DISCRETE APPROXIMATIONS OF A CONTROLLED SWEEPING PROCESS

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Abstract. The paper is devoted to the study of a new class of optimal control problems governed by the classical Moreau sweeping process with the new feature that the polyhedral moving set is not fixed while controlled by time-dependent functions. The dynamics of such problems is described by dissipative non-Lipschitzian differential inclusions with state constraints of equality and inequality types. It makes challenging and difficult their analysis and optimization. In this paper we establish some existence results for the sweeping process under consideration and develop the method of discrete approximations that allows us to strongly approximate, in the $W^{1,2}$ topology, optimal solutions of the continuous-type sweeping process by their discrete counterparts.

Key words: optimal control, sweeping process, moving controlled polyhedra, dissipative differential inclusions, discrete approximations, variational analysis.

AMS subject classifications: 49J52, 49J53, 49K24, 49M25, 90C30.

1 Introduction

In the 1970s, Jean Jacques Moreau introduced a class of mathematical models in mechanics named *sweeping process* ("processus du rafle" in French); see [18, 19, 20] and the book [14] for more details and references. Over the years, besides mechanical and other applications, the sweeping process theory has become an important area of nonlinear and variational analysis with serious mathematical achievements discussed in the recent surveys [2, 8, 11].

The sweeping process was introduced by Moreau in the form

$$\dot{x}(t) \in -N\big(x(t); C(t)\big) \quad \text{a.e.} \quad t \in [0, T] \tag{1.1}$$

via the negative normal cone to a moving closed and convex set $C(t)$ continuously depending

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on t. We can see that the right-hand side of the differential inclusion in (1.1) is given by a dissipative and discontinuous mapping of the time variable.

It has been well recognized in the sweeping process theory that the Cauchy problem for (1.1) with $x(0) = x_0$ admits a *unique* solution under natural assumptions on the moving set $C(t)$. This excludes considering optimization of (1.1) with the given initial point x_0 and the fixed set $C(t)$, which would place such a problem in the usual framework of optimal control of differential inclusions with fixed right-hand sides; see, e.g., [17, 22, 25].

In our first joint paper on this subject [6], a new viewpoint on the study and optimization of the sweeping process has been suggested, which essence consists of *controlling* the moving sets $C(t)$ by some time-dependent control actions in our possession. Those can be chosen in such a way that, by changing the shape of $C(t)$ and hence the right-hand side of (1.1), we are in a position to optimize a given cost functional to achieve the best sweeping performance.

This approach was partially examined in [6] in the case when the sweeping process was generated by a moving affine hyperplane whose normal direction and boundary were acting as control functions. The study in [6] was confined to considering cost functionals independent of time, control, and control velocities with imposing a rather restrictive assumption on the uniform Lipschitzian continuity of feasible controls, which ensures the possibility to truncate the original unbounded differential inclusion to a bounded one. We established in [6] the existence of optimal controls and analyzed the continuous-time optimization problem by using discrete approximations under the aforementioned assumption.

In this paper we investigate a significantly more adequate sweeping control model from both mathematical and mechanical viewpoints. Namely, we consider the optimal control problem (P) of minimizing the cost functional of the Bolza type

$$J[x,u,b]\colon = \varphi\big(x(T)\big) + \int_0^T \ell\big(t, x(t), u(t), b(t), \dot{x}(t), \dot{u}(t), \dot{b}(t)\big)\,dt \tag{1.2}$$

with an extended-real-valued terminal cost $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ and a running cost $\ell\colon [0,T] \times \mathbb{R}^{2(n+nm+m)} \to \overline{\mathbb{R}}$ over the controlled sweeping dynamics described by the following differential inclusion on the finite interval $[0,T]$ with the control and state constraints

$$\begin{cases} \dot{x}(t) \in -N\big(x(t); C(t)\big) \ \text{ a.e. } \ t \in [0,T], \quad x(0) := x_0 \in C(0) \\ \text{with } \ C(t) := \big\{x \in \mathbb{R}^n \big|\ \langle u_i(t), x\rangle \le b_i(t),\ i = 1, \ldots, m\big\} \ \text{ and} \\ \|u_i(t)\| = 1 \ \text{ for all } \ t \in [0,T],\ i = 1, \ldots, m, \end{cases} \tag{1.3}$$

where the controls functions $u(\cdot) = \big(u_1(\cdot), \ldots, u_m(\cdot)\big)$ and $b(\cdot) = \big(b_1(\cdot), \ldots, b_m(\cdot)\big)$ are merely absolutely continuous on $[0,T]$ without any uniform bounds, and where the absolutely continuous solutions $x(t)$ of (1.3) are understood in the standard sense of Carathéodory.

It is essential to observe that the state constraints on $x(t)$ are intrinsically presented in (1.3) due to the fact that $N(x; \Omega) = \emptyset$ when $x \in \Omega$ for any nonempty set Ω. Thus, denoting

$$C(u,b) := \big\{x \in \mathbb{R}^n \big|\ \langle u_i, x\rangle \le b_i,\ i = 1, \ldots, m\big\}, \tag{1.4}$$

we actually have in (1.3) the "hidden" constraint

$$x(t) \in C\big(u(t), b(t)\big) \ \text{ for all } \ t \in [0,T]. \tag{1.5}$$

Note also that, for each fixed $u = (u_1, \ldots, u_m) \in \mathbb{R}^{nm}$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$, the set $C(u, b)$ in (1.4) is a convex polyhedron. Polyhedral descriptions of moving sets in the (uncontrolled) sweeping process were explored and applied in the literature; see, e.g., [10, 12, 13] with interesting applications to models of elastoplasticity and hysteresis.

To the best of our knowledge, the class of optimal control problems of type (P) is new in optimal control theory and, in particular, for optimal control of differential inclusions; see more discussion in Section 3. It is worth mentioning that the recent paper [4], which studies a different class of optimal control problems with an equivalent variational inequality description of the sweeping process of the rate-independent hysteresis type, where the convex moving set is fixed while controls appear in an associated ordinary differential equation. Some of the results obtained in [4] have been further extended and numerically implemented in the forthcoming paper [1].

The main goal of this paper is to develop the *method of discrete approximations* in the line of [15] considered there for Lipschitzian differential inclusions; see also [17, 22] and the references therein. However, Lipschitzian assumptions and the like essentially used in the previous results are dramatically violated in the framework of (1.3). Thus implementing this method in the setting of (P) requires the development of a significantly more elaborated technique of discrete approximations different also from our first attempt in [6].

Prior to optimization in (1.2)–(1.3), we need to ensure that, given any feasible control pair $(u(t), b(t))$ of absolutely continuous functions, the Cauchy problem for the sweeping inclusion in (1.3) admits a solution $x(t)$ absolutely continuous on $[0, T]$. It does not follow from the classical existence results of the sweeping process theory since the moving set $C(t)$ in (1.3) is generally unbounded and may not be Lipschitz continuous or even absolutely continuous on $[0, T]$. Answering this question, we prove in Section 2, under natural and unrestrictive assumptions, the existence of absolutely continuous sweeping trajectories $x(t)$ on $[0, T]$ corresponding to any feasible control pair $(u(t), b(t))$. We also discuss therein the existence of Lipschitz continuous solutions of (1.3) under additional requirements on the initial data. The obtained existence results are of their own interest while lay the foundation for further developments in this paper concerning well-posedness and convergence of optimal solutions for discrete approximations. Note that we do not address here the existence of optimal solutions to (P). In fact, such an existence theorem can be established similarly to [6, Theorem 3.3] under the convexity of the running cost ℓ in (1.2) with respect to the velocity variables as well as the corresponding coercivity conditions. Note also that related existence results for various classes of evolution systems, including the sweeping process but with no controls appearing in the moving sets, can be found in [4, 21, 24].

The subsequent Sections 3 and 4 deal with discrete approximations of the sweeping control problem (P). In Section 3 we construct a discrete approximation of the controlled constrained sweeping inclusion in (1.3) formulated in an equivalent differential inclusion form and justify the strong (in the norm topology of $W^{1,2}$) approximation of any feasible trajectory of (1.3) by the corresponding feasible solutions to the discrete inclusions, which are piecewise linearly extended on the whole interval $[0, T]$.

