Disjunctive Cuts for Cross-Sections of the Second-Order Cone

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Abstract

In this paper we provide a unified treatment of general two-term disjunctions on cross-sections of the second-order cone. We derive a closed-form expression for a convex inequality that is valid for such a disjunctive set and show that this inequality is sufficient to characterize the closed convex hull of all two-term disjunctions on ellipsoids and paraboloids, and split disjunctions on all cross-sections of the second-order cone. Our approach extends the work of Kılınç-Karzan and Yıldız on general two-term disjunctions for the second-order cone.

Keywords: Mixed-integer conic programming, second-order cone programming, cutting planes, disjunctive cuts

1 Introduction

In this paper we consider the mixed-integer second-order conic programming (MISOCP) problem

$$\sup\{d^\top x : \ Ax = b, \ x \in \mathbb{L}^n, \ x_j \in \mathbb{Z} \ \forall j \in J\} \tag{1}$$

where $\mathbb{L}^n$ is the n-dimensional second-order cone $\mathbb{L}^n := \{x \in \mathbb{R}^n : \ \|(x_1; \ldots; x_{n-1})\|_2 \leq x_n\}$, A is an $m \times n$ real matrix of full row rank, d and b are real vectors of appropriate dimensions, and $J \subseteq \{1, \ldots, n\}$. The set S of feasible solutions to this problem is called a *mixed-integer second-order conic set*. Because the structure of S can be very complicated, a first approach to solving (1) entails solving the relaxed problem obtained after dropping the integrality requirements on the variables:

$$\sup\{d^\top x : \ Ax = b, \ x \in \mathbb{L}^n\}.$$

The set of feasible solutions

$$C := \{x \in \mathbb{L}^n : \ Ax = b\}$$

to this relaxed problem is called the *continuous relaxation* of S. Unfortunately, the continuous relaxation is often a poor approximation to the mixed-integer conic set, and tighter formulations are needed for the development of practical strategies for solving (1). An effective way to improve the approximation quality of the continuous relaxation C is to strengthen it with additional inequalities that are valid for S but not for the whole of C. Such valid inequalities can be derived by exploiting the integrality of the variables x_j, $j \in J$, and enhancing C with linear *two-term disjunctions* $l_1^\top x \geq l_{1,0} \vee l_2^\top x \geq l_{2,0}$ that are satisfied by all solutions in S. Valid inequalities that are obtained from disjunctions using this approach are known as *disjunctive*

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cuts. In this paper we study two-term disjunctions on the set C and give closed-form expressions for the strongest disjunctive cuts that can be obtained from such disjunctions.

Disjunctive cuts were introduced by Balas in the context of mixed-integer linear programming (MILP) [3] and have since been the cornerstone of theoretical and practical achievements in integer programming. There has been a lot of recent interest in extending disjunctive cutting-plane theory from the domain of MILP to that of mixed-integer conic programming [1, 2, 4–16, 18]. Several papers in the last few years have focused on deriving closed-form expressions for convex inequalities that fully describe the convex hull of a disjunctive second-order conic set in the space of the original variables [1, 2, 4, 5, 8, 10, 14–16]. In this paper, we pursue a similar goal and study general two-term disjunctions on a cross-section C of the second-order cone. Given a disjunction $l_1^\top x \geq l_{1,0} \vee l_2^\top x \geq l_{2,0}$ on C, we let

$$C_1 := \{x \in C : \ l_1^\top x \geq l_{1,0}\} \quad \text{and} \quad C_2 := \{x \in C : \ l_2^\top x \geq l_{2,0}\}.$$

In order to derive the tightest disjunctive cuts that can be obtained for S from the disjunction $C_1 \cup C_2$, we study the closed convex hull $\overline{\text{conv}}(C_1 \cup C_2)$. In particular, we are interested in convex inequalities that may be added to the description of C to obtain a characterization of $\overline{\text{conv}}(C_1 \cup C_2)$. Our approach extends [14] and provides a unified treatment of *general* two-term disjunctions on *all* cross-sections of the second-order cone. Such cross-sections include ellipsoids, paraboloids, and hyperboloids as special cases. This generalizes the work of [10, 15] on split disjunctions on cross-sections of the second-order cone and [4] on disjoint two-term disjunctions on ellipsoids. We note here that similar results on disjoint two-term disjunctions on cross-sections of the second-order cone were recently obtained independently in [8].

We first show in Section 2 that the continuous relaxation C can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. In Section 3, we prove our main theorem (Theorem 3) characterizing $\overline{\text{conv}}(C_1 \cup C_2)$.

Throughout the paper, we use $\text{conv}\, K$, $\overline{\text{conv}}\, K$, $\text{cone}\, K$, and $\text{span}\, K$ to refer to the convex hull, closed convex hull, conical hull, and linear span of a set K, respectively. We also use $\text{bd}\, K$ and $\text{int}\, K$ to refer the boundary and interior of K. Given a vector $u \in \mathbb{R}^n$, we let $\tilde{u} := (u_1; \ldots ; u_{n-1})$ denote the subvector obtained by dropping its last entry.

2 Intersection of the Second-Order Cone with an Affine Subspace

Let $E := \{x \in \mathbb{R}^n : \ Ax = b\}$ so that $C = \mathbb{L}^n \cap E$. We are going to use the following lemma to simplify our analysis. See also Section 2.1 of [5] for a similar result.

