On the effects of combining objectives in multi-objective optimization

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Abstract

In multi-objective optimization, one considers optimization problems with more than one objective function, and in general these objectives conflict each other. As the solution set of a multiobjective problem is often rather large and contains points of no interest to the decision-maker, strategies are sought that reduce the size of the solution set. One such strategy is to combine several objectives with each other, i.e. by summing them up, before employing tools to solve the resulting multiobjective optimization problem. This approach can be used to reduce the dimensionality of the solution set as well as to discarde certain unwanted solutions, especially the 'extreme' ones found by minimizing just one of the objectives given in the classical sense while disregarding all others. In this paper, we discuss in detail how the strategy of combining objectives linearly influences the set of optimal, i.e. efficient solutions.

1 Introduction

In multi-objective optimization, one considers optimization problems with more than one objective function, and in general these objectives conflict each other. Such optimization problems arise in many applications; in most of them the vectors of different objective function values are compared componentwise. Using the classical optimality concepts, any feasible point minimizing one of the concurrent objective functions is thus considered to be at least a so-called weakly optimal solution of the multi-objective optimization problem.
As the solution set of a multiobjective problem is often rather large and contains points of no interest to the decision-maker, strategies are sought that reduce the size of the solution set. One such strategy is to combine several objectives with

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each other, i.e. by summing them up, before employing tools to solve the resulting multiobjective optimization problem. This approach can be used to reduce the size of the solution set by discarding unwanted solutions, especially the 'extreme' ones found by minimizing just one of the objectives given in the classical sense while disregarding all others. For instance, in a recent application [8, Example 1], the following approach was used: one of the objective functions was replaced by a weighted sum of all objectives. This eliminated the 'unwanted' minima of the specific objective replaced in the overall solution set. Also, this strategy can be used to reduce the dimensionality of the objective space —an important consideration, as the time needed for solving multiobjective optimization problems (i.e. generating an approximation of the set of solution points) in general grows exponentially with the number of objective functions. Also, of course, reducing the dimension of the objective space to 2 or 3 has immediate impact on the employment of visualization strategies.

In this paper, we discuss in detail how the strategy of combining objectives linearly influences the set of optimal, i.e. efficient solutions. In contrast to classical perturbation analysis, our approach is a global one, i.e. we are interested in how the whole solution set changes when the structure of the objective function vector is changed in such a radical way. The rest of this paper is as follows. In Section 2 we provide the necessary notations and definitions. The main results are presented in Section 3.

2 Notation and basic definitions

We study in this paper a multi-objective optimization problem with m not necessarily convex objective functions $f_i \colon S \to \mathbb{R}$ defined on a nonempty closed set $S \subset \mathbb{R}^n$:

$$\min_{x \in S} \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} . \tag{MOP}$$

To define what constitutes an optimal solution, we assume that the image space $\mathbb{R}^m$ of the problem is partially ordered. Let $K \subset \mathbb{R}^m$ be a convex cone which defines this partial ordering. Then for all $x, y \in \mathbb{R}^m$,

$$x \leq_K y \ :\Leftrightarrow \ y - x \in K \ ,$$

and it is this partial order that defines solutions of the given multiobjective problem, see Definition 2.2 below. (Recall that a set $K \subset \mathbb{R}^m$ is a cone if $\lambda x \in K$ for all $\lambda \geq 0$ and all $x \in K$ and a cone is convex if $x + y \in K$ for all $x, y \in K$.)

In the following, $\mathrm{int}(K)$ denotes the interior of the cone K. We will also consider elements from the dual cone of K. The dual cone is defined by

$$K^* := \{y \in \mathbb{R}^m \mid y^\top x \geq 0 \ \text{ for all } x \in K\} \ .$$

For an arbitrary set $M \subset \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{p \times m}$, we write $AM := \{y \in \mathbb{R}^p \mid y = Ax, x \in M\}$ and $f(M) := \{f(x) \in \mathbb{R}^m \mid x \in M\}$ for $f \colon M \to \mathbb{R}^m$.

In applications, the most widely used cone is the convex cone $K = \mathbb{R}^m_+$ which defines the componentwise (natural) partial ordering in $\mathbb{R}^m$. For any $x, y \in \mathbb{R}^m$

$$x \leq_{\mathbb{R}^m} y \;\Leftrightarrow\; y - x \in \mathbb{R}^m_+ \;\Leftrightarrow\; x_i \leq y_i \text{ for all } i = 1, \ldots, m\ .$$

Such partial orderings lead to the optimality concepts given in Definition 2.2. For defining proper optimal solution (in the sense of Borwein [1]) we need the definition of a contingent cone.

Definition 2.1. *The tangent cone $T_Z(x^0)$ at $x^0 \in Z \subset \mathbb{R}^m$ is the cone*

$$T_Z(x^0) := \{d \in \mathbb{R}^m \mid \ \exists\ \{x^k\}_{k=1}^{\infty} \subset Z \text{ converging to } x^0,\ \exists\ \{t_k\}_{k=1}^{\infty} \downarrow 0 \\ \text{with } d = \lim_{k\to\infty} \tfrac{1}{t_k}(x^k - x^0)\}.$$

The normal cone to the set Z at $x^0 \in Z$ is defined by

$$N_Z(x^0) := \{d \in \mathbb{R}^m \mid d^\top v \geq 0\ \forall\ v \in T_Z(x^0)\}.$$