Section 4 concerns discrete approximations of a given optimal solution to the sweeping optimal control problem (P). In fact, we consider not just a global solution to (P) but a local minimizer of the so-called intermediate type, which lies between weak and strong local minima in variational and control problems. Imposing a mild constraint qualification condition, which takes into account the specific structure of the polyhedral constraints in (1.5), we first justify the existence of optimal solutions to discrete approximations and then prove their $W^{1,2}$-strong convergence to the given local minimizer for (P).

In the concluding Section 5 we discuss some directions of our ongoing and future research. Besides certainly being of its own interest and providing approximate solutions (due to the established well-posedness and strong convergence) to the original sweeping control problem (P), the method of discrete approximations can be viewed as a driving force to derive necessary optimality conditions for continuous-time control problems. In the framework of (P), this requires deriving necessary optimality conditions for the (nonsmooth) discrete approximation problems and then passing there to the limit with the decreasing step of discretization. This is the subject of our current research project [7].

The notation of this paper is basically standard in variational analysis and optimal control; see, e.g., [17, 25]. Recall that $\mathbb{B}$ stands for the closed unit ball in the space in question and that the symbol "cone" signifies the conic hulls of a set.

2 Existence of Sweeping Trajectories

We start this section by observing that the unbounded polyhedral moving set $C(t)$ in (1.3) is not Hausdorff absolutely continuous (not even talking about its Lipschitz continuity) on $[0,T]$. This means that the classical sweeping theory (see, e.g. [8]) does not allow us to claim the existence of sweeping trajectories corresponding to feasible controls $(u(\cdot), b(\cdot))$ in (P). Nevertheless, in what follows we justify the required existence under some assumptions imposed on the general closed and convex moving set $C(t)$ and verify their validity in the polyhedral case under consideration.

Given a closed and convex set $C(t) \subset \mathbb{R}^n$ for $t \in [0,T]$, denote by $v(t) := \pi_{C(t)}(0)$ the unique *projection* of the origin onto $C(t)$ and consider the shifted set $K(t) := C(t) - v(t)$.

Theorem 2.1 (existence of absolutely continuous sweeping trajectories for general moving sets). *Let the projection $v\colon [0,T] \to \mathbb{R}^n$ be absolutely continuous on $[0,T]$. We assume that for every $r > 0$ and $\varepsilon > 0$ there is $\delta = \delta(r,\varepsilon) > 0$ such that*

$$\sum_{i=1}^{l} \max_{z \in K(\alpha_i) \cap r\mathbb{B}} \operatorname{dist}\big(z; K(\beta_i)\big) \le \varepsilon. \tag{2.1}$$

for every collection of mutually disjoint subintervals

$$\big\{[\alpha_i, \beta_i] \big|\, i = 1, \ldots, l\big\} \;\; \text{of} \;\; [0,T] \;\; \text{with} \;\; \sum_{i=1}^{l} |\beta_i - \alpha_i| \le \delta.$$

Then there exists an absolutely continuous solution of the Cauchy problem

$$\dot{x}(t) \in -N\big(x(t); C(t)\big), \quad x(0) = x_0 \in C(0). \tag{2.2}$$

Proof. For $m \in \mathbb{N}$ and $t_m^i := iT/m$ as $0 \le i \le m$ we set

$$v_m^i := \pi_{C(t_m^i)}(0) \text{ and } K_m^i := C(t_m^i) - v_m^i.$$

By the assumed absolute continuity of $v(\cdot)$, it is of bounded variation on $[0, T]$, i.e., there is $V > 0$ not depending on m and such that

$$\sum_{k=0}^{i-1} \|v_m^{k+1} - v_m^k\| \le V. \tag{2.3}$$

It follows from the construction above that $0 \in K_m^i$. Consider now the discretization algorithm of the *catching up type* defined by (cf. [19])

$$x_m^0 := x_0,\ x_m^i := \pi_{C(t_m^i)}(x_m^{i-1}), \text{ and } x_m(t) := x_m^i + \frac{m}{T}(t - t_m^{i-1})(x_m^i - x_m^{i-1}) \tag{2.4}$$

for $t \in [t_m^{i-1}, t_m^i)$ and denote $y_m^i := \pi_{\{K_m^i + v_m^{i-1}\}}(x_m^{i-1})$ for all $m \in \mathbb{N}$, $1 \le i \le m$. First we prove that there is a constant R depending only on x_0 and V so that

$$\|v_m^i - x_m^i\| \le R \text{ whenever } 1 \le i \le m. \tag{2.5}$$

To verify this, define inductively $\{z_{i,j}\}_{0 \le i \le m,\, 0 \le j \le i}$ by

$$z_{i,i} := v_m^i \text{ for } 1 \le i \le m \text{ and } z_{i,j} := \pi_{C(t_m^i)}(z_{i-1,j}) \text{ for } 0 \le j \le i-1.$$

Since the projection onto a convex set is Lipschitz continuous with modulus $L = 1$, we have

$$\|z_{i,j-1} - z_{i,j}\| \le \|v_m^{j-1} - v_m^j\| \text{ for all } 1 \le i \le m,\ 1 \le j \le i,$$

$$\|z_{i,0} - x_m^i\| \le \|v_m^0 - x_0\| \le \|v_m^0\| + \|x_0\| \le 2\|x_0\|$$

by taking into account that $x_0 \in C(0)$. This and (2.3) give us the estimate

$$\begin{aligned}
\|v_m^i - x_m^i\| = \|z_{i,i} - x_m^i\| &\le \sum_{k=0}^{i-1} \Big\| z_{i,k+1} - z_{i,k} \Big\| + \|z_{i,0} - x_m^i\| \\
&\le \sum_{k=0}^{i-1} \|v_m^{k+1} - v_m^k\| + 2\|x_0\| \\
&\le V + 2\|x_0\| =: R,
\end{aligned}$$

and so (2.5) is verified. Deduce further from the definitions of x_m^i and y_m^i that

$$\|x_m^i - y_m^i\| \le \|v_m^i - v_m^{i-1}\|, \quad 1 \le i \le m. \tag{2.6}$$

Moreover, we get from the above constructions the fulfillment of

$$\max_{x\in K_m^{i-1}\cap R\mathbb{B}} \operatorname{dist}\big(x;K_m^i\big)=\max_{x\in C(t_m^{i-1})\cap B(v_m^{i-1},R)} \operatorname{dist}(x;K_m^i+v_m^{i-1}). \tag{2.7}$$

It follows from $x_m^{i-1}\in C(t_m^{i-1})$ and $\|x_m^{i-1}-v_m^{i-1}\|\le R$ that $x_m^{i-1}\in C(t_m^{i-1})\cap B(v_m^{i-1},R)$ while from $y_m^i=\pi_{\{K_m^i+v_m^{i-1}\}}(x_m^{i-1})$ and (2.7) that

$$\|y_m^i-x_m^{i-1}\|\le \max_{x\in K_m^{i-1}\cap R\mathbb{B}} \operatorname{dist}\big(x;K_m^i\big),\quad 1\le i\le m. \tag{2.8}$$

Combining now (2.6) and (2.8) tells us that

$$\begin{aligned}\|x_m^i-x_m^{i-1}\| &\le \|x_m^i-y_m^i\|+\|y_m^i-x_m^{i-1}\|\\ &\le \|v_m^i-v_m^{i-1}\|+\max_{x\in K_m^{i-1}\cap R\mathbb{B}} \operatorname{dist}\big(x;K_m^i\big).\end{aligned} \tag{2.9}$$

Using (2.1), (2.9) and the absolute continuity of $v(\cdot)$ on $[0,T]$, we conclude due to [11, Theorem 1] that the functions $x_m(\cdot)$ from (2.4) converge along a subsequence to some function $x(\cdot)$ absolutely continuous on $[0,T]$, which is a solution of the Cauchy problem (2.2). This completes the proof of the theorem. △

The following consequence of Theorem 2.1 directly concerns the controlled sweeping process (1.3) under consideration in this paper.

