Calmness of linear programs under perturbations of all data: characterization and modulus*

M.J. Cánovas[†]· A. Hantoute[‡]· J. Parra[†]· F.J. Toledo[†]

Abstract

This paper provides operative point-based formulas (only involving the nominal data, and not data in a neighborhood) for computing or estimating the calmness modulus of the optimal set (argmin) mapping in linear optimization under uniqueness of nominal optimal solutions. Our analysis is developed in two different parametric settings. First, in the framework of canonical perturbations (i.e., perturbations of the objective function and the right-hand-side of the constraints), the paper provides a computationally tractable formula for the calmness modulus, which goes beyond some preliminary results of the literature. Second, in the framework of perturbations of all coefficients, the paper provides a characterization of the calmness property for the optimal set mapping, as well as an operative upper bound for the corresponding calmness modulus. Illustrative examples are provided.

Keywords. Variational analysis · Calmness · Linear programming · Calmness modulus

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1 Introduction

The present paper is focussed on quantifying the stability of (finite) linear optimization problems, through the analysis of the *calmness* property of the

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†Center of Operations Research, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain (canovas@umh.es, parra@umh.es, javier.toledo@umh.es).

‡Departamento de Ingeniería Matemática, Centro de Modelamiento Matemático (CMM), Universidad de Chile, Santiago, Chile (ahantoute@dim.uchile.cl).

optimal set mapping (also called *argmin mapping*) and the computation or estimation of the corresponding *calmness modulus.* Our linear optimization problem is expressed in the form

$$P(c,a,b): \quad \begin{array}{ll} \text{minimize} & c'x \\ \text{subject to} & a_t'x \leq b_t, \ \ t \in T := \{1,2,...,m\}, \end{array} \tag{1}$$

where $x \in \boldsymbol{R}^n$ is the vector of decision variables, and $c \in \boldsymbol{R}^n$, $a \equiv (a_t)_{t\in T} \in (\boldsymbol{R}^n)^T$, and $b \equiv (b_t)_{t\in T} \in \boldsymbol{R}^T$ are the problem's data. All elements in $\boldsymbol{R}^n$ are regarded as column-vectors and y' denotes the transpose of $y \in \boldsymbol{R}^n$.

We consider two parameterized families of linear optimization problems: the first one, $\left\{P(c,\overline{a},b) : (c,b) \in \boldsymbol{R}^n \times \boldsymbol{R}^T\right\}$, corresponds to the framework of *canonical perturbations*; i.e., perturbations fall on the objective function coefficient vector, c, together with the right-hand-side of the constraints, b, while the left hand side, $\overline{a} \equiv (\overline{a}_t)_{t\in T}$, is considered to be fixed at its nominal value. The second family, which corresponds to the context of perturbations of *all data,* is of the form $\left\{P(c,a,b) : (c,a,b) \in \boldsymbol{R}^n \times (\boldsymbol{R}^n)^T \times \boldsymbol{R}^T\right\}$. Associated with this second family, we consider the corresponding *optimal set mapping*, $\mathcal{S} : \boldsymbol{R}^n \times (\boldsymbol{R}^n)^T \times \boldsymbol{R}^T \rightrightarrows \boldsymbol{R}^n$, defined by

$$\mathcal{S}(c,a,b) := \{x \in \boldsymbol{R}^n \mid x \text{ is an optimal solution of } P(c,a,b)\}.$$

Here, the parameter space $\boldsymbol{R}^n \times (\boldsymbol{R}^n)^T \times \boldsymbol{R}^T$ is endowed with the norm

$$\|(c,a,b)\| := \max\left\{\|c\|_*, \|(a,b)\|_\infty\right\}, \tag{2}$$

where $\boldsymbol{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\|\leq 1} |u'x|$, and $\|(a,b)\|_\infty := \max_{t\in T} \left\|\binom{a_t}{b_t}\right\|$, where

$$\left\|\begin{pmatrix} a_t \\ b_t \end{pmatrix}\right\| = \max\left\{\|a_t\|_*, |b_t|\right\}.$$

When confined to the particular case of canonical perturbations, the associated optimal set mapping $\mathcal{S}_{\overline{a}} : \boldsymbol{R}^n \times \boldsymbol{R}^T \rightrightarrows \boldsymbol{R}^n$ is given by

$$\mathcal{S}_{\overline{a}}(c,b) = \mathcal{S}(c,\overline{a},b), \text{ for all } (c,b) \in \boldsymbol{R}^n \times \boldsymbol{R}^T.$$

This paper provides, in Theorem 3.1, an operative formula for the calmness modulus of $\mathcal{S}_{\overline{a}}$ under uniqueness of the nominal optimal solution by combining some results traced out from [5] and [7]. However, the main difficulties tackled in the paper are related to the context of perturbations of all data. At this moment we point out that mapping $\mathcal{S}_{\overline{a}}$ is always calm at any point of its graph, as a consequence of a classical result by Robinson [20], since the Karush-Kunh-Tucker conditions allow us to express the graph of $\mathcal{S}_{\overline{a}}$ as a finite union of polyhedral sets. This is no longer the case for $\mathcal{S}$ in the framework of perturbations of all data. In relation to this last framework, the paper establishes in Theorem 4.1 a characterization for the calmness of $\mathcal{S}$ and provides, in Theorem 4.2, an operative upper bound for the corresponding calmness modulus.

In the next paragraphs we recall some definitions related to a generic mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with distances denoted indistinctly by d). $\mathcal{M}$ is said to be *calm* at $(\overline{y}, \overline{x}) \in \text{gph}\mathcal{M}$ (the graph of $\mathcal{M}$) if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$d\left(x, \mathcal{M}\left(\overline{y}\right)\right) \leq \kappa d\left(y, \overline{y}\right) \tag{3}$$

whenever $x \in \mathcal{M}(y) \cap U$ and $y \in V$; where, as usual, $d(x, \Omega)$ is defined as $\inf\{d(x,z) \mid z \in \Omega\}$ for $\Omega \subset \boldsymbol{R}^n$, and $d(x, \emptyset) := +\infty$.

It is well-known that the calmness of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ is equivalent to the *metric subregularity* of $\mathcal{M}^{-1}$ at $(\overline{x}, \overline{y})$ (see, for instance, [9, Theorem 3H.3 and Exercise 3H.4]). Recall that $\mathcal{M}^{-1}$ (given by $y \in \mathcal{M}^{-1}(x) \Leftrightarrow x \in \mathcal{M}(y)$) is *metrically subregular* at $(\overline{x}, \overline{y})$ if there exist a constant $\kappa \geq 0$ and a (possibly smaller) neighborhood U of $\overline{x}$ such that

$$d\left(x, \mathcal{M}\left(\overline{y}\right)\right) \leq \kappa d\left(\overline{y}, \mathcal{M}^{-1}\left(x\right)\right), \text{ for all } x \in U. \tag{4}$$

The infimum of those $\kappa \geq 0$ for which (3) –or (4)– holds (for some associated neighborhoods) is called the *calmness modulus* of $\mathcal{M}$ at $(\overline{y}, \overline{x})$ and it is denoted by $\text{clm}\mathcal{M}(\overline{y}, \overline{x})$. The case when $\mathcal{M}$ is not calm at $(\overline{y}, \overline{x})$ corresponds to $\text{clm}\mathcal{M}(\overline{y}, \overline{x}) = +\infty$.

For comparative purposes, recall that $\mathcal{M}$ satisfies the *Aubin* property (also called pseudo-Lipschitz or Lipschitz-like) at $(\overline{y}, \overline{x})$ when (3) –or (4)– are valid when replacing $\overline{y}$ with an arbitrary $\widetilde{y}$ in some neighborhood V of $\overline{y}$. The corresponding infimum of all κ's is then called Lipschitz modulus and denoted by $\text{lip}\mathcal{M}(\overline{y}, \overline{x})$. Obviously,

$$\text{clm}\mathcal{M}\left(\overline{y}, \overline{x}\right) \leq \text{lip}\mathcal{M}\left(\overline{y}, \overline{x}\right). \tag{5}$$

Calmness and Aubin properties play an important role in relation to issues from optimization (theory and algorithms). Comprehensive studies of these properties can be traced out from the monographs [9, 15, 19, 21]. One can find in the literature deep contributions to the analysis of calmness for constraint systems in the context of canonical perturbations (see, e.g., [10, 13, 16, 17]). The reader is addressed to [1, 18] for the analysis of this property in relation to *local error bounds.* Subdifferential approaches to calmness/local error bounds can be found in [1, 12, 14, 18].

