Projection Methods: An Annotated Bibliography of Books and Reviews

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Abstract

Projections onto sets are used in a wide variety of methods in optimization theory but not every method that uses projections really belongs to the class of projection methods as we mean it here. Here *projection methods* are iterative algorithms that use projections onto sets while relying on the general principle that when a family of (usually closed and convex) sets is present then projections (or approximate projections) onto the given individual sets are easier to perform than projections onto other sets (intersections, image sets under some transformation, etc.) that are derived from the given family of individual sets. Projection methods employ projections (or approximate projections) onto convex sets in various ways. They may use different kinds of projections and, sometimes, even use different projections within the same algorithm. They serve to solve a variety of problems which are either of the feasibility or the optimization types. They have different algorithmic structures, of which some are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behavioral patterns. This class of algorithms has witnessed great progress in recent years and its member algorithms have been applied with success to many scientific, technological, and mathematical problems. This annotated bibliography includes books and review papers on, or related to, projection methods that we know about, use, and like. If you know of books or review papers that should be added to this list please contact us.

Keywords and phrases: Projection methods, annotated bibliography, convex feasibility, variational inequalities, von Neumann, Kaczmarz, Cimmino, fixed points, row-action methods.

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1 Introduction

Projection methods. Projections onto sets are used in a wide variety of methods in optimization theory but not every method that uses projections really belongs to the class of projection methods as we mean it here. Here *projection methods* are iterative algorithms that use projections onto sets while relying on the general principle that when a family of (usually closed and convex) sets is present then projections (or approximate projections) onto the given individual sets are easier to perform than projections onto other sets (intersections, image sets under some transformation, etc.) that are derived from the given family of individual sets.

A projection algorithm reaches its goal, related to the whole family of sets, by performing projections onto the individual sets. Projection algorithms employ projections (or approximate projections) onto convex sets in various ways. They may use different kinds of projections and, sometimes, even use different projections within the same algorithm. They serve to solve a variety of problems which are either of the feasibility or the optimization types. They have different algorithmic structures, of which some are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behavioral patterns in some significant fields of applications.

Apart from theoretical interest, the main advantage of projection methods, which makes them successful in real-world applications, is computational. They commonly have the ability to handle huge-size problems of dimensions beyond which other, more sophisticated currently available, methods start to stutter or cease to be efficient. This is so because the building bricks of a projection algorithm are the projections onto the given individual sets (assumed and actually easy to perform) and the algorithmic structures are either sequential or simultaneous or in-between, such as in the block-iterative projection (BIP) methods or in the more recent string-averaging projection (SAP) methods. An advantage of projection methods is that they work with initial data and do not require transformation of, or other operations on, huge and sparse matrices describing the problem.

Purpose of the paper. We present here an annotated bibliography of books and review papers on projection methods. This should be helpful to researchers, veterans or newcomers, in the field by directing them to the many existing resources. The vast amount of research papers in the field of projection methods makes it sometimes difficult to master even within a specific sub-area. On the other hand, projection methods send branches both into fields of applications wherein real-world problems are solved and into theoretical areas in mathematics such as, but not only, fixed point theory and variational inequalities. Researchers in each of these, seemingly perpendicular, directions might wish to consult the bibliography presented here and check the annotated items' references lists for further information.

We emphasize that we had no intention to create a bibliography of the research itself in the field of projection methods. This would have been a humongous job beyond our plans. So, we stick with books, book chapters and review papers, except for a few glitches that made us unable to refrain from mentioning a (very) few articles which are not strictly review papers but contain beside their research contents also some worthwhile review-like material.

An apology. Oversight and lack of knowledge are human traits which we are not innocent of. Therefore, we apologize for omissions and other negligence and lacunas in this paper. We kindly ask our readers to communicate to us any additional items and informations that fit the structure and spirit of the paper and we will gladly consider those for inclusion in future revisions, extensions and updates of the paper. Such updates will be posted on www.arxiv.org.

Organization of the paper. After paying tribute to some of the early workers who laid the foundations (Section 2) we cite and annotate books in Section 3, followed by annotated citations of review papers in Section 4. We go alphabetically through the cited items in each section.