Corollary 2.2 (existence of sweeping trajectories for moving polyhedra). *Let $C(t)$ be generated in* (1.3) *by absolutely continuous controls* $(u(\cdot),b(\cdot))$ *on* $[0,T]$. *Then the corresponding Cauchy problem in* (2.2) *admits an absolutely continuous solution on* $[0,T]$.

Proof. We need to check that all the assumptions of Theorem 2.1 hold in our polyhedral case. For simplicity let us do it for $m=1$ in (1.3). Then $v(t)=\max\{b(t),0\}u(t)$, and hence this projection is absolutely continuous on $[0,T]$. It remains to verify condition (2.1), which means in our polyhedral case that

$$\max_{z\in K(\alpha_i)\cap r\mathbb{B}} \operatorname{dist}\big(z;K(\beta_i)\big)\le r\|u_1(\beta_i)-u_1(\alpha_i)\|, \tag{2.10}$$

where $K(\alpha_i)=\{x\in\mathbb{R}^n|\ \langle u(\alpha_i),x\rangle\le 0\}$ and $K(\beta_i)=\{x\in\mathbb{R}^n|\ \langle u(\beta_i),x\rangle\le 0\}$. To proceed, take any $x\in r\mathbb{B}$ such that $x\in K(\alpha_i)$ but $x\notin K(\beta_i)$. We have $\|x\|\le r$, $\langle u(\alpha_i),x\rangle\le 0$, $\langle u(\beta_i),x\rangle>0$, and so

$$\begin{aligned}\operatorname{dist}\big(x;K(\beta_i)\big) =&\frac{|\langle u(\beta_i),x\rangle|}{\|u(\beta_i)\|}=\langle u(\beta_i),x\rangle\\ =&\langle u(\beta_i)-u(\alpha_i),x\rangle+\langle u(\alpha_i),x\rangle\\ \le&\langle u(\beta_i)-u(\alpha_i),x\rangle\le r\|u(\beta_i)-u(\alpha_i)\|,\end{aligned}$$

which verifies (2.10) and thus completes the proof of the corollary. △

Following the same procedure as in the proof of Theorem 2.1, we can establish the existence of Lipschitzian solutions to the Cauchy problem in (2.2) under additional assumptions on the initial data of the general sweeping process.

Proposition 2.3 (Lipschitzian sweeping trajectories). *In the notation of Theorem 2.1, let $v(\cdot)$ be Lipschitz continuous on $[0,T]$, and let for each $r>0$ there exist $L_r>0$ such that*

$$\max_{z\in K(t)\cap r\mathbb{B}} \operatorname{dist}\big(z;K(s)\big)\le L_r|t-s| \quad \textit{whenever} \quad t,s\in[0,T]. \tag{2.11}$$

Then the Cauchy problem in (2.2) *admits a solution that is Lipschitz continuous on* $[0,T]$.

Proof. It follows from (2.11) and the constructions above that there are constants $L_v>0$ and $L_R>0$ for which we have

$$\|v_m^i-v_m^{i-1}\|\le L_v\frac{T}{m} \quad \text{and} \quad \max_{x\in K_m^{i-1}\cap R\mathbb{B}} \operatorname{dist}\big(x;K_m^i\big)\le L_R\frac{T}{m}.$$

Thus estimate (2.9) that holds in this setting tells us that

$$\|x_m^i-x_m^{i-1}\|\le (L_v+L_R)T/m \quad \text{for all} \quad i=1,\ldots,m.$$

It implies that the functions $x_m(\cdot)$ from (2.4) are Lipschitz continuous on $[0,T]$ with the uniform Lipschitz constant L_v+L_R. Following the proof of Theorem 2.1, we conclude that the limiting function $x(\cdot)$, which is a solution to the Cauchy problem (2.2) is also Lipschitz continuous on $[0,T]$ with the the same Lipschitz constant. $\triangle$

The next proposition is of its own interest. Its partial version was essentially used in the discrete approximation method developed in [6].

Proposition 2.4 (truncation to bounded differential inclusions for uniformly Lipschitzian controls). *Suppose that feasible controls in* (1.3) *are uniformly Lipschitzian on* $[0,T]$ *with the moduli* L_u *and* L_b*, respectively, and with the fixed initial conditions. Then there exists a constant* $M>0$ *dependent only on* T, x_0, u_0, b_0, L_u, *and* L_b *such that the sweeping process in* (1.3) *is equivalent to the bounded differential inclusion*

$$-\dot{x}(t)\in N\big(x(t);C(t)\big)\cap M\mathbb{B} \quad \textit{a.e.} \quad t\in[0,T], \quad x(0)=x_0\in C(0).$$

Proof. To verify this statement, it suffices to show the Lipschitz constants L_v and L_R in the proof of Proposition 2.3 can be expressed via the given bounds L_u and L_b of the uniformly Lipschitzian controls. As in the proof of Corollary 2.2, we only consider the case of $m=1$. It follows from (2.10) the constant L_R can be chosen as $L_R=L_u$. Furthermore, we get from the proof of Corollary 2.2 that $v(t)=\max\{b(t),0\}u(t)$. Then the estimates

$$b(t)\le b(s)+L_b|t-s|\le \max\big\{b(s),0\big\}+L_b|t-s| \quad \text{for all} \quad t,s\in[0,T]$$

ensure that $\max\{b(t),0\}\le\max\{b(s),0\}+L_b|t-s|$. Similarly we get

$$\max\big\{b(s),0\big\}\le\max\big\{b(t),0\big\}+L_b|t-s|$$

and thus arrive at the uniform estimate

$$\big|\max\big\{b(t),0\big\}-\max\big\{b(s),0\big\}\big|\le L_b|t-s|, \quad t,s\in[0,T].$$

This tells us that the function $\max\{b(t), 0\}$ is Lipschitz continuous on $[0, T]$ with constant L_b. The function $v(t)$ defined above as the product of two Lipschitz continuous functions is also Lipschitz continuous on $[0, T]$ with modulus

$$L_v := \max_{t\in[0,T]} |b(t)|L_u + \max_{t\in[0,T]} |u(t)|L_b.$$

Observing finally the obvious estimates

$$\max_{t\in[0,T]} |b(t)| \leq |b_0| + L_b T \quad \text{and} \quad \max_{t\in[0,T]} |u(t)| \leq |u_0| + L_u T,$$

we arrive at the conclusion of this proposition. △

Note that such a truncation was first established by Thibault [23] for a sweeping process with an absolutely continuous moving set $C(t)$, which is not the case here. Actually in this paper, in contrast to [6], we prefer not to impose uniform Lipschitzian conditions on feasible controls due to the discretization procedure to approximate intermediate local minimizers of (P) developed in Section 4 under less restrictive assumptions. Furthermore, absolute continuity has been well recognized as a natural regularity of minimizers of integral functionals in variational problems including those considered in this paper.

3 Discrete Approximations of the Sweeping Inclusion

The main goal of this section is to construct a well-posed sequence of discrete approximations for the controlled sweeping inclusion (1.3) with the constraints on the control and state variables, but without considering so far the whole optimization problem (P). We first rewrite the control system (1.3) in the form of a differential inclusion with respect to a new variable unifying all the control and state variables in the original problem.

Given $x \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m) \in \mathbb{R}^{nm}$, and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$, consider the triple $z := (x, u, b)$ and define the set-valued mapping $F\colon \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ by

$$F(z) := -N\big(x; C(u, b)\big) \quad \text{with} \quad C(u, b) = \big\{x \in \mathbb{R}^n \big|\ \langle u_i, x\rangle \leq b_i,\ i = 1, \ldots, m\big\}. \tag{3.1}$$

Then the sweeping system in (1.3) can be rewritten as

$$\dot{z}(t) \in G\big(z(t)\big) := F\big(z(t)\big) \times \mathbb{R}^{nm} \times \mathbb{R}^m \quad \text{a.e.} \quad t \in [0, T] \tag{3.2}$$

with the initial condition $z(0) = (x_0, u(0), b(0))$ such that $\langle u_i(0), x_0\rangle \leq b_i(0)$ as $i = 1, \ldots, m$. It follows from Corollary 2.2 that the Cauchy problem for differential inclusion (3.2) admits a (feasible) solution in the class of absolutely continuous functions $z(t)$ on $[0, T]$.

