Justification of Constrained Game Equilibrium Models[1]

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Abstract

We consider an extension of a noncooperative game where players have joint binding constraints. In this model, the constrained equilibrium can not be implemented within the same noncooperative framework and requires some other additional regulation procedures. We consider several approaches to resolution of this problem. In particular, a share allocation method is presented and substantiated. We also show that its regularization leads to a decomposable penalty method.

Key words: Noncooperative games; joint constraints; generalized equilibrium points; share allocation; decomposable penalty method.

1 Introduction

Let us first consider the well-known *l-person noncooperative game*, where the i-th player has a strategy set $X_i \subseteq \mathbb{R}^{n_i}$ and a payoff (utility) function $f_i : X \to \mathbb{R}$ with

$$X = X_1 \times \ldots \times X_l.$$

That is, each i-th player selects an element $x_i \in X_i$ for $i = 1, \ldots, l$ and receives the utility $f_i(x)$ at the situation $x = (x_1, \ldots, x_l)^\top \in X$. In the classical noncooperative game framework, all the players are supposed to be equal and independent in the sense that each player has some (complete or incomplete) information about other strategy sets, but is not able to forecast strategy choices, which is very essential, because his/her utility function value depends on the strategies of all the players. Moreover, it is supposed that all the players make their choices simultaneously; see e.g. [1]. This *basic information scheme* of the noncooperative game imposes certain restrictions on its solution concepts and implementation mechanisms.

The most popular solution concept for this problem was suggested by Nash [2]. The *Nash equilibrium problem* (NEP) consists in finding a point $x^* = (x_1^*, \ldots, x_l^*)^\top \in X$ such that

$$f_i(x_{-i}^*, v_i) \leq f_i(x^*) \quad \forall v_i \in X_i, i = 1, \ldots, l. \tag{1}$$

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Here and below, we set $(x_{-i}, v_i) = (x_1, \ldots, x_{i-1}, v_i, x_{i+1}, \ldots, x_l)$ for brevity. The conditions in (1) means that any single player can not increase his/her utility from that at the equilibrium point by taking any unilateral strategy deviation. We would like to note that the Nash equilibrium concept corresponds to the above basic information scheme of the noncooperative game. Next, following [3], we can set

$$\Psi(x, v) = -\sum_{i=1}^{l} f_i(x_{-i}, v_i), \tag{2}$$

$v = (v_1, \ldots, v_l)^\top$ with $n = \sum\limits_{i=1}^{l} n_i$ and

$$\Phi(x, y) = \Psi(x, y) - \Psi(x, x), \tag{3}$$

then NEP (1) becomes equivalent to the general *equilibrium problem* (EP) which is to find a point $x^* \in X$ such that

$$\Phi(x^*, v) \geq 0 \quad \forall v \in X. \tag{4}$$

The above noncooperative game and Nash equilibrium solution concepts admit various modifications and extensions. Investigations of noncooperative games subject to joint binding constraints dates back to early works [4]–[8] and this topic is among the most popular now; see e.g. [9]–[11] and the references therein. We now turn to this extension, taking the formulation from [5] as a basis.

In this *constrained l-person noncooperative game*, all the players together with the above utility functions and strategy sets have the joint constraint set

$$V = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{l} h_i(x_i) \leq b \right\} = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{l} h_{ij}(x_i) \leq b_j, \ j = 1, \ldots, m \right\},$$

where $h_i(x_i) = (h_{i1}(x_i), \ldots, h_{im}(x_i))^\top$, $h_{ij} : \mathbb{R}^{n_i} \to \mathbb{R}$, $j = 1, \ldots, m$, $i = 1, \ldots, l$ are given functions, $b \in \mathbb{R}^m$ is a fixed vector in $\mathbb{R}^m$, as an addition to the set X. That is, they have the common feasible set

$$D = X \bigcap V, \tag{5}$$

A point $x^* = (x_1^*, \ldots, x_l^*)^\top \in D$ is said to be an equilibrium point for this game, if

$$f_i(x_{-i}^*, v_i) \leq f_i(x^*) \quad \forall (x_{-i}^*, v_i) \in D, \ i = 1, \ldots, l. \tag{6}$$

Utilizing the functions Ψ and Φ from (2)–(3), we can also consider the general EP: Find a point $x^* \in D$ such that

$$\Phi(x^*, v) \geq 0 \quad \forall v \in D, \tag{7}$$

cf. (4). Its solutions are called *normalized equilibrium points*. Clearly, each normalized equilibrium point is an equilibrium point, but the reverse assertion is not true in general. The main question consists in implementation of the constrained equilibrium concept within the same noncooperative framework.

In fact, the presence of the joint binding constraints requires concordant actions of all the players, but this contradicts the basic information scheme of the noncooperative game. We observe that certain dependence of players is natural for Stackelberg type leader-follower games, but here all the players are supposed to be equal. These drawbacks were discussed e.g. in [8, 12, 13]. Indeed, models of real systems may involve additional constraints which are absent in the known classical models, however, they should be taken into account in properly modified solution concepts. In case of contradictions with the previous mechanisms, additional regulation procedures, which allow elements to attain the corresponding state, are necessary. Otherwise the model seems incomplete or non-adequate. The concordance of a control mechanism with the basic information scheme is very essential for complex systems with active elements, because each of them may have his/her own interests and activity (strategy) sets. For instance, it was shown in [14] that incorporation of joint constraints in a multi-product auction model leads to necessity to change the directions of information flows and some information content between elements.