Lemma 1. *Let V be a p-dimensional linear subspace of $\mathbb{R}^n$. The intersection $\mathbb{L}^n \cap V$ is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^p$.*

Proof. Let $D \in \mathbb{R}^{n \times p}$ be a matrix whose columns form an orthonormal basis for V. By simple linear algebra, we can write

$$\begin{aligned} \mathbb{L}^n \cap V &= \{x \in \mathbb{L}^n : \ x = Dy \ \text{ for some } y \in \mathbb{R}^p\} \\ &= D\mathbb{K}^p \quad \text{where} \quad \mathbb{K}^p := \{y \in \mathbb{R}^p : \ Dy \in \mathbb{L}^n\}. \end{aligned}$$

Let $D = \begin{pmatrix} \tilde{D} \\ d_n^\top \end{pmatrix}$. Via the definition of $\mathbb{L}^n$ and the orthonormality of the columns of D,

$$\mathbb{K}^p = \{y \in \mathbb{R}^p : \ y^\top (I - 2 d_n d_n^\top) y \leq 0, \ d_n^\top y \geq 0\}.$$

The matrix $I - 2d_n d_n^\top$ has at most one nonpositive eigenvalue. In particular, it is positive semidefinite if $2\|d_n\|_2^2 \le 1$ and indefinite with exactly one negative eigenvalue otherwise. In the first case, $\mathbb{K}^p$ reduces to the origin or a half-line and so does $\mathbb{L}^n \cap V = D\mathbb{K}^p$. In the second case, $I - 2d_n d_n^\top$ admits a spectral decomposition of the form $Q^\top \operatorname{Diag}(\lambda) Q$ where $\lambda_1 \ge \ldots \ge \lambda_{p-1} > 0 > \lambda_p$ and Q is an orthonormal matrix whose last column is $d_n / \|d_n\|_2$. Using this decomposition, we can write

$$\begin{aligned}\mathbb{K}^p &= \{y \in \mathbb{R}^p : \, y^\top Q^\top \operatorname{Diag}(\lambda) Q y \le 0, \, d_n^\top y \ge 0\} \\ &= \{y \in \mathbb{R}^p : \, \operatorname{Diag}\left(\lambda_1^{1/2} \, \ldots \, \lambda_{p-1}^{1/2} \, \ |\lambda_p|^{1/2}\right) Q^\top y \in \mathbb{L}^p\}.\end{aligned}$$

The matrix $\operatorname{Diag}\left(\lambda_1^{1/2} \, \ldots \, \lambda_{p-1}^{1/2} \, \ |\lambda_p|^{1/2}\right) Q^\top$ has an inverse H because the entries of $\left(\lambda_1^{1/2} \, \ldots \, \lambda_{p-1}^{1/2} \, \ |\lambda_p|^{1/2}\right)$ are all positive. Thus $\mathbb{K}^p = H\mathbb{L}^p$. The set $\mathbb{L}^n \cap V = DH\mathbb{L}^p$ is an injective linear transformation of $\mathbb{L}^p$ because $Z := DH$ has full column rank. Furthermore, Z admits a pseudo-inverse $Z^+ := (Z^\top Z)^{-1} Z^\top$ and $\mathbb{L}^p = Z^+(\mathbb{L}^n \cap V)$. This implies a linear bijection between $\mathbb{L}^p$ and $\mathbb{L}^n \cap V$. □

Lemma 1 implies that, when $b = 0$, C is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m}$. The closed convex hull $\overline{\operatorname{conv}}(C_1 \cup C_2)$ can be described easily when C is a single point or a half-line. Furthermore, the problem of characterizing $\overline{\operatorname{conv}}(C_1 \cup C_2)$ when C is a bijective linear transformation of $\mathbb{L}^{n-m}$ can be reduced to that of convexifying an associated two-term disjunction on $\mathbb{L}^{n-m}$. We refer the reader to [14] for a study of disjunctive cuts that are obtained from two-term disjunctions on the second-order cone.

In the remainder, we focus on the case $b \ne 0$. Note that, whenever this is the case, we can permute and normalize the rows of (A, b) so that its last row reads $(a_m^\top, 1)$, and subtracting a multiple of $(a_m^\top, 1)$ from the other rows if necessary, we can write the remaining rows of (A, b) as $(\tilde{A}, 0)$. Therefore, we can assume that all components of b are zero except the last one. Isolating the last row of (A, b) from the others, we can then write

$$E = \{x \in \mathbb{R}^n : \, \tilde{A}x = 0, \, a_m^\top x = 1\}.$$

Let $V := \{x \in \mathbb{R}^n : \, \tilde{A}x = 0\}$. By Lemma 1, $\mathbb{L}^n \cap V$ is the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m+1}$. Again, the first two cases are easy and not of interest in our analysis. In the last case, we can find a matrix D whose columns form an orthonormal basis for V and define a nonsingular matrix H such that $\{y \in \mathbb{R}^{n-m+1} : \, Dy \in \mathbb{L}^n\} = H\mathbb{L}^{n-m+1}$ as in the proof of Lemma 1. Then we can represent C equivalently as

$$\begin{aligned}C &= \{x \in \mathbb{L}^n : \, x = Dy, \, a_m^\top x = 1\} \\ &= D\{y \in \mathbb{R}^{n-m+1} : \, Dy \in \mathbb{L}^n, \, a_m^\top Dy = 1\} \\ &= D\{y \in \mathbb{R}^{n-m+1} : \, y \in H\mathbb{L}^{n-m+1}, \, a_m^\top Dy = 1\} \\ &= DH\{z \in \mathbb{L}^{n-m+1} : \, a_m^\top DHz = 1\}.\end{aligned}$$