Note that the resulting inclusion in (3.2) written in the conventional form of the theory of differential inclusions is highly irregular in the sense that its right-hand sides do not possess any Lipschitzian or even continuity properties. Furthermore, (3.2) implicitly contain the state constraints on $z(t) = (x(t), u(t), b(t))$ given by the bilinear inequalities in (1.5) together with the other part of the equality type

$$\|u_i(t)\| = 1 \quad \text{for all} \quad t \in [0, T],\ i = 1, \ldots, m.$$

The following theorem ensures the *strong approximation* (in the norm of $W^{1,2}[0,T]$) of any given feasible trajectory for the sweeping process in (3.2) by a sequence of feasible solutions to its finite-difference/discrete counterparts. It differs in several significant aspects and is more involved than the approximation result established in [6, Theorem 6.1] in the case of the halfspace moving control set $C(t)$. Furthermore, it is essentially different from all the previous developments in this direction obtained for compact-valued differential inclusions satisfying the Lipschitzian [15, 17, 22] and so-called modified one-sided Lipschitzian (MOSL) [9] properties, which both are dramatically violated here.

Theorem 3.1 **($W^{1,2}$-strong discrete approximation of sweeping trajectories).** *Fix an arbitrary feasible solution $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ to the control sweeping process in* (3.2) *with $x(\cdot) \in W^{2,\infty}([0,T];\mathbb{R}^n$ and $(\bar{u}(\cdot), b(\cdot)) \in W^{1,2}([0,T];\mathbb{R}^{nm+m})$. Consider a sequence of arbitrary discrete partitions of $[0,T]$ denoted by*

$$\Delta_k := \big\{0 = t_0^k < t_1^k < \ldots < t_k^k = T\big\} \ \text{ with } \ h_k \equiv t_{j+1}^k - t_j^k \downarrow 0 \tag{3.3}$$

as $k \to \infty$ and assume that the inclusion (3.2) *holds on Δ_k for every k. Then there is a sequence of piecewise linear functions $z^k(t) := (x^k(t), u^k(t), b^k(t))$ on $[0,T]$ with*

$$\big(x^k(0), u^k(0), b^k(0)\big) = \big(x_0, \bar{u}(0), \bar{b}(0)\big), \tag{3.4}$$

$$\|u_i^k(t_j^k)\| = 1 \ \text{ for } \ i = 1, \ldots, m \tag{3.5}$$

satisfying the discretized inclusions

$$x^k(t) = x^k(t_j^k) + (t - t_j^k)v_j^k, \quad t_j^k \le t \le t_{j+1}^k, \quad j = 0, \ldots, k-1, \tag{3.6}$$

with $v_j^k \in F(z^k(t_j^k))$ on Δ_k and such that

$$z^k(t) \to \bar{z}(t) \ \text{ uniformly on } \ [0,T] \ \text{ and } \ \int_0^T \|\dot{z}^k(t) - \dot{\bar{z}}(t)\|^2\, dt \to 0 \ \text{ as } \ k \to \infty. \tag{3.7}$$

Proof. Define $y^k(\cdot) := \big(y_1^k(\cdot), y_2^k(\cdot), y_3^k(\cdot)\big)$ to be piecewise linear on $[0,T]$ such that

$$\big(y_1^k(t_j^k), y_2^k(t_j^k), y_3^k(t_j^k)\big) := \big(\bar{x}(t_j^k), \bar{u}(t_j^k), \bar{b}(t_j^k)\big),$$

for all $j = 0, \ldots, k$. We also define $w^k := \dot{y}^k$ with $w^k = \big(w_1^k, w_2^k, w_3^k\big)$ and observe that

$$\begin{aligned}
&\big[\langle \bar{x}(t_j^k), \bar{u}_i(t_j^k)\rangle = \bar{b}_i(t_j^k)\big] \Longrightarrow \big[\langle y_1^k(t_j^k), y_{2i}^k(t_j^k)\rangle = y_{3i}^k(t_j^k)\big],\\
&\big[\langle \bar{x}(t_j^k), \bar{u}_i(t_j^k)\rangle < \bar{b}_i(t_j^k)\big] \Longrightarrow \big[\langle y_1^k(t_j^k), y_{2i}^k(t_j^k)\rangle < y_{3i}^k(t_j^k)\big]
\end{aligned}$$

on the mesh Δ_k for all $j = 1, \ldots, k-1$ and $i = 1, \ldots, m$. It follows by construction that

$$y^k(\cdot) := \big(y_1^k(\cdot), y_2^k(\cdot), y_3^k(\cdot)\big) \to \bar{z}(\cdot) \ \text{ uniformly on } \ [0,T],$$

$$w^k(\cdot) := \big(w_1^k(\cdot), w_2^k(\cdot), w_3^k(\cdot)\big) \to \dot{\bar{z}}(\cdot) \ \text{ in norm of } \ L^2([0,T];\mathbb{R}^{n+nm+m}).$$

Next define $u^k(t) := y_2^k(t)$ on $t \in [0,T]$ and get $u^k(0) = y_2^k(0) = \bar{u}(0)$. We can clearly choose $\|u_i^k(t)\| = 1$ on Δ_k for all $i = 1,\ldots,m$ and $k \in \mathbb{N}$ to satisfy constraints (3.5).

Fix further k, denote $t_j := t_j^k$ as $j = 1,\ldots,k-1$, and construct the claimed trajectories $x^k(t)$ of (3.6) as follows. First put $(x^k(0), b^k(0)) := (x_0, \bar{b}(0))$ and, proceeding by induction, suppose that the value of $x^k(t_j)$ is known. We need to construct the functions $x^k(\cdot)$ and $b^k(\cdot)$ at the remaining points of the interval $[t_j, t_{j+1}]$ as required by (3.6). To do it, define

$$b_i^k(t_j) := \langle x^k(t_j), u_i^k(t_j)\rangle \ \text{ if } \ y_{3i}^k(t_j) = \langle y_1^k(t_j), y_{2i}^k(t_j)\rangle$$

for every index $i = 1,\ldots,m$ and choose $b_i^k(t_j)$ satisfying

$$b_i^k(t_j) > \langle x^k(t_j), u_i^k(t_j)\rangle \ \text{ if } \ y_{3i}^k(t_j) > \langle y_1^k(t_j), y_{2i}^k(t_j)\rangle.$$

We can always obtain this selection in such a way that

$$b_i^k(t_j) - \bar{b}_i(t_j) = b_i^k(t_j) - y_{3i}^k(t_j) = \langle x^k(t_j), u_i^k(t_j)\rangle - \langle y_1^k(t_j), y_{2i}^k(t_j)\rangle = \langle x^k(t_j) - y_1^k(t_j), \bar{u}_i(t_j)\rangle.$$

Next take the unique projection v_j^k in the convex set from (3.1) as

$$v_j^k = \pi_{F(x^k(t_j),u^k(t_j),b^k(t_j))}(w_{1j}^k) \tag{3.8}$$

and define the functions $x^k(t)$ on the continuous-time interval $[t_j, t_{j+1}]$ by (3.6). Since $v_j^k \in F\big(x^k(t_j), u^k(t_j), b^k(t_j)\big)$, the discrete inclusion (3.6) is satisfied at all $t_j \in \Delta_k$ on the discrete mesh (3.3). Furthermore, due to the normal cone structure of the sets $F(z)$ in (3.1), it follows from (3.8) that the constructed discrete process $z^k(t_j) = (x^k(t_j), u^k(t_j), b^k(t_j))$ satisfies the state constraints generated by the sets $C(u,b)$ in (3.1):

$$x^k(t_j) \in C\big(u^k(t_j), b^k(t_j)\big) \ \text{ for all } \ t_j \in \Delta_k.$$

