denote by F_1^d and F_2^d with the following property: They both contain a feasible solution of the $(d, d-1)$ -AP on the subproblem induced by the index set $\{1, 2\}^d$ and all other elements of the two solutions will be the same. If F_1^d and F_2^d exist, then we are done. We now explain how F_1^d and F_2^d can be constructed recursively from a feasible solution of the $(d-1, d-2)$ -AP, which we denote by F^{d-1} . We assume that F^{d-1} will

also contain a feasible solution on the subproblem of size 2 which is induced by the index set $\{1, 2\}^{d-1}$. The feasible solutions F_j^d , $j = 1, 2$ are defined as follows:

$$F_j^d = \{(i, a_1, a_2, \dots, a_{d-2}, \phi_i^j(a_{d-1})) : (a_1, \dots, a_{d-1}) \in F^{d-1}, i = 1, \dots, n\}$$

where for $j = 1, 2$ the set of the n permutations ϕ_i^j , $i = 1, \dots, n$, are chosen to be mutually disjoint. (Two permutations α and β are disjoint if $\alpha(i) \neq \beta(i)$ for all i .) We choose ϕ_i^1 and ϕ_i^2 such that they coincide except for the images of 1 and 2, i.e., $\phi_1^1(1) = 1$, $\phi_1^1(2) = 2$, $\phi_2^1(1) = 2$, $\phi_2^1(2) = 1$ contrary to $\phi_1^2(1) = 2$, $\phi_1^2(2) = 1$, $\phi_2^2(1) = 1$, $\phi_2^2(2) = 2$. To show that such two sets of permutations exist, we represent them as two $n \times n$ Latin squares. For $j = 1, 2$, let the j -th table contain the integer $\phi_r^j(s)$ in the row r and column s . The resulting tables will be two Latin squares of order n which are identical except in the 2×2 upper-left corner. That corner is filled with two different Latin squares of order 2, respectively. It is well known that for $n \geq 4$ such Latin squares exist, see [17].

From Observation 2.1 (ii) we get that F_j^d are indeed feasible solutions. Finally, by assumption the objective values of F_1^d and F_2^d are equal, hence the equation (8) is obtained which completes the proof. \square

We believe that the result in Proposition 2.5 also holds for $n = 3$, but failed to generalize our proof for $d = 3$ and $d = 4$ to the case $d \geq 5$.

We are now ready to formulate and prove the COVP characterization for the $(d, d - 1)$ -AP.

Theorem 2.6. *An instance of the $(d, d - 1)$ -AP with cost array C has the constant objective value property (COVP) if and only if C is a sum-decomposable array with parameters d and $d - 1$.*

Proof. If the cost array C is sum-decomposable, then it is straightforward to see that every feasible solution has the same objective value.

Conversely, assume that every feasible solution has the same objective value. For the case $n \neq 3$ the statement follows from Proposition 2.5 and Lemma 2.4.

For the remaining case $n = 3$ we make use of the same technique that has been used in [11, 14] to obtain a COVP type characterization for the TSP.

Let $\mathcal{C}(d, n)$ denote the collection of all d -dimensional $n \times n \times \dots \times n$ cost arrays C for which all feasible solutions of the $(d, d - 1)$ -AP have the same objective value. Clearly $\mathcal{C}(d, n)$ is a linear subspace of the set of all d -dimensional $n \times n \times \dots \times n$ arrays.

Our goal is to prove that

$$\mathcal{C}(d, 3) = \text{SAVS}(d, d - 1, 3). \quad (9)$$

To that end, we consider all feasible solutions of the $(d, d-1)$ -AP for $n = 3$. Next we build up the 0-1 matrix M_d where the rows of M_d correspond to the feasible solutions and the columns correspond to the d -tuples over $\{1, 2, 3\}$. The entry of M_d that corresponds to the feasible solution F and the d -tuple (i_1, i_2, \dots, i_d) is set to 1 if and only if $(i_1, i_2, \dots, i_d) \in F$.