The structure of the paper is as follows: Section 2 provides the necessary notation and preliminary results. Section 3 presents the aforementioned tractable formula for computing the calmness modulus of $\mathcal{S}_{\overline{a}}$ under the uniqueness of nominal optimal solution. Moreover, a compraràtive analysis between calmness and Lipschitz moduli is carried out. Illustrative examples are provided. Section 4 tackles, in a first stage, the characterization of the calmess property of $\mathcal{S}$ at a given point of its graph, again under the assumption of uniqueness of nominal optimal solution. In a second stage, Section 4 provides an operative upper bound on the calmness modulus of $\mathcal{S}$, as well as some examples showing that this upper bound may be attained or not. Examples are also intended to show some perturbation strategies underlying the referred upper bound.

2 Preliminaries

In this section we introduce some additional notation and preliminary results which are needed later on. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by conv$X$ and cone$X$ the *convex hull* and the *conical convex hull* of X, respectively. It is assumed that cone$X$ always contains the zero-vector 0_k, in particular cone(?) = $\{0_k\}$. If X is a subset of any topological space, int$X$, cl$X$ and bd$X$ stand, respectively, for the interior, the closure, and the boundary of X.

We begin this section with a proposition which comes straightforwardly from [8, Theorem 5]. This result allows us to develop Section 4 under perturbations of *all* parameters, using as a starting point some results given in Section 3 in the framework of canonical perturbations. From now on, we denote by $\mathcal{F}_{\overline{a}}$ and $\mathcal{F}$ the *feasible set mappings* corresponding, respectively, to the settings of canonical perturbations and perturbations of all data. Formally,

$$\mathcal{F}_{\overline{a}}(b) := \mathcal{F}(\overline{a}, b), \ b \in \mathbb{R}^T,$$

where $\mathcal{F} : (\mathbb{R}^n)^T \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$ is given by

$$\mathcal{F}(a, b) := \left\{ x \in \mathbb{R}^n \mid a_t' x \leq b_t, \ t \in T \right\}, \ (a, b) \in (\mathbb{R}^n)^T \times \mathbb{R}^T.$$

Proposition 2.1 (see [8, Theorem 5]) *Let* $((\overline{a}, \overline{b}), \overline{x}) \in \text{gph}\mathcal{F}$. *Then*

$$\text{clm}\mathcal{F}((\overline{a}, \overline{b}), \overline{x}) = (\|\overline{x}\| + 1)\, \text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}).$$

Remark 2.1 As a consequence of the previous proposition, the involved calmness moduli are both finite (i.e., both mappings are calm at the corresponding points of their graphs) due to the finiteness of T, since the calmness modulus in the right-hand-side is finite according to the above mentioned result by Robinson [20] (because $\mathcal{F}_{\overline{a}}$ has a polyhedral graph). Observe that gph$\mathcal{F}$ may not be written as a finite union of polyhedral sets (just consider the case of a single inequality in $\boldsymbol{R}$), so that the calmness of $\mathcal{F}$ at $((\overline{a}, \overline{b}), \overline{x})$ does not follow from Robinson's result. In summary, at this moment we know that mappings $\mathcal{F}_{\overline{a}}$, $\mathcal{F}$, and $\mathcal{S}_{\overline{a}}$ are calm at any point of their grahps. We will see in Section 4 that this is not the case for $\mathcal{S}$.

Throughout the paper, we appeal to the *set of active indices* at $x \in \mathcal{F}(a, b)$, denoted by $T_{a,b}(x)$ and defined as

$$T_{a,b}(x) := \left\{ t \in T \mid a_t' x = b_t \right\}.$$

The next result follows directly from [5, Proposition 4.1] and constitutes a key tool in the present paper since it provides a point-based expression (i.e., just involving the nominal elements, and not elements in a neighborhood) for $\text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x})$ assuming $\mathcal{F}(\overline{a}, \overline{b}) = \{\overline{x}\}$. Such an assumption may seem too restrictive, but it is not so when applied to mappings $\mathcal{L}_D$ defined later. The reader can easily check that the assumption $\mathcal{F}(\overline{a}, \overline{b}) = \{\overline{x}\}$ entails $0_n \in \text{int conv} \left\{ \overline{a}_t, t \in T_{\overline{a},\overline{b}}(\overline{x}) \right\}$, since otherwise the separation theorem would provide a nonzero feasible direction of $\mathcal{F}(\overline{a}, \overline{b})$ at $\overline{x}$.

Proposition 2.2 [5, Proposition 4.1] *Let* $(\overline{a}, \overline{b}) \in (\boldsymbol{R}^n)^T \times \boldsymbol{R}^T$ *and assume* $\mathcal{F}(\overline{a}, \overline{b}) = \{\overline{x}\}$. *Then*

$$\text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x}) = \frac{1}{d_* \left(0_n, \text{bd conv} \left\{ \overline{a}_t, t \in T_{\overline{a},\overline{b}}(\overline{x}) \right\} \right)},$$

where d_* *stands for the distance in* $\boldsymbol{R}^n$ *associated with* $\|\cdot\|_*$.

Remark 2.2 More in detail, the previous expression for $\text{clm}\mathcal{F}_{\overline{a}}(\overline{b}, \overline{x})$ comes from [18, Theorem 1] and [5, Theorem 3.1] (which are the basis for the referred [5, Proposition 4.1]). The first of these results provides a subdifferential approach to the computation of local error bounds (closely related to calmness moduli) and the second establishes a key result for deriving a point-based formula; specifically, [5, Theorem 3.1] states, for any convex finite function $f : R^n \longrightarrow R$ at any given $\overline{x} \in R^n$,

$$\mathrm{bd}\partial f(\overline{x}) = \limsup_{x \to \overline{x},\ x \neq \overline{x}} \partial f(x).$$

For the sake of simplicity, from now on we abbreviate our nominal parameter as $\overline{p}$; i.e.,

$$\overline{p} := \left(\overline{c}, \overline{a}, \overline{b}\right) \in R^n \times (R^n)^T \times R^T.$$

The next proposition comes directly from [7, Corollary 8] and constitutes our starting point in Section 3. In it, associated with a given $(\overline{p}, \overline{x}) \in \mathrm{gph}\mathcal{S}$, we appeal to the following family of index subsets associated with the Karush-Kuhn-Tucker (KKT) conditions (hereafter referred to as *KKT index sets*)

$$\mathcal{K}_{\overline{p}}(\overline{x}) = \left\{ D \subset T_{\overline{a},\overline{b}}(\overline{x}) \middle| \, |D| \leq n \text{ and } -\overline{c} \in \mathrm{cone}\left\{\overline{a}_t,\ t \in D\right\} \right\},$$

where $|D|$ stands for the cardinality of D and condition $|D| \leq n$ comes from Caratheodory's Theorem. For any $D \in \mathcal{K}_{\overline{p}}(\overline{x})$ we consider the mapping $\mathcal{L}_D : (R^n)^T \times R^T \times (R^n)^D \times R^D \rightrightarrows R^n$ given by

$$\mathcal{L}_D(a, b, u, d) := \left\{ x \in R^n \mid a_t' x \leq b_t,\ t \in T;\ u_t' x \leq d_t,\ t \in D \right\}, \qquad (6)$$

and, using the notation $\overline{a}_D = (\overline{a}_t)_{t \in D}$, $\overline{b}_D = \left(\overline{b}_t\right)_{t \in D}$, we also define

$$\mathcal{L}_{D,\overline{a},-\overline{a}_D}(b, d) := \mathcal{L}_D(\overline{a}, b, -\overline{a}_D, d) \text{ for } (b, d) \in R^T \times R^D. \qquad (7)$$

Observe that all preliminary results for feasible set mappings $\mathcal{F}$ and $\mathcal{F}_{\overline{a}}$ may be specified for $\mathcal{L}_D$ and $\mathcal{L}_{D,\overline{a},-\overline{a}_D}$, respectively, which are nothing else but feasible set mappings associated with enlarged systems.