It is clear from the above constructions that the initial condition (3.4) also holds for the discrete trajectory $z^k(\cdot)$. Thus it remains to justify the claimed $W^{1,2}$-convergence (3.7) of the extended discrete trajectories $z^k(t)$ on $[0,T]$ using the additional assumption $\bar{x}(\cdot) \in W^{2,\infty}([0,T];\mathbb{R}^n$ on the given feasible trajectory $\bar{z}(\cdot)$. In preparation to this, observe that

$$\Big\|\frac{x^k(t_{j+1}) - x^k(t_j)}{t_{j+1} - t_j} - \frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j}\Big\| = \Big\|v_j^k - \frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j}\Big\| = \|v_j^k - w_{1j}^k\|.$$

The choice of $b^k(t_j)$ and the fact that $u^k(t_j) = \bar{u}(t_j)$ imply that $F\big(x^k(t_j), u^k(t_j), b^k(t_j)\big) = F(\bar{x}(t_j), \bar{u}(t_j), \bar{b}(t_j))$ and therefore $v_j^k = \pi_{F(\bar{x}(t_j),\bar{u}(t_j),\bar{b}(t_j))}(w_{1j}^k)$. Hence the condition

$$\begin{aligned}\sum_{j=0}^{k-1} \|v_j^k - w_{1j}^k\| &= \sum_{j=0}^{k-1} \text{dist}\Big(\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j}; F\big(\bar{x}(t_j), \bar{u}(t_j), \bar{b}(t_j)\big)\Big)\\ &\le \sum_{j=0}^{k-1} \Big\|\frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{t_{j+1} - t_j} - \dot{\bar{x}}(t_j)\Big\|^2 \le M\end{aligned} \tag{3.9}$$

holds for a suitable constant M, which exists due to $\bar{x}(\cdot)\in W^{2,\infty}([0,T];\mathbb{R}^n$. Furthermore, it follows from the construction above that for all $j=0,\ldots,k-1$ we have the relationships

$$\begin{aligned}
&\Big|\frac{b_i^k(t_{j+1})-b_i^k(t_j)}{t_{j+1}-t_j}-\frac{\bar{b}_i(t_{j+1})-\bar{b}_i(t_j)}{t_{j+1}-t_j}\Big|=\Big|\frac{b_i^k(t_{j+1})-\bar{b}(t_{j+1})}{t_{j+1}-t_j}-\frac{b_i^k(t_j)-\bar{b}(t_j)}{t_{j+1}-t_j}\Big|\\
&=\Big|\Big\langle\frac{x^k(t_{j+1})-\bar{x}(t_{j+1})}{t_{j+1}-t_j},\bar{u}_i(t_{j+1})\Big\rangle-\Big\langle\frac{x^k(t_j)-\bar{x}(t_j)}{t_{j+1}-t_j},\bar{u}_i(t_j)\Big\rangle\Big|\\
&=\Big|\Big\langle\frac{x^k(t_{j+1})-\bar{x}(t_{j+1})}{t_{j+1}-t_j}-\frac{x^k(t_j)-\bar{x}(t_j)}{t_{j+1}-t_j},\bar{u}_i(t_{j+1})\Big\rangle-\Big\langle\frac{x^k(t_j)-\bar{x}(t_j)}{t_{j+1}-t_j},\bar{u}_i(t_{j+1})-\bar{u}_i(t_j)\Big\rangle\Big|\\
&\le\Big\|\frac{x^k(t_{j+1})-x^k(t_j)}{t_{j+1}-t_j}-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{t_{j+1}-t_j}\Big\|+\Big\|\frac{x^k(t_j)-\bar{x}(t_j)}{t_{j+1}-t_j}\Big\|\cdot\|\bar{u}_i(t_{j+1})-\bar{u}_i(t_j)\|.
\end{aligned}\tag{3.10}$$

Now we are ready to justify the $W^{1,2}$-strong convergence in (3.7). Observe that we have to prove only the claimed convergence of x^k and b^k, since the corresponding convergence of u^k comes automatically from the above constructions. It follows from (3.9) that

$$\int_0^T\|\dot{x}^k(t)-w_1^k(t)\|^2dt=\sum_{j=0}^{k-1}(t_{j+1}-t_j)\|v_{1j}^k-w_j^k\|^2\le\sum_{j=0}^{k-1}(t_{j+1}^k-t_j^k)\Big\|\frac{\bar{x}(t_{j+1}^k)-\bar{x}(t_j^k)}{t_{j+1}^k-t_j^k}-\dot{\bar{x}}(t_j^k)\Big\|^2\downarrow 0,$$

which gives us the L^2-convergence of $\dot{x}^k$ to $\dot{\bar{x}}$. For the case of b^k we get from (3.10) that

$$\int_0^T\|\dot{b}^k(t)-w_3^k(t)\|^2dt\le 2\int_0^T\|\dot{x}^k(t)-w_1^k(t)\|^2dt+2M^2\sum_{j=0}^{k-1}(t_{j+1}-t_j)\|\bar{u}_i(t_{j+1})-\bar{u}_i(t_j)\|^2$$

with a possibly different constant M. This estimate and the absolute continuity of $\bar{u}(\cdot)$ on $[0,T]$ ensure the L^2-convergence of $\dot{b}^k$ to $\dot{\bar{b}}$. Taking into account that the initial conditions for $u^k(0)$ and $b^k(0)$ are fixed in (3.4), we complete the proof of the theorem. △

4 Strong Convergence of Discrete Optimal Solutions

In this section we construct a sequence of discrete approximations for the whole optimal control problem (P) of minimizing the cost functional (1.2) over the constrained sweeping system (1.3). Our goal is to build a sequence of discrete optimization problems (P_k), which admit optimal solutions such that their piecewise linear extensions on $[0,T]$ converge in the $W^{1,2}$ norm topology to the given local optimal solution of the original problem (P). Local minima for (P) are understood in the following sense of "intermediate local minimizers" (i.l.m.) introduced by Mordukhovich [15] in the theory of differential inclusions. This notion obviously covers strong local minimizers (corresponding to $\alpha=0$ in Definition 4.1) and in general occupies an intermediate position between weak and strong minimizers in dynamic optimization; see [15] and [17, Chapter 6] for more details and references.

Definition 4.1 (intermediate local minimizers). *We say that a feasible solution $\bar{z}(\cdot)=\big(\bar{x}(\cdot),\bar{u}(\cdot),\bar{b}(\cdot)\big)\in W^{1,2}([0,T];\mathbb{R}^{n+nm+m})$ to (P) is an* INTERMEDIATE LOCAL MINIMIZER *for this problem if there are numbers $\alpha \geq 0$ and $\varepsilon > 0$ such that $J[\bar{z}] \leq J[z]$ for any feasible solution $z(\cdot) = \big(x(\cdot), u(\cdot), b(\cdot)\big)$ to (P) with*

$$\big\|\big(x(t), u(t), b(t)\big) - \big(\bar{x}(t), \bar{u}(t), \bar{b}(t)\big)\big\| < \varepsilon \quad as \quad t \in [0, T] \quad and$$

$$\alpha \int_0^T \Big(\Big\|\dot{x}(t) - \dot{\bar{x}}(t)\Big\|^2 + \Big\|\dot{u}(t) - \dot{\bar{u}}(t)\Big\|^2 + \Big\|\dot{b}(t) - \dot{\bar{b}}(t)\Big\|^2\Big)\, dt < \varepsilon. \tag{4.1}$$

It is clear that the general setting of $\alpha \geq 0$ in (4.1) reduces to the cases when either $\alpha = 1$ or $\alpha = 0$. Without loss of generality we consider the case of $\alpha = 1$ in what follows.