Let us observe how the matrices M_d look like. Every row of the matrix M_{d+1} is obtained from three disjoint rows of matrix M_d . For every row r_1 of the matrix M_d there are exactly two rows r_2, r_3 disjoint with it, and r_2 and r_3 are also mutually disjoint. Therefore $r_1 r_2 r_3$ and $r_1 r_3 r_2$ are rows of M_{d+1} . Hence the matrix M_{d+1} has twice as many rows as M_d . This corresponds to the fact that for $n = 3$ the number of feasible solutions doubles when moving from the planar d -dimensional assignment problem to the $(d+1)$ -dimensional one.

It is easy to see that M_d is a $3 \cdot 2^{d-1} \times 3^d$ matrix. Below the matrices M_d for $d = 1, 2$ are provided as illustration.

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array} \right)$$

$\mathcal{C}(d, 3)$ is the solution space of the linear equation system with coefficient matrix M_d and a constant right hand side vector. Thus we obtain

$$\dim \mathcal{C}(d, 3) = 3^d + 1 - \text{rank}(M_d)$$

From Proposition 2.2 (iv) we know that $\dim(\text{SAVS}(d, d-1, 3)) = 3^d - 2^d$. Hence in order to prove that (9) holds, we need to show that $\text{rank}(M_d) = 2^d + 1$. Observe that in fact it suffices to show that

$$\text{rank}(M_d) \geq 2^d + 1 \tag{10}$$

since obviously $\text{SAVS}(d, d-1, 3) \subseteq \mathcal{C}(d, 3)$. The validity of (10) follows from Lemma 2.7 below. This concludes the proof. \square

Lemma 2.7. *Let M_d be the matrix constructed above. We have*

$$\text{rank}(M_d) \geq 2^d + 1.$$

Proof. We start with observing the following recursive structure of M_d . Define

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad C_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and recursively for $k \geq 1$

$$A_k = \begin{pmatrix} A_{k-1} & B_{k-1} & C_{k-1} \\ A_{k-1} & C_{k-1} & B_{k-1} \end{pmatrix} \quad B_k = \begin{pmatrix} B_{k-1} & C_{k-1} & A_{k-1} \\ B_{k-1} & A_{k-1} & C_{k-1} \end{pmatrix}$$

$$C_k = \begin{pmatrix} C_{k-1} & A_{k-1} & B_{k-1} \\ C_{k-1} & B_{k-1} & A_{k-1} \end{pmatrix}$$

A_k, B_k and C_k are $3 \cdot 2^k \times 3^{k+1}$ matrices. It is easy to see that

$$M_d = A_{d+1} \quad \text{for } d \geq 1. \quad (11)$$

Next we will exhibit a regular $(2^d + 1) \times (2^d + 1)$ submatrix M'_d of M_d which will settle the lemma. We construct new matrices A'_k, B'_k and C'_k from A_k, B_k and C_k as follows: First, remove all columns with indices $\geq 2 \cdot 3^k + 1$. Next, remove all rows and columns with indices that are divisible by 3. It is straightforward to observe that the recursive structure survives this construction. More specifically we have

$$A'_k = \begin{pmatrix} A'_{k-1} & B'_{k-1} \\ A'_{k-1} & C'_{k-1} \end{pmatrix} \quad B'_k = \begin{pmatrix} B'_{k-1} & C'_{k-1} \\ B'_{k-1} & A'_{k-1} \end{pmatrix} \quad C'_k = \begin{pmatrix} C'_{k-1} & A'_{k-1} \\ C'_{k-1} & B'_{k-1} \end{pmatrix} \quad (12)$$

for $k \geq 1$ and

$$A'_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B'_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C'_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrices A'_k, B'_k and C'_k have 2^{k+1} rows and 2^{k+1} columns.

We obtain our target matrix M'_d from the matrix A'_{d+1} by re-inserting row 3 and column 3 of the matrix A_{d+1} . In order to show that M'_d is regular, we will calculate its determinant by a recursive approach.