Proposition 2.3 [7, Corollary 8] *Let* $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in R^n \times (R^n)^T \times R^T$ *and assume* $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. *Then*

$$\mathrm{clm}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \mathrm{clm}\mathcal{L}_{D,\overline{a},-\overline{a}_D}\left(\left(\overline{b}, -\overline{b}_D\right), \overline{x}\right).$$

Remark 2.3 Observe that $\mathcal{L}_D\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right)$ is the set of KKT points of problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$ associated with D as the KKT index set. Under our current assumption $\mathcal{S}(\overline{p}) = \{\overline{x}\}$, we have

$$\mathcal{L}_D\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right) = \{\overline{x}\} \text{ for all } D \in \mathcal{K}_{\overline{p}}(\overline{x}).$$

3 Calmness modulus vs. Lipschitz modulus under canonical perturbations

The following theorem provides the announced expression for $\operatorname{clm}\mathcal{S}_{\overline{a}}$, only involving the nominal point $\overline{x}$ and the nominal problem's data $\left(\overline{c},\overline{a},\overline{b}\right)$.

Theorem 3.1 *Let* $\overline{p}=\left(\overline{c},\overline{a},\overline{b}\right)\in R^n\times(R^n)^T\times R^T$ *and assume* $\mathcal{S}(\overline{p})=\{\overline{x}\}$. *Then*

$$\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=\max_{D\in\mathcal{K}_{\overline{p}}(\overline{x})}\frac{1}{d_*\left(0_n,\operatorname{bd}\operatorname{conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(\overline{x});-\overline{a}_t,\ t\in D\right\}\right)}.$$

Proof The result follows by combining Propositions 2.2 and 2.3, and Remark 2.3. □

Remark 3.1 For problem (1), given any $(\overline{p},\overline{x})\in\operatorname{gph}\mathcal{S}$ without requiring $\mathcal{S}(\overline{p})=\{\overline{x}\}$, and denoting $\mathcal{S}_{\overline{c},\overline{a}}(b):=\mathcal{S}_{\overline{a}}(\overline{c},b)$ (which equals $\mathcal{S}(\overline{c},\overline{a},b)$) for $b\in R^T$, [7, Theorem 7] establishes

$$\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=\operatorname{clm}\mathcal{S}_{\overline{c},\overline{a}}\left(\overline{b},\overline{x}\right);$$

i.e., perturbations of $\overline{c}$ are negligible when computing the calmness modulus of $\mathcal{S}_{\overline{a}}$ at $\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$, and, therefore, only perturbations of $\overline{b}$ are needed.

For comparative purposes, in the following proposition we recall the expression of $\operatorname{lip}\mathcal{S}_{\overline{a}}(\left(\overline{c},\overline{b}\right),\overline{x})$, provided that it is finite, where we use the notation

$$\mathcal{T}_{\overline{p}}(\overline{x})=\{D\in\mathcal{K}_{\overline{p}}(\overline{x})\,||D|=n\text{ and }A_D\text{ is nonsingular}\},$$

with A_D denoting the matrix whose rows are $\overline{a}_t',t\in D$ (given in some prefixed order).

Remark 3.2 The assumption $\mathcal{S}(\overline{p})=\{\overline{x}\}$ entails $-\overline{c}\in\operatorname{int}\operatorname{cone}\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(\overline{x})\}$, which implies $\mathcal{T}_{\overline{p}}(\overline{x})\neq\emptyset$. The assumption $\mathcal{S}(\overline{p})=\{\overline{x}\}$ also implies that 0_n belongs to $\operatorname{int}\operatorname{cone}\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(\overline{x});\ \overline{c}\}$, which is contained in $\operatorname{int}\operatorname{cone}\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(\overline{x});\ -\overline{a}_t,\ t\in D\}$ for any $D\in\mathcal{K}_{\overline{p}}(\overline{x})$, and, consequently,

$$0_n\in\operatorname{int}\operatorname{conv}\left\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(\overline{x});-\overline{a}_t,\ t\in D\right\}\text{ for all }D\in\mathcal{K}_{\overline{p}}(\overline{x}).$$

Accordingly, the denominator appearing in Theorem 3.1 is always positive.

We also appeal to the following concepts:

- The *Slater constraint qualification* (SCQ) holds at parameter $\left(\overline{a}, \overline{b}\right) \in (\mathbb{R}^n)^T \times \mathbb{R}^T$ if there exists $\widehat{x} \in \mathbb{R}^n$ (called Slater point) such that $\overline{a}_t' \widehat{x} < \overline{b}_t$ for all $t \in T$.

- The *Nürnberger condition* holds at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \mathrm{gph}\mathcal{S}_{\overline{a}}$ if the SCQ verifies at $\left(\overline{a}, \overline{b}\right)$ and
$$\mathcal{T}_{\overline{p}}\left(\overline{x}\right) = \mathcal{K}_{\overline{p}}\left(\overline{x}\right).$$

Proposition 3.1 *Let $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \in \mathrm{gph}\mathcal{S}_{\overline{a}}$. Then $\mathcal{S}_{\overline{a}}$ satisfies the Aubin property at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$, i.e., $\mathrm{lip}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) < +\infty$, if and only if the Nürnberger condition holds at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$. In this case,*

$$\mathrm{lip}\mathcal{S}_{\overline{a}}(\left(\overline{c}, \overline{b}\right), \overline{x}) = \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \left\| A_D^{-1} \right\|. \tag{8}$$

Remark 3.3 The previous characterization of the Aubin property for $\mathcal{S}_{\overline{a}}$ can be found in [6, Theorem 16], although the name 'Nürnberger condition' appeared for the first time in [4] (extended to the convex case). It can be easily seen that the Nürnberger condition at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ entails $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. Expression (8) comes from [3, Corollary 2]. For $D \in \mathcal{T}_{\overline{p}}\left(\overline{x}\right)$, we can identify matrix A_D with the 'endomorphism' $\mathbb{R}^n \ni x \mapsto A_D x \in \mathbb{R}^D$, where $\mathbb{R}^n$ is equipped with an arbitrary norm $\|\cdot\|$ and $\mathbb{R}^D$ is endowed with the supremum norm $\|\cdot\|_\infty$. For our choice of norms we have

$$\left\| A_D^{-1} \right\| := \max_{\|y\|_\infty \leq 1} \left\| A_D^{-1} y \right\| = \frac{1}{d_* \left(0_n, \mathrm{bd\,conv}\left\{\pm \overline{a}_t, t \in D\right\}\right)}, \tag{9}$$

where the last equality is a straightforward consequence of [2, Corollary 3.2] together with the fact that $\left\| A_D^{-1} \right\|$ coincides with the Lipschitz modulus of A_D^{-1} at any point of its graph. Moreover, with the only assumption that $\mathcal{S}(\overline{p}) = \{\overline{x}\}$, [7, Theorem 13] shows that

$$\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right) \leq \max_{D \in \mathcal{T}_{\overline{p}}(\overline{x})} \left\| A_D^{-1} \right\| \tag{10}$$

without requiring the Nürnberger condition; i.e., the right-hand-side of (10) is finite and still constitutes an upper bound on the calmness modulus when the Lipschitz modulus is infinite.

The next example comes from [7, Example 2] and shows that inequality (10) may be strict even under the Nürnberger condition. In [7, Example 2], *ad hoc* arguments were used to obtain $\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$. Now Theorem 3.1 provides a direct way to compute that modulus, as the subsequent figure (Figure 1) illustrates.