In this paper, we analyze several approaches to resolution of this problem for constrained noncooperative games. In particular, a share allocation one is presented and substantiated. We also show that its regularization leads to a decomposable penalty method procedure.

2 General Approaches

We first consider an example of applications of a constrained noncooperative game, which represents a multi-product oligopoly market and follows the lines of those in [15, Sections 2.1-2.3]. Some other examples can be e.g. found in [9, 16, 10, 17]. In what follows, the model will be used for illustration of regulation methods.

Example 2.1 (Water quality management) *Consider a system of l industrial firms which utilize common plants for treatment of their wastes containing m polluted substances. Let $x_i = (x_{i1}, \dots, x_{im})^\top \in \mathbb{R}^m$ is the vector of wastes of the i-th firm for a fixed time period T. Then $\mu_i(x_i)$ denotes the benefit of this firm (e.g. the amount saved by not treating it themselves). Next, suppose that the unit treatment charge φ_j for the j-th substance depends on the total volume of this substance*

$$\sigma_j(x) = \sum_{s=1}^{l} x_{sj},$$

that is $\varphi_j = \varphi_j[\sigma_j(x)]$ with $x = (x_1, \dots, x_l)^\top$. Further, we denote by $X_i \subseteq \mathbb{R}^m$ the feasible waste vector set of the i-th firm and suppose that the total pollution volumes *within the period T must be bounded above by the fixed vector $b \in \mathbb{R}^m$. Then, considering firms as independent players, we obtain clearly a constrained l-person noncooperative game, where the i-th player has the pure strategy set X_i and a payoff (utility) function*

$$f_i(x) = \mu_i(x_i) - \sum_{j=1}^{m} x_{ij}\varphi_j[\sigma_j(x)],$$

the joint constraint set is defined as

$$V = \left\{ x \in \mathbb{R}^m \;\middle|\; \sum_{s=1}^{l} x_s \leq b \right\}.$$

Then we can in principle utilize equilibrium point concepts from (5)–(6) or (5), (7) in order to forecast waste vectors of each firm.

So, we consider the general constrained l-person noncooperative game, which is represented first by the problem (5)–(6) and the related problem (5),(7). In what follows, we suppose that each strategy set X_i is convex and closed and that each utility function f_i is concave and lower semicontinuous in its i-th variable x_i for $i = 1, \dots, l$. Also we suppose that $h_{ij} : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, m$, $i = 1, \dots, l$ are convex functions, and that the common feasible set D is nonempty. It follows that $\Phi : X \times X \to \mathbb{R}$ in (2)–(3), (7) is an equilibrium bi-function, i.e., $\Phi(x, x) = 0$ for every $x \in X$, besides, $\Phi(x, \cdot)$ is convex and lower semicontinuous for each $x \in X$. In addition, we suppose also that $\Phi(\cdot, v)$ is upper semicontinuous for each $v \in X$.

In what follows, we need some monotonicity properties. We recall that an equilibrium bi-function $\Phi : X \times X \to \mathbb{R}$ is said to be:

(a) *monotone* if, for each pair of points $x', x'' \in X$, we have

$$\Phi(x', x'') + \Phi(x'', x') \leq 0;$$

(b) *strongly monotone* with constant $\varkappa > 0$ if, for each pair of points $x', x'' \in X$, we have

$$\Phi(x', x'') + \Phi(x'', x') \leq -\varkappa \|x'' - x'\|^2.$$

We intend to find additional regulation procedures which allow elements (players) to attain game equilibrium states in the presence of joint binding constraints without destroying the basic information scheme of noncooperative games.

There exist a number of iterative solution methods for finding constrained game equilibria of form (5)–(6); see e.g. [9, 10, 11, 18, 13, 19] and the references therein. Most of them represent modifications of solution methods for EP (7); see e.g. [6, 7], [20]–[27] and the references therein. They generate iterative sequences tending to constrained game equilibrium points under suitable assumptions. However, most of these methods treat the joint constraint set V explicitly, i.e. assume tacitly existence of some players concordance (treaty) on these constraints and can not be used for suggesting an implementable regulation mechanism.

Let us describe for example a **general penalty method** for finding constrained game equilibrium points, taking the normalized EP (5),(7) as a basis. First we take a general penalty function for the set V defined by

$$\tilde{P}(x) \begin{cases} = 0, & x \in V, \\ > 0, & x \notin V; \end{cases}$$

which is usually supposed to be convex and lower semicontinuous on X. The most popular choice is

$$\tilde{P}(x) = \sum_{j=1}^{m} \left(\max \left\{ \sum_{i=1}^{l} h_{ij}(x_i) - b_j, 0 \right\} \right)^{\sigma}, \ \sigma \geq 1.$$

Next, given a number $\tau > 0$, we consider the problem of finding a point $x(\tau) \in X$ such that

$$\Phi(x(\tau), v) + \tau[\tilde{P}(v) - \tilde{P}(x(\tau))] \geq 0 \quad \forall v \in X. \tag{8}$$

Under suitable coercivity assumptions one can prove convergence of the auxiliary solution sequence $\{x(\tau)\}$ to a solution of EP (5),(7) if $\tau \nearrow +\infty$ (or even large enough if $\sigma = 1$); see e.g. [21]–[23], [18, 28]. Note that each auxiliary EP (8) has the same Cartesian product feasible set X, but can not be reduced to NEP due to the non-separability of the penalty function $\tilde{P}$. A modification of this approach was proposed in [19]. Then the common penalty function term $\tilde{P}(x)$ is inserted into each particular problem (1), i.e. the i-th player has the utility function $f_i(x) - \tau\tilde{P}(x)$. This means that each player must have additional charges after any common violation of the total constraints regardless of individual contributions, which does not seem fair. Therefore, implementation of this mechanism within a noncooperative game framework may meet serious difficulties.