Definition 2.2. *(a) A point $\bar{x} \in S$ is an* optimal solution *of (MOP) if*

$$(\{f(\bar{x})\} - K) \cap f(S) = \{f(\bar{x})\}\ . \tag{1}$$

(b) Let int$(K) \neq \emptyset$. A point $\bar{x} \in S$ is a weakly optimal solution *of (MOP) if*

$$(\{f(\bar{x})\} - \mathrm{int}(K)) \cap f(S) = \emptyset\ . \tag{2}$$

(c) A point $\bar{x} \in S$ is a properly optimal solution *of (MOP) if it is an optimal solution of (MOP) and if*

$$(-K) \cap (T_{f(S)+K}(f(\bar{x}))) = \{0_{\mathbb{R}^m}\}\ . \tag{3}$$

We denote the set of optimal and of weakly optimal solutions of (MOP) w.r.t. the convex cone K by $\mathcal{M}(\mathrm{MOP}, K)$ and $\mathcal{M}_w(\mathrm{MOP}, K)$, respectively.
For $K = \mathbb{R}^m_+$, (1) is equivalent to that there exists no $x \in S$ with

$$\begin{array}{rcll} f_i(x) & \leq & f_i(\bar{x}), & i = 1, \ldots, m, \\ f_j(x) & < & f_j(\bar{x}), & \text{for at least one } \ j \in \{1, \ldots, m\} \end{array}$$

and (2) is equivalent to that there exists no $x \in S$ with

$$f_i(x) \ < \ f_i(\bar{x}), \quad i = 1, \ldots, m\ .$$

The individual minima of each objective function are weakly optimal solutions of (MOP) w.r.t. $K = \mathbb{R}^m_+$, cf. [3, Lemma 2.12] or Lemma 3.8:

Remark 2.3. *We have*

$$\mathrm{argmin}\{f_i(x) \mid x \in S\} \subseteq \mathcal{M}_w(MOP, \mathbb{R}^m_+) \ \text{ for all } i = 1, \ldots, m\ .$$

In the following we study how the solution sets $\mathcal{M}_w(\text{MOP}, K)$ and $\mathcal{M}(\text{MOP}, K)$ change when we modify the given multi-objective optimization problem as follows. Let A be a real $p \times m$ matrix with rows $a^j \in \mathbb{R}^m$ and consider the multi-objective optimization problem given by

$$\min_{x \in S} Af(x) = \begin{pmatrix} (a^1)^\top f(x) \\ \vdots \\ (a^p)^\top f(x) \end{pmatrix} . \tag{A-MOP}$$

We denote the set of optimal and of weakly optimal solutions of (A-MOP) w.r.t. a convex cone C by $\mathcal{M}(\text{A-MOP}, C)$ and $\mathcal{M}_w(\text{A-MOP}, C)$, respectively. Obviously, $\mathcal{M}(\text{A-MOP}, K) = \mathcal{M}(\text{MOP}, K)$ and $\mathcal{M}_w(\text{A-MOP}, K) = \mathcal{M}_w(\text{MOP}, K)$ if $p = m$ and A is a positive definite diagonal matrix.
Note again that we are not interested in local perturbation analysis, i.e. what happens when entries of A (or the identity matrix implicitly assumed in (MOP)) are perturbed by small values. Instead, our focus is on a global analysis of the change of the solution set for arbitrary A and arbitrary p.

Example 2.4. *In [8, Example 1] a multi-objective optimization problem with two objective functions f_1 and f_2 over some feasible set S is considered. However, instead of solving $\min_{x \in S}(f_1(x), f_2(x))$ directly, the modified problem*

$$\min_{x \in S} \begin{pmatrix} f_1(x) \\ f_1(x) + 140 f_2(x) \end{pmatrix}$$

w.r.t. the natural ordering cone, i.e. $K = \mathbb{R}^m_+$, was solved. In this case,

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 140 \end{pmatrix} .$$

3 Combining Objectives

3.1 Relationships between sets of optimal points

We begin our analysis by establishing some results on the relationships between the set of solutions to the original problem, $\mathcal{M}(\text{MOP}, K)$, and the set of solutions of the multi-objective problem where the objectives are linearly transformed by a matrix, $\mathcal{M}(\text{A-MOP}, C)$.

Theorem 3.1. *Let A be a $p \times m$ matrix, let $C \subset \mathbb{R}^p$ be an arbitrary convex cone and define $K := \{y \in \mathbb{R}^m \mid Ay \in C\}$.*

1. *We have*

$$\mathcal{M}(MOP, K) \subseteq \mathcal{M}(A\text{-}MOP, C)$$

and if, in addition, $\operatorname{int}(K) = \{y \in \mathbb{R}^m \mid Ay \in int(C)\}$, we also have

$$\mathcal{M}_w(MOP, K) = \mathcal{M}_w(A\text{-}MOP, C) .$$

2. Let A have rank m. We then have

$$\mathcal{M}(MOP,K)=\mathcal{M}(A\text{-}MOP,C)\ .$$

Proof. We have that $\bar{x}\in\mathcal{M}(\mathrm{MOP},K)$ holds if and only if

$$\begin{array}{ll} & (\{f(\bar{x})\}-K)\cap f(S)=\{f(\bar{x})\} \\ \Leftrightarrow & \nexists\ x\in S \text{ with } f(x)\neq f(\bar{x}) \text{ and } f(\bar{x})-f(x)\in K \\ \Leftrightarrow & \nexists\ x\in S \text{ with } f(x)\neq f(\bar{x}) \text{ and } A(f(\bar{x})-f(x))\in C \\ \Leftrightarrow & \forall\ x\in S \text{ it holds } f(x)-f(\bar{x})=0_{\mathbb{R}^m} \text{ or } A(f(\bar{x})-f(x))\notin C \\ \overset{(*)}{\Rightarrow} & \forall\ x\in S \text{ it holds } A(f(x)-f(\bar{x}))=0_{\mathbb{R}^p} \text{ or } A(f(\bar{x})-f(x))\notin C \\ \Leftrightarrow & \nexists\ x\in S \text{ with } Af(x)\neq Af(\bar{x}) \text{ and } A(f(\bar{x}))-A(f(x))\in C \\ \Leftrightarrow & (\{Af(\bar{x})\}-C)\cap(Af(S))=\{Af(\bar{x})\}, \end{array}$$

which shows the first relationship of part 1. If A has rank m, then in (*) above the backward direction also holds, which shows part 2.
Similar, if $\mathrm{int}(K)=\{y\in\mathbb{R}^m\mid Ay\in\mathrm{int}(C)\}$, we obtain the following: $\bar{x}\in\mathcal{M}_w(\mathrm{MOP},K)$ holds if and only if

$$\begin{array}{ll} & (\{f(\bar{x})\}-\mathrm{int}(K))\cap f(S)=\emptyset \\ \Leftrightarrow & \nexists\ x\in S \text{ with } f(\bar{x})-f(x)\in\mathrm{int}(K) \\ \Leftrightarrow & \nexists\ x\in S \text{ with } A(f(\bar{x})-f(x))\in\mathrm{int}(C) \\ \Leftrightarrow & \nexists\ x\in S \text{ with } Af(x)\in A(f(\bar{x}))-\mathrm{int}(C) \\ \Leftrightarrow & (\{Af(\bar{x})\}-\mathrm{int}(C))\cap(Af(S))=\emptyset\ . \end{array}$$

□

This result generalizes the results given in [13, Lemma 2.3.4], [5, Cor. 4.1] and [3, Lemma 1.18] for $C=\mathbb{R}^p_+$.