Example 3.1 Consider the linear optimization problem $P\left(\overline{c},\overline{a},\overline{b}\right)$, in $\boldsymbol{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} \text{minimize} & x_1+\frac{1}{3}x_2 & \\ \text{subject to} & -x_1\leq 0, & (t=1) \\ & -x_1-\frac{1}{2}x_2\leq 0, & (t=2) \\ & -x_1-x_2\leq 0, & (t=3) \\ & -x_1+x_2\leq 0, & (t=4) \end{array}$$

whose unique optimal solution is $\bar{x}=0_2$, and where

$$\mathcal{K}_{\overline{p}}\left(\bar{x}\right)=\mathcal{T}_{\overline{p}}\left(\bar{x}\right)=\left\{\left\{1,2\right\},\left\{1,3\right\},\left\{2,4\right\},\left\{3,4\right\}\right\}.$$

According to Theorem 3.1, the reader can easily check that the corresponding maximum over $D\in\mathcal{K}_{\overline{p}}\left(\bar{x}\right)$ is attained at both $D=\{1,2\}$ and $D=\{1,3\}$, and therefore $\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)$ coincides, if for instance we choose $D=\{1,2\}$, with

$$\frac{1}{d_*\left(0_2,\operatorname{bd}\operatorname{conv}\left\{\pm\begin{pmatrix}-1\\0\end{pmatrix},\pm\begin{pmatrix}-1\\-1/2\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\right\}\right)}=\sqrt{5}.$$

The next figure illustrates this example. We can see that the distance in the previous denominator is attained at $(1/5,-2/5)'$ and equals $1/\sqrt{5}$. In the same figure we can also check that the distance from the origin to the segments with discontinuous trace is $1/\sqrt{17}$, which coincides with $\left\|A_{\{1,2\}}^{-1}\right\|^{-1}$ according to (9). Hence,

$$\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=\sqrt{5}<\operatorname{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=\sqrt{17}.$$

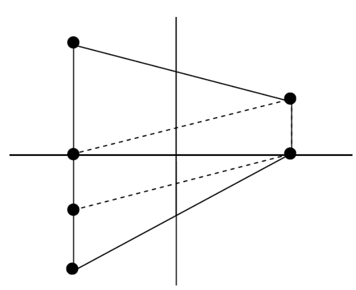

Figure 1: Illustration of Example 3.1

The following example can be traced out from [6, Example 6]. In this example we can see that $\mathrm{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=+\infty$. Now we are able to compute the calmness modulus.

Example 3.2 Consider $P\left(\overline{c},\overline{a},\overline{b}\right)$, in R^2 endowed with the Euclidean norm,

$$\begin{array}{lll} \text{minimize} & x_1 & \\ \text{subject to} & -x_1+x_2\leq 0, & (t=1) \\ & -x_1-x_2\leq 0, & (t=2) \\ & -x_1\leq 0, & (t=3) \end{array}$$

whose unique optimal solution is $\bar{x}=0_2$, and where

$$\mathcal{K}_{\overline{p}}\left(\overline{x}\right)=\left\{\left\{1,2\right\},\left\{1,3\right\},\left\{2,3\right\},\left\{3\right\}\right\},$$

whereas $\mathcal{T}_{\overline{p}}\left(\overline{x}\right)=\mathcal{K}_{\overline{p}}\left(\overline{x}\right)\setminus\left\{\left\{3\right\}\right\}$. In this case, the maximum considered in Theorem 3.1 is attained at any element of $\left\{\left\{1,3\right\},\left\{2,3\right\},\left\{3\right\}\right\}$, yielding $\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=\sqrt{5}$. Also observe that in this case $\max_{D\in\mathcal{T}_{\overline{p}}(\overline{x})}\left\|A_D^{-1}\right\|=\sqrt{5}$, although $\mathrm{lip}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right)=+\infty$ since the Nürnberger condition fails (see Proposition 3.1).

4 Calmness under perturbation of all coefficients

As we show in this section, devoted to characterize the calmness of $\mathcal{S}$ and to provide an upper estimate of $\mathrm{clm}\mathcal{S}\left(\overline{p},\overline{x}\right)$, the case of perturbations of all data is notably different from the one of canonical perturbations. To start with, we point out the fact that the finiteness of T is no longer a sufficient condition for the calmness of $\mathcal{S}$.

Fact 4.1 *$\mathcal{S}$ may be not calm at some $\left(\left(\overline{c},\overline{a},\overline{b}\right),\overline{x}\right)\in\mathrm{gph}\mathcal{S}$ with $\mathcal{S}\left(\overline{c},\overline{a},\overline{b}\right)=\left\{\overline{x}\right\}$.*

Proof Consider $P\left(\overline{c},\overline{a},\overline{b}\right)$, in R^2 endowed with the Euclidean norm, given by

$$\begin{array}{lll} \text{minimize} & x_1 & \\ \text{subject to} & -x_1\leq 0, & (t=1) \\ & -x_2\leq 0, & (t=2) \\ & x_2\leq 0, & (t=3) \end{array} \tag{11}$$

whose unique optimal solution is $\bar{x} = 0_2$. In order to show that $\text{clm}\mathcal{S}(\overline{p}, \overline{x}) = +\infty$, let us consider (a^r, b^r), for $r = 1, 2, ...$, defined by

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} := \begin{cases} \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix}, \ t = 1, 2, \\ \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} - \begin{pmatrix} (1/r^2, 0)' \\ 1/r^3 \end{pmatrix}, \ t = 3. \end{cases}$$

Then we have $\mathcal{S}(\overline{c}, a^r, b^r) = \{x^r\}$, with $x^r := \left(\frac{1}{r}, 0\right)'$, and

$$\text{clm}\mathcal{S}(\overline{p}, \overline{x}) \geq \lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|(\overline{c}, a^r, b^r) - \left(\overline{c}, \overline{a}, \overline{b}\right)\right\|} = \lim_{r\to\infty} \frac{1/r}{1/r^2} = +\infty.$$

□

Observe that SCQ fails in the previous counterexample. The next theorem characterizes the calmness property of $\mathcal{S}$ under the uniqueness of nominal optimal solution.

Theorem 4.1 *Let* $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in R^n \times (R^n)^T \times R^T$ *and assume that* $\mathcal{S}(\overline{p}) = \{\overline{x}\}$. *Then the following statements are equivalent:*
(i) $\mathcal{S}$ *is calm at* $(\overline{p}, \overline{x})$;
(ii) Either the SCQ holds at $\left(\overline{a}, \overline{b}\right)$ *or* $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$;
(iii) $0_n \notin \text{bd}\,\text{conv}\left\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\right\}$.

Proof $(i) \Rightarrow (ii)$. Reasoning by contradiction, assume that the SCQ fails at $\left(\overline{a}, \overline{b}\right)$, which entails (applying [11, Theorem 6.9(v)])

$$0_{n+1} \in \text{conv}\left\{\begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix},\ t \in T\right\}, \tag{12}$$

(which in our finite setting is the same as $0_n \in \text{conv}\left\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\right\}$) and assume also that $\mathcal{S}(\overline{p})$ & $\mathcal{F}\left(\overline{a}, \overline{b}\right)$ (in particular, $\overline{c} \neq 0_n$), ensuring the existence of some sequence $\{x^r\} \subset \mathcal{F}\left(\overline{a}, \overline{b}\right) \setminus \mathcal{S}(\overline{p})$ converging to $\overline{x}$. Now define a sequence of perturbed problems in the following way:

$$\begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} := \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} - \|x^r - \overline{x}\|^2 \begin{pmatrix} \overline{c} \\ \overline{c}' x^r \end{pmatrix}, \text{ for } r = 1, 2, ... \text{ and all } t \in T.$$

From (12), let us write

$$0_{n+1} = \sum_{t\in T} \lambda_t \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix},$$

with $\lambda_t \geq 0$ for all $t \in T$ and $\sum_{t\in T} \lambda_t = 1$. Then, for each r, we have

$$\sum_{t\in T} \lambda_t \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} = -\left\| x^r - \overline{x} \right\|^2 \begin{pmatrix} \overline{c} \\ \overline{c}' x^r \end{pmatrix}.$$

In this way, an standard argument of linear programming yields

$$x^r \in \mathcal{S}\left(\overline{c}, a^r, b^r\right).$$

More in detail, for each r, the inequality '$\overline{c}'x \geq \overline{c}'x^r$' is a consequence of the system $\left\{ (a_t^r)' x \leq b_t^r,\ t \in T \right\}$, according to the Farkas Lemma. Then,

$$\begin{aligned} \mathrm{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right) &\geq \lim_{r\to\infty} \frac{\left\| x^r - \overline{x} \right\|}{\left\| \left(\overline{c}, a^r, b^r\right) - \left(\overline{c}, \overline{a}, \overline{b}\right) \right\|} \\ &= \lim_{r} \frac{\left\| x^r - \overline{x} \right\|}{\left\| x^r - \overline{x} \right\|^2 \max\left\{ \left\| \overline{c} \right\|_*, \left| \overline{c}' x^r \right| \right\}} = +\infty. \end{aligned}$$