Let us now describe the well-known **Lagrangian method**. Here and below $\mathbb{R}^m_+$ denotes the non-negative orthant in $\mathbb{R}^m$, i.e.,

$$\mathbb{R}^m_+ = \{y \in \mathbb{R}^n \mid y_j \geq 0 \quad j = 1, \dots, m\}.$$

Then we can replace EP (5),(7) with the primal-dual system: Find a pair $(x^*, y^*) \in X \times \mathbb{R}^m_+$ such that

$$\Phi(x^*, v) + \sum_{i=1}^{l} \langle y^*, h_i(v_i) - h_i(x_i^*) \rangle \geq 0 \quad \forall v \in X, \tag{9}$$

$$\langle b - \sum_{i=1}^{l} h_i(x_i^*), y - y^* \rangle \geq 0 \quad \forall y \in \mathbb{R}^m_+. \tag{10}$$

That is, if (x^*, y^*) solves (9)– (10), then x^* is a solution to EP (5),(7). Conversely, if x^* is a solution to EP (5),(7), then there exists $y^* \in \mathbb{R}^m_+$ such that (x^*, y^*) solves (9)–(10), under a proper constraint qualification; see e.g. [22, 23]. Note that problem (9) is equivalent to the usual NEP of form (1), where the the i-th player has the utility function

$$\tilde{f}_i(x; y^*) = f_i(x) - \langle y^*, h_i(x_i)\rangle.$$

One can find a solution of system (9)– (10) by dual type iterative methods; see e.g. [29]–[31]. They combine a sequential solution of problems of form (9) (i.e. (1)) with $y^* = y^k$ and an adjustment process for the dual iterates y^k. Convergence requires certain monotonicity properties of Φ. For instance, we describe an analog of the Uzawa method.

Given a point $\tilde{y} \in \mathbb{R}^m_+$, we first set

$$X(\tilde{y}) = \left\{ \tilde{x} \in X \;\middle|\; \Phi(\tilde{x}, x) + \sum_{i=1}^{l} \langle \tilde{y}, h_i(v_i) - h_i(\tilde{x}_i)\rangle \geq 0, \quad \forall x \in X \right\}$$

and

$$W(\tilde{y}) = \left\{ g \in \mathbb{R}^m \;\middle|\; g = b - \sum_{i=1}^{l} h_i(\tilde{x}_i),\ \tilde{x} \in X(\tilde{y}) \right\}.$$

Given a starting point $y^0 \in \mathbb{R}^m_+$, the method consists in sequential explicit projection iterates

$$y^{k+1} = \pi_+ \left[y^k + \lambda_k W(y^k) \right], \lambda_k > 0, \tag{11}$$

where $\pi_+ [\cdot]$ denotes the projection mapping onto $\mathbb{R}^m_+$. Convergence of method (11) was established under the strong monotonicity of Φ, a combined regularization type method enables us to attain convergence under the usual monotonicity; see e.g. [29]–[31].

Anyway, these procedures seem concordant with the noncooperative game setting, because they only include an additional upper control level. In fact, the dual variable y is treated as prices (charges vector) for the joint constraints. After a regulator selection of the current prices y^k, one should find an element $x^k \in X(y^k)$, i.e. solve the NEP of form (1), where the i-th player has the utility function $\tilde{f}_i(x; y^k)$. The vector $W(y^k)$ then gives the dis-balance at the Nash equilibrium point x^k, it allows the regulator to find the next prices, and so on.

However, the process may also meet implementation difficulties. Taking Example 2.1, we see that this process forces firms to pay double pollution charges, namely, each utility function $f_i(x)$ already contains charges for the individual wastes, but the new utility function $\tilde{f}_i(x; y^k)$ nevertheless contains charges for approximation of the upper pollution limits by total wastes, which seems superfluous.

Therefore, we need additional flexible control procedures satisfying the basic information scheme of noncooperative games and being applicable for rather general classes of applications.

3 Decomposable Penalty Method

We first consider the simple transformation of the joint constraint set by inserting auxiliary variables:

$$V = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists u \in \mathbb{R}^{ml}, \sum_{i=1}^{l} u_i = b,\ h_i(x_i) \le u_i, i = 1, \dots, l \right\},$$

where $u = (u_1, \dots, u_l)^\top$, $u_i \in \mathbb{R}^m$, $i = 1, \dots, l$. These variables u_i determine a partition of the right-hand side vector b, i.e. give explicit shares of players. We can now separate the constraints and consider the auxiliary penalty problem: Find a pair $w(\tau) = (x(\tau), u(\tau)) \in X \times U$, $\tau > 0$ such that

$$\Phi_\tau(w(\tau), w) = \Phi(x(\tau), x) + \tau[P(w) - P(w(\tau))] \ge 0 \quad \forall w = (x, u) \in X \times U, \tag{12}$$

where

$$P(w) = \sum_{i=1}^{l} P_i(x_i, u_i),$$

$$P_i(x_i, u_i) = \sum_{j=1}^{m} \left(\max\left\{h_{ij}(x_i) - u_{ij}, 0\right\}\right)^\sigma, i = 1, \dots, l,\ \sigma \ge 1;$$

$$U = \left\{ u \in \mathbb{R}^{ml} \;\middle|\; \sum_{i=1}^{l} u_i = b \right\}.$$

We observe that such decomposable penalty methods were suggested for completely separable optimization problems in [32, 33]. Here we obtain an auxiliary equilibrium problem with separable feasible set, rather that a completely separable equilibrium problem.