2 Early beginnings: von Neumann, Kaczmarz and Cimmino

In telegraphic language we recognize von Neumann, Kaczmarz and Cimmino as the forefathers. John von Neumann's 1933 [54] method of alternating projections (MAP) is a projection method for finding the projection of a given point onto the intersection of two subspaces in Hilbert space. Stefan Kaczmarz, in a three pages paper published in 1937 [47], (posthumous translation into English in [48]) presented a sequential projections method for solving a (consistent) system of linear equations. Historical information about his life can be found in the papers of Maligranda [52] and Parks [55].

Gianfranco Cimmino proposed in [28], published in 1938, a simultaneous projection method for the same problem in which one projects the current iterate simultaneously on all hyperplanes, representing the linear equations of the system, and then takes a convex combination to form the next iterate. A historical account of Cimmino's work was published by Benzi [10].

In 1954 Agmon [1] and Motzkin and Schoenberg [53] generalized the sequential projections method from hyperplanes to half-spaces, and then Eremin [35] in 1965, Bregman [14] in 1965, and Gubin, Polyak and Raik in 1967 [42] generalized it farther to convex sets. In 1970 Auslender [3, page 78] generalized Cimmino's simultaneous projection method to convex sets. Arriving from a different perspective, one must also mention here the seminal papers of Amemiya and Ando [2] and of Halperin [43] that discuss products of contractions and products of projection operators, respectively, in Hilbert space.

A significant early contribution was the work of Lev Bregman who introduced a new "distance" that gave rise to new "projections" that include as special cases the Euclidean distance and projection as well as the Kullback-Leibler "distance" and entropic projections. Bregman's 1967 seminal paper [15], based on his doctoral dissertation, had absolutely no follow-up in the literature until the appearance, 14 years later, of [25]. Nowadays, Bregman projections and projection algorithms that employ them are abundant in the literature.

These were the early beginnings that paved the way for the subsequent "explosion" of research in this field that continues to this day and covers many aspects. These include, but are not limited to, developments of new algorithmic structures for projection methods, usage of different types of projections, application of projection methods to new types of feasibility, optimization, or variational inequalities problems, investigations of the above in various spaces, branching into fixed point theory and other mathematical areas, and using projection methods in significant real-world problems with real data sets of humongous dimensions, and more.

The interface between projection methods and significant real-world problems was and still is a fertilizer for both. In 1970 Gordon, Bender and Herman [41] published an algebraic reconstruction technique (ART) for three-dimensional electron microscopy and x-ray photography. It was recognized later on that their ART is an independent rediscovery of Kaczmarz's sequential projections method for solving a system of linear equations. The first CT (computerized tomography) scanner by EMI (Electric & Musical Industries Ltd., London, England, UK), invented by G.N. Hounsfield [45], used a variant of ART, i.e., of Kaczmarz's sequential projections method. For his pioneering invention, Hounsfield shared a Nobel prize with A.M. Cormack in 1979. Read more on this field in [44].

3 Books on projection methods

[3] **A. Auslender, *Optimisation: Méthodes Numériques*, Masson, Paris, France, 1976 (in French).**

This book on optimization is one of the earliest books to include a subsection on the convex feasibility problem in the finite-dimensional Euclidean space R^n (in Chapter V) and to discuss orthogonal (least Euclidean distance) projections onto convex sets. To our knowledge, the algorithm in Equations (1.6)–(1.7) on page 78 therein is the first published version of Cimmino's simultaneous projections algorithm extended to handle convex sets.

[6] **H.H. Bauschke, R.S. Burachik, P.L. Combettes, V. Elser, D.R. Luke and H. Wolkowicz (Editors), *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, Springer, 2011.**

This book presents recent work in variational and numerical analysis. The contributions provide state-of-the-art theory and practice in first-order fixed-point algorithms, identify emerging problems driven by applications, and discuss new approaches for their solution. The book is a compendium of topics explored at the Banff International Research Station "Interdisciplinary Workshop on Fixed-Point Algorithms for Inverse Problems in Science and Engineering" in November 2009. The link to projection methods is due to the mathematical topics discussed which include: Bregman distances, feasibility problems, the common fixed point problem, the Douglas–Rachford algorithm, monotone operators, proximal splitting methods, nonexpansive mappings and more, all of which are highly relevant to projection methods.