Now we construct discrete approximation problems (P_k), $k \in \mathbb{N}$, involving the given i.l.m. $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ for the original problem (P). Let Δ_k be the discrete mesh (3.3) with discretization step $h_k \downarrow 0$ as $k \to \infty$. For each $k \in \mathbb{N}$ the problem (P_k) is defined by:

$$\text{minimize} \quad J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\Big(t_j^k, x_j^k, u_j^k, b_j^k, \frac{x_{j+1}^k - x_j^k}{h_k}, \frac{u_{j+1}^k - u_j^k}{h_k}, \frac{b_{j+1}^k - b_j^k}{h_k}\Big)$$

$$+ \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big(\Big\|\frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t)\Big\|^2 + \Big\|\frac{u_{j+1}^k - u_j^k}{h_k} - \dot{\bar{u}}(t)\Big\|^2 + \Big\|\frac{b_{j+1}^k - b_j^k}{h_k} - \dot{\bar{b}}(t)\Big\|^2\Big)\, dt$$

over elements $z^k := (x_0^k, \ldots, x_k^k, u_0^k, \ldots, u_k^k, b_0^k, \ldots, b_k^k)$ with $u_j^k \in \mathbb{R}^{nm}$, $u_j^k := (u_{j1}^k, \ldots, u_{jm}^k)$ for every $j = 0, \ldots, k$ subject to the constraints

$$x_{j+1}^k \in x_j^k + h_k F(x_j^k, u_j^k, b_j^k) \quad \text{for} \quad j = 0, \ldots, k-1 \quad \text{with} \quad x_0^k = x_0, \tag{4.2}$$

$$\langle u_{ki}^k, x_k^k \rangle \leq b_{ki}^k \quad \text{for} \quad i = 1, \ldots, m, \tag{4.3}$$

$$\big\|u_{ji}^k\big\| = 1 \quad \text{for} \quad j = 0, \ldots, k-1 \quad \text{and} \quad i = 1, \ldots, m, \tag{4.4}$$

$$\big\|(x_j^k, u_k^k, b_j^k) - \big(\bar{x}(t_j^k), \bar{u}(t_j^k), \bar{b}(t_j^k)\big)\big\| \leq \varepsilon/2 \quad \text{for} \quad j = 0, \ldots, k, \tag{4.5}$$

$$\sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big(\Big\|\frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t)\Big\|^2 + \Big\|\frac{u_{j+1}^k - u_j^k}{h_k} - \dot{\bar{u}}(t)\Big\|^2 + \Big\|\frac{b_{j+1}^k - b_j^k}{h_k} - \dot{\bar{b}}(t)\Big\|^2\Big)\, dt \leq \frac{\varepsilon}{2}, \tag{4.6}$$

where $\varepsilon > 0$ is from Definition 4.1 with $\alpha = 1$. Note that the index j plays a role of the discrete time in (P_k) and that inclusions (4.2) correspond to (3.6) on Δ_k.

To proceed further, we have to make sure that problems (P_k) admit optimal solutions. This is done below under an additional qualification condition imposed on the given i.l.m. $\bar{z}(\cdot)$. Consider the set of active constraint indices

$$I(x, u, b) := \big\{i \in \{1, \ldots, m\} \big|\ \langle u_i, x \rangle = b_i\big\} \tag{4.7}$$

associated with the constraint system $C(u, b)$ in (3.1) and recall that the positive linear independence of vectors is defined similarly to their standard linear independence but involving only nonnegative coefficients in linear combinations.

Definition 4.2 (constraint qualification). *The* POSITIVE LINEAR INDEPENDENCE CONSTRAINT QUALIFICATION (PLICQ) *holds for the intermediate local minimizer $\bar{z}(\cdot)$ of (P) if the vectors $\{\bar{u}_i(t)|\ i \in I(\bar{x}(t), \bar{u}(t), \bar{b}(t))\}$ are positively linearly independent for all $t \in [0, T]$.*

An important feature of the introduced qualification condition used in what follows is its robustness with respect to perturbations described in the next proposition.

Proposition 4.3 (robustness of PLICQ). *If the PLICQ condition holds at $\bar{z}(\cdot)$, then there is a number $\eta > 0$ such that whenever $(x, u, b) \in \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^m$ with*

$$\|(x, u, b) - (\bar{x}(t), \bar{u}(t), \bar{b}(t)\| < \eta \ \text{ for some } \ t \in \Delta := \{\Delta_k |\ k \in \mathbb{N}\}$$

we have that the vectors $\{u_i \in \mathbb{R}^n |\ i \in I(x, u, b)\}$ are positively linearly independent.

Proof. By the classical Gordan theorem (see, e.g., [5, Theorem 2.4.3]), the PLICQ condition at (x, u, b) is equivalent to the existence of some direction $d \in \mathbb{R}^n$ for which

$$\langle u_i, d \rangle < 0 \ \text{ for all } \ i \in I(x, u, b). \tag{4.8}$$

If the triple (x', u', b') is close to (x, u, b), we still have the strict inequality in (4.8) with the replacement of u by u' therein. Since $I(x', u', b') \subset I(x, u, b)$, this yields the validity of PLICQ at (x', u', b') and thus ensures the claimed robustness. △

Next we establish the existence of optimal solutions to each discrete problem (P_k) whenever $k \in \mathbb{N}$ is sufficiently large. Observe that the existence of optimal solutions immediately follows from the classical Weierstrass existence theorem in the case of discrete approximations of uniformly bounded differential inclusions. However, it is not the case for problems (P_k) associated with the intrinsically unbounded sweeping control problem (P). We prove this result under the PLICQ assumption on $\bar{z}(\cdot)$ by using Attouch's theorem of the subdifferential convergence and the extremal principle of variational analysis. To the best of our knowledge, the usage of constraint qualifications in establishing the existence of optimal solutions (not for optimality conditions) is new in optimization theory.

We suppose in what follows that $\varepsilon \leq 2\eta$ in the construction of problems (P_k), where the number $\eta > 0$ is taken from Proposition 4.3 for the i.l.m. under consideration.

Theorem 4.4 (existence of discrete optimal solutions). *Let the cost functions φ and $\ell(t, \cdot, \cdot, \cdot, \cdot, \cdot)$ for all $t \in [0, T]$ be lower semicontinuous around the given i.l.m. $\bar{z}(\cdot)$ satisfying the PLICQ condition. Then for all $k \in \mathbb{N}$ sufficiently large there exist optimal solutions to the discrete problems (P_k).*

Proof. It follows from Theorem 3.1 that the set of feasible solutions z^k to (P_k) is nonempty for large k. Furthermore, the constraint (4.5) ensures that it is bounded. By the Weierstrass existence theorem it remains to show that this set is closed. To proceed, take a sequence $z_\nu = (x_0^\nu, \ldots, x_k^\nu, u_0^\nu, \ldots, u_k^\nu, b_0^\nu, \ldots, b_k^\nu)$ that converges to some vector $z = (x_0, \ldots, x_k, u_0, \ldots, u_k, b_0, \ldots, b_k)$ as $\nu \to \infty$. We need to verify that z is feasible to (P_k)

if all z_ν have this property. It only requires checking that the components of the limiting vector z satisfy inclusions (4.2) for all $j = 0, \ldots, k-1$. It follows from the construction of (P_k) and the choice of $\varepsilon \le 2\eta$ that the limiting vector z belongs to the η-neighborhood of the i.l.m. $\bar{z}(\cdot)$ from Proposition 4.3, and so we can freely use the PLICQ condition at z.

Taking into account the normal cone (i.e., the subdifferential of the indicator function) structure of the right-hand side of (4.2), we can apply Attouch's theorem on the graphical convergence of the subdifferentials from the epigraphical convergence of extended-real-valued lower semicontinuous convex functions; see, e.g., [3, Theorem 12.35]. It follows now from the composite form of the indicator function to the moving sets $C(u, b)$ in (3.1) and the robustness property of PLICQ from Proposition 4.3 that the claimed closedness of the feasible set to each (P_k) with large k is a consequence of the following statement:

Let $(x_\nu, u_\nu, b_\nu) \to (\bar{x}, \bar{u}, \bar{b})$ *as* $\nu \to \infty$, *and let*

$$x_\nu \in C_\nu := \big\{x \in \mathbb{R}^n \big|\ \langle u_{\nu i}, x\rangle \le b_{\nu i},\ i = 1, \ldots, m\big\} \ \text{ for all }\ \nu \in \mathbb{N}.$$

Assume that the vectors $\{\bar{u}_i |\ i \in I(x, \bar{u}, \bar{b})\}$ *are positively linearly independent on the set*

$$C := \big\{x \in \mathbb{R}^n \big|\ \langle \bar{u}_i, x\rangle \le \bar{b}_i,\ i = 1, \ldots, m\big\}.$$

Then the sets C_ν *graphically converge to the set* C *as* $\nu \to \infty$.