In order to prove convergence of the penalty method we will take the usual coercivity condition:

(C) *There exists a point* $\tilde{x} \in V$ *such that*

$$\Phi(x, \tilde{x}) \to -\infty \quad \textit{as} \quad \|x - \tilde{x}\| \to \infty,\ x \in X;$$

see e.g. [21, 22]. Then problem (5),(7) must have a solution; see e.g. [28, Proposition 2]. Further, **(C)** implies the same coercivity condition for problem (12). Now there exists a point $\tilde{u} \in U$ such that $h_i(\tilde{x}_i) \le \tilde{u}_i$, $i = 1, \dots, l$. Set $\tilde{w} = (\tilde{x}, \tilde{u})$, then, for any $\tau > 0$ it holds that

$$\Phi_\tau(w, \tilde{w}) \to -\infty \quad \text{as} \quad \|w - \tilde{w}\| \to \infty,\ w = (x, u) \in X \times U. \tag{13}$$

In fact,

$$\Phi_\tau(w, \tilde{w}) = \Phi(x, \tilde{x}) - \tau \sum_{i=1}^{l} P_i(x_i, u_i) \le \Phi(x, \tilde{x}).$$

If $\|w - \tilde{w}\| \to \infty$, then only two cases are possible.

Case 1: $\|x - \tilde{x}\| \to \infty$. Now the above inequality implies (13).

Case 2: $\|x - \tilde{x}\| \leq C \leq \infty$. It follows that $\|u - \tilde{u}\| \to \infty$. Since $u \in U$, there exists at least one pair of indices s and t such that $u_{st} \to -\infty$, hence $P_s(x_s, u_s) \to +\infty$, which also implies (13).

It follows that problem (12) has a solution for any $\tau > 0$. The convergence proof follows the lines of Theorem 1 in [28] and is omitted, we give now the precise formulation of these results.

Proposition 3.1 *Suppose that* **(C)** *is fulfilled, the sequence* $\{\tau_k\}$ *satisfies*

$$\{\tau_k\} \nearrow +\infty.$$

Then:

(i) EP (5),(7) has a solution;

(ii) EP (12) has a solution for each $\tau > 0$*;*

(iii) Each sequence $\{w(\tau_k)\}$ *of solutions of EP (12) with* $w(\tau_k) = (x(\tau_k), u(\tau_k))$ *has limit points and all limit points of the sequence* $\{x(\tau_k)\}$ *are solutions of EP (5),(7).*

We see that the set U still contains coupling constraints for auxiliary variables and we need special methods for solving EP (12). Nevertheless, this approach may indicate a new way of creation of more efficient mechanisms.

4 Share Allocation Problem

If we know all the shares u_i, $i = 1, \ldots, l$, of players, both the problems (5),(7) and (12) reduce to NEPs. Hence, an additional upper control level for these shares assignment leads to the desired procedures conforming to the basic information scheme of noncooperative games. It seems close to the Lagrangian method. In fact, for the fully separable optimization problems they are known as price (Dantzig-Wolfe) and right-hand side (Kornai-Liptak) decomposition methods, respectively; see e.g. [34, 35]. Rather recently, the right-hand side decomposition was extended for variational inequalities [36], thus allowing further extensions.

So, given a feasible partition $u \in U$, we can consider the reduced EP: Find a point $x(u) = (x_1(u), \ldots, x_l(u))^\top \in D(u)$ such that

$$\Phi(x^*, v) \geq 0 \quad \forall v \in D(u), \tag{14}$$

where

$$D(u) = D_1(u_1) \times \ldots \times D_l(u_l),$$

$D_i(u_i) = \{x_i \in X_i \,|\, h_i(x_i) \leq u_i\}$, $i = 1, \ldots, l$; cf. (4) and (7). Clearly, (14) is equivalent to the NEP:

$$f_i(x_{-i}(u), v_i) \leq f_i(x(u)) \quad \forall v_i \in D_i(u_i), i = 1, \ldots, l. \tag{15}$$

We denote by U_i the set of points u_i such that $D_i(u_i)$ is nonempty. It was shown in [36, Proposition 4.1], that

$$U_i = \{u_i \in \mathbb{R}^m \mid \xi_i(u_i) \leq 0\},$$

where

$$\xi_i(u_i) = \min_{x_i \in X_i} \max_{j=1,\dots,m} \{h_{ij}(x_i) - u_{ij}\}$$

is a convex function. We now suppose that there exists a convex set $\tilde{U} \subseteq U$ and $\tilde{U} \subseteq U_1 \times \ldots \times U_l$, such that for each $u \in \tilde{U}$, there exists a pair $(x(u), y(u)) \in X \times \mathbb{R}^{ml}_+$ such that

$$\Phi(x(u), v) + \sum_{i=1}^{l} \langle y_i(u), h_i(v_i) - h_i(x_i(u)) \rangle \geq 0 \quad \forall v \in X, \tag{16}$$

$$\langle u_i - h_i(x_i(u)), y_i - y_i(u) \rangle \geq 0, \quad \forall y_i \in \mathbb{R}^m_+, \ i = 1, \dots, l; \tag{17}$$

where $y(u) = (y_1(u), \dots, y_l(u))^\top$. Clearly, if $(x(u), y(u))$ solves (16)–(17), then $x(u)$ is a solution to (15).