$(ii) \Rightarrow (i)$. The case when SCQ holds at $\left(\overline{a}, \overline{b}\right)$ follows from [5, Theorem 4.1], which works in a more general semi-infinite setting. Specifically, assuming SCQ at $\left(\overline{a}, \overline{b}\right)$ and $\mathcal{S}\left(\overline{p}\right) = \{\overline{x}\}$, [5, Theorem 4.1] yields

$$\mathrm{clm}\mathcal{S}\left(\overline{p}, \overline{x}\right) \leq \frac{\left(\left\| \overline{x} \right\| + 1 \right) \max\left\{ 1, \mathrm{clm}\vartheta\left(\overline{p}\right) \right\}}{d_*\left(0_n, \mathrm{bd}\,\mathrm{conv}\left\{ \overline{c};\ \overline{a}_t,\ t \in T_{\overline{a},\overline{b}}\left(\overline{x}\right) \right\} \right)}. \tag{13}$$

Observe that the right-hand-side of (13) is finite since the denominator is positive as a consequence of the fact that $-\overline{c} \in \mathrm{int}\,\mathrm{cone}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}\left(\overline{x}\right)\}$ (see Remark 3.2), which entails $0_n \in \mathrm{int}\,\mathrm{conv}\left\{ \overline{c};\ \overline{a}_t,\ t \in T_{\overline{a},\overline{b}}\left(\overline{x}\right) \right\}$, and $\mathrm{clm}\vartheta\left(\overline{p}\right)$ is the calmness modulus of the (real-extended) optimal value function associated with problem (1) at the nominal parameter $\overline{p}$, which is also finite. Indeed, under the current assumptions, ϑ turns out to be Lipschitz continuous (and, hence, calm) at $\overline{p}$ (see [5, Remark 4.1] and references therein).

The calmness of $\mathcal{S}$ at $\left(\overline{p}, \overline{x}\right)$ when $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$ follows from the calmness of $\mathcal{F}$ at $\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right)$ together with the obvious fact that $\mathcal{S}\left(c, a, b\right) \subset \mathcal{F}\left(a, b\right)$ for every $\left(c, a, b\right) \in \boldsymbol{R}^n \times \left(\boldsymbol{R}^n\right)^T \times \boldsymbol{R}^T$.

$(ii) \Leftrightarrow (iii)$ comes from the facts that: SCQ holds at $\left(\overline{a}, \overline{b}\right)$ if and only if $0_n \notin \mathrm{conv}\left\{ \overline{a}_t,\ t \in T_{\overline{a},\overline{b}}\left(\overline{x}\right) \right\}$ (see again [11, Theorem 6.9(v)], mentioned at the beginning of '$(i) \Rightarrow (ii)$' in the current proof), and $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$ if and only if $0_n \in \mathrm{int}\,\mathrm{conv}\left\{ \overline{a}_t,\ t \in T_{\overline{a},\overline{b}}\left(\overline{x}\right) \right\}$ (see the paragraph preceding Proposition 2.2 for the direct implication; the converse is evident). $\square$

A key point in the counterexample given to establish Fact 4.1 is that $-\overline{c} \in \text{cone}\{a_2^r, a_3^r\}$, for all r, while $-\overline{c} \notin \text{cone}\{\overline{a}_2, \overline{a}_3\}$. The following Lemma shows SCQ precludes this situation.

Lemma 4.1 *Let* $(\overline{p}, \overline{x}) \in \text{gph}\mathcal{S}$, *with* $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right)$ *and assume that the SCQ holds at* $\left(\overline{a}, \overline{b}\right)$. *Consider any sequence* $\{(p^r, x^r)\} \subset \text{gph}\mathcal{S}$, *with* $p^r = (c^r, a^r, b^r)$, *converging to* $(\overline{p}, \overline{x})$. *For each* r, *let* $D^r \subset T_{a^r, b^r}(x^r)$, *such that* $|D^r| \leq n$ *and*

$$-c^r \in \text{cone}\{a_t^r,\ t \in D^r\}. \tag{14}$$

Then, there exists a subsequence $\{(p^{r_k}, x^{r_k})\}$ *of* $\{(p^r, x^r)\}$ *such that the corresponding* $\{D^{r_k}\}$ *is constant and, denoting* $D^{r_k} = \widehat{D}$ *for all* k, *we have*

$$\widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x}).$$

Proof Consider sequences $\{(p^r, x^r)\} \subset \text{gph}\mathcal{S}$ and $\{D^r\}$ as in the statement of the lemma. The finiteness of T allows us to consider a constant subsequence $\{D^{r_k}\}$. Write $D^{r_k} = \widehat{D}$ for all k. Our assumption $\widehat{D} \subset T_{a^{r_k}, b^{r_k}}(x^{r_k})$ clearly implies $\widehat{D} \subset T_{\overline{a}, \overline{b}}(\overline{x})$ by letting $k \to \infty$. From (14), we can write, for each k,

$$-c^{r_k} = \sum_{t \in \widehat{D}} \lambda_t^{r_k} a_t^{r_k}, \tag{15}$$

for some $\lambda_t^{r_k} \geq 0$, $t \in \widehat{D}$. Note that the sequence $\{\gamma_k\}$, where $\gamma_k := \sum_{t \in \widehat{D}} \lambda_t^{r_k}$ for all k, is bounded; since otherwise, dividing both members of (15) by γ_k and letting $k \to \infty$, we would obtain

$$0_n \in \text{conv}\left\{\overline{a}_t,\ t \in T_{\overline{a}, \overline{b}}(\overline{x})\right\},$$

which represents a contradiction with the SCQ (see, e.g., [11, Theorem 6.9(v)]).

The boundedness of $\{\gamma_k\}$ yields the existence of some subsequence of $k's$ (denoted in the same way for simplicity) such that, for each $t \in \widehat{D}$, the sequence $\{\lambda_t^{r_k}\}_{k \in \mathbb{N}}$ converges to some $\lambda_t \geq 0$. Then, from (15) we conclude

$$-\overline{c} = \sum_{t \in \widehat{D}} \lambda_t \overline{a}_t \in \text{cone}\left\{\overline{a}_t,\ t \in \widehat{D}\right\}.$$

This implies $\widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x})$. □

Now we are able to provide an upper bound on the calmness modulus of $\mathcal{S}$ under the uniqueness of nominal optimal solution. We point out the fact that the right-hand-side of both inequalities in (i) and (ii) below is always finite when $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ (see Remark 3.2), although obviously these inequalities are not true when $\mathcal{S}$ is not calm at $(\overline{p}, \overline{x})$.

Theorem 4.2 *Let* $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right) \in R^n \times (R^n)^T \times R^T$. *Assume that* $\mathcal{S}(\overline{p}) = \{\overline{x}\}$ *and that* $\mathcal{S}$ *is calm at* $(\overline{p}, \overline{x})$. *The following assertions are true:*

(*i*) *If the SCQ holds at* $\left(\overline{a}, \overline{b}\right)$, *then*

$$\operatorname{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd}\operatorname{conv}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x}); -\overline{a}_t,\ t \in D\}\right)}. \tag{16}$$

(*ii*) *If* $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \{\overline{x}\}$, *then*

$$\operatorname{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \operatorname{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right) = \frac{\|\overline{x}\| + 1}{d_*\left(0_n, \operatorname{bd}\operatorname{conv}\{\overline{a}_t,\ t \in T_{\overline{a},\overline{b}}(\overline{x})\}\right)}.$$

Proof (*i*) First note that the right-hand-side of (16) may be written as

$$\max_{D \in \mathcal{K}_{\overline{p}}(\overline{x})} \operatorname{clm}\mathcal{L}_D\left(\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right), \overline{x}\right)$$

as an application of Propositions 2.1 and 2.2. Recall from its definition (6) that $\mathcal{L}_D$ is nothing else but the feasible set mapping associated with a certain enlarged system, whose parameter size is measured by

$$\|(a, b, u, d)\| := \max\left\{\max_{t \in T}\left\|\begin{pmatrix} a_t \\ b_t \end{pmatrix}\right\|, \max_{t \in D}\left\|\begin{pmatrix} u_t \\ d_t \end{pmatrix}\right\|\right\}.$$

Set

$$\operatorname{clm}\mathcal{S}(\overline{p}, \overline{x}) = \lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|}, \tag{17}$$

for certain sequences of parameters $p^r = (c^r, a^r, b^r)$ and points $x^r \in \mathcal{S}(p^r)$ such that $(p^r, x^r) \to (\overline{p}, \overline{x})$ with $p^r \neq \overline{p}$. By the KKT conditions (together with Caratheodory's Theorem) , for each r there exists $D^r \subset T_{a^r,b^r}(x^r)$ such that $|D^r| \leq n$ and

$$-c^r \in \operatorname{cone}\{a_t^r,\ t \in D^r\}.$$

Applying the previous lemma we may assume w.l.o.g. that $D^r = \widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x})$ for all r.