Remark 3.2. *A simple sufficient condition for* $\mathrm{int}(K)=\{y\in\mathbb{R}^m\mid Ay\in\mathrm{int}(C)\}$ *is* $\mathrm{int}(K)\neq\emptyset\neq\{y\in\mathbb{R}^m\mid Ay\in\mathrm{int}(C)\}$*, as Theorem 6.7 from Rockafellar [12] shows.*

Note that the assumption that A has rank m in the theorem above corresponds to pointedness of the cone K, as the following lemma shows. Recall that a cone $K\subset\mathbb{R}^m$ is pointed if $K\cap(-K)=\{0_{\mathbb{R}^m}\}$ and the binary relation $\leq_K$ is anti-symmetric if and only if K is pointed.

Lemma 3.3. *Let A be a $p\times m$ matrix, let $C\subset\mathbb{R}^p$ be an arbitrary pointed convex cone and define $K:=\{y\in\mathbb{R}^m\mid Ay\in C\}$. Then the following are equivalent:*

(i) K is pointed;

(ii) $K\setminus\{0_{\mathbb{R}^m}\}=\{y\in\mathbb{R}^m\mid Ay\in C\setminus\{0_{\mathbb{R}^p}\}\}$;

(iii) $rank(A)=m$.

Proof. We first show that (i) implies (ii). For that, assume there exists $y \in K\setminus\{0_{\mathbb{R}^m}\}$ with $Ay = 0_{\mathbb{R}^p}$. Then $A(-y) = 0_{\mathbb{R}^p}$ and hence $-y \in K$ in contradiction to K being pointed. Next we assume (ii) holds and we assume there is some $y \in K \cap (-K)$ with $y \neq 0_{\mathbb{R}^m}$. Then $z := Ay \in C \setminus \{0_{\mathbb{R}^p}\}$ and also $-z = A(-y) \in C \setminus \{0_{\mathbb{R}^p}\}$ in contradiction to C being pointed. Hence, (ii) implies (i). Next, (ii) implies

$$y = 0_{\mathbb{R}^m} \;\Leftrightarrow\; Ay = 0_{\mathbb{R}^p}$$

which is equivalent to $\operatorname{rank}(A) = m$. Finally, assume (iii) and suppose there is a $y \in K$, $y \neq 0_{\mathbb{R}^m}$. Then $Ay \neq 0_{\mathbb{R}^p}$, from which (ii) follows. □

The result generalizes a result given in [5] for $C = \mathbb{R}^p_+$.
In case the matrix A is a componentwise non-negative (i.e. it's rows can be interpreted as rows of weights of the objective functions) $m \times m$ matrix and $C = \mathbb{R}^m_+$, we have $\mathbb{R}^m_+ \subseteq K = \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^m_+\}$. It is a well known fact that if $K^1, K^2 \subset \mathbb{R}^m$ are two convex cones with $K^1 \subseteq K^2$, then $\mathcal{M}(\mathrm{MOP}, K^2) \subseteq \mathcal{M}(\mathrm{MOP}, K^1)$ and $\mathcal{M}_w(\mathrm{MOP}, K^2) \subseteq \mathcal{M}_w(\mathrm{MOP}, K^1)$ (in case $\operatorname{int}(K^1) \neq \emptyset$). This implies together with the preceding theorem and Remark 3.2 with $K := \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^m_+\}$ the following corollary.

Corollary 3.4. *Let A be regular and nonnegative. Then*

$$\mathcal{M}(A\text{-}MOP, \mathbb{R}^m_+) \subseteq \mathcal{M}(MOP, \mathbb{R}^m_+) \;\; and \;\; \mathcal{M}_w(A\text{-}MOP, \mathbb{R}^m_+) \subseteq \mathcal{M}_w(MOP, \mathbb{R}^m_+)\ .$$

Example 3.5. *We consider the same setting as in Example 2.4. The implied polyhedral convex cone K is thus*

$$K = \left\{y \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_1 + 140 y_2 \geq 0\right\}$$

which contains the non-negative orthant $\mathbb{R}^2_+$. Thus, one has to expect that the solution set of the modified multi-objective optimization problem (A-MOP) is a proper subset of the solution set of the original multi-objective optimization problem (MOP).

The following example shows that a modification of the multi-objective optimization problem with a matrix A as discussed above may not change the optimal solution set at all:

Example 3.6. *Let $C = \mathbb{R}^2_+$, A be a regular non-negative 2×2 matrix. Moreover, let $S = [1,2] \times [1,2]$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x) = x$ for all $x \in \mathbb{R}^2$.*
Then by Lemma 3.3 and Corollary 3.4 the cone K is pointed, $\mathbb{R}^2_+ \subseteq K$ and

$$\mathcal{M}(MOP, K) = \mathcal{M}(A\text{-}MOP, \mathbb{R}^2_+) \subseteq \mathcal{M}(MOP, \mathbb{R}^2_+).$$

We have $\mathcal{M}(MOP, \mathbb{R}^2_+) = \{(1,1)\}$. As there exists no $d \neq 0_{\mathbb{R}^2}$ with $d \in K \cap (-\mathbb{R}^2_+)$ as K is pointed this leads to

$$\mathcal{M}(MOP, K) = \mathcal{M}(A\text{-}MOP, \mathbb{R}^2_+) = \mathcal{M}(MOP, \mathbb{R}^2_+) = \{(1,1)\}$$

for any non-negative regular matrix A. An essential condition for this result is uniqueness of the optimal solution.

However, $\mathcal{M}_w(MOP, \mathbb{R}^2_+) = \operatorname{conv}\{(1,1),(1,2)\} \cup \operatorname{conv}\{(1,1),(2,1)\}$. *But, for each* $\alpha > 0$ *and*

$$A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

we have $\mathcal{M}_w(A\text{-}MOP, \mathbb{R}^2_+) = \operatorname{conv}\{(1,1),(1,2)\}$.