The set $C = \mathbb{L}^n \cap E$ is a bijective linear transformation of $\{z \in \mathbb{L}^{n-m+1} : \, a_m^\top DHz = 1\}$. Furthermore, the same linear transformation maps any two-term disjunction in $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$ to a two-term disjunction in C. Thus, without any loss of generality, we can take $m = 1$ in (1) and study the problem of describing $\overline{\operatorname{conv}}(C_1 \cup C_2)$ where

$$\begin{aligned}C &= \{x \in \mathbb{L}^n : \, a^\top x = 1\}, \\ C_1 &= \{x \in C : \, l_1^\top x \ge l_{1,0}\}, \quad \text{and} \quad C_2 = \{x \in C : \, l_2^\top x \ge l_{2,0}\}.\end{aligned} \tag{2}$$

3 Two-Term Disjunctions on Cross-Sections of the Second-Order Cone

3.1 Preliminaries

Consider C, C_1, and C_2 defined as in (2). The set C is an ellipsoid when $a \in \operatorname{int} \mathbb{L}^n$, a paraboloid when $a \in \operatorname{bd} \mathbb{L}^n$, a hyperboloid when $a \notin \pm\mathbb{L}^n$, and empty when $a \in -\mathbb{L}^n$.

When $C_1 \subseteq C_2$, we have $\overline{\text{conv}}(C_1 \cup C_2) = C_2$. Similarly, when $C_1 \supseteq C_2$, we have $\overline{\text{conv}}(C_1 \cup C_2) = C_1$. In the remainder we focus on the case where $C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.

Assumption 1. *$C_1 \not\subseteq C_2$ and $C_1 \not\supseteq C_2$.*

We also make the following technical assumption.

Assumption 2. *C_1 and C_2 are both strictly feasible.*

The following simple observation underlies our approach.

Observation 1. *Let C, C_1, and C_2 be defined as in (2). Then $C_1 = \{x \in C : (\beta_1 l_1 + \gamma_1 a)^\top x \geq \beta_1 l_{1,0} + \gamma_1\}$ for any $\beta_1 > 0$ and $\gamma_1 \in \mathbb{R}$. Similarly, $C_2 = \{x \in C : (\beta_2 l_2 + \gamma_2 a)^\top x \geq \beta_2 l_{2,0} + \gamma_2\}$ for any $\beta_2 > 0$ and $\gamma_2 \in \mathbb{R}$.*

Observation 1 allows us to conclude

$$C_1 = \{x \in C : \ (l_1 - l_{1,0} a)^\top x \geq 0\} \quad \text{and} \quad C_2 = \{x \in C : \ (l_2 - l_{2,0} a)^\top x \geq 0\}.$$

Recall that by Assumption 1 we have $C_1, C_2 \subsetneq C$ and by Assumption 2 the sets C_1 and C_2 are both strictly feasible. This implies $l_i - l_{i,0} a \notin \pm\mathbb{L}^n$, or equivalently $\left\| \tilde{l}_i - l_{i,0} \tilde{a} \right\|_2^2 > (l_{i,n} - l_{i,0} a_n)^2$, for $i \in \{1, 2\}$. Let

$$\lambda_i := \frac{1}{\sqrt{\left\| \tilde{l}_i - l_{i,0} \tilde{a} \right\|_2^2 - (l_{i,n} - l_{i,0} a_n)^2}} \quad \text{and} \quad c_i := \lambda_i (l_i - l_{i,0} a) \quad \text{for} \quad i \in \{1, 2\}. \tag{3}$$

Because $\lambda_1, \lambda_2 > 0$, we can write

$$C_1 = \{x \in C : \ c_1^\top x \geq 0\} \quad \text{and} \quad C_2 = \{x \in C : \ c_2^\top x \geq 0\}.$$

This scaling ensures

$$\|\tilde{c}_1\|_2^2 - c_{1,n}^2 = \|\tilde{c}_2\|_2^2 - c_{2,n}^2 = 1. \tag{4}$$

In particular, it has the following consequences.

Remark 1. *Let c_1 and c_2 satisfy (4). Then*

$$\mathcal{M} := \|\tilde{c}_1\|_2^2 - c_{1,n}^2 - (\|\tilde{c}_2\|_2^2 - c_{2,n}^2) = 0,$$
$$\mathcal{N} := \|\tilde{c}_1 - \tilde{c}_2\|_2^2 - (c_{1,n} - c_{2,n})^2 = 2 - 2(\tilde{c}_1^\top \tilde{c}_2 - c_{1,n} c_{2,n}).$$