Since $\widehat{D} \subset T_{a^r,b^r}(x^r)$ and $x^r \in \mathcal{F}(a^r, b^r)$, we have

$$x^r \in \mathcal{L}_{\widehat{D}}\left(a^r, b^r, -a^r_{\widehat{D}}, -b^r_{\widehat{D}}\right),\ r = 1, 2, ...$$

Then, since $\left\|\left(a^r, b^r, -a^r_{\widehat{D}}, -b^r_{\widehat{D}}\right) - \left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right)\right\| = \left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|$, we have (applying the convention $0/0 := 0$ if necessary)

$$\begin{aligned} \operatorname{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right), \overline{x}\right) &\geq \limsup_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\|} \\ &\geq \lim_{r \to \infty} \frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|} = \operatorname{clm}\mathcal{S}(\overline{p}, \overline{x}). \end{aligned}$$

Note that $\mathcal{L}_{\widehat{D}}\left(\overline{a},\overline{b},-\overline{a}_{\widehat{D}},-\overline{b}_{\widehat{D}}\right)=\{\overline{x}\}$, which comes from the uniqueness of nominal optimal solution, has been appealed to in the first inequality of the previous chain.

(ii) It follows from the facts that $\mathcal{S}(\overline{p})=\mathcal{F}\left(\overline{a},\overline{b}\right)=\{\overline{x}\}$ and $\mathcal{S}(p)\subset\mathcal{F}(a,b)$ for every $p=(c,a,b)\in \mathbb{R}^n\times(\mathbb{R}^n)^T\times\mathbb{R}^T$. More specifically, if $\mathrm{clm}\mathcal{S}(\overline{p},\overline{x})$ is written as (17), then

$$\mathrm{clm}\mathcal{S}(\overline{p},\overline{x})\leq\limsup_{r\to\infty}\frac{\|x^r-\overline{x}\|}{\left\|(a^r,b^r)-\left(\overline{a},\overline{b}\right)\right\|}\leq\mathrm{clm}\mathcal{F}\left(\left(\overline{a},\overline{b}\right),\overline{x}\right).$$

Finally, the expression of $\mathrm{clm}\mathcal{F}\left(\left(\overline{a},\overline{b}\right),\overline{x}\right)$ comes directly from Propositions 2.1 and 2.2. □

Remark 4.1 In case (i) of the previous theorem, and recalling Theorem 3.1, inequality (16) may be read as

$$\mathrm{clm}\mathcal{S}(\overline{p},\overline{x})\leq(\|\overline{x}\|+1)\,\mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),\overline{x}\right), \tag{18}$$

which in the case $\overline{x}=0_n$ holds as an equality; i.e.,

$$\begin{aligned}\mathrm{clm}\mathcal{S}(\overline{p},0_n) &= \mathrm{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c},\overline{b}\right),0_n\right)\\ &= \max_{D\in\mathcal{K}_{\overline{p}}(0_n)}\frac{1}{d_*\left(0_n,\mathrm{bd\,conv}\{\overline{a}_t,\ t\in T_{\overline{a},\overline{b}}(0_n);-\overline{a}_t,\ t\in D\}\right)}\end{aligned}$$

as a direct consequence of the fact that $\mathrm{clm}\mathcal{S}_{\overline{a}}(\left(\overline{c},\overline{b}\right),\overline{x})\leq\mathrm{clm}\mathcal{S}(\overline{p},\overline{x})$, which follows immediately from the definitions.

Fact 4.2 *The upper bound on* $\mathrm{clm}\mathcal{S}(\overline{p},\overline{x})$ *provided in Theorem 4.2(i) may be strict when* $\overline{x}\neq 0_n$ *(see Remark 4.1). Accordingly, inequality (18) may be strict for* $\overline{x}\neq 0_n$.

Proof Let us consider the following example, where we also intend to illustrate some underlying geometrical perturbation ideas. Consider, in the context of parameterized linear optimization problems $P(c,a,b)$ of the form (1), the nominal problem, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} P\left(\overline{c},\overline{a},\overline{b}\right): & \text{minimize} & 10x_1 \\ & \text{subject to} & -x_1+x_2\leq -1\ \ (t=1),\\ & & -2x_1-2x_2\leq -6\ \ (t=2),\\ & & -x_1\leq -2\ \ (t=3),\end{array} \tag{19}$$

whose unique optimal solution is $\overline{x} = (2,1)'$. Set once more $\overline{p} = \left(\overline{c}, \overline{a}, \overline{b}\right)$. The reader can check the following:

$D \in \mathcal{K}_{\overline{p}}(\overline{x})$	$\text{clm}\mathcal{L}_D\left(\left(\overline{a}, \overline{b}, -\overline{a}_D, -\overline{b}_D\right), \overline{x}\right)$
$\{3\}$, $\{1,3\}$	$5+\sqrt{5} \approx 7.2361$
$\{1,2\}$	$\sqrt{10}\left(1+\sqrt{5}\right)/4 \approx 2.5583$
$\{2,3\}$	$\sqrt{13}\left(1+\sqrt{5}\right)/2 = 5.8339$

Hence, the maximum in the right-hand-side of (16) equals $5+\sqrt{5} \approx 7.2361$ and is attained at both $D = \{3\}$ and $D = \{1,3\}$. Let us write $\text{clm}\mathcal{S}(\overline{p}, \overline{x})$ in the form (17) and assume w.l.o.g. that the associated sequence $\{D^r\}_{r\in\mathbb{N}}$ is constant, say $D^r = \widehat{D} \in \mathcal{K}_{\overline{p}}(\overline{x})$ for all r, according to the lines after (17) in the proof of Theorem 4.2. Next we are going to show that

$$\text{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \frac{1}{10}\sqrt{820\sqrt{5}+3142} \approx 7.0538, \tag{20}$$

and therefore (16) holds as a strict inequality in this example.

Looking at the end of the proof of Theorem 4.2(i), we realize that $\text{clm}\mathcal{S}(\overline{p}, \overline{x}) \leq \text{clm}\mathcal{L}_{\widehat{D}}\left(\left(\overline{a}, \overline{b}, -\overline{a}_{\widehat{D}}, -\overline{b}_{\widehat{D}}\right), \overline{x}\right)$, so that our claim (20) holds automatically if $\widehat{D}$ equals either $\{1,2\}$ or $\{2,3\}$.