[7] **H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, NY, USA, 2011.**

This outstanding book presents relations among convex analysis, monotone operators and nonexpansive operators. The fact that the metric projection operator is both monotone and nonexpansive in not only reason for including this book here. More important is the fact that the metric projection, as a firmly nonexpansive operator, is the resolvent of a maximal monotone operator. Moreover, finding a zero of a monotone operator is equivalent to finding a fixed point of its resolvent. The authors present a wide spectrum of properties of all these operators and apply these properties to many methods which use projections onto closed convex subsets.

[9] **A. Ben-Israel and T. Greville, *Generalized Inverses: Theory and Applications*, 2nd Edition, Springer-Verlag, New York, NY, USA, 2003.**

This fundamental work on generalized inverses discusses orthogonal projections and projectors (Chapter 2, Section 7) and projectors associated with essentially strictly convex norms (Chapter 3, Section 7) and relates them to generalized inverses.

[11] **V. Berinde, *Iterative Approximation of Fixed Points*, Lecture Notes in Mathematics, Vol. 1912, Springer-Verlag, Berlin, Heidelberg, Germany, 2007.**

This book deals with convergence and with stability of fixed point iterative procedures in Banach spaces. Most of these procedures are defined by Pickard iteration, Krasnosel'skiĭ iteration, Mann iteration, Ishikawa iteration, viscosity approximation, or their modifications. The convergence theorems presented in the book can be in particular applied to various projection methods.

[12] **E. Blum and W. Oettli, *Mathematische Optimierung: Grundlagen und Verfahren*, Springer-Verlag, Berlin, Heidelberg, Germany, 1975.**

This 1975 German language book on mathematical optimization contains a chapter (Kapitel 6) on projections and contractive methods. In it, the projected gradient minimization method is studied (under the name "the method of Uzawa") and Fejér-contractive mappings are handled. Those are then used to describe an iterative method of Eremin and Mazurov.

[13] **J.M. Borwein and Q.J. Zhu, *Techniques of Variational Analysis*, Springer-Verlag, New York, NY, USA, 2005. Paperback 2010.**

Variational arguments are classical techniques whose use can be traced back to the early development of the calculus of variations and further. The book discusses various forms of variational principles in Chapter 2 and then discusses applications of variational techniques in different areas in Chapters 3–7. Section 4.5 in Chapter 4 is devoted to Convex Feasibility Problems since they can be viewed as, and handled by, techniques of variational analysis.

[16] **C. Brezinski, *Projection Methods for Systems of Equations*, Elsevier Science Publishers, Amsterdam, The Netherlands, 1997.**

The main part of this book is devoted to iterative solution of systems of linear equations in a Euclidean space. Most of the methods described in the book employ projections (orthogonal or oblique) onto a sequence of hyperplanes. The author gives a general model and shows that many projection methods known from the literature are special cases of it. The book also describes how to accelerate the convergence of these methods.

[18] **D. Butnariu, Y. Censor and S. Reich (Editors), *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, Elsevier Science Publishers, Amsterdam, The Netherlands, 2001.**

This 504 pages book contains papers presented at the "Research Workshop on Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications", held March 13–16, 2000, jointly at the University of Haifa and the Technion in Haifa, Israel. Most of the 27 papers in it are about projection methods or closely related to them and the focus is on parallel feasibility and optimization algorithms and their applications, in particular on inherently parallel algorithms. By this term one means algorithms which are logically (i.e., in their mathematical formulations) parallel, not just parallelizable under some conditions, such as when the underlying problem is decomposable in a certain manner.

[19] **C.L. Byrne, *Applied Iterative Methods*, AK Peters, Wellsely, MA, USA, 2008.**

The author describes in this book a huge number of iterative methods for several optimization problems in a Euclidean space. The main part of the book is devoted to projection methods. The author presents many methods and their variants known from the literature, as well as their applications.

[20] **C.L. Byrne, *Iterative Optimization in Inverse Problems*, Chapman and Hall/CRC Press, Boca Raton, FL, USA, 2014.**

The books is devoted to iterative methods for inverse problems in a Euclidean space, in which projection methods play an important role. In Chapters 6 and 7 the author presents properties of several classes of operators containing the metric projections onto closed convex subsets. In Chapters 8 and 9 these properties are applied to projection methods for the convex feasibility problem and for the split feasibility problem.