Remark 2. *Let C_1 and C_2, defined as in (2), satisfy Assumption 1. Let c_1 and c_2 be defined as in (3). By Assumption 1 we have $c_1 - c_2 \notin \pm\mathbb{L}^n$. Indeed, $c_1 - c_2 \in \mathbb{L}^n$ implies that $(c_1 - c_2)^\top x \geq 0$ for all $x \in \mathbb{L}^n$, and this implies $C_1 \subseteq C_2$; similarly $c_2 - c_1 \in \mathbb{L}^n$ implies $C_2 \subseteq C_1$. Hence,*

$$\mathcal{N} = \|\tilde{c}_1 - \tilde{c}_2\|_2^2 - (c_{1,n} - c_{2,n})^2 > 0.$$

Let Q_1 and Q_2 be the relaxations of C_1 and C_2 to the whole cone $\mathbb{L}^n$:

$$Q_1 := \{x \in \mathbb{L}^n : \; c_1^\top x \geq 0\} \quad \text{and} \quad Q_2 := \{x \in \mathbb{L}^n : \; c_2^\top x \geq 0\}. \tag{5}$$

It is clear that Q_1 and Q_2 satisfy Assumptions 1 and 2 whenever C_1 and C_2 do. Furthermore, Q_1 and Q_2 are closed, convex, pointed cones, so $\text{conv}(Q_1 \cup Q_2)$ is always closed.

The following results from [14] are useful in proving our results.

Theorem 1. *[[14], Theorem 1 and Remark 3] Let Q_1 and Q_2, defined as in (5), satisfy Assumptions 1 and 2. Then the inequality*

$$-(c_1 + c_2)^\top x \leq \sqrt{\left((c_1 - c_2)^\top x\right)^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|^2\right)} \tag{6}$$

is valid for $\text{conv}(Q_1 \cup Q_2)$. Furthermore, this inequality is convex in $\mathbb{L}^n$.

This result implies in particular that (6) is valid for $\overline{\text{conv}}(C_1 \cup C_2)$. The next proposition shows that (6) can be written in conic quadratic form in $\mathbb{L}^n$ except in the region where both clauses of the disjunction are satisfied. Its proof is a simple extension of the proofs of Propositions 2 and 3 in [14] and therefore omitted. Let

$$r := \begin{pmatrix} \tilde{c}_1 - \tilde{c}_2 \\ -c_{1,n} + c_{2,n} \end{pmatrix}.$$

Proposition 1. *[[14], Propositions 2 and 3] Let Q_1 and Q_2, defined as in (5), satisfy Assumptions 1 and 2. Let $x' \in \mathbb{L}^n$ be such that $c_1^\top x' \leq 0$ or $c_2^\top x' \leq 0$. Then the following statements are equivalent:*

i) x' satisfies (6).

ii) x' satisfies the conic quadratic inequality

$$\mathcal{N}x' - 2(c_1^\top x')r \in \mathbb{L}^n. \tag{7}$$

iii) x' satisfies the conic quadratic inequality

$$\mathcal{N}x' + 2(c_2^\top x')r \in \mathbb{L}^n. \tag{8}$$

Remark 3. *When c_1 and c_2 satisfy (4), the inequalities (7) and (8) describe a cylindrical second-order cone whose lineality space contains $\text{span}\{r\}$. This follows from Remark 1 by observing that*

$$\mathcal{N} = 2 - 2(\tilde{c}_1^\top \tilde{c}_2 - c_{1,n}c_{2,n}) = 2c_1^\top r = -2c_2^\top r.$$

The next theorem shows that a single inequality of the form (6) is in fact sufficient to describe $\text{conv}(Q_1 \cup Q_2)$. This settles a case left open by Kılınç-Karzan and Yıldız [14].

Theorem 2. *Let Q_1 and Q_2, defined as in (5), satisfy Assumptions 1 and 2. Assume without any loss of generality that c_1 and c_2 have been scaled so that they satisfy (4). Then*

$$\text{conv}(Q_1 \cup Q_2) = \left\{ x \in \mathbb{L}^n : -(c_1 + c_2)^\top x \leq \sqrt{\left((c_1 - c_2)^\top x\right)^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|^2\right)} \right\}. \tag{9}$$

Proof. Let D denote the set on the right-hand side of (9). We already know that

$$-(c_1+c_2)^\top x \le \sqrt{((c_1-c_2)^\top x)^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|^2\right)} \tag{10}$$

is valid for $\operatorname{conv}(Q_1 \cup Q_2)$. Hence, $\operatorname{conv}(Q_1 \cup Q_2) \subseteq D$. Let $x' \in D$. If $x' \in Q_1 \cup Q_2$, then clearly $x' \in \operatorname{conv}(Q_1 \cup Q_2)$. Therefore, suppose $x' \in \mathbb{L}^n \setminus (Q_1 \cup Q_2)$ is a point that satisfies (10). By Proposition 1, x' satisfies

$$\mathcal{N}x' - 2(c_1^\top x')r \in \mathbb{L}^n \quad \text{and} \quad \mathcal{N}x' + 2(c_2^\top x')r \in \mathbb{L}^n.$$

We are going to show that x' belongs to $\operatorname{conv}(Q_1 \cup Q_2)$.