Let us consider now the case $\widehat{D} = \{3\}$ and set $\varepsilon_r := \|p^r - \overline{p}\|$ for all r, with $p^r = (c^r, a^r, b^r)$. Next, we relax the constraints determining $\mathcal{L}_{\{3\}}\left(a^r, b^r, -a^r_{\{3\}}, -b^r_{\{3\}}\right)$ (which contains x^r) around x^r in an appropriate way. Specifically, we are going to show that x^r is a solution of the following system:

$$\left.\begin{array}{rl}\left(-1-2\alpha_r/\sqrt{5}\right)x_1 + \left(1-\alpha_r/\sqrt{5}\right)x_2 & \leq -1+\alpha_r, \\ \left(-2-2\alpha_r/\sqrt{5}\right)x_1 + \left(-2-\alpha_r/\sqrt{5}\right)x_2 & \leq -6+\alpha_r, \\ \left(-1-2\alpha_r/\sqrt{5}\right)x_1 + \left(-\alpha_r/\sqrt{5}\right)x_2 & \leq -2+\alpha_r, \\ \left(1-\alpha_r\right)x_1 - \dfrac{\alpha_r\left(1+\alpha_r\right)}{10}x_2 & \leq 2+\alpha_r,\end{array}\right\} \tag{21}$$

with $\alpha_r := \dfrac{\|\overline{x}\| + \beta\varepsilon_r}{\|\overline{x}\| - \beta\varepsilon_r}\varepsilon_r$, where we have preferred to write $\|\overline{x}\|$ instead of $\sqrt{5}$, a scalar $\beta > 5+\sqrt{5}$ is arbitrarily chosen, and r is assumed to be large enough to ensure $\|\overline{x}\| - \beta\varepsilon_r > 0$.

The reader can check via a routinary computation that, if $\widetilde{x}^r$ stands for the furthest solution of (21) with respect to $\overline{x}$, then one has

$$\|\widetilde{x}^r - \overline{x}\| \approx \frac{\sqrt{820\sqrt{5}+3142}}{10}\alpha_r$$

(i.e., $\lim_{r\to\infty} \|\widetilde{x}^r - \overline{x}\| / \alpha_r = (1/10)\sqrt{820\sqrt{5}+3142} \approx 7.0538$), which together with the obvious fact that $\|x^r - \overline{x}\| \leq \|\widetilde{x}^r - \overline{x}\|$ –since x^r is a solution of (21)– yields (20) by taking into account that $\alpha_r \approx \varepsilon_r$ as $r \to \infty$.

Now let us go to the details. The first three inequalities of (21) may be written in the form

$$\begin{pmatrix} \overline{a}_t - \alpha_r \overline{x}/\|\overline{x}\| \\ \overline{b}_t + \alpha_r \end{pmatrix}' \begin{pmatrix} x \\ -1 \end{pmatrix} \leq 0, \text{ for } t = 1, 2, 3, \tag{22}$$

recalling that x, $\overline{x}$, and $\overline{a}_t$ are regarded as column vectors (in $\mathbb{R}^2$). Let us see that $x = x^r$ verifies (22). The fact that $x^r \in \mathcal{F}(a^r, b^r)$ allows us to write, for $t = 1, 2, 3$, and recalling our choice of norms in Section 1,

$$\begin{aligned} &\begin{pmatrix} \overline{a}_t - \alpha_r \overline{x}/\|\overline{x}\| \\ \overline{b}_t + \alpha_r \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} \\ \leq\ & \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix}' \begin{pmatrix} x^r \\ -1 \end{pmatrix} + \left\| \begin{pmatrix} \overline{a}_t \\ \overline{b}_t \end{pmatrix} - \begin{pmatrix} a_t^r \\ b_t^r \end{pmatrix} \right\| (\|x^r\| + 1) - \alpha_r \left(\frac{\overline{x}' x^r}{\|\overline{x}\|} + 1 \right) \\ \leq\ & 0 + \varepsilon_r (\|\overline{x}\| + \|x^r - \overline{x}\| + 1) - \alpha_r (\|\overline{x}\| - \|x^r - \overline{x}\| + 1), \end{aligned} \tag{23}$$

where we have appealed to $\overline{x}' x^r = \|\overline{x}\|^2 + \overline{x}'(x^r - \overline{x}) \geq \|\overline{x}\|^2 - \|\overline{x}\| \|x^r - \overline{x}\|$.

From now on, let us assume that $r \in \mathbb{N}$ is large enough to ensure $\|x^r - \overline{x}\| < \beta\varepsilon_r$ –recall (16), (17), and the definition of ε_r–. Then, taking the definition of α_r into account, the last member of chain (23) may be bounded from above by

$$\varepsilon_r (\|\overline{x}\| + \beta\varepsilon_r) - \alpha_r (\|\overline{x}\| - \beta\varepsilon_r) + \varepsilon_r - \alpha_r = \varepsilon_r - \alpha_r \leq 0.$$

In this way we have proved that x^r verifies the three first inequalities of (21).

In order to check the fourth inequality of (21) at $x = x^r$, and recalling our choice of norms in Section 1, we proceed as follows: according to the KKT conditions (with $\widehat{D} = \{3\}$ acting as the KKT index set), write $-c^r = \lambda_3^r a_3^r$ with $\lambda_3^r \geq 0$ and set $a_3^r = (a_{31}^r, a_{32}^r)'$. Then one has, with d standing for the Euclidean distance in $\mathbb{R}^2$ and recalling that $\overline{c} = (10, 0)'$ and $\overline{a}_3 = (-1, 0)'$,

$$\varepsilon_r \geq \|c^r - \overline{c}\| \geq d(\overline{c}, \mathbb{R}a_3^r) = \frac{10\,|a_{32}^r|}{\|a_3^r\|} \geq \frac{10\,|a_{32}^r|}{1 + \varepsilon_r},$$

and, accordingly $|a_{32}^r| \leq \dfrac{\varepsilon_r(1+\varepsilon_r)}{10} \leq \dfrac{\alpha_r(1+\alpha_r)}{10}$. Finally, the fact that $x^r \in \mathcal{L}_{\widehat{D}}\left(a^r, b^r, -a_{\widehat{D}}^r, -b_{\widehat{D}}^r\right)$ entails $(a_3^r)' x^r = b_3^r$, and clearly we may assume that both coordinates of $x^r = (x_1^r, x_2^r)'$ are positive, provided that r is large enough. Consequently, recalling that $\varepsilon_r = \|p^r - \overline{p}\| \leq \alpha_r$ (strictly, indeed) and writing $\overline{a}_3 = (\overline{a}_{31}, \overline{a}_{32})'$, we have

$$-\overline{b}_3 + \alpha_r \geq -b_3^r = -a_{31}^r x_1^r - a_{32}^r x_2^r \geq (-\overline{a}_{31} - \alpha_r)\, x_1^r - \frac{\alpha_r\,(1+\alpha_r)}{10} x_2^r, \quad (24)$$

which is nothing else but the fourth inequality of (21) at $x = x^r$.

Now we tackle the remaining case $\widehat{D} = \{1, 3\}$. Let us write, for each r, $-c^r = \lambda_1^r a_1^r + \lambda_3^r a_3^r$ with $\lambda_1^r, \lambda_3^r \geq 0$ and $a_i^r = (a_{i1}^r, a_{i2}^r)'$ for $i = 1, 3$. The proof of the fulfillment of the three first inequalities of (21) is the same as in the previous case. Let us distinguish two subcases: $a_{32}^r > 0$ for infinitely many r's (w.l.o.g. for all r) and the subcase $a_{32}^r \leq 0$ (again, w.l.o.g., for all r). In the first one, for r large enough to ensure that $a_{11}^r < 0$, $a_{12}^r > 0$, $a_{31}^r < 0$, and $-a_{31}^r / \|a_3^r\| > -a_{11}^r / \|a_1^r\|$ (i.e., the acute angle of a_3^r with the abscissas axis is smaller than the one of a_1^r), we have

$$\varepsilon_r \geq \|c^r - \overline{c}\| \geq d\,(-\overline{c}, \operatorname{cone}\{a_1^r, a_3^r\}) = \frac{10\,|a_{32}^r|}{\|a_3^r\|} \geq \frac{10\,|a_{32}^r|}{1+\varepsilon_r},$$

so that the fourth inequality of (21) holds at $x = x^r$ and the subcase $a_{32}^r > 0$ works exactly as the previous case. The subcase $a_{32}^r \leq 0$ is also very similar, but replacing in an appropriate way the fourth inequality of (21). Indeed, instead of (24) we have

$$-\overline{b}_3 + \alpha_r \geq -b_3^r = -a_{31}^r x_1^r - a_{32}^r x_2^r \geq (-\overline{a}_{31} - \alpha_r)\, x_1^r.$$

In this way, replacing the fourth inequality of (21) with $(1 - \alpha_r)\, x_1 \leq 2 + \alpha_r$, and denoting by $\widetilde{x}^r$ the furthest solution of the corresponding counterpart of (21) with respect to $\overline{x}$, the reader can check that

$$\|\widetilde{x}^r - \overline{x}\| \approx \sqrt{8\sqrt{5} + 30}\ \alpha_r$$

with $\sqrt{8\sqrt{5} + 30} \approx 6.9202$, leading again to (20). $\square$

The next example shows that the upper bound on $\operatorname{clm}\mathcal{S}\,(\overline{p}, \overline{x})$ provided in Theorem 4.2(i) may be attained for $\overline{x} \neq 0_n$. It is basically the same example as (19), with the only difference that the objective function coefficient vector $\overline{c}$ is shorter here. Observe that in the case of canonical perturbations the size of $\overline{c}$ has no effect on $\operatorname{clm}\mathcal{S}_{\overline{a}}\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ (see Theorem 3.1).