[21] **A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Lecture Notes in Mathematics, Vol. 2057, Springer-Verlag, Berlin, Heidelberg, Germany, 2012.**

The book deals with iteration methods for solving fixed points problems in a Hilbert space. The presentation consolidates many methods which apply nonexpansive and quasi-nonexpansive operators. The author gives a wide spectrum of properties of several classes of operators: quasi-nonexpansive, strongly quasi-nonexpansive, nonexpansive and averaged. All these classes contain the metric projections onto closed convex subsets and are closed under convex combination and composition. The properties of these classes, as well as some general convergence theorems, enable to prove the convergence for members of a large class of projection methods for solving fixed point problems.

[27] **Y. Censor and S.A. Zenios, *Parallel Optimization: Theory, Algorithms, and Applications*, Oxford University Press, New York, NY, USA, 1997.**

This book presents Bregman distances and Bregman projections along with Csiszar φ-divergences (Chapter 2). Many projection methods are then described along with their convergence analyses. Chapter 5 is about iterative methods for convex feasibility problems, all of which are projection methods. Chapter 6 deals with iterative algorithms for linearly constrained optimization problems, all of which are again from the family of projection methods and embedded in the more general framework of Bregman projections (of which orthogonal projections are a special case). The third part of the book is devoted to applications and has chapters for matrix estimation problems, image reconstruction from projections, the inverse problem of radiation therapy treatment planning, multicommodity network flow problems and planning under uncertainty.

[29] **J.W. Chinneck, *Feasibility and Infeasibility in Optimization: Algorithms and Computational Methods*, Springer, New York, NY, USA, 2007.**

Part I of this book is entitled: “Seeking Feasibility”, Part II is entitled: “Analyzing Infeasibility” and Part III is devoted to applications. With emphasis on the computational aspects and without shying from heuristic algorithmic modifications that extend the algorithms' efficiency, the book summarizes the state-of-the-art at the interface of optimization and feasibility. Anyone interested in feasibility-seeking methods that work or anyone looking for algorithms that have not yet been fully explored mathematically will find here interesting materials.

[33] **F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, NY, USA, 2001.**

Frank Deutsch's book is a fundamental resource for all who study the theory of projection methods in Hilbert spaces. The core of the book is the metric projection onto a closed convex subset and its properties. Chapters 9-10 are devoted to projection methods. The author applies the properties of the metric projection to von Neumann's alternating projection method and to Halperin's cyclic projection method for solving the best approximation problem for a finite family of closed subspaces. Further, the Dykstra algorithm is presented for solving the best approximation problem for a finite family of closed convex subsets. The book contains many examples, where the presented methods are applied to linear equations, linear inequalities, isotone regression, convex regression and to the shape-preserving interpolation. Every chapter in the book ends with interesting exercises.

[36] **I.I. Eremin, *Theory of Linear Optimization*, VSP – International Science Publishers, Zeist, The Netherlands, 2002.**

This monograph is dedicated to the basic component of the theory of linear optimization: systems of linear inequalities. Chapter 6 is devoted to methods of projection in linear programming and deals with Fejér mappings and processes in that context. This is a topic on which the author has published research papers, as mentioned in Section 2.

[37] **R. Escalante and M. Raydan, *Alternating Projection Methods*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA, 2011.**

Starting from von Neumann's 1933 method of alternating projections (MAP) for finding the projection of a given point onto the intersection of two subspaces in Hilbert space, Escalante and Raydan meticulously make their way to row-action methods, Bregman projections, the convex feasibility problem, Dykstra's algorithm for the best approximation problem (BAP) and matrix problems. Needless to say that projection methods are the backbone of this book which appeared in SIAM's series on "Fundamentals of Algorithms".

[38] **F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Volume I and Volume II, Springer-Verlag, New York, NY, USA, 2003.**

This over 1200 pages two volumes book is a comprehensive compendium on theory and methods for solving variational inequalities in Euclidean spaces. In particular, Chapter 12 is devoted to iterative methods for monotone problems where the metric projections play an important role. The authors present many methods: fixed point iterations, extragradient methods, hyperplane projection methods, regularization algorithms, all of which employ metric projections (orthogonal or oblique) in each iteration. The rest of this chapter is devoted to more general methods: proximal point methods and splitting methods.