By Remarks 2 and 3, $0 < \mathcal{N} = 2c_1^\top r = -2c_2^\top r$. Let

$$\begin{aligned} \alpha_1 &:= \frac{-c_1^\top x'}{c_1^\top r}, \quad & \alpha_2 &:= \frac{-c_2^\top x'}{c_2^\top r}, \\ x_1 &:= x' + \alpha_1 r, \quad & x_2 &:= x' + \alpha_2 r. \end{aligned} \tag{11}$$

It is not difficult to see that $c_1^\top x_1 = c_2^\top x_2 = 0$. Furthermore, $x' \in \operatorname{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. Therefore, the only thing we need to show is $x_1, x_2 \in \mathbb{L}^n$. By Remark 3

$$\mathcal{N}r - 2(c_1^\top r)r = \mathcal{N}r + 2(c_2^\top r)r = 0.$$

Hence,

$$\begin{aligned} &\mathcal{N}x_1 - 2(c_1^\top x_1)r = \mathcal{N}x' - 2(c_1^\top x')r \in \mathbb{L}^n \quad \text{and} \\ &\mathcal{N}x_2 + 2(c_2^\top x_2)r = \mathcal{N}x' + 2(c_2^\top x')r \in \mathbb{L}^n. \end{aligned}$$

Now observing that $c_1^\top x_1 = c_2^\top x_2 = 0$ and $\mathcal{N} > 0$ shows $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in Q_1$ and $x_2 \in Q_2$. □

In the next section we show that the inequality (6) is also sufficient to describe $\overline{\operatorname{conv}}(C_1 \cup C_2)$ when the sets C_1 and C_2 satisfy certain conditions.

3.2 The Disjunctive Cut

In Theorem 3 we present the main result of this paper. Its proof requires the following technical lemma.

Lemma 2. *Let C_1 and C_2, defined as in (2), satisfy Assumptions 1 and 2. Let c_1 and c_2 be defined as in (3). Suppose $a^\top r \neq 0$, and let $x^* := \frac{r}{a^\top r}$. Let $x' \in C \setminus (C_1 \cup C_2)$ satisfy (6).*

a) If $a^\top r > 0$, then $c_1^\top (x' - x^*) < 0$. If in addition

$$(a + \operatorname{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \text{or} \quad (-a + \operatorname{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \text{or} \\ (-a + \operatorname{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset, \tag{12}$$

then $c_2^\top (x' - x^*) \geq 0$.

b) If $a^\top r < 0$, then $c_2^\top(x' - x^*) < 0$. If in addition

$$(a + \operatorname{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \quad (-a + \operatorname{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \\ (-a + \operatorname{cone}\{c_1\}) \cap -\mathbb{L}^n \neq \emptyset, \tag{13}$$

then $c_1^\top(x' - x^*) \geq 0$.

Proof. By Remarks 2 and 3, $\mathcal{N} = 2c_1^\top r = -2c_2^\top r > 0$. From this, we get

$$\mathcal{N}x^* - 2(c_1^\top x^*)r = \frac{1}{a^\top r}\left(\mathcal{N} - 2c_1^\top r\right) r = 0, \tag{14}$$

$$\mathcal{N}x^* + 2(c_2^\top x^*)r = \frac{1}{a^\top r}\left(\mathcal{N} + 2c_2^\top r\right) r = 0. \tag{15}$$

Furthermore, $a^\top x' = a^\top x^* = 1$.

a) Having $x' \notin C_1$ implies $c_1^\top x' < 0$. Furthermore, it follows from $c_1^\top r = \frac{\mathcal{N}}{2} > 0$ that

$$c_1^\top x^* = \frac{c_1^\top r}{a^\top r} > 0.$$

Thus, we get $c_1^\top(x' - x^*) < 0$.

Now suppose $(a + \operatorname{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset$. Then there exist $\lambda \geq 0$ and $0 \leq \theta \leq 1$ such that $a + \lambda(\theta c_1 + (1-\theta)c_2) \in \mathbb{L}^n$. The point x' does not belong to either C_1 or C_2 and satisfies (6). By Proposition 1, it satisfies (8) as well. Using (15), we can write

$$\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r \in \mathbb{L}^n. \tag{16}$$

Because $\mathbb{L}^n$ is self-dual, we get

$$\begin{aligned}
0 &\leq (a + \lambda(\theta c_1 + (1-\theta)c_2))^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda(\theta c_1 + (1-\theta)c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(c_1 - c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) + \lambda c_2^\top(x' - x^*)(\mathcal{N} + 2c_2^\top r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(c_1 - c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(\mathcal{N}(c_1 - c_2)^\top(x' - x^*) + 2c_2^\top(x' - x^*)(c_1 - c_2)^\top r) \\
&= 2c_2^\top(x' - x^*)a^\top r + \lambda\theta(\mathcal{N}(c_1 + c_2)^\top(x' - x^*)) \\
&= (2a^\top r + \lambda\theta\mathcal{N})c_2^\top(x' - x^*) + \lambda\theta\mathcal{N}c_1^\top(x' - x^*)
\end{aligned}$$

where we have used $a^\top(x' - x^*) = 0$ to obtain the first equality, $\mathcal{N} + 2c_2^\top r = 0$ to obtain the third equality, and $(c_1 - c_2)^\top r = \mathcal{N}$ to obtain the fifth equality. Now it follows from $2a^\top r + \lambda\theta\mathcal{N} > 0$, $c_1^\top(x' - x^*) < 0$, and $\lambda\theta\mathcal{N} \geq 0$ that $c_2^\top(x' - x^*) \geq 0$.