If the set of optimal solutions of the multi-objective optimization problem is unbounded, the modified multi-objective optimization problem may have no (weak) optimal solution.

Example 3.7. *Let* $S = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0\}$ *and* $f(x) = x$ *for all* $x \in \mathbb{R}^2$. *Then,* $\mathcal{M}(MOP, \mathbb{R}^2_+) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}$. *Consider the regular non-negative matrix*

$$A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \qquad \text{with} \qquad \alpha \geq 0$$

Then,

$$\mathcal{M}(A\text{-}MOP, \mathbb{R}^2_+) = \begin{cases} \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}, & \text{if } 0 \leq \alpha < 1 \\ \emptyset, & \text{if } \alpha \geq 1 \end{cases}$$

and

$$\mathcal{M}_w(A\text{-}MOP, \mathbb{R}^2_+) = \begin{cases} \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}, & \text{if } 0 \leq \alpha \leq 1 \\ \emptyset, & \text{if } \alpha > 1 \end{cases} .$$

Next we examine some classical scalarization techniques, and how changing the objectives to linear combinations of the given objectives changes the set of solutions that can be recovered by scalarizations.

3.2 Linear Scalarizations

For the study of linear scalarizations, we need the following two basic lemmas.

Lemma 3.8. *(a) If there exists some* $w \in K^* \setminus \{0_{\mathbb{R}^m}\}$ *with* $\bar{x} \in \operatorname{argmin}\{w^\top f(x) \mid x \in S\}$, *then* $\bar{x} \in \mathcal{M}_w(MOP, K)$. *If even* $w \in \operatorname{int}(K^*)$ *then* $\bar{x} \in \mathcal{M}(MOP, K)$.

(b) If $\bar{x} \in \mathcal{M}_w(MOP, K)$ *and* $f(S) + K$ *is convex, then* $\bar{x} \in \operatorname{argmin}\{w^\top f(x) \mid x \in S\}$ *for some* $w \in K^* \setminus \{0_{\mathbb{R}^m}\}$.

(c) If there exists some $w \in \operatorname{int}(K^*)$ *with* $\bar{x} \in \operatorname{argmin}\{w^\top f(x) \mid x \in S\}$, *then* $\bar{x}$ *is a properly optimal solution of (MOP). If* $\bar{x}$ *is a properly optimal solution of (MOP) and* $f(S) + K$ *is convex, then* $\bar{x} \in \operatorname{argmin}\{w^\top f(x) \mid x \in S\}$ *for some* $w \in \operatorname{int}(K^*)$.

A proof of this lemma can be found in various places, see, e.g., [7, 2, 9].

Lemma 3.9. *Let* A *be a regular* $m \times m$ *matrix,* C *a closed convex cone and let the convex cone* $K = \{y \in \mathbb{R}^m \mid Ay \in C\} = A^{-1}C$ *be given. The dual cone of* K *is*

$$K^* = \{z \in \mathbb{R}^m \mid (A^{-1})^\top z \in C^*\} .$$

Proof. We have

$$\begin{aligned} K^* &= \{z \in \mathbb{R}^m \mid z^\top y \geq 0 \;\forall\; y \in K\} \\ &= \{z \in \mathbb{R}^m \mid z^\top A^{-1} s \geq 0 \;\forall\; s \in C\} \\ &= \{z \in \mathbb{R}^m \mid ((A^{-1})^\top z)^\top s \geq 0 \;\forall\; s \in C\} \\ &= \{z \in \mathbb{R}^m \mid (A^{-1})^\top z \in C^*\}. \end{aligned}$$

□

Corollary 3.10. *Let A be a regular $m \times m$ matrix and let the convex cone $K = \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^m_+\} = A^{-1}\mathbb{R}^m_+$ be given. The dual cone of K is*

$$K^* = \{z \in \mathbb{R}^m \mid (A^{-1})^\top z \in \mathbb{R}^m_+\} \ .$$

If A is regular and nonnegative, then $\mathbb{R}^m_+ \subseteq K = \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^m_+\}$ and thus $K^* \subseteq \mathbb{R}^m_+$.

Lemma 3.11. *Let $K = \{y \in \mathbb{R}^m \mid Ay \in C\}$, $f(S) + K$ be convex and $\operatorname{int}(K) = \{y \in \mathbb{R}^m \mid Ay \in \operatorname{int}(C)\}$. Then*

$$\begin{aligned} \mathcal{M}_w(MOP, K) = \mathcal{M}_w(A\text{-}MOP, C) &= \textstyle\bigcup_{w \in K^* \setminus \{0_{\mathbb{R}^m}\}} \operatorname{argmin}\{w^\top f(x) \mid x \in S\} \\ \mathcal{M}_w(MOP, C) &= \textstyle\bigcup_{w \in C^* \setminus \{0_{\mathbb{R}^m}\}} \operatorname{argmin}\{w^\top f(x) \mid x \in S\} \ . \end{aligned}$$

If A has additionally rank m, then

$$\mathcal{M}(MOP, K) = \mathcal{M}(A\text{-}MOP, C) \supseteq \bigcup_{w \in \operatorname{int}(K^*)} \operatorname{argmin}\{w^\top f(x) \mid x \in S\} \qquad (4)$$

$$\mathcal{M}(MOP, C) \supseteq \bigcup_{w \in \operatorname{int}(C^*)} \operatorname{argmin}\{w^\top f(x) \mid x \in S\} \qquad (5)$$

Proof. This follows directly from Lemma 3.8 and Theorem 3.1. □

As $\mathcal{M}(\text{A-MOP}, C) \subseteq \mathcal{M}_w(\text{A-MOP}, C)$ and $\mathcal{M}(\text{MOP}, C) \subseteq \mathcal{M}_w(\text{MOP}, C)$ the above lemma provides upper and lower bounds for the sets of optimal solutions. It also characterizes those optimal solutions which are not found considering the problem (A-MOP) w.r.t. the componentwise ordering (i.e. $C = \mathbb{R}^m_+$) instead of the original problem (MOP) w.r.t. the componentwise ordering in case the matrix A is a nonnegative regular matrix, see Corollary 3.4.
Lemma 3.8 also implies the following result:

Lemma 3.12. *Let $K = \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^m_+\}$ and A be regular. Let $\bar{x}^i$ denote a minimum of the objective function f_i, $i = 1, \ldots, m$. If $e^i \in K^*$, i.e. $(A^{-1})^\top e^i \in \mathbb{R}^m_+$, then $\bar{x}^i \in \mathcal{M}_w(MOP, K)$. If, in addition, $e^i \in \operatorname{int}(K^*)$ or $\operatorname{int}(K) = \{y \in \mathbb{R}^m \mid Ay \in \operatorname{int}(\mathbb{R}^m_+)\}$, then $\bar{x}^i \in \mathcal{M}(A\text{-}MOP, \mathbb{R}^m_+)$.*

Example 3.13. *Consider again the same setting as in Example 2.4. For the given choice of A,*

$$K^* = \left\{ z \in \mathbb{R}^2 \,\middle|\, \begin{pmatrix} 1 & -1/140 \\ 0 & 1/140 \end{pmatrix} z \in \mathbb{R}^2_+ \right\} \ .$$

Hence, in case $f(S)+K$ is convex, only those weakly optimal solutions $\bar{x}$ of the original problem $\mathcal{M}_w(MOP, \mathbb{R}^m_+)$ are found by determining the weakly optimal solutions of (A-MOP) for which $\bar{x} \in argmin\{w^\top f(x) \mid x \in S\}$ for some $w \in \mathbb{R}^2 \setminus \{(0,0)\}$ with $w_1 \geq w_2/140 \geq 0$.
As $e^2 \notin K^*$, $\bar{x}^2$, i.e. any minimal solution of $\min_{x\in S} f_2(x)$, is an optimal solution of (MOP) but might not be found by solving (A-MOP) instead.

In what follows, we use the following notation:

$$\mathcal{E}(\mathrm{MOP}, K) := \{f(x) \in \mathbb{R}^m \mid x \in \mathcal{M}(\mathrm{MOP}, K)\}$$

and

$$\mathcal{E}^m(\mathrm{MOP}, K) := \{(f_1(x), \ldots, f_{m-1}(x)) \in \mathbb{R}^{m-1} \mid x \in \mathcal{M}(\mathrm{MOP}, K)\}.$$

Moreover, let K have a nonempty interior and let $u : \mathcal{E}^m(\mathrm{MOP}, K) \rightrightarrows \mathbb{R}$ denote the set-valued map with

$$\mathrm{graph}(u) := \{(y, z) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid u(y) = z,\ y \in \mathcal{E}^m(\mathrm{MOP}, K)\} = \mathcal{E}(\mathrm{MOP}, K).$$

We use the notation y^m for $y^m := (y_1, \ldots, y_{m-1})$ for some point $y \in \mathbb{R}^m$.

Theorem 3.14. *Let K be a convex closed pointed cone with $\mathbb{R}^m_+ \subseteq K$, f_i, $i = 1, \ldots, m$ be convex, S be convex and compact, $\mathcal{E}^m(MOP, K)$ be convex and let the single-valued map u, defined as above, be convex. Let $\bar{x} \in \mathcal{M}(MOP, K)$ be a properly optimal solution with $f^m(\bar{x}) \in int(\mathcal{E}^m(MOP, K))$ and u be continuously partial differentiable in $f^m(\bar{x})$. Then there exists a unique $w \in int(K^*)$ with $\|w\| = 1$ and*

$$\bar{x} \in argmin\{w^\top f(x) \mid x \in S\}.$$

Proof. First, note that in case of $e^m \in K \cup (-K)$ (what can also be assumed by re-sorting the functions f_i), and hence in case of $\mathbb{R}^m_+ \subseteq K$, the map u is in fact single-valued. To see this, assume there exist $y \in \mathcal{E}^m(\mathrm{MOP}, K)$ and $z^1, z^1 \in \mathbb{R}$, $z^1 \neq z^2$ with $(y, z^1), (y, z^2) \in \mathcal{E}(\mathrm{MOP}, K)$. This implies

$$(0_{\mathbb{R}^{m-1}}, z) := (0_{\mathbb{R}^{m-1}}, z^1 - z^2) = (y, z^1) - (y, z^2) \notin K \cup (-K)$$

and thus

$$e^m = \frac{1}{|z|}(0_{\mathbb{R}^{m-1}}, z) \notin K \cup (-K)\ ,$$

which is a contradiction to the assumption.
Under the assumptions of the theorem, the subdifferential of u in $\bar{y}^m$ with $\bar{y} := f(\bar{x})$ equals $\{\nabla u(\bar{y}^m)\}$. For any $\bar{x} \in \mathcal{M}(\mathrm{MOP}, K)$ which is properly optimal there exists by Lemma 3.8(c) at least one $w \in \mathrm{int}(K^*)$ with $\|w\| = 1$ and

$$w^\top f(x) - w^\top f(\bar{x}) \geq 0 \qquad \forall\ x \in S.$$

Note that $w \neq 0_{\mathbb{R}^m}$ as otherwise $K^* = \mathbb{R}^m$ and then $K = \{0_{R^m}\}$. Moreover, as $\mathbb{R}^m_+ \subseteq K$, we have $\mathrm{int}(K^*) \subseteq K^* \subseteq \mathbb{R}^m_+$. Let such a w be chosen. Then $w_m > 0$. Under the assumptions of the theorem, for any $y \in f(S)$ there exists some $\hat{y} \in \mathcal{E}(\mathrm{MOP}, K)$ and some $k \in K$ with $y = \hat{y} + k$, see [13, Theorem 3.2.9], and as $w \in K^*$ we obtain $w^\top y = w^\top \hat{y} + w^\top k \geq w^\top \hat{y}$ and thus it holds

$$\begin{array}{rrcll}
 & w^\top(f(x) - f(\bar{x})) & \geq & 0 & \forall\, x \in S \\
\Leftrightarrow & w^\top(y - \bar{y}) & \geq & 0 & \forall\, y \in f(S) \\
\Leftrightarrow & w^\top(y - \bar{y}) & \geq & 0 & \forall\, y \in \mathcal{E}(\mathrm{MOP}, K) \\
\Leftrightarrow & (w^m)^\top(y^m - \bar{y}^m) & \geq & -w_m(y_m - \bar{y}_m) & \forall\, y \in \mathcal{E}(\mathrm{MOP}, K) \\
\Leftrightarrow & -\frac{1}{w_m}(w^m)^\top(y^m - \bar{y}^m) & \leq & y_m - \bar{y}_m & \forall\, y \in \mathcal{E}(\mathrm{MOP}, K) \\
\Leftrightarrow & -\frac{1}{w_m}(w^m)^\top(y^m - \bar{y}^m) & \leq & u(y^m) - u(\bar{y}^m) & \forall\, y \in \mathcal{E}(\mathrm{MOP}, K)
\end{array}$$