We denote by $T(u)$ the set of all the solution points $-y(u)$ (with the negative sign). Treating $T(u)$ as values of the set-valued mapping T with $T(u) = \varnothing$ at each $u \notin \tilde{U}$, we can define the variational inequality (VI): Find a point $u^* \in U$ such that

$$\exists t^* \in T(u^*), \ \langle t^*, u - u^* \rangle \geq 0, \quad \forall u \in U. \tag{18}$$

Thus, we think that the *master VI* (18) yields the optimal shares of common constraints among players in the sense of problem (5),(7). We now substantiate the validity of VI (18).

Theorem 4.1 *If a point u^* solves VI (18), the corresponding solution $x(u^*)$ in (16)–(17) is a solution of EP (5),(7).*

Proof. Let a point u^* solve VI (18). Then there exists the corresponding vector $t^* \in T(u^*)$, such that u^* is a solution of the optimization problem:

$$\min \to \sum \langle t^*, u \rangle \quad \text{s.t.} \quad \sum_{i=1}^{l} u_i = b.$$

By duality, there exists a vector $w^* \in \mathbb{R}^m$ such that

$$t_i^* + w^* = \mathbf{0}, \quad i = 1, \dots, l, \tag{19}$$

$$\sum_{i=1}^{l} u_i^* = b. \tag{20}$$

It follows that $w^* \in \mathbb{R}^m_+$. By using (19)–(20) in (16)–(17) we see that there exists a point $x(u^*) \in X$ such that

$$\Phi(x(u^*), v) + \sum_{i=1}^{l} \langle w^*, h_i(v_i) - h_i(x_i(u^*)) \rangle \geq 0 \quad \forall v \in X,$$
$$\langle u_i^* - h_i(x_i(u^*)), y - w^* \rangle \geq 0, \quad \forall y \in \mathbb{R}^m_+, \ i = 1, \dots, l.$$

Summing the second series of relations over $i = 1, \dots, l$ and taking into account (20) gives (9)–(10). It follows that $x(u^*)$ is a solution of EP (5),(7). □

We conclude that solution of VI (18) based on the parametric NEPs (15) enables us to find the right share allocation values. The Lagrange multipliers $y_i(u)$ in (16)–(17) can be treated as validity estimates of the particular constraints $h_i(x_i) \leq u_i$, $i = 1, \dots, l$. Hence, the solution concept in (18) has a rather simple interpretation. A system regulator assigns share allocation values u_i for players, they determine the corresponding Nash equilibrium point together with their validity share constraint estimates. Due to (19)–(20), the share assignment becomes optimal and yields the constrained equilibrium solution if all the estimates coincide.

VI (18) admits a suitable re-formulation. Note that U is the shifted linear subspace

$$U_0 = \left\{ u \in \mathbb{R}^{ml} \;\middle|\; \sum_{i=1}^{l} u_i = \mathbf{0} \right\}.$$

For brevity, denote by $\pi(u)$ the projection of a point u onto U_0. It is given by the explicit formula

$$[\pi(u)]_i = u_i - (1/l) \sum_{s=1}^{l} u_s \quad \forall i = 1, \dots, l.$$

Then, VI (18) becomes equivalent to the following inclusion:

$$u^* \in \tilde{U}, \ \mathbf{0} \in \bar{T}(y^*), \tag{21}$$

where $\bar{T}(u) = \pi[T(u)]$.

5 Iterative Allocation Methods

In order to obtain a convergent iterative procedures based on solution of problems (18) or (21), we need additional properties of the set-valued mapping T. We recall that a set-valued mapping $Q : V \rightrightarrows E$ is said to be *monotone*, iff, for all $v, w \in V$ and for all $q' \in Q(v)$, $q'' \in Q(w)$, we have $\langle q' - q'', v - w \rangle \geq 0$.

Proposition 5.1 *Suppose the bi-function Φ is monotone. Then:*
(i) System (16)–(17) has convex and closed solution set for each $u \in \tilde{U}$;
(ii) The mapping T is monotone;
(iii) The mapping T has nonempty, convex, and closed values at each point $u \in \tilde{U}$.

Proof. First we note that summing all the relations in (16)–(17), we obtain the EP: Find $\bar{w} = (\bar{x}, \bar{y}) = (x(u), y(u)) \in X \times \mathbb{R}^{ml}_+$ such that

$$\Delta(\bar{w}, w) \geq 0 \quad \forall w = (v, x) \in X \times \mathbb{R}^{ml}_+,$$

with

$$\Delta(\bar{w}, w) = \Phi(\bar{x}, v) + \sum_{i=1}^{l} [\langle \bar{y}_i, h_i(v_i) - u_i \rangle + \langle y_i, u_i - h_i(\bar{x}_i) \rangle].$$

Since

$$\Delta(\bar{w}, w) + \Delta(w, \bar{w}) = \Phi(\bar{x}, v) + \Phi(v, \bar{x}) \leq 0,$$

we see that Δ is monotone. Besides, it is upper semicontinuous in the first variable, convex and lower semicontinuous in the second variable, and is an equilibrium bifunction. Such EPs must have a convex and closed solution set; see, e.g., [31]. Hence, (i) holds true.