Now suppose $(-a + \operatorname{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset$, and let $\lambda \geq 0$ and $0 \leq \theta \leq 1$ be such that $-a + \lambda(\theta c_1 + (1-\theta)c_2) \in \mathbb{L}^n$. By Proposition 1, x' satisfies (7), and using (14), we can write

$$\mathcal{N}(x' - x^*) - 2c_1^\top(x' - x^*)r \in \mathbb{L}^n. \tag{17}$$

As before, because $\mathbb{L}^n$ is self-dual, we get

$$0 \leq (-a + \lambda(\theta c_1 + (1-\theta)c_2))^\top(\mathcal{N}(x' - x^*) - 2c_1^\top(x' - x^*)r).$$

The right-hand side of this inequality is identical to

$$(2a^\top r + \lambda(1-\theta)\mathcal{N})c_1^\top(x' - x^*) + \lambda(1-\theta)\mathcal{N}c_2^\top(x' - x^*).$$

It follows from $2a^\top r+\lambda(1-\theta)\mathcal{N} > 0$, $c_1^\top(x'-x^*) < 0$, and $\lambda(1-\theta)\mathcal{N} \geq 0$ that $c_2^\top(x'-x^*) \geq 0$. Finally suppose $(-a + \operatorname{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset$, and let $\theta \geq 0$ be such that $-a + \theta c_2 \in -\mathbb{L}^n$. Then using (16),

$$\begin{aligned} 0 &\geq (-a + \theta c_2)^\top(\mathcal{N}(x' - x^*) + 2c_2^\top(x' - x^*)r) \\ &= -2c_2^\top(x' - x^*)a^\top r + \theta c_2^\top(x' - x^*)(\mathcal{N} + 2c_2^\top r) \\ &= -2c_2^\top(x' - x^*)a^\top r. \end{aligned}$$

It follows from $a^\top r > 0$ that $c_2^\top(x' - x^*) \geq 0$.

b) If $a^\top r < 0$, then $a^\top(-r) > 0$. Since $-r := \begin{pmatrix} \tilde{c}_2 - \tilde{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}$, b) follows from a) by interchanging the roles of C_1 and C_2.

□

Remark 4. *The conditions* (12) *and* (13) *are directly related to the sufficient conditions that guarantee the closedness of the convex hull of a two-term disjunction on* $\mathbb{L}^n$ *explored in [14]. In particular, one can show that the convex hull of the disjunction* $h_1^\top x \geq h_{1,0} \vee h_2^\top x \geq h_{2,0}$ *on* $\mathbb{L}^n$ *is closed if*

i) $h_{1,0} = h_{2,0} \in \{\pm 1\}$ *and there exists* $0 < \mu < 1$ *such that* $\mu h_1 + (1-\mu)h_2 \in \mathbb{L}^n$*, or*

ii) $h_{1,0} = h_{2,0} = -1$ *and* $h_1, h_2 \in -\operatorname{int}\mathbb{L}^n$*.*

Letting $h_i := a + \theta_i c_i$ *and* $h_{i,0} := 1$ *(*$h_i := -a + \theta_i c_i$ *and* $h_{i,0} := -1$*) for some* $\theta_i > 0$ *for both* $i \in \{1,2\}$ *leads to the conditions* (12) *and* (13)*.*

In the next result we show that the inequality (6) is sufficient to describe $\overline{\operatorname{conv}}(C_1 \cup C_2)$ when the conditions (12) and (13) hold.

Theorem 3. *Let* C_1 *and* C_2*, defined as in* (2)*, satisfy Assumptions 1 and 2. Let* c_1 *and* c_2 *be defined as in* (3)*. Suppose also that one of the following conditions is satisfied:*

a) $a^\top r = 0$*,*

b) $a^\top r > 0$ *and* (12) *holds,*

c) $a^\top r < 0$ *and* (13) *holds.*

Then

$$\overline{\operatorname{conv}}(C_1 \cup C_2) = \{x \in C : \ x \text{ satisfies } (6)\}. \tag{18}$$

Proof. Let D denote the set on the right-hand side of (18). The inequality (6) is valid for $\overline{\operatorname{conv}}(C_1 \cup C_2)$ by Theorem 1. Hence, $\overline{\operatorname{conv}}(C_1 \cup C_2) \subseteq D$. Let $x' \in D$. If $x' \in C_1 \cup C_2$, then clearly $x' \, \overline{\operatorname{conv}}(C_1 \cup C_2)$. Therefore, suppose $x' \in C \setminus (C_1 \cup C_2)$ is a point that satisfies (6). By Proposition 1, it satisfies (7) and (8) as well. We are going to show that in each case x' belongs to $\overline{\operatorname{conv}}(C_1 \cup C_2)$.

a) Suppose $a^\top r = 0$. By Remarks 2 and 3, $\mathcal{N} = 2c_1^\top r = -2c_2^\top r > 0$. Define α_1, α_2, x_1, and x_2 as in (11). It is not difficult to see that $a^\top x_1 = a^\top x_2 = 1$ and $c_1^\top x_1 = c_2^\top x_2 = 0$. Furthermore, $x' \in \operatorname{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. One can show that $x_1, x_2 \in \mathbb{L}^n$ using the same arguments as in the proof of Theorem 2. This proves $x_1 \in C_1$ and $x_2 \in C_2$.

b) Suppose $a^\top r > 0$ and (12) holds. Let $x^* := \frac{r}{a^\top r}$. Then by Lemma 2, $c_1^\top (x' - x^*) < 0$ and $c_2^\top (x' - x^*) \geq 0$.