Thus, since u is convex,

$$-\frac{1}{w_m}(w^m) \in \partial u(\bar{y}) = \{\nabla u(\bar{y}^m)\}.$$

Hence there is only one weight vector w with $\|w\| = 1$. □

Let S be compact. Then, for each $w \in \mathrm{int}(K^*)$ an optimal solution of the corresponding scalarized problem, i.e. a point in $\mathcal{M}(\mathrm{MOP}, K)$ exists. Moreover, under the assumptions above, for each properly optimal solution the corresponding $w \in \mathrm{int}(K^*)$ is uniquely defined up to multiplication by a positive scalar. Thus, we have a one-to-one correspondence between weight vectors and optimal solutions, and if $\mathrm{int}(K^*)$ is smaller than $\mathrm{int}(\mathbb{R}^m_+)$, the solution sets become smaller as well. Convex multicriterial optimization problems have connected sets of (weakly) optimal solutions. If the multiobjective optimization problem is a linear one, we obtain an even more helpful characterization of the set of optimal solutions.

Theorem 3.15. *Consider the linear multicriterial optimization problem, i.e. let S be a convex polyhedron and $f(x) = Cx$ for a $m \times n$ matrix C. If $\mathcal{E}(MOP, \mathbb{R}^m_+) \neq f(S)$ and* $\mathrm{int\,conv}\,\mathcal{E}(MOP, \mathbb{R}^m_+) \neq \emptyset$*, then $\mathcal{E}(MOP, \mathbb{R}^m_+)$ is equal to a connected union of facets of the set $f(S)$.*

Here, a facet of the set $f(S) \subset \mathbb{R}^m$ is a face with dimension $m - 1$.

Proof. For each $x \in \mathcal{M}(\mathrm{MOP}, \mathbb{R}^m_+)$ there exists $\lambda \geq 0, \lambda \neq 0$ such that x is an optimal solution of the problem

$$\min\{\lambda^\top Cx : x \in S\}. \tag{6}$$

The set of optimal solutions $\Psi(\lambda)$ is equal to a face of the set S. Hence, $\Psi(\Lambda) := \bigcup_{\lambda \in \Lambda} \Psi(\lambda)$ with $\Lambda = \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1\}$ is equal to the union of faces of the set S. These faces are convex polyhedra of dimension zero (i.e. vertices) up to dimension n, which is then equal to the set S itself. The last case is possible only if the interior of S is not empty. By parametric linear optimization [11], this set is connected. This implies that the convex hull of $\mathcal{E}(\mathrm{MOP}, \mathbb{R}^n_+)$ is equal to a convex polyhedron, all vertices of this polyhedron are vertices of $f(S)$, too. By [10, Theorem 3.5, page 91] and the assumption $\operatorname{int}\operatorname{conv}\mathcal{E}(\mathrm{MOP}, \mathbb{R}^m_+) \neq \emptyset$, the set $\mathcal{E}(\mathrm{MOP}, \mathbb{R}^m_+)$ has a unique minimal representation (to within scalar multiplication) as solution set of a system of finitely many linear inequalities $a^i z \leq b_i,\ i = 1, \ldots, p$ and for each facet (which is a face of dimension $m-1$) F_i of $\operatorname{conv}\mathcal{E}(\mathrm{MOP}, \mathbb{R}^n_+)$ there is an inequality representing this facet:

$$F_i = \{z \in \operatorname{conv}\mathcal{E}(\mathrm{MOP}, \mathbb{R}^m_+) : a^i z = b_i\}.$$

As intersection of convex polyhedra, F_i is a convex polyhedron and all its vertices are obviously vertices of $\mathcal{E}(\mathrm{MOP}, \mathbb{R}^m_+) \subset f(S)$. For each facet F_i there exists $y^i \in F_i$ with $y^i \notin F_j,\ j \neq i$. If $y^i \in \mathcal{M}(\mathrm{MOP}, \mathbb{R}^m_+)$ then, F_i is a facet of $\mathcal{M}(\mathrm{MOP}, \mathbb{R}^m_+)$ and the theorem is true. □

Note that the vector a^i used in the proof of Theorem 3.15 is unique up to scalar multiplication, it is a normal vector of F_i. Since a vertex $\overline{z} \in F_i$ is equal to $\overline{z} = C\overline{x}$, where $\overline{x}$ solves

$$\min\{\lambda^\top Cx : x \in S\}$$

for some $\lambda \in \Lambda$ we can assume that $\frac{a^i}{\|a^i\|} \in \Lambda$. This implies that the cone V of all nonnegative linear combinations of the normals of all the facets F_i of $\mathcal{E}(\mathrm{MOP}, \mathbb{R}^m_+)$ is a subset of $\mathbb{R}^m_+$. If some of the optimal solutions $\widehat{x}$ of the linear multiobjective optimization problem is desired not to belong the set of optimal solutions of $\mathcal{M}(\mathrm{MOP}, K)$, the cone K can be obtained by deleting some normal vector a^i from the computation of the cone V. At the same time this can be used to decide if it is possible to avoid the computation of $\widehat{x}$.
We close this section with a new characterization of weakly optimal solutions which is related to the properly optimal solutions as defined in Definition 3.