[39] **A. Galántai, *Projectors and Projection Methods*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2004.**

With idempotent matrices as projectors, this book is mostly linear algebra oriented in its presentation and the techniques that it uses. As such it helps to look at projection methods from that perspective which is complementary to the analysis perspective exercised by other books mentioned in this paper. Chapter 4 on iterative projection methods for linear algebraic systems includes descriptions of the methods of Kaczmarz, Cimmino, Altman, Gastinel, Householder and Bauer. The 369 items long references list is a treasure chest for anyone interested in the field.

[40] **N. Gastinel, *Linear Numerical Analysis*, Hermann, Paris, France, 1970. Translated from the original French text: *Analyse Numérique Linéaire*, Hermann, Paris, France, 1966.**

Limited to linear numerical analysis, Gastinel's 1970 book contains a chapter (Chapter 5) dedicated to "indirect methods" for solving linear systems. Orthogonal (least Euclidean distance) projections onto hyperplanes are considered and Kaczmarz's method and Cimmino's method for linear systems are studied. We think that this is the first appearance of these methods in a textbook, although Kaczmarz's method and its applications were described by Tompkins in Chapter 18 of an edited book [60] published in 1956.

[44] **G.T. Herman, *Fundamentals of Computerized Tomography: Image Reconstruction from Projections*, Springer-Verlag, London, UK, 2nd Edition, 2009.**

This is a revised and updated version of the successful 1980 first edition. Warning: the term "projections" in "image reconstruction from projections" has a different meaning than "projections onto sets", as used elsewhere in this paper. The book is devoted to the fundamentals of the field of image reconstruction from projections with particular emphasis on the computational and mathematical procedures underlying the data collection, image reconstruction, and image display in the practice of computerized tomography. It is written from the point of view of the practitioner: points of implementation and application are carefully discussed and illustrated. The major emphasis of the book is on reconstruction methods; these are thoroughly surveyed. Chapter 11 on algebraic reconstruction techniques and Chapter 12 on quadratic optimization methods are related to projection methods as they are used in this field.

[46] **A.S. Householder, *The Theory of Matrices in Numerical Analysis*, Dover Publications, Inc., New York, NY, USA, 1975.**

Originally published in 1964, like good wine, aging does not diminish the value and beauty of this book. The Preface opens with: "This book represents an effort to select and present certain aspects of the theory of matrices that are most useful in developing and appraising computational methods for solving systems of linear equations...". We think that the book lives up to this, particularly since Chapter 4 on "The Solution of Linear Systems: Methods of Successive Approximation" includes the Subsection 4.2 on "Methods of Projection" which makes this book eligible for inclusion here. Besides Kaczmarz's method it mentions also an 1951 projection method due to de la Garza.

[50] **I. Konnov, *Combined Relaxation Methods for Variational***

***Inequalities*, Lecture Notes in Economics and Mathematical Systems, Vol. 495, Springer-Verlag, Berlin, Heidelberg, Germany, 2001.**

The combined relaxation (CR) methods of Konnov do not fall into our definition of "projection methods". On the contrary, they use a projection step onto a set within a more general two-level algorithmic structure. However, the activity in CR methods to replace projections onto a set by projections onto separating hyperplanes has something in common with similar activities in "projection methods" as we mean them here. Besides, the careful treatment of such algorithms for variational inequalities and the research possibilities opened by looking at those from the point of view of projection methods tilted our decision towards including the book here.

[51] **N.S. Kurpel', *Projection-Iterative Methods for Solution of Operator Equations*, Translations of Mathematical Monographs, Vol. 46, American Mathematical Society, Providence, RI, USA, 1976.**

This is a 196 pages monograph with 238 bibliographic items many of which are not cited anymore in current days literature. Iterative methods for solution of operator equation are studied here in a general setting. Abstract metric and normed spaces (in Chapter 1) as well as Banach spaces (in Chapter 2) are the background against which iterative methods, employing general algorithmic operators, are investigated. Specialization of those general algorithmic operators to projection operators leads to projection methods.