To prove (ii), take arbitrary points $u', u'' \in \tilde{U}$. Then there exist the corresponding solutions (x', y') and (x'', y'') in (16)–(17). We can determine $-y' = t' \in T(u')$ and $-y'' = t'' \in T(u'')$, respectively. It follows from (16) that

$$\Phi(x', x'') + \sum_{i=1}^{l} \langle y'_i, h_i(x''_i) - h_i(x'_i) \rangle \geq 0,$$
$$\Phi(x'', x') + \sum_{i=1}^{l} \langle y''_i, h_i(x'_i) - h_i(x''_i) \rangle \geq 0;$$

hence

$$\sum_{i=1}^{l} \langle t''_i - t'_i, h_i(x''_i) - h_i(x'_i) \rangle = \sum_{i=1}^{l} \langle y''_i - y'_i, h_i(x'_i) - h_i(x''_i) \rangle$$
$$\geq -[\Phi(x', x'') + \Phi(x'', x')] \geq 0$$

since Φ is monotone. At the same time, (17) gives

$$\langle h_i(x'_i) - u'_i, y'_i - y''_i \rangle \geq 0 \ \text{ and } \langle h_i(x''_i) - u''_i, y''_i - y'_i \rangle \geq 0;$$

hence, by the above,

$$\sum_{i=1}^{l} \langle t''_i - t'_i, u''_i - u'_i \rangle = \sum_{i=1}^{l} \langle y'_i - y''_i, u''_i - u'_i \rangle$$
$$\geq \sum_{i=1}^{l} \langle t''_i - t'_i, h_i(x''_i) - h_i(x'_i) \rangle \geq 0.$$

This means that T is monotone and part (ii) is true. Part (iii) follows from (i). □

We also recall that a set-valued mapping $Q : V \rightrightarrows E$ is said to be *closed* on W, iff, for each pair of sequences $\{v^k\} \to v$, $\{q^k\} \to q$ such that $v^k \in W$ and $q^k \in Q(v^k)$, we have $q \in Q(v)$.

Suppose that either the set X is bounded or condition **(C)** holds, then for each $u \in \tilde{U}$ there exists a solution x^* of (14); see e.g. [28, Proposition 2]. Besides, all these solutions are uniformly bounded, which implies the closedness of T.

Proposition 5.2 *If either the set X is bounded or* **(C)** *is fulfilled, then the mapping T is closed.*

Being based on these results, we can make use of several iterative methods for finding a solution of VI (18). A specialization of the combined relaxation (CR) method for set-valued problems was suggested in [28]. Here, for the sake of simplicity, we consider the averaging method [37], despite its rather low convergence. Its "pure" version applied to inclusion (21) can be described as follows.

Averaging method. Choose a point $u^0 \in \tilde{U}$ and a positive sequence $\{\alpha_k\}$ such that

$$\sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty.$$

Set $z^0 = u^0$, $\beta_0 = \alpha_0$. At the k-th iteration, $k = 0, 1, \ldots$, set

$$\begin{array}{rcl} \beta_{k+1} & = & \beta_k + \alpha_{k+1}, \tau_{k+1} = \alpha_{k+1}/\beta_{k+1}; \\ u^{k+1} & = & u^k + \alpha_k t^k, t^k \in \bar{T}(u^k); \\ z^{k+1} & = & \tau_{k+1} u^{k+1} + (1 - \tau_{k+1}) z^k. \end{array}$$

This method will generate a convergent sequence since $\bar{T}$ is monotone. However, T may be empty at some points out of $\tilde{U}$. Then we should replace T with the subdifferential of the proper constraint function, see e.g. [28, Section 7].

The above solution process has also a rather simple interpretation. The system regulator assigns sequentially share allocation values u^k and report them to the players, with taking into account his/her previous decisions. For each current value, they determine the corresponding Nash equilibrium point and their validity share constraint estimates t^k. The system regulator receives the current values t^k and changes the share assignment, and so on. This procedure seems rather natural.

6 Regularization

In this section, we intend to consider a way to replace the set-valued problems (18) and (21) with their approximations having better properties. We follow [28], where a regularization approach was suggested for decomposable variational inequalities. In fact, the mapping T is set-valued, can have empty values out of $\tilde{U}$ and does not possesses strengthened monotonicity properties.

Choose a number $\varepsilon \geq 0$. Then, for each $u \in \mathbb{R}^{ml}$, there exists a unique pair $w^{\varepsilon}(u) = (x^{\varepsilon}(u), y^{\varepsilon}(u)) \in X \times \mathbb{R}^{ml}_{+}$ such that

$$\Phi(x^{\varepsilon}(u), v) + \varepsilon\langle x^{\varepsilon}, v - x^{\varepsilon}\rangle + \sum_{i=1}^{l}\langle y_i^{\varepsilon}(u), h_i(v_i) - h_i(x_i^{\varepsilon}(u))\rangle \geq 0 \ \forall v \in X, \quad (22)$$

$$\langle u_i - h_i(x_i^{\varepsilon}(u)) + \varepsilon y_i^{\varepsilon}(u), y_i - y_i^{\varepsilon}(u)\rangle \geq 0, \ \forall y_i \in \mathbb{R}^{m}_{+}, \ i = 1, \dots, l. \quad (23)$$

In fact, this system represents a regularization of problem (16)–(17), which corresponds to $\varepsilon = 0$. Due to Proposition 5.1 (ii), the cost bi-function in EP (22)–(23) is strongly monotone, and the assertion is true; see e.g. [31].