Theorem 3.16. *Let* $\operatorname{int}(K) \neq \emptyset$. *Then* $\bar{x} \in \mathcal{M}_w(MOP, K)$ *if and only if*

$$(\{f(\bar{x})\} - \operatorname{int}(K)) \cap (\{f(\bar{x})\} + T_{f(S)+K}(f(\bar{x}))) = \emptyset. \tag{7}$$

Proof. First assume $\bar{x} \in \mathcal{M}_w(\mathrm{MOP}, K)$ and assume that there exists $d \in (\{f(\bar{x})\} - \operatorname{int}(K)) \cap (\{f(\bar{x})\} + T_{f(S)+K}(f(\bar{x}))) \neq \emptyset$. Then, there exist sequences $\{y^k\}_{k=1}^\infty \subseteq f(S)$, $\{w^k\}_{k=1}^\infty \subseteq K$ and $\{t_k\}_{k=1}^\infty \downarrow 0$ such that

$$d = f(\bar{x}) + \lim_{k\to\infty} \frac{1}{t_k}(y^k + w^k - f(\bar{x})).$$

Since $d \in \{f(\bar{x})\} - \operatorname{int}(K)$ and this set is open, we derive that

$$f(\bar{x}) + \frac{1}{t_k}(y^k + w^k - f(\bar{x})) =: z^k \in (\{f(\bar{x})\} - \operatorname{int}(K))\ \forall\ k \geq \bar{k}$$

and sufficiently large $\bar{k}$, i. e.

$$z^k - f(\bar{x}) \in -\operatorname{int}(K).$$

This implies

$$\frac{1}{t_k}(y^k + w^k - f(\bar{x})) \in -\mathrm{int}(K)$$

or

$$\frac{1}{t_k}(y^k - f(\bar{x})) \in \{-\frac{1}{t_k}w^k\} - \mathrm{int}(K) \subseteq \mathrm{int}(K)$$

which results in

$$y^k \in (\{f(\bar{x})\} - \mathrm{int}(K)) \cap f(S).$$

Hence $f(x^k) \in (\{f(\bar{x})\} - \mathrm{int}(K)) \cap f(S)$ for some x^k and sufficiently large k. This contradicts $\bar{x} \in \mathcal{M}_w(\mathrm{MOP}, K)$ and thus (7) holds.
Next assume that (7) holds but $\bar{x} \notin \mathcal{M}_w(\mathrm{MOP}, K)$. Then there exists $\hat{y} := f(\hat{x}) \in f(S) \setminus \{f(\bar{x})\}$ and $d \in \mathrm{int}(K)$ such that

$$f(\bar{x}) = \hat{y} + d.$$

Let $t_k = \frac{1}{k}$, $d_k := (1 - \frac{1}{k})\, d \in \mathrm{int}(K)$ and

$$y_k := \hat{y} + d_k \in \{\hat{y}\} + \mathrm{int}(K) \subseteq f(S) + K$$

for all $k \in \mathbb{N}$. Then it follows

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \hat{y} + d_k = \lim_{k\to\infty} (f(\bar{x}) - \frac{1}{k}d) = f(\bar{x})$$

and

$$\lim_{k\to\infty} \frac{1}{t_k}\,(y_k - f(\bar{x})) = \lim_{k\to\infty} k\,(\hat{y} + d_k - f(\bar{x})) = \lim_{k\to\infty} k\,(d_k - d) = -d \in T_{f(S)+K}(f(\bar{x})).$$

Hence

$$f(\bar{x}) = \hat{y} + d \in \{f(\hat{x})\} + \mathrm{int}(K)$$

and

$$f(\hat{x}) = f(\bar{x}) - d = \{f(\bar{x})\} + T_{f(S)+K}(f(\bar{x}))$$

which contradicts (7). □

Hence, as long as $\mathrm{int}(K) \neq \emptyset$, then $\bar{x} \in \mathcal{M}_w(\mathrm{MOP}, K)$ if and only if

$$(-\mathrm{int}(K)) \cap T_{f(S)}(\bar{x}) = \emptyset.$$

Thus, given a cone K, the set of optimal solutions $\mathcal{M}_w(\mathrm{MOP}, K)$ and some point $\bar{x} \in M_w(\mathrm{MOP}, K)$, we can ask ourselves the following question: how do we need to modify K to some cone C such that $\bar{x} \notin \mathcal{M}_w(\mathrm{MOP}, C)$? The investigation of this question is an interesting topic for future research but is beyond the scope of paper.

3.3 Tschebyscheff-Scalarizations

In this section, which is on the Tschebyscheff-scalarization as used in [8], we need the notion of cone-monotone functions. For a cone $C \subseteq \mathbb{R}^m$, a function $g\colon \mathbb{R}^m \longrightarrow \mathbb{R}$ is called *C-monotone*, if

$$x \leq_C y \quad \Longrightarrow g(x) \leq g(y)$$

holds for all $x, y \in \mathbb{R}^m$.

Let us assume that all objective functions f_i are bounded below, and assume without loss of generality that 0 is a strict bound for all objectives f_i $(i = 1, \dots, m)$ and that the cones C and K which we are considering in the following are such that $f(S) \subset \{0_{\mathbb{R}^m}\} + \mathrm{int}(C)$ as well as $Af(S) \subset \{0_{\mathbb{R}^m}\} + \mathrm{int}(K)$. With Ω denoting a positive definite diagonal matrix, the corresponding Tschebyscheff-scalarization of (MOP) is given by

$$\min_{x \in S} \|\Omega f(x)\|_\infty. \tag{8}$$

Likewise, the corresponding scalarization of (A-MOP) is

$$\min_{x \in S} \|\Omega A f(x)\|_\infty. \tag{9}$$

These scalarization problems have a reformulation by introducing an additional variable $t \in \mathbb{R}$ as follows and are thus related to the Pascoletti-Serafini scalarization as discussed in [4]. A point $\bar{x}$ is a minimal solution of (8) (with Ω having positive diagonal entries $\omega_1, \dots, \omega_m$) if and only if there exists a $\bar{t} \in \mathbb{R}$ such that $(\bar{t}, \bar{x})$ is an optimal solution of

$$\begin{gathered} \min\ t \\ \text{subject to the constraints} \\ \tfrac{1}{\omega_i} t - f_i(x) \geq 0, \qquad i = 1, \dots, m, \\ t \in \mathbb{R},\ x \in S\ . \end{gathered}$$