   First, suppose $c_2^\top (x' - x^*) > 0$, and let

$$\begin{aligned} \alpha_1 &:= \frac{-c_1^\top x'}{c_1^\top (x' - x^*)}, & \alpha_2 &:= \frac{-c_2^\top x'}{c_2^\top (x' - x^*)}, \\ x_1 &:= x' + \alpha_1 (x' - x^*), & x_2 &:= x' + \alpha_2 (x' - x^*). \end{aligned} \tag{19}$$

   As in part a), $a^\top x_1 = a^\top x_2 = 1$, $c_1^\top x_1 = c_2^\top x_2 = 0$, and $x' \in \operatorname{conv}\{x_1, x_2\}$ because $\alpha_1 < 0 < \alpha_2$. To show $x_1, x_2 \in \mathbb{L}^n$, first note $\mathcal{N} x^* - 2(c_1^\top x^*) r = \mathcal{N} x^* + 2(c_2^\top x^*) r = 0$ as in (14) and (15). Using this and $c_1^\top x_1 = c_2^\top x_2 = 0$, we get

$$\begin{aligned} \mathcal{N} x_1 &= \mathcal{N} x_1 - 2(c_1^\top x_1) r = (1 + \alpha_1)(\mathcal{N} x' - 2(c_1^\top x') r), \\ \mathcal{N} x_2 &= \mathcal{N} x_2 + 2(c_2^\top x_2) r = (1 + \alpha_2)(\mathcal{N} x' + 2(c_2^\top x') r). \end{aligned}$$

   Clearly, $1 + \alpha_2 > 0$, so $\mathcal{N} x_2 \in \mathbb{L}^n$. Furthermore,

$$1 + \alpha_1 = \frac{-c_1^\top x^*}{c_1^\top (x' - x^*)} = \frac{-c_1^\top r}{(a^\top r) c_1^\top (x' - x^*)} = \frac{-\mathcal{N}}{2(a^\top r) c_1^\top (x' - x^*)} > 0$$

   where we have used the relationships $\mathcal{N} > 0$, $a^\top r > 0$, and $c_1^\top (x' - x^*) < 0$ to reach the inequality. It follows that $\mathcal{N} x_2 \in \mathbb{L}$ as well. Because $\mathcal{N} > 0$, we get $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in C_1$ and $x_2 \in C_2$.

   Now suppose $c_2^\top (x' - x^*) = 0$, and define α_1 and x_1 as in (19). All of the arguments that we have just used to show $\alpha_1 < 0$ and $x_1 \in C_1$ continue to hold. Using $\mathcal{N} x^* + 2c_2^\top x^* r = 0$, we can write

$$\mathcal{N}(x' - x^*) = \mathcal{N}(x' - x^*) + 2c_2^\top (x' - x^*) r \in \mathbb{L}^n.$$

   Because $\mathcal{N} > 0$, we get $x' - x^* \in \mathbb{L}^n$. Together with $c_2^\top (x' - x^*) = 0$ and $a^\top (x' - x^*) = 0$, this implies $x' - x^* \in \operatorname{rec} C_2$. Then $x' = x_1 - \alpha_1 (x' - x^*) \in C_1 + \operatorname{rec} C_2$ because $\alpha_1 < 0$. The claim now follows from the fact that the last set is contained in $\overline{\operatorname{conv}}(C_1 \cup C_2)$ (see, e.g., [17, Theorem 9.8]).

c) Suppose $a^\top r < 0$ and (13) holds. Since $-r := \begin{pmatrix} \tilde{c}_2 - \tilde{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}$, c) follows from b) by interchanging the roles of C_1 and C_2.

□

The following result shows that when C is an ellipsoid or a paraboloid, any two-term disjunction can be convexified by adding the cut (6) to the description of C.

Corollary 1. *Let C_1 and C_2, defined as in* (2), *satisfy Assumptions 1 and 2. Let c_1 and c_2 be defined as in* (3). *If $a \in \mathbb{L}^n$, then* (18) *holds.*

Proof. The result follows from Theorem 3 after observing that conditions (12) and (13) are trivially satisfied for any c_1 and c_2 when $a \in \mathbb{L}^n$. □

The case of a split disjunction is particularly relevant in the solution of MISOCP problems, and it has been studied by several groups recently, in particular Dadush et al. [10], Andersen and Jensen [1], Belotti et al. [4], Modaresi et al. [15]. Theorem 3 has the following consequence for a split disjunction.

Corollary 2. *Consider C_1 and C_2 defined by a split disjunction on C as in* (2). *Suppose Assumptions 1 and 2 hold. Let c_1 and c_2 be defined as in* (3). *Then* (18) *holds.*

Proof. Let $l_1^\top x \geq l_{1,0} \vee l_2^\top x \geq l_{2,0}$ define a split disjunction on C with $l_2 = -tl_1$ for some $t > 0$. Then we have $tl_{1,0} > -l_{2,0}$ so that $C_1 \cup C_2 \neq C$. Let λ_1, λ_2, c_1, and c_2 be defined as in (3). Let $\theta_2 := \frac{1}{\lambda_2(tl_{1,0}+l_{2,0})}$ and $\theta_1 := \frac{t\lambda_2\theta_2}{\lambda_1}$. Then

$$a + \theta_1 c_1 + \theta_2 c_2 = a + \lambda_2\theta_2(t(l_1 - l_{1,0}a) + (l_2 - l_{2,0}a)) = 0 \in \mathbb{L}^n.$$

The result now follows from Theorem 3 after observing that $\theta_1, \theta_2 \geq 0$ imply that conditions (12) and (13) are satisfied. □

When the sets C_1 and C_2 do not intersect, except possibly on their boundary, Proposition 1 says that (6) can be expressed in conic quadratic form and directly implies the following result.