Example 4.1 Consider the nominal problem obtained from (19) by just replacing $\overline{c}$ therein with $(1, 0)'$. Then (16) holds as an equality. Just consider the perturbed parameter $p^r = (c^r, a^r, b^r)$ given by

$$\begin{pmatrix} a_1^r \\ b_1^r \end{pmatrix} = \begin{pmatrix} \overline{a}_1 - \frac{1}{r}\frac{\overline{x}}{\|\overline{x}\|} \\ \overline{b}_1 + \frac{1}{r} \end{pmatrix}, \quad \begin{pmatrix} a_2^r \\ b_2^r \end{pmatrix} = \begin{pmatrix} \overline{a}_2 \\ \overline{b}_2 \end{pmatrix},$$
$$\begin{pmatrix} a_3^r \\ b_3^r \end{pmatrix} = \begin{pmatrix} \overline{a}_3 + \frac{1}{r}\frac{\overline{x}}{\|\overline{x}\|} \\ \overline{b}_3 - \frac{1}{r} \end{pmatrix}, \quad c^r = -a_3^r. \tag{25}$$

The reader can check that

$$x^r := \frac{1}{1 - 4/\left(r\sqrt{5}\right)} \begin{pmatrix} 2 - 3/\left(r\sqrt{5}\right) + 1/r \\ 1 + 6/\left(r\sqrt{5}\right) + 2/r \end{pmatrix} \in \mathcal{S}\left(p^r\right), \tag{26}$$

indeed $x^r \in \mathcal{L}_{\{3\}}\left(a^r, b^r, -a^r_{\{3\}}, -b^r_{\{3\}}\right)$, and

$$\lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\|p^r - \overline{p}\|} = 5 + \sqrt{5}.$$

Remark 4.2 Coming back to example (19), if we perturbed there the constraint system as in (25), then the minimum perturbation of $\overline{c} = (10, 0)'$ (yielding a perturbed c^r) making point x^r in (26) belong to $\mathcal{S}\left(p^r\right)$ would satisfy

$$\|c^r - \overline{c}\| = d\left(-\overline{c}, \operatorname{cone}\{a_1^r, a_3^r\}\right) \approx 2\sqrt{5}/r,$$

while $\left\|(a^r, b^r) - \left(\overline{a}, \overline{b}\right)\right\| = 1/r$ and, accordingly we would obtain the smaller ratio $\|x^r - \overline{x}\| / \|p^r - \overline{p}\| \approx \left(5 + \sqrt{5}\right) / \left(2\sqrt{5}\right) \approx 1.618$.

The upper bound given in Theorem 4.2(*ii*) may also not be attained, even with $\overline{x} = 0_n$ as the following example shows.

Example 4.2 Consider the nominal problem $P\left(\overline{c}, \overline{a}, \overline{b}\right)$, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{lll} \text{minimize} & x_1 + x_2 & \\ \text{subject to} & -x_1 \leq 0, & (t = 1) \\ & -x_2 \leq 0, & (t = 2) \\ & x_1 + x_2 \leq 0, & (t = 3) \end{array}$$

so that $\mathcal{F}\left(\overline{a}, \overline{b}\right) = \mathcal{S}\left(\overline{c}, \overline{a}, \overline{b}\right) = \{\overline{x}\}$ with $\overline{x} = 0_2$. Then, appealing to [8, Theorems 4 and 5], the reader can check that $\operatorname{clm}\mathcal{F}\left(\left(\overline{a}, \overline{b}\right), \overline{x}\right) = \sqrt{5}$. On the other hand, if $\widetilde{\mathcal{S}}$ denotes the optimal set mapping obtained by removing the last constraint $(t = 3)$ from the parameterized problem, and we denote as $\left(a_{\widetilde{T}}, b_{\widetilde{T}}\right)$ the restriction of parameter (a,b) to $\widetilde{T} := \{1,2\}$, then the reader can easily check that

$$\begin{aligned} \operatorname{clm}\mathcal{S}\left(\left(\overline{c},\overline{a},\overline{b}\right),\overline{x}\right) &\leq \operatorname{clm}\widetilde{\mathcal{S}}\left(\left(\overline{c},\overline{a}_{\widetilde{T}},\overline{b}_{\widetilde{T}}\right),\overline{x}\right) = \sqrt{2} && (27) \\ &< \sqrt{5} = \operatorname{clm}\mathcal{F}\left(\left(\overline{a},\overline{b}\right),\overline{x}\right), && (28) \end{aligned}$$

where the first inequality in (27) comes from the fact that $\mathcal{S}\left(\overline{c},\overline{a},\overline{b}\right) = \widetilde{\mathcal{S}}\left(\overline{c},\overline{a}_{\widetilde{T}},\overline{b}_{\widetilde{T}}\right) = \{\overline{x}\}$ and $\mathcal{S}(c,a,b) \subset \widetilde{\mathcal{S}}\left(c,a_{\widetilde{T}},b_{\widetilde{T}}\right)$ for (c,a,b) close enough to $\left(\overline{c},\overline{a},\overline{b}\right)$; the last equality in (27) comes straightforwardly from Remark 4.1, and the equality in (28) comes from [8, Theorems 4 and 5].

We finish the paper with a last example, which shows that the upper bound given in Theorem 4.2(*ii*) may be attained and be strictly larger than the right-hand-side of (16). This example comes from adding a new constraint to example (11) in order to provoke the uniqueness of nominal optimal solution.

Example 4.3 Consider $P\left(\overline{c},\overline{a},\overline{b}\right)$, in $\mathbb{R}^2$ endowed with the Euclidean norm,

$$\begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & -x_1 \leq 0, \quad (t=1) \\ & -x_2 \leq 0, \quad (t=2) \\ & x_2 \leq 0, \quad\;\; (t=3) \\ & \frac{1}{2}x_1 \leq 0. \quad (t=4) \end{array}$$

whose unique feasible solution is $\bar{x} = 0_2$. Then, the reader can chech that $\{1\} \in \mathcal{K}_{\overline{p}}(\overline{x})$ and $t=1$ must belong to any other $D \in \mathcal{K}_{\overline{p}}(\overline{x})$, entailing that the right-hand-side of (16) equals $\sqrt{2}$. On the other hand, let us consider, for each $r = 1,2,...$, the perturbed problem $P(\overline{c},a^r,b^r)$ given by

$$\begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & -x_1 \leq 0, \\ & -x_2 \leq -\frac{1}{r}, \\ & -\frac{1}{r^2}x_1 + x_2 \leq -\frac{2}{r^3} + \frac{1}{r}, \\ & \frac{1}{2}x_1 \leq \frac{1}{r}. \end{array}$$

Then we have $\mathcal{S}(\overline{c},a^r,b^r) = \{x^r\}$, with $x^r := \left(\frac{2}{r},\frac{1}{r}\right)'$, and

$$\begin{aligned} \operatorname{clm}\mathcal{S}(\overline{p},\overline{x}) &\geq \lim_{r\to\infty} \frac{\|x^r - \overline{x}\|}{\left\|(\overline{c},a^r,b^r) - \left(\overline{c},\overline{a},\overline{b}\right)\right\|} \\ &= \lim_{r\to\infty} \frac{\sqrt{5}/r}{1/r} = \sqrt{5} = \operatorname{clm}\mathcal{F}\left(\left(\overline{a},\overline{b}\right),\overline{x}\right), \end{aligned}$$

where the last equality comes from [8, Theorems 4 and 5].

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