To justify this statement, note first that the convergence $(u_\nu, b_\nu) \to (\bar{u}, \bar{b})$ and $x_\nu \to \bar{x}$ with $x_\nu \in C_\nu$ ensures that $\bar{x} \in C$. It remains to show that for every $\widetilde{x} \in C$ there is a sequence $x_\nu \in C_\nu$ with $x_\nu \to \widetilde{x}$ as $\nu \to \infty$. Picking such a vector, define the closed sets

$$\Omega_i := \big\{x \big|\ \langle \bar{u}_i, x\rangle \le \bar{b}_i\big\}, \quad i = 1, \ldots, m,$$

and observe that $\widetilde{x} \in C = \bigcap_{i=1}^m \Omega_i$. Let us verify that, under the positive linear independence assumption, $\widetilde{x}$ is not a locally extremal point of the set system $\{\Omega_1, \ldots, \Omega_m\}$ in the sense of [16, Definition 2.1]. This means that for every neighborhood U of x_0 and for any sequences $a_{i\nu} \to 0$ from $\mathbb{R}^n$ as $\nu \to \infty$ and $i = 1, \ldots, m$ there is a subsequence $\{a_{i\nu_k}\}_{k\in\mathbb{N}}$ with

$$\bigcap_{i=1}^m \Big(\Omega_i - a_{i\nu_k}\Big) \cap U \ne \emptyset \ \text{ for all }\ k \in \mathbb{N}. \tag{4.9}$$

Assuming the contrary, we employ the extremal principle from [16, Theorem 2.8] and find normals $x_i^* \in N(x_0, \Omega_i)$ for $i = 1, \ldots, m$ such that

$$x_1^* + \ldots + x_m^* = 0 \ \text{ and }\ \|x_1^*\| + \ldots + \|x_m^*\| = 1. \tag{4.10}$$

Exploiting the explicit structure of the normal cones

$$N(\widetilde{x}, \Omega_i) = \begin{cases} \big\{\lambda \bar{u}_i \big|\ \lambda \ge 0\big\} \ \text{ if }\ \langle \widetilde{x}, \bar{u}_i\rangle = \bar{b}_i, \\ \{0\} \text{ otherwise}, \quad i = 1, \ldots, m, \end{cases}$$

to the sets Ω_i at $\widetilde{x}$ ensures the existence of nonnegative numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\sum_{i \in I(\widetilde{x}, \bar{u}, \bar{b})} \lambda_i \bar{u}_i = 0 \quad \text{and} \quad \sum_{i \in I(\widetilde{x}, \bar{u}, \bar{b})} \lambda_i = 1.$$

Thus the vectors $\{u_i |\; i \in I(\widetilde{x}, \bar{u}, \bar{b})\}$ are positively linearly dependent, which contradicts the PLICQ and shows that $\widetilde{x}$ is not a locally extremal point of the set system $\{\Omega_1, \ldots, \Omega_m\}$.

Using this, for every $i = 1, \ldots, m$ we choose $a_{i\nu}$ so that

$$\big(\Omega_i \cap U\big) - a_{i\nu} \subset \Omega_{i\nu} := \big\{x \in \mathbb{R}^n \big|\; \langle x, u_{i\nu} \rangle \le b_{i\nu}\big\}. \tag{4.11}$$

It follows from the convergence $(u_{i\nu}, b_{i\nu}) \to (\bar{u}_i, \bar{b}_i)$ that $a_{i\nu} \to 0$ as $\nu \to \infty$. Combining (4.9) and (4.11) gives us a sequence of vectors

$$x_\nu \in C_\nu := \big\{x \big|\; \langle x, u_{i\nu} \rangle \le b_{i\nu},\; i = 1, \ldots, m\big\} = \bigcap_{i=1}^{m} \Omega_{i\nu},$$

which converges to $\widetilde{x}$. This completes the proof of the theorem. △

Now let us construct an example showing that the sets of feasible solutions to (P_k) may *not be closed* without the PLICQ condition of Theorem 4.4.

Example 4.5 (PLICQ is essential for closedness). It is sufficient to show that we do not have the closedness of the normal cone mapping to the moving sets $C(u, b)$ if the PLICQ condition fails. Based on this example, it is not hard to observe that optimal solutions to problems (P_k) may not exist without imposing the PLICQ assumption.

Take $n = m = 2$ in problem (P_k) for each fixed $k \in \mathbb{N}$ and consider the sequences

$$u_{1\nu} := \Big(\frac{-\nu}{\sqrt{\nu^2+1}}, \frac{1}{\sqrt{\nu^2+1}}\Big),\; u_{2\nu} := \Big(\frac{\nu}{\sqrt{\nu^2+1}}, \frac{1}{\sqrt{\nu^2+1}}\Big),$$

and $b_{1\nu} = b_{2\nu} := \frac{1}{\sqrt{\nu^2+1}}$ for every $\nu \in \mathbb{N}$. Defining the sets

$$C_\nu := \big\{x \in \mathbb{R}^2 \big|\; \langle u_{i\nu}, x \rangle \le b_{i\nu},\; i = 1, 2\big\},$$

we get that $(0,1) \in C_\nu$ and $(0,1) \in N((0,1); C_\nu)$ for all $\nu \in \mathbb{N}$. Then

$$u_1 = \lim_{\nu\, to \infty} u_{1\nu} = (-1, 0),\; u_2 := \lim_{\nu \to \infty} u_{2\nu} = (1, 0),\; b_i = \lim_{\nu \to \infty} b_{i\nu} = 0 \text{ as } \; i = 1, 2.$$

Thus it gives us the limiting set and the normal cone to it as follows

$$C := \big\{x \in \mathbb{R}^2 \big|\; \langle u_i, x \rangle \le b_i,\; i = 1, 2\big\} = \big\{(0, r) \big|\; r \in \mathbb{R}\big\} \;\text{ and }\; N\big((0,1); C\big) = \big\{(r, 0) \big|\; r \in \mathbb{R}\big\}.$$

Therefore we have $(0,1) \notin N((0,1); C)$, which shows the failure of the closedness property for normals to the moving convex sets under consideration. It is easy to see that the limiting control vectors u_1 and u_2 are positively linearly dependent and that the sets C_ν do not converge to C graphically.

To establish the final result of this section, we need one more definition concerning local minimizers under consideration. Along with the original problem (P) we consider its relaxation constructed as follows. Denote by $\ell_F(t,x,u,b,v,w.\nu)$ the *convexification* of the integrand in (1.2) on the set $F(x,u,b)$ from (3.1) with respect to the *velocity* variables (v,w,ν) for all (t,x,u,b), i.e., the largest convex and l.s.c. function majorized by $\ell(t,x,u,b,\cdot,\cdot,\cdot)$ on this set. Define the *relaxed sweeping problem* (R) by

$$\text{minimize } \widehat{J}[z] := \varphi\big(x(T)\big) + \int_0^T \widehat{\ell}_F\big(t,x(t),u(t),b(t),\dot{x}(t),\dot{u}(t),\dot{b}(t)\big)\,dt \tag{4.12}$$

over the triples $z(t) = (x(t),u(t),b(t))$ with absolutely continuous trajectories $x(t)$ controls $u(t),b(t)$ on $[0,T]$ satisfying the constraints in (1.3). Of course, there is no difference between problems (P) and (R) if the integrand ℓ is convex and l.s.c. with respect to the velocity variables (v,w,ν). In the nonconvex case we say that $\bar{z}(\cdot)$ is a *relaxed intermediate local minimizer* (r.i.l.m.) for (P) if it is an i.l.m. for this problem and $J[\bar{z}] = \widehat{J}[\bar{z}]$. It obviously implies that $\bar{z}(\cdot)$ is an i.l.m. for the relaxed problem (R).