Corollary 3. *Let C_1 and C_2, defined as in* (2), *satisfy Assumptions 1 and 2. Let c_1 and c_2 be defined as in* (3). *Suppose that one of the conditions a), b), or c) of Theorem 3 holds. Suppose, in addition, that*

$$\{x \in C : \ c_1^\top x > 0, \, c_2^\top x > 0\} = \emptyset.$$

Then

$$\begin{aligned}\overline{\text{conv}}(C_1 \cup C_2) &= \{x \in C : \ x \ \textit{satisfies} \ (7)\} \\ &= \{x \in C : \ x \ \textit{satisfies} \ (8)\}.\end{aligned}$$

3.3 Two Examples

In this section we illustrate Theorem 3 with two examples.

3.3.1 A Two-Term Disjunction on a Paraboloid

Consider the disjunction $-2x_1 - x_2 - 2x_4 \geq 0 \vee x_1 \geq 0$ on the paraboloid $C := \{x \in \mathbb{L}^4 : x_1 + x_4 = 1\}$. Let $C_1 := \{x \in C : -2x_1 - x_2 - 2x_4 \geq 0\}$ and $C_2 := \{x \in C : x_1 \geq 0\}$. Noting that C is a paraboloid and C_1 and C_2 are disjoint, we can use Corollary 3 to characterize $\overline{\text{conv}}(C_1 \cup C_2)$ with a conic quadratic inequality:

$$\overline{\text{conv}}(C_1 \cup C_2) = \left\{x \in C : \ 3x + x_1(-3; -1; 0; 2) \in \mathbb{L}^4\right\}.$$

Figure 1 depicts the paraboloid C in mesh and the disjunction $C_1 \cup C_2$ in blue. The conic quadratic disjunctive cut added to convexify this set is shown in red.

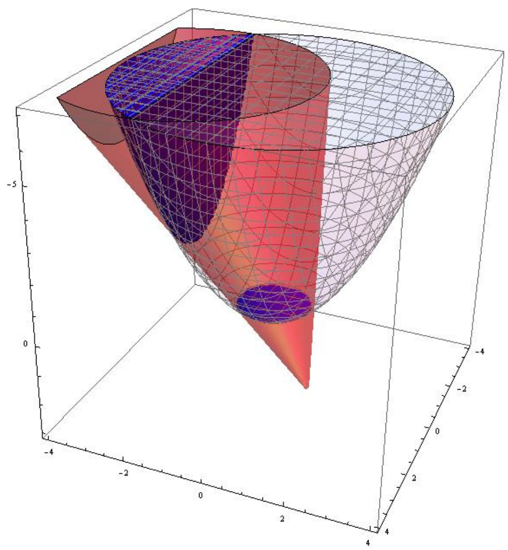

Figure 1: The disjunctive cut obtained from a two-term disjunction on a paraboloid.

3.3.2 A Two-Term Disjunction on a Hyperboloid

Consider the disjunction $-2x_1 - x_2 \geq 0 \vee \sqrt{2}x_1 - x_3 \geq 0$ on the hyperboloid $C := \{x \in \mathbb{L}^3 : x_1 = 2\}$. Let $C_1 := \{x \in C : -2x_1 - x_2 \geq 0\}$ and $C_2 := \{x \in C : \sqrt{2}x_1 - x_3 \geq 0\}$. Note that, in this setting,

$$a^\top r = \frac{1}{10}(1;0;0)^\top \left(-2\sqrt{5} + 5\sqrt{2}; -\sqrt{5}; -5\right) < 0,$$

but none of the conditions (13) are satisfied. The conic quadratic inequality

$$(5 + 2\sqrt{10})x + (\sqrt{2}x_1 - x_3)\left(-2\sqrt{5} + 5\sqrt{2}; -\sqrt{5}; -5\right) \in \mathbb{L}^3 \tag{20}$$

of Theorem 3 is valid for $C_1 \cup C_2$ but not sufficient to describe its closed convex hull. Indeed, the inequality $x_2 \leq 2$ is valid for $\overline{\text{conv}}(C_1 \cup C_2)$ but is not implied by (20). Figure 2 depicts the hyperboloid C in mesh and the disjunction $C_1 \cup C_2$ in blue. The conic quadratic disjunctive cut (20) is shown in red.

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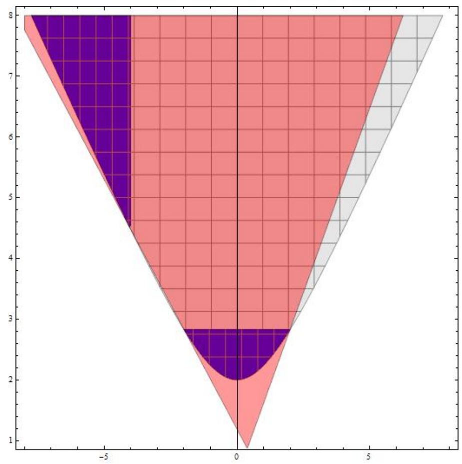

Figure 2: The disjunctive cut obtained from a two-term disjunction on a hyperboloid.

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