We can set $F^{\varepsilon}(u) = -y^{\varepsilon}(u)$ for $\varepsilon > 0$ and consider it as an approximation of $T(u)$ when $\varepsilon \approx 0$. First we conclude that the mapping F^{ε} is single-valued and defined throughout $\mathbb{R}^{ml}$.

System (22)–(23) admits a suitable re-formulation. Indeed, (23) is a complementarity problem, and we can write its solution explicitly as

$$y_i^{\varepsilon}(u) = (1/\varepsilon)\pi_{+}[h_i(x_i^{\varepsilon}(u)) - u_i], \quad \forall i = 1, \dots, l. \quad (24)$$

The corresponding substitution in (22) leads to the problem of finding $x^{\varepsilon}(u) \in X$ such that

$$\begin{array}{l} \Phi(x^{\varepsilon}(u), v) + \varepsilon\langle x^{\varepsilon}, v - x^{\varepsilon}\rangle \\ +(1/\varepsilon)\displaystyle\sum_{i=1}^{l}\langle \pi_{+}[h_i(x_i^{\varepsilon}(u)) - u_i], h_i(v_i) - h_i(x_i^{\varepsilon}(u))\rangle \geq 0 \ \forall v \in X. \end{array} \quad (25)$$

In turn, under the above assumptions problem (25) is equivalent to the following EP: Find $x^{\varepsilon}(u) \in X$ such that

$$\begin{array}{l} \Phi(x^{\varepsilon}(u), v) + (\varepsilon/2)\displaystyle\sum_{i=1}^{l}\left(\|v_i\|^2 - \|x_i^{\varepsilon}\|^2\right) \\ +(1/(2\varepsilon))\displaystyle\sum_{i=1}^{l}\left(\|\pi_{+}[h_i(v_i) - u_i]\|^2 - \|\pi_{+}[h_i(x_i^{\varepsilon}(u)) - u_i]\|^2\right) \geq 0 \ \forall v \in X. \end{array} \quad (26)$$

However, this is nothing but the auxiliary problem of the decomposable regularized penalty method applied to EP (14). Note that the second term in (26) can be dropped if Φ is strongly monotone, then (26) reduces to the auxiliary problem (12) of the decomposable penalty method with $\tau = 1/(2\varepsilon)$, $\sigma = 2$, and fixed u. Moreover, (26) is equivalent to the NEP: Find $x^{\varepsilon}(u) \in X$ such that

$$\tilde{f}_i^{\varepsilon}(x_{-i}^{\varepsilon}(u), v_i) \leq \tilde{f}_i^{\varepsilon}(x^{\varepsilon}(u)) \quad \forall v_i \in X_i, \ i = 1, \dots, l. \quad (27)$$

where the the i-th player has the utility function

$$\tilde{f}_i^{\varepsilon}(x) = f_i(x) - (\varepsilon/2)\|x_i\|^2 - (1/(2\varepsilon))\|\pi_{+}[h_i(x_i) - u_i]\|^2. \quad (28)$$

Therefore, we have proved the basic equivalence result.

Proposition 6.1 *System (22)–(23) is equivalent to EP (26) or to NEP (27)–(28), where $y^\varepsilon(u)$ can be then found from (24).*

This means that the calculation of $F^\varepsilon(u)$ also reduces to the NEP.

Proposition 6.2 *Suppose the bi-function Φ is monotone. Then, for any fixed $\varepsilon > 0$, F^ε is co-coercive with constant ε, i.e.*

$$\langle u' - u'', F^\varepsilon(u') - F^\varepsilon(u'')\rangle \geq \varepsilon \|F^\varepsilon(u') - F^\varepsilon(u'')\|^2$$

for all $u', u'' \in \mathbb{R}^{ml}$.

Proof. Take arbitrary points $u', u'' \in \mathbb{R}^{ml}$. Then there exist the corresponding unique solutions (x', y') and (x'', y'') of (22)–(23). It follows from (22) that

$$\Phi(x', x'') + \varepsilon\langle x', x'' - x'\rangle + \sum_{i=1}^{l} \langle y'_i, h_i(x''_i) - h_i(x'_i)\rangle \geq 0,$$
$$\Phi(x'', x') + \varepsilon\langle x'', x' - x''\rangle + \sum_{i=1}^{l} \langle y''_i, h_i(x'_i) - h_i(x''_i)\rangle \geq 0;$$

hence

$$\sum_{i=1}^{l} \langle y''_i - y'_i, h_i(x'_i) - h_i(x''_i)\rangle$$
$$\geq -[\Phi(x', x'') + \Phi(x'', x')] + \varepsilon\|x'' - x'\|^2 \geq \varepsilon\|x'' - x'\|^2$$

since Φ is monotone. At the same time, (23) gives

$$\langle h_i(x'_i) - u'_i - \varepsilon y', y'_i - y''_i\rangle \geq 0 \ \text{ and } \langle h_i(x''_i) - u''_i - \varepsilon y'', y''_i - y'_i\rangle \geq 0;$$

hence,

$$\langle h_i(x'_i) - h_i(x''_i), y'_i - y''_i\rangle \geq \langle u'_i - u''_i, y'_i - y''_i\rangle + \varepsilon\|u''_i - u'_i\|^2$$

for each $i = 1, \dots, l$. Adding these inequalities and combining with the above, we obtain

$$\sum_{i=1}^{l} \langle y'_i - y''_i, u''_i - u'_i\rangle \geq \varepsilon(\|x'' - x'\|^2 + \|u'' - u'\|^2)$$

therefore,

$$\langle u'' - u', F^\varepsilon(u'') - F^\varepsilon(u')\rangle \geq \varepsilon\|u'' - u'\|^2 = \varepsilon\|F^\varepsilon(u'') - F^\varepsilon(u')\|^2.$$