[56] **C. Popa, *Projection Algorithms - Classical Results and Developments: Applications to Image Reconstruction*, Lambert Academic Publishing - AV Akademikerverlag GmbH & Co. KG, Saarbrücken, Germany, 2012.**

As a textbook, this book provides a short and useful introduction into the field of projection based solvers. It is at the same time also a research monograph that describes some of the author's research results in the field. Set in the finite-dimensional Euclidean space context, the theoretical discussions provide ideas for further developments and research. Besides introductory material, the book includes chapters on extensions to inconsistent least squares problems, oblique and generalized oblique projections, constraining strategies, and some special projection-based algorithms.

[57] **Y. Saad, *Iterative Methods for Sparse Linear Systems*, Second Edition, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA, 2003.**

The book, freely available from the author's homepage on the Internet, declares its intention to provide up-to-date coverage of iterative methods for solving large sparse linear systems. It focuses on practical methods that work for general sparse matrices rather than for any specific class of problems. Although the very definition of "projection methods", in Chapter 5 ("Projection Methods") is not identical with what we mean by this term here, the relations and connections between them cannot be mistaken.

[58] **H.D. Scolnik, A.R. De Pierro, N.E. Echebest and M.T. Guardarucci (Guest Editors), *Special Issue of International Transactions in Operational Research*, Volume 16, Issue 4, July 2009.**

This special issue is devoted to projection methods. From block-iterative algorithms, through the string-averaging method for sparse common fixed-point problems, and from convergence of the method of alternating projections through perturbation-resilient block-iterative projection methods with application to image reconstruction from projections – this special issue is a focused source for novel ideas and for literature coverage of the subject. It is available at: http://onlinelibrary.wiley.com/doi/10.1111/itor.2009.16.issue-4/issuetoc.

[59] **H. Stark and Y. Yang, *Vector Space Projections: A Numerical Approach to Signal and Image Processing, Neural Nets, and Optics*, John Wiley & Sons, Inc. New York, NY, USA, 1998.**

Using the term "vector-space projections", this book presents a nice blend of theory (Chapters 1–5) and applications (Chapters 6–9) of the "projections onto convex sets" (POCS) methods which is the name adopted mostly by the engineering community for projection methods. The book reflects the growing interest in the application of these methods to problem solving in science and engineering. It brings together material previously scattered in disparate papers, book chapters, and articles, and offers a systematic treatment of vector space projections.

4 Review papers on projection methods

[4] **H.H. Bauschke and J.M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Review* 38 (1996), 367–426.**

This review paper, based on the first author's Ph.D. work, has not lost its vitality to this day. With 109 items in its bibliography and a subject index, it is a treasure of knowledge central to the field. The abstract says: "Due to their extraordinary utility and broad applicability in many areas of classical mathematics and modern physical sciences (most notably, computerized tomography), algorithms for solving convex feasibility problems continue to receive great attention. To unify, generalize, and review some of these algorithms, a very broad and flexible framework is investigated. Several crucial new concepts which allow a systematic discussion of questions on behavior in general Hilbert spaces and on the quality of convergence are brought out. Numerous examples are given."

[5] **H.H. Bauschke, J.M. Borwein and A.S. Lewis, The method of cyclic projections for closed convex sets in Hilbert space, *Contemporary Mathematics* 204 (1997), 1–38.**

From the Abstract: "Although in many applications [...] the convex constraint sets do not necessarily intersect, the method of cyclic projections is still employed. Results on the behaviour of the algorithm for this general case are improved, unified, and reviewed. The analysis relies on key concepts from convex analysis and the theory of nonexpansive mappings." Meticulously written, this paper contains a wealth of material that both reviews and extends many important results.

[8] **H.H. Bauschke and V.R. Koch, Projection methods: Swiss army knives for solving feasibility and best approximation problems with halfspaces, in: S. Reich and A. Zaslavski (Editors), Proceedings of the workshop on Infinite Products of Operators and Their**

Applications, Haifa, Israel, 2012, *Contemprary Mathematics,* accepted for publication.

Although geared toward solving the specific problem of automated design of road alignments, this paper is written with an eye on reviewing the field. It provides a selection of state-of-the-art projection methods, superiorization algorithms, and best approximation algorithms. Various observations on the algorithms and their relationships are given along with broad numerical experiments introducing performance profiles for projection methods.