We will make use of the observation that the upper left and lower left block are identical in the matrices A'_k, B'_k and C'_k . This will allow us to create a zero block as lower left block of a reduced matrix which has the same determinant as A'_k . This results in

$$\det A'_k = \det A'_{k-1} \det (C'_{k-1} - B'_{k-1}) \quad (13)$$

for $k \geq 1$. An analogous argument yields

$$\det (C'_k - B'_k) = \det (C'_{k-1} - B'_{k-1}) \det (B'_{k-1} + C'_{k-1} - 2A'_{k-1}) \quad (14)$$

and

$$\det (B'_k + C'_k - 2A'_k) = \det (B'_{k-1} + C'_{k-1} - 2A'_{k-1}) \det 3 (B'_{k-1} - C'_{k-1}) \quad (15)$$

for $k \geq 1$. Furthermore observe that

$$\det 3 (B'_{k-1} - C'_{k-1}) = 3^{2^k} \det (C'_{k-1} - B'_{k-1}) \quad (16)$$

as the involved matrices are of size $2^k \times 2^k$.

Let

$$z_k = \det A'_k, u_k = \det (C'_k - B'_k), v_k := \det (B'_k + C'_k - 2A'_k).$$

By explicit calculations we get the initial values

$$z_0 = 1, u_0 = -1, v_0 = 3.$$

From (13)–(16) we obtain the following recursions for $k \geq 1$

$$z_k = z_{k-1}u_{k-1}, \quad u_k = u_{k-1}v_{k-1}, \quad v_k = 3^{2^k} v_{k-1}u_{k-1}. \quad (17)$$

By combining the second and third equation in (17) we obtain $v_k = 3^{2^k} u_k$ which allows to eliminate v_k . We obtain the new system of recursions

$$z_k = z_{k-1}u_{k-1}, \quad u_k = 3^{2^{k-1}} u_{k-1}^2 \quad \text{for } k \geq 1 \quad (18)$$

This already implies that all matrices A'_k are regular, but for the sake of completeness we provide the solution for the recursion above. It is not hard to show that

$$u_k = 3^{k2^{k-1}}, \quad z_k = 3^{(k-2)2^{k-1}+1}$$

provides a solution to the system (18) with the initial conditions $z_0 = 1$ and $u_0 = -1$.

As a consequence thereof we get $\det A'_{d+1} = 3^{d2^{d+1}+1}$. Note that M'_d differs from A'_{d+1} only in its additional row and additional column. The additional column (the third column) of M'_d corresponds to the third unit vector. By developing the determinant of M'_d with respect to this column, we obtain

$$\det M'_d = \det A'_{d+1} = 3^{d2^{d+1}+1}$$

which implies that M'_d is regular and hence $\text{rank}(M_d) \geq 2^d + 1$. \square

Let us mention that $\text{rank} M_d = 2^d + 1$ can be shown by calculating the reduced row echelon form of matrix M_d . For our purposes it sufficed to show a weaker result which could be obtained more elegantly.

2.5 The COVP for the general case: (d, s)-AP

Theorems 2.3 and 2.6 suggest the following conjecture for the (d, s)-AP.

Conjecture 2.8. *A feasible instance of the (d, s)-AP with cost array C of size n has the constant objective value property (COVP) if and only if C is a sum-decomposable array with parameters d and s.*

Theorems 2.3 and 2.6 imply that Conjecture 2.8 is true for the following cases: (2, 1)-AP, (3, 1)-AP, (3, 2)-AP, (4, 1)-AP and (4, 3)-AP. So this leaves us with the (4, 2)-AP as smallest unsettled case. This is also the smallest case for which it is not guaranteed that a feasible solution exists for all $n \geq 2$.

We managed to disprove Conjecture 2.8 by coming up with the following counterexample for the (4, 2)-AP with $n = 3$.

Counterexample 2.9. There are 36 Graeco-Latin squares of size 3 which leads to 72 feasible solutions for the (4, 2)-AP with $n = 3$.