Similarly, a point $\bar{x}$ is a minimal solution of (9) (with Ω having positive diagonal entries $\omega_1, \dots, \omega_p$) if and only if there exists a $\bar{t} \in \mathbb{R}$ such that $(\bar{t}, \bar{x})$ is an optimal solution of

$$\begin{gathered} \min\ t \\ \text{subject to the constraints} \\ \tfrac{1}{\omega_i} t - (a^i)^\top f(x) \geq 0, \qquad i = 1, \dots, p, \\ t \in \mathbb{R},\ x \in S\ . \end{gathered}$$

It is well-known that $\|\Omega \cdot \|_\infty$ is $\mathbb{R}^m_+$-monotone for all positive definite diagonal matrices Ω, i. e. for all $y^1, y^2 \in \mathbb{R}^m$

$$y^2 - y^1 \in \mathbb{R}^m_+ \ \Rightarrow\ \|\Omega y^1\|_\infty \leq \|\Omega y^2\|_\infty\ .$$

Thus (8) is an adequate scalarization to characterize the minimal solutions of (MOP) w.r.t. the convex cone $C = \mathbb{R}^m_+$ while (9) is an adequate scalarization to characterize the minimal solutions of (A-MOP) w.r.t. the convex cone $C = \mathbb{R}^m_+$. This also implies for a regular matrix A that $\|\Omega A y^1\|_\infty \leq \|\Omega A y^2\|_\infty$ is satisfied whenever

$$A(y^2 - y^1) = Ay^2 - Ay^1 \in \mathbb{R}^m_+ \ \Leftrightarrow\ y^2 - y^1 \in A^{-1}\mathbb{R}^m_+\ .$$

Naturally, we arrive thus at

$$K := A^{-1}\mathbb{R}^m_+ = \{y \in \mathbb{R}^m \mid Ay \in \mathbb{R}^m_+\}$$

for the functional $\|\Omega A \cdot \|_\infty$ to be K-monotone and hence (9) can also be used to characterize the optimal solutions w.r.t. K of (MOP) in case A is a regular matrix. This corresponds to the result of Theorem 3.1 for a regular matrix A, $\mathcal{M}(\text{MOP}, K) = \mathcal{M}(\text{A-MOP}, \mathbb{R}^m_+)$.
We conclude this section by an illustration on the impact of numerical approximations as it was also realized in [8]. The approximation can for instance be calculated by using the Pascoletti-Serafini reformulations as discussed above.

Example 3.17. *We consider the nonconvex multi-objective optimization problem*

$$\min \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\textit{such that}$$
$$x_1 \leq 1, \; x_2 \leq 1,$$
$$x_1^2 + x_2^2 \geq 1$$

and the version (A-MOP) of it using the same nonnegative regular matrix A as in Example 2.4 w.r.t. the natural ordering cone. Generating an approximation of $\mathcal{E}(A\text{-}MOP, \mathbb{R}^2_+)$ leads to Figure 1(a). The same approximation points x re-mapped for the original problem (MOP) lead to Figure 1(b). These points form an approximation of the set $\mathcal{E}(MOP, K)$.

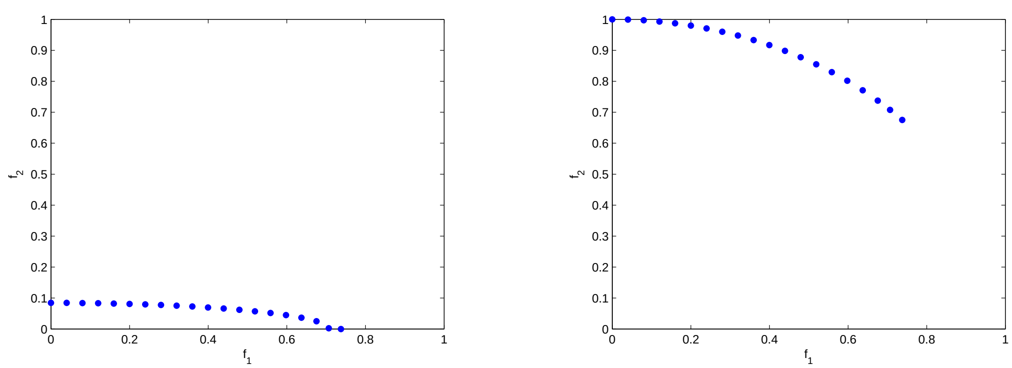

Figure 1: (a) Approximation of $\mathcal{E}(\text{A-MOP}, \mathbb{R}^m_+)$ (b) approximation points x re-mapped for the original problem (MOP)

References

[1] J.M. Borwein, "Proper efficient points for maximizations with respect to cones", *SIAM J. Control Optim.* 15 (1977) 57–63.

[2] M. Ehrgott, *Multicriteria Optimization* (Springer, Heidelberg, 2005).

[3] G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization* (Springer, Heidelberg, 2008).

[4] G. Eichfelder, "Scalarizations For Adaptively Solving Multi-Objective Optimization Problems", *Computational Optimization and Applications* 44 (2009) 249–273.

[5] A. Engau and M.M. Wiecek, "Cone characterizations of approximate solutions in real vector optimization", *J. Optim. Theory Appl.* 134 (2007) 499–513.

[6] A. Göpfert and R. Nehse, *Vektoroptimierung. Theorie, Verfahren und Anwendungen* (Teubner, 1990).

[7] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions*, 2nd edition (Springer, Heidelberg, 2011).

[8] C.Y. Kaya and H. Maurer, *A Numerical Method for Finding the Pareto Front of Nonconvex Multiobjective Optimal Control Problems*, *Computational Optimization and Applications*, September 2013, DOI 10.1007/s10589-013-9603-2.

[9] K.M. Miettinen, *Nonlinear Multiobjective Optimization* (Kluwer Academic Publishers, Boston, 1999).

[10] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization* (John Wiley & Sons, New York, 1988).

[11] F. Nožička, J. Guddat, H. Hollatz and B. Bank, *Theorie der linearen parametrischen Optimierung* (Akademie-Verlag, Berlin, 1974).

[12] R. T. Rockafellar, *Convex Analysis* (Princeton University Press, 1970).

[13] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization* (Academic Press, London, 1985).

[14] M. Tanaka, "GA-based decision support system for multi-criteria optimization", *Proceedings of the international Conference on Systems, Man and Cybernetics* 2 (1995) 1556–1561.