[10] **M. Benzi, Gianfranco Cimmino's contributions to numerical mathematics, in: *Atti del Seminario di Analisi Matematica*, Dipartimento di Matematica dell'Universita' di Bologna. Volume Speciale: Ciclo di Conferenze in Memoria di Gianfranco Cimmino, Marzo-Aprile 2004, Tecnoprint, Bologna, Italy (2005), pp. 87–109.**

The abstract of this paper precisely describes its contents. It says: "Gianfranco Cimmino (1908–1989) authored several papers in the field of numerical analysis, and particularly in the area of matrix computations. His most important contribution in this field is the iterative method for solving linear algebraic systems that bears his name, published in 1938. This paper reviews Cimmino's main contributions to numerical mathematics, together with subsequent developments inspired by his work. Some background information on Italian mathematics and on Mauro Picone's Istituto Nazionale per le Applicazioni del Calcolo, where Cimmino's early numerical work took place, is provided. The lasting importance of Cimmino's work in various application areas is demonstrated by an analysis of citation patterns in the broad technical and scientific literature."

[17] **R.E. Bruck, On the random product of orthogonal projections in Hilbert space II, *Contemporary Mathematics* 513 (2010), 65–98.**

This paper is mainly concerned with certain abstract properties of products of linear orthogonal projections onto closed subspaces of a Hilbert space. We mention it here because its first section presents a brief, but very readable and informative, history of the study of convergence of infinite products of such projections, as well as of their nonlinear counterparts.

[22] **Y. Censor, Row-action methods for huge and sparse systems and their applications, *SIAM Review* 23 (1981), 444–466.**

This early (1981) review paper brings together and discusses theory and applications of methods, identified and labelled as row-action methods, for linear feasibility problems, linearly constrained optimization problems and some interval convex programming problems. The main feature of row-action methods is that they are iterative procedures which, without making any changes to the original matrix A, use the rows of A, one row at a time. Such methods are important and have demonstrated effectiveness for problems with large or huge matrices which do not enjoy any detectable or usable structural pattern, apart from a high degree of sparseness. Fields of application where row-action methods are used in various ways include image reconstruction from projection, operations research and game theory, learning theory, pattern recognition and transportation theory. A row-action method for the nonlinear convex feasibility problem is also presented.

[23] **Y. Censor, Iterative methods for the convex feasibility problem, *Annals of Discrete Mathematics* 20 (1984), 83–91.**

Abstract: "The problem of finding a point in the intersection of a finite family of closed convex sets in the Euclidean space is considered here. Several iterative methods for its solution are reviewed and some connections between them are pointed out."

[24] **Y. Censor, W. Chen, P.L. Combettes, R. Davidi and G.T. Herman, On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints, *Computational Optimization and Applications* 51 (2012), 1065–1088.**

Besides interesting experimental results, this paper contains (in its introduction) many pointers to applications wherein projection methods were used. Section 3 contains a brief glimpse into some recently published results that show the efficacy of projection methods for some large problems, and their use in commercial devices.

[26] **Y. Censor and A. Segal, Iterative projection methods in biomedical inverse problems, in: Y. Censor, M. Jiang and A.K. Louis (Editors), *Mathematical Methods in Biomedical Imaging and***

***Intensity-Modulated Radiation Therapy (IMRT)*, Edizioni della Normale, Pisa, Italy, 2008, pp. 65–96.**

In this paper on projection methods the authors review Bregman projections and the following algorithmic structures: sequential projection algorithms, string-averaging algorithmic structures, block-iterative algorithmic schemes with underrelaxed Bregman projections, component averaging (CAV)). Seminorm-induced oblique projections for sparse nonlinear convex feasibility problems and BICAV (Block-iterative component averaging) are discussed, followed by a review of subgradient projections and perturbed projections for the multiple-sets split feasibility problem. Finally, algorithms for the quasi-convex feasibility problem are presented.

[30] **P.L. Combettes, The foundations of set-theoretic estimation, *Proceedings of the IEEE* 81 (1993), 182–208.**

The paper eloquently presents the rational of posing problems as feasibility problems rather than optimization problems. Well-connected to the practical aspects of estimation problems, the discussion starts with a study of various sets to define solutions and continues with mathematical methods consistent or inconsistent feasibility problems. A comprehensive section on connections with other estimation procedures closes this interesting (equipped with 223 references) paper.