We consider the system of linear equations that is obtained by requiring that all 72 feasible solutions have the same objective function value. The dimension of the solution space of this system of equations and thus the dimension of the space of cost arrays with the COVP is 49 which can easily be calculated by a computer algebra system. By Proposition 2.2 one gets that the dimension of SAVS(4,2,3) is 33. Hence, there exists a cost array with the COVP that is not sum-decomposable. Now we provide one such array.

Let C be the $3 \times 3 \times 3 \times 3$ array where $c(1, 1, 1, 2)$, $c(1, 1, 2, 1)$, $c(1, 2, 1, 1)$, $c(1, 2, 2, 2)$, $c(2, 1, 1, 1)$, $c(2, 1, 2, 2)$, $c(2, 2, 1, 2)$, $c(2, 2, 2, 1)$ and $c(3, 3, 3, 3)$ have value 1 and all other entries have value 0. All 72 feasible solutions of the (4,2)-AP with this cost array have the objective value 1, and it is easy to check that C is not sum-decomposable.

We tried to find counterexamples for the (4, 2)-AP for $n \geq 4$, but failed. For $n = 4$ and $n = 5$ we could on the contrary prove Conjecture 2.8 by computational means. For $n = 6$ there are no feasible solutions and for $n = 7$ the number of feasible solutions got too large.

Conjecture 2.10. *Conjecture 2.8 is true for the (4, 2)-AP with $n \neq 3$.*

We believe that this restricted version of Conjecture 2.8 is true and that there is some hope to prove it. We believe that the counterexample for $n = 3$ exists as for $n = 3$ the number of feasible solutions is relatively small while it then grows very fast.

Note that for larger values of n even the number of Graeco-Latin squares is unknown. This outlaws explicit proof approaches as the set of feasible solutions is not known. A proof would need to exploit the structure of the set of Graeco-Latin squares.

For $d = 5$ we managed to check that Conjecture 2.8 holds for the (5, 2)-AP for $n = 4$. For $n = 2, 3, 6$ there are no feasible solutions. For $n = 5$ the set of feasible solutions became too large for our straightforward computational approach. The same happened for the (5, 3)-AP for the $n = 4$ case; for $n = 2, 3$ the problem is again infeasible.

The motivation behind our experiments was our wish to obtain a feeling whether it could be that Conjecture 2.8 becomes true when it needs to hold

only for values of $n \geq n_0$ where n_0 might depend on d and s . Unfortunately, we could handle only very small cases and do not have enough material to support this type of conjecture.

3 The COVP for d -dimensional transportation problems

In this section we deal with the COVP for d -dimensional transportation problems. Specifically, we show that our COVP characterization for the axial d -dimensional assignment problem carries over to the axial d -dimensional transportation problem while this approach fails for the more involved planar case.

Multi-dimensional transportation problems are known in the literature under diverse names. Alternative names are for example multi-index resp. d -index transportation problems, d -fold transportation problems and multi-way resp. d -way transportation problems, see e.g. [8, 16].

d -dimensional transportation problems can be defined along the lines of the definition of the d -dimensional assignment problem (d, s) -AP. We are given a d -dimensional $n_1 \times n_2 \times \dots \times n_d$ cost array C . While in the assignment case the right hand of all equality constraints is equal to one, in the transportation case we are additionally given an s -dimensional array B^Q for each set $Q \in \mathcal{Q}_s$ of fixed indices which provides the right hand side values for this group of constraints induced by the set Q . The arrays B^Q can be viewed as marginals for the transportation array $X = (x(i_1, i_2, \dots, i_d))$. We refer to the resulting transportation problem as (d, s) -TP.

Like for the assignment case, we obtain the *axial d -dimensional transportation problem* for the case $s = 1$ and the *planar d -dimensional transportation problem* for the case $s = d - 1$.