□

Remark 6.1 *From the proof of Proposition 6.2 it follows that the mapping F^ε remains co-coercive with constant ε even if we remove the regularization term in (22), i.e. in the case of the usual decomposable penalty method. However, we can not then guarantee the existence and uniqueness for $x^\varepsilon(u)$ at each point $u \in \mathbb{R}^{ml}$ unless the bi-function Φ is strongly monotone.*

Now we establish the basic approximation property.

Theorem 6.1 *Suppose that the bi-function Φ is monotone, the set $T(u)$ is non-empty at some point $u \in \mathbb{R}^{ml}$, and that we take any sequence $\{\varepsilon_k\} \searrow 0$. Then, for $z^k = (x^k, y^k)$, $x^k = x^{\varepsilon_k}(u)$, $y^k = y^{\varepsilon_k}(u)$ it holds that*

$$\lim_{k\to\infty} z^k = z_n^*, \tag{29}$$

where z_n^* is the minimal norm solution of system (16)–(17).

Proof. For brevity, denote by $Z^*(u)$ the solution set of system (16)–(17). Due to Proposition 5.1 (i), $Z^*(u)$ is convex and closed. By assumption, it is nonempty and z_n^* exists. Let $\bar{z} = (\bar{x}, \bar{y})$ be an arbitrary element of $Z^*(u)$. Then, by definition, we have

$$\begin{gathered} \Phi(\bar{x}, x^k) + \sum_{i=1}^{l} \langle \bar{y}_i, h_i(x_i^k) - h_i(\bar{x}_i) \rangle \geq 0, \\ \langle u_i - h_i(\bar{x}_i), y_i^k - \bar{y}_i \rangle \geq 0, \ i = 1, \dots, l; \end{gathered}$$

and

$$\begin{gathered} \Phi(x^k, \bar{x}) + \varepsilon_k \langle x^k, \bar{x} - x^k \rangle + \sum_{i=1}^{l} \langle y_i^k, h_i(\bar{x}_i) - h_i(x_i^k) \rangle \geq 0, \\ \langle u_i - h_i(x_i^k) + \varepsilon_k y_i^k, \bar{y}_i - y_i^k \rangle \geq 0, \ i = 1, \dots, l; \end{gathered}$$

Summing these inequalities and using the monotonicity of Φ gives

$$\varepsilon_k \langle z^k, \bar{z} - z^k \rangle \geq [\Phi(x^k, \bar{x}) + \Phi(\bar{x}, x^k)] + \varepsilon_k \langle z^k, \bar{z} - z^k \rangle \geq 0,$$

hence $\langle z^k, \bar{z} - z^k \rangle \geq 0$ and

$$\|z^k\| \leq \|z_n^*\|$$

since $\bar{z}$ was taken arbitrarily. It follows that the sequence $\{z^k\}$ is bounded and has limit points. Taking the limit $\{\varepsilon_k\} \searrow 0$ in (16)–(17) with $\varepsilon = \varepsilon_k$ gives that all these limit points points belong to $Z^*(u)$. However, due to the above inequality and the uniqueness of z_n^*, we obtain (29). □

Thus, we can approximate an element of the set $T(u)$ with values of $F^{\varepsilon}(u)$, which possesses better properties. We can make use of various iterative methods for finding a solution of the problem: Find a point $u^* \in U$ such that

$$\langle F^{\varepsilon}(u^*), u - u^* \rangle \geq 0, \quad \forall u \in U;$$

where $\varepsilon > 0$ is small enough. Clearly, this is an approximation of the master VI (18). It seems also better to consider the equation

$$u^* \in U, \; \bar{F}^{\varepsilon}(u^*) = \mathbf{0},$$

where $\bar{F}^{\varepsilon}(u) = \pi[F^{\varepsilon}(u)]$; cf. (21). A specialization of the other (CR) method was suggested in [28] for this problem. However, we can also apply the averaging method, described in Section 5. We only should now take $t^k = \bar{F}^{\varepsilon}(u^k)$ at each iteration. In such a way, we obtain an approximation of the initial constrained equilibrium solution.

7 Conclusions

We considered the solution concept problem for noncooperative games with joint binding constraints for players. In order to attain the constrained game equilibrium, certain concordant actions of all the players become necessary, but this contradicts to the basic information scheme of the usual noncooperative game, in particular to the independence of players. By using the share allocation method, we suggested to reduce the constrained game equilibrium problem to a sequence of pure Nash equilibrium problems, though with an additional upper control level for the shares assignment. The corresponding two-level iteration procedure indicates a dynamic process for attaining constrained game equilibrium points. We also showed that its regularization leads to a decomposable penalty method with better algorithmic properties.

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