[31] **P.L. Combettes, The convex feasibility problem in image recovery, in: *Advances in Imaging and Electron Physics*, vol. 95, (P. Hawkes, Editor), pp. 155–270, Academic Press, New York, NY, USA, 1996.**

Image recovery is a broad discipline that encompasses the large body of inverse problems in which an image is to be inferred from the observation of data consisting of signals physically or mathematically related to it. Image restoration and image reconstruction from projections are two main subbranches of image recovery. The traditional approach has been to use a criterion of optimality, which usually leads to a single "best" solution. An alternative approach is to use feasibility as an acceptance test, in which compliance with all prior information and the data defines a set of equally acceptable solutions. This framework, which underlies the feasibility-seeking approach, is discussed in this survey. It contains an overview of convex set theoretic image recovery, of construction of property sets, of solving the convex feasibility problem, and numerical examples.

[32] **P.L. Combettes and J.-C. Pesquet, Proximal splitting methods in signal processing, in: *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, (H.H. Bauschke, R.S. Burachik, P.L. Combettes, V. Elser, D.R. Luke and H. Wolkowicz, Editors), Springer, New York, NY, USA, 2011, pp. 185–212.**

The proximity operator of a convex function is a natural extension of the notion of a projection operator onto a convex set. This tool, which plays a central role in the analysis and the numerical solution of convex optimization problems, has recently been introduced in the arena of inverse problems and, especially, in signal processing, where it has become increasingly important. In this paper, the authors review the basic properties of proximity operators which are relevant to signal processing and present optimization methods based on these operators. These proximal splitting methods are shown to capture and extend several well-known algorithms in a unifying framework, covering several important projection methods.

[34] **F. Deutsch and H. Hundal, Arbitrarily slow convergence of sequences of linear operators: An updated survey, in: S. Reich and A. Zaslavski (Editors), Proceedings of the Workshop on Infinite Products of Operators and Their Applications, Haifa, Israel, 2012, *Contemporary Mathematics*, accepted for publication.**

This is an updated survey on the slowest possible rate of convergence of a sequence of linear operators that converges pointwise to a linear operator. Although written in the general context, it contains special sections that relate the general results to projections: Section 9 is on Application to Cyclic Projections, and Scetion 10 is on Application to Intermittent Projections**.**

[61] **D.C. Youla, Mathematical theory of image restoration by the method of convex projections, in: *Image Recovery: Theory and Applications* (H. Stark, editor), Academic Press, Orlando, FL, USA (1987), pp. 29–78.**

Projection methods are frequently called POCS methods ("projections onto convex sets") particularly in the engineering community. In this 1978 article, Youla suggested that many problems in image restoration could be formulated in terms of linear subspaces and orthogonal projections in Hilbert space. He noted that the linear formulation can result in loss of important information and lead to an ill-posed restoration problem, and offered orthogonal projections onto closed convex sets as a way to smooth the restoration problem and reintroduce information about the image to be restored. Because Youla's objective is to familiarize the image processing community with the fundamentals of convex feasibility, asymptotically regular nonexpansive mappings, and fixed point theory, this paper is a relatively self-contained tutorial introduction to these topics and to the results known up to 1987.

[62] **E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory. Part I. Projections on convex sets, in: *Contributions to Nonlinear Functional Analysis*, E.H. Zarantonello (Editor), Academic Press, New York, NY, USA, 1971, pp. 239–341.**

Everything you wanted probably to know about projections on convex sets in 1971 is probably in this Part I (out of two parts) paper. Its sections are: 1. Projections, basic properties; 2. Vertices and faces; 3. The range of $I - P_K$; 4.Translation sets, parallel convex sets, differentiability; 5. The algebra of projections.

5 The end is open

In a recent paper [49] it is shown that a variant of ART (i.e., of Kaczmarz's sequential projections method) can be used for crystal lattice orientation distribution function estimation from diffraction data. One of the problems discussed in [49] has 1,372,000,000 unknowns and the number of equations is potentially infinite. They are randomly generated and a projection step can be carried out as soon as a new equation is available (an ideal use of a sequential projection method of the row-action type, see [22]). The result reported in the paper for that problem is obtained after 1,000,000,000 such projection steps. As for all methodologies, projection methods are not necessarily the approach of choice in all applications. However, in important applications in biomedicine, image processing, and many other fields, see [24], projection methods work well and have been used successfully for a long time.

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