As we will need the axial d -dimensional transportation problem below, we provide its explicit formulation. We are given an $n_1 \times n_2 \times \dots \times n_d$ cost array $C = (c(i_1, i_2, \dots, i_d))$ and d supply-demand vectors B_1, \dots, B_d , where the k -th vector $B_k = (b_k(i))$ is an n_k -dimensional vector over the nonnegative integers. Furthermore we assume $\sum_{i=1}^{n_1} b_1(i) = \sum_{i=1}^{n_2} b_2(i) = \dots = \sum_{i=1}^{n_d} b_d(i)$. Let $I_r = \{1, \dots, n_r\}$ be the index set for i_r , $r = 1, \dots, d$. We obtain the following formulation for the $(d, 1)$ -TP.

$$\begin{aligned}
& \min \sum_{i_1 \in I_1} \sum_{i_2 \in I_2} \dots \sum_{i_d \in I_d} c(i_1, i_2, \dots, i_d) x(i_1, i_2, \dots, i_d) \\
& \text{s.t.} \quad \sum_{\substack{i_1 \in I_1, \dots, i_d \in I_d \\ \text{s.t. } i_k = j}} x(i_1, i_2, \dots, i_d) = b_k(j) \text{ for all } k \in \{1, \dots, d\}, j \in \{1, \dots, n_k\} \\
& \quad x(i_1, i_2, \dots, i_d) \geq 0 \quad \text{for all } i_r = 1, \dots, n_r, r = 1, \dots, d.
\end{aligned}$$

If $X = (x(i_1, \dots, i_n))$ has to be integral, the problem above becomes NP-hard for $d \geq 3$. For $d = 2$ the well-known classical Hitchcock transportation problem arises.

Theorem 3.1. *An instance of the axial d -dimensional transportation problem with cost array C has the COVP if and only if it is sum-decomposable with parameters d and $s = 1$, i.e. if and only if there exist d real vectors $V_1 = (v_1(i)), \dots, V_d = (v_d(i))$ such that*

$$c(i_1, i_2, \dots, i_d) = v_1(i_1) + v_2(i_2) + \dots + v_d(i_d).$$

Proof. Any instance of the integral axial d -dimensional transportation problem can be transformed into an equivalent instance of the axial d -dimensional assignment problem. To that end, we replace every supply/demand facility that has a supply/demand value $t > 1$ by t facilities with identical transportation costs that have supply/demand value 1. In this manner we get an equivalent problem with a blown up $n \times n \times \dots \times n$ cost array where $n = \sum_{i=1}^{n_1} b_1(i)$ and all supplies/demands being 1. Thus the newly obtained problem is the $(d, 1)$ -AP.

For the integral version of the $(d, 1)$ -TP we can apply the COVP characterization from Theorem 2.3 directly. For the non-integral version observe that the transformed problem with unit supplies and demands is a relaxation of the $(d, 1)$ -AP which results if the integrality constraints on X are dropped. In this case it follows from Theorem 2.3 that the set of instances with the COVP is a subspace of $\text{SAVS}(d, 1, n)$. Therefore, it is equal to $\text{SAVS}(d, 1, n)$.

Note that the transformation that blows up the cost array and the inverse transformation preserve the sum-decomposability property of the cost array. \square

Note that setting $d = 2$ in Theorem 3.1 implies that an instance of the classical transportation problem with cost matrix C has the COVP if and only if it is a sum matrix. The proof of the Theorem 3.1 provides the connection to assignment problems and further to Berenguer's COVP characterization for the TSP, cf. Theorem 1.1. As a by-product this reveals the nature of the connection between a result of Klinz and Woeginger [13] on the optimality of the North-West corner rule and Theorem 3.1 and thus answers an open problem mentioned in [13].

At first sight one might expect that Theorem 2.6 for the planar d -dimensional assignment problem $(d, d - 1)$ -AP carries over to the planar d -dimensional transportation problem $(d, d - 1)$ -TP. However several difficulties arise in this case.

First, note that the blow-up technique to transform the transportation problem to a (continuous) assignment problem does not work in general in the planar setting.

The second and probably bigger obstacle to a COVP type characterization for the planar case comes from the fact that for $d \geq 3$ the d -dimensional planar transportation problem does not necessarily have feasible solutions (not even in the non-integral case). Due to the universality result of de Loera and Onn [8] checking feasibility for the 3-dimensional planar (integer) transportation problem is as hard as deciding whether a general linear (integer) program has a feasible solution (the result already holds for a fixed third dimension, i.e., for $n_3 = 3$). This means that there is not much hope to be able to provide a nice sufficient and necessary condition for the set of instances with the COVP property for the 3-dimensional planar transportation problem and even less hope for cases with $d > 3$.

4 The COVP for spanning tree, shortest path and matching problems

In this section we provide COVP type characterization for the minimum spanning tree problem, the shortest path problem and the minimum weight maximum cardinality matching problem.

4.1 The COVP for the minimum spanning tree problem

In the minimum spanning (MST) problem we are given a connected, undirected graph $G = (V, E)$ and each edge $(i, j) \in E$ is assigned a weight w_{ij} . The task is to find a spanning tree for which the sum of the weights of the tree edges is minimal.

Theorem 4.1 characterizes when the COVP holds for the MST for graphs that contain a Hamiltonian cycle. Note that the case where G is complete and has $n \geq 3$ vertices is a special case.

Theorem 4.1. *Let an instance I of the MST problem with graph G and weights w be given such that G contains a Hamiltonian cycle. I has the COVP if and only if all edges have the same weight.*

Proof. The if direction is trivial. For the other direction, assume that every spanning tree has the same weight. Let H be a Hamiltonian cycle in G . By removing any edge from H we obtain a spanning tree. Regardless of which edge we delete, all resulting trees have the same weight which implies that that every edge in H has the same weight. Let e be an edge not in H . As the edge set of $H \cup \{e\}$ contains two cycles, we can remove two edges $e', e'' \neq e$ from H such that a spanning tree results. Hence e needs to have the same weight as the edges in H . Thus all edges have the same weight. \square

4.2 The COVP for the shortest path problem

Given a weighted graph (undirected or directed) with n vertices the shortest path problem is the problem of finding a path from vertex 1 to vertex n such that the sum of the edge weights along the path is minimized.

In the following we consider both the undirected and the directed version of the shortest path problem in a complete graph and provide COVP type characterizations.

Theorem 4.2. *Let G be the complete undirected graph with $n \geq 3$ vertices and let $w(i, j) \geq 0$ denote the weight of the edge (i, j) . This instance of the undirected shortest path problem has the COVP if and only if the weights are of the following form*

$$w(i, j) = w(j, i) = \begin{cases} a & \text{if } i = 1, j \neq n, \\ b & \text{if } i \neq 1, j = n, \\ a + b & \text{if } i = 1, j = n, \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

for some non-negative reals a and b .

Proof. Assume that every path from 1 to n has the same weight. For $n = 3$ the result is straightforward. Assume $n \geq 4$ and take two distinct vertices i and j such that $1 < i, j < n$. Consider the five paths from vertex 1 to vertex n that only go through a subset of the vertices $\{1, i, j, n\}$. By our assumption we get the following equations

$$\begin{aligned} w(1, n) &= w(1, i) + w(i, j) + w(j, n) \\ &= w(1, j) + w(j, i) + w(i, n) \\ &= w(1, i) + w(i, n) \\ &= w(1, j) + w(j, n). \end{aligned}$$

By adding and subtracting appropriate equations we get that $w(i, j) = 0$, $w(1, i) = w(1, j)$, $w(i, n) = w(j, n)$ from where (19) follows.

Note that (19) is also a sufficient condition for the COVP to hold which concludes the proof. \square

Theorem 4.3. *Let $G = (V, E)$ be the directed complete acyclic graph on n vertices, i.e., $V = \{1, 2, \dots, n\}$ and $E = \{(i, j) \in V \times V : i < j\}$. Let $w(i, j)$ denote the weight of edge (i, j) . This instance of the directed shortest path problem has the COVP if and only there exists a real vector a such that*

$$w(i, j) = a_j - a_i \quad \text{for all } i, j \in \{1, \dots, n\}, i < j \quad (20)$$

Proof. Assume that every path from vertex 1 to vertex n has the same weight. Consider the path composed of the edges $(1, i)$, (i, j) and (j, n) . It has the same weight as the path composed of the edges $(1, j)$ and (j, n) . It follows $w(i, j) = w(1, j) - w(1, i)$. By defining the vector $a_i := w(1, i)$ for $i = 1, \dots, n$ it follows that $w(i, j) = a_j - a_i$.

Now assume that for all $i < j$ the weight of (i, j) can be represented by (20) for some vector a . Consider an arbitrary path from vertex 1 to n , and let $1 = v_1 < v_2 < \dots < v_k = n$ be all vertices of that path. The weight of the path is

$$\sum_{i=1}^{k-1} w(v_i, v_{i+1}) = \sum_{i=1}^{k-1} a_{v_{i+1}} - a_{v_i} = a_{v_k} - a_{v_1} = a_n - a_1,$$

which is equal for every path from vertex 1 to n . Hence, (20) is also sufficient, the statement holds. \square

4.3 The COVP for the minimum weight maximum cardinality matching problem

In the minimum weight maximum cardinality matching problem we are given an undirected graph $G = (V, E)$ and edge weights $w(i, j)$ for the edges $(i, j) \in E$. Our goal is to find among all matchings of maximal cardinality a matching for which the sum of the edge weights is minimized.

Theorem 4.4. *Let an instance I of the minimum weight maximum cardinality matching on the complete undirected graph G with n vertices and edge weights $w(i, j)$ be given.*

- (i) *If n is odd, the instance I has the COVP if and only if all edge weights are equal.*
- (ii) *If n is even, the instance I has the COVP if and only if there exists a real vector $A = (a_i)$ such that*

$$w(i, j) = a_i + a_j \quad \text{for all } i \neq j. \quad (21)$$

Proof. Let n be odd. Suppose that every maximum cardinality matching has the same weight. Let $i, j, k \in V$ be three distinct vertices. Let M be a maximum cardinality matching on the vertex set $V \setminus \{i, j, k\}$. By adding an arbitrary edge from the triangle defined by i, j and k to M we obtain a maximum cardinality matching on the initial instance. By assumption it follows that every edge in the triangle defined by i, j and k has the same weight. Hence the statement follows.

Let n be even. Assume that every perfect matching has the same weight. Since each of the 2 pairs of edges (i, j) , (k, l) and (i, l) , (j, k) respectively can be identically extended to a perfect matching it follows that

$$w(i, j) + w(k, l) = w(i, l) + w(j, k)$$

for all distinct integers i, j, k, l . Hence, there exist two real vectors $U = (u_i)$ and $V = (v_i)$ such that $w(i, j) = u_i + v_j$ for all $i \neq j$. Since the weight matrix has to be symmetric (G is undirected), there exists a real vector $A = (a_i)$ such that

$$w(i, j) = a_i + a_j \tag{22}$$

for all $i \neq j$. Note that this is also a sufficient condition for the COVP. \square

5 Conclusion

In this paper we dealt with the question to characterize the set of all instances of a given combinatorial optimization problem for which every feasible solution has the same objective value. We answered this question for the following problems: the axial and the planar d -dimensional assignment problem, the axial d -dimensional transportation problem, the minimum spanning tree problem in Hamiltonian graphs, the shortest path problem in undirected and directed complete graphs and the minimum weight cardinality matching problem in complete graphs.

For the general d -dimensional assignment problem (d, s) -AP we failed to obtain a COVP characterization for the cases with $s \in \{2, \dots, d - 2\}$. We provided a counterexample with $d = 4$, $s = 2$ and $n = 3$ to the conjecture that sum-decomposability provides a COVP characterization for all feasible cases of the (d, s) -AP. It remains as challenging open question to answer the question whether sum-decomposability characterizes the COVP for all cases of the (d, s) -AP for which feasible solutions exist and where n is appropriately large (for example $n \geq 4$ for the $(4, 2)$ -AP).

Another interesting question for future research is to find combinatorial optimization problems for which a nice COVP characterization can be obtained where the underlying cost structure is not essentially sum-decomposable.

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