It is worth mentioning that the above r.i.l.m. notion can be treated as an intermediate local counterpart of the *relaxation stability* of (P) meaning that $J_P = \widehat{J}_R$ for the optimal values of the cost functionals in (P) and (R), respectively. It has been well recognized in control theory that rather general nonconvex continuous-time systems enjoy a certain "hidden convexity" (Bogolyubov-type and Lyapunov theorems) that ensure, in particular, relaxation stability in minimizing *nonconvex* integral functionals over trajectories of differential inclusions satisfying the aforementioned Lipschitzian or MOSL conditions; see [9, 15, 17] for precise results and discussions. As we know, such conditions do not hold for the heavily non-Lipschitzian sweeping inclusion (1.1). Thus it is still an open question about the validity of the relaxation stability property for the sweeping control problem (P) with nonconvex integrands in velocities.

The next theorem makes a bridge between optimal solutions to the continuous-time and discrete-time sweeping control problems under consideration.

Theorem 4.6 ($W^{1,2}$-strong convergence of discrete optimal solutions). *Let $\bar{z}(\cdot) = (\bar{x}(\cdot),\bar{u}(\cdot),\bar{b}(\cdot))$ be an r.i.l.m. for problem (P). In addition to the assumptions of Theorem 4.4, suppose that both terminal and running costs in (1.2) are continuous at $\bar{x}(T)$ and at $(\bar{z}(t),\dot{\bar{z}}(t))$ for a.e. $t \in [0,T]$, respectively. Then any sequence of piecewise linearly extended optimal solutions $\bar{z}^k(t) = (\bar{x}^k(t),\bar{u}^k(t),\bar{b}^k(t))$, $t \in T$, of the discrete problems (P_k) converges to $\bar{z}(t)$ in the norm topology of $W^{1,2}([0,T];\mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^m)$.*

Proof. The claimed convergence easily follows from

$$\lim_{k\to\infty} \int_0^T \left(\left\|\dot{\bar{x}}^k(t) - \dot{\bar{x}}(t)\right\|^2 + \left\|\dot{\bar{u}}^k(t) - \dot{\bar{u}}(t)\right\|^2 + \left\|\dot{\bar{b}}^k(t) - \dot{\bar{b}}(t)\right\|^2\right) dt = 0. \tag{4.13}$$

Suppose that (4.13) does not hold, i.e., the limit along a subsequence therein (no relabeling) equals to some $\gamma > 0$. By the weak compactness of the unit ball in $L^2 := L^2([0,T];\mathbb{R}^n \times$

$\mathbb{R}^{nm} \times \mathbb{R}^m)$ we find a triple $(v(\cdot), w(\cdot), \nu(\cdot)) \in L^2([0,T];\mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^m)$ and yet another subsequence of $\{\bar{z}^k(\cdot\}$ such that

$$\big(\dot{\bar{x}}^k(\cdot), \dot{\bar{u}}^k(\cdot), \dot{\bar{b}}^k(\cdot)\big) \to \big(v(\cdot), w(\cdot), \nu(\cdot)\big) \;\text{ weakly in }\; L^2.$$

Define the absolutely continuous function $\widetilde{z}(\cdot) := (\widetilde{x}(\cdot), \widetilde{u}(\cdot), \widetilde{b}(\cdot))\colon [0,T] \to \mathbb{R}^{n+nm+m}$ by

$$\widetilde{z}(t) := \big(x_0, \bar{u}(0), \bar{b}(0)\big) + \int_0^t \big(v(s), w(s), \nu(s)\big)\, ds, \quad t \in [0,T],$$

for which $\dot{\widetilde{z}}(t) = (v(t), w(t), \nu(t))$ a.e. on $[0,T]$. The Mazur weak closure theorem allows us to find a sequence of convex combinations of $(\dot{\bar{x}}^k(\cdot), \dot{\bar{u}}^k(\cdot), \dot{\bar{b}}^k(\cdot))$, which converges to $(v(\cdot), w(\cdot), \nu(\cdot))$ strongly in L^2 and thus a.e. on $[0,T]$ along a subsequence. It follows from the constraint structure (4.2)–(4.6) of (P_k) and the convex-valuedness of F that $\widetilde{z}(\cdot)$ satisfies all the constraints in (1.3) and belongs to the prescribed ε-neighborhood (in $W^{1,2}$) of the i.l.m. $\bar{z}(\cdot)$ from Definition 4.1. Moreover, the construction of ℓ_F and its convexity in the velocity variables implies the inequality

$$\begin{aligned}&\int_0^T \widehat{\ell}_F\big(t, \widetilde{x}(t), \widetilde{u}(t), \widetilde{b}(t), \dot{\widetilde{x}}(t), \dot{\widetilde{u}}(t), \dot{\widetilde{b}}(t)\big)\, dt\\ &\le \liminf_{k\to\infty} \Big\{ h_k \sum_{j=0}^{k-1} \ell\Big(t_j^k, \bar{x}_j^k, \bar{u}_j^k, \bar{b}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{h_k}, \frac{\bar{u}_{j+1}^k - \bar{u}_j^k}{h_k}, \frac{\bar{b}_{j+1}^k - \bar{b}_j^k}{h_k}\Big)\Big\}.\end{aligned}$$

The passage to the limit in the cost functional of (P_k) and the definition of $\gamma > 0$ yield

$$\widehat{J}[\widetilde{z}] + \gamma = \varphi\big(\widetilde{x}(T)\big) + \int_0^T \widehat{\ell}_F\big(t, \widetilde{x}(t), \widetilde{u}(t), \widetilde{b}(t), \dot{\widetilde{x}}(t), \dot{\widetilde{u}}(t), \dot{\widetilde{b}}(t)\big)\, dt + \gamma \le \liminf_{k\to\infty} J_k[\bar{z}^k]. \tag{4.14}$$

Now we apply Theorem 3.1 to the local minimizer $\bar{z}(\cdot)$ under consideration and find the sequence $\{z^k(\cdot)\}$ of feasible solutions to (P_k), piecewise linearly extended to the whole interval $[0,T]$, which strongly approximates $\bar{z}(\cdot)$ in the $W^{1,2}$ topology. Since $\bar{z}_k(\cdot)$ is an optimal solution to (P_k), we have

$$J_k[\bar{z}^k] \le J_k[z^k] \;\text{ for each }\; k \in \mathbb{N}. \tag{4.15}$$

It follows from the structure of the cost functions in (P_k), the strong $W^{1,2}$-convergence in Theorem 3.1, and the continuity assumptions on φ and ℓ imposed above that $J_k[z^k] \to J[\bar{z}]$ as $k \to \infty$. This implies by taking (4.15) into account that

$$\limsup_{k\to\infty} J_k[\bar{z}^k] \le J[\bar{z}]. \tag{4.16}$$

Combining the relationships in (4.14) and (4.16) with the definition of r.i.l.m., we get

$$\widehat{J}[\widetilde{z}] + \gamma \le J[\bar{z}] = \widehat{J}[\bar{z}], \;\text{ i.e., }\; \widehat{J}[\widetilde{z}] < \widehat{J}[\bar{z}],$$

which clearly contradicts the fact that $\bar{z}(\cdot)$ is a r.i.l.m. for problem (P). This justifies the validity of (4.13) and thus completes the proof of the theorem. $\triangle$

5 Concluding Remarks

The paper justifies well-posedness of a new and pretty general optimal control problem for the basic version of the (heavily non-Lipschitzian) sweeping process with a controlled moving set and establishes the $W^{1,2}$-strong convergence of optimal solutions for discrete approximation problems to the given intermediate-type local minimizer of the original continuous-time control problem for the constrained sweeping process. Thus allows us to treat optimal solutions to the finite-dimensional problems of discrete optimization as suboptimal solutions to the original infinite-dimensional one. On the other hand, the method of discrete approximations can be viewed as a vehicle to obtain necessary optimality conditions in the generalized Bolza problem for the controlled sweeping process by deriving first such conditions for discrete approximation problems and then by passing to the limit therein when the discretization step tends to zero.

This is the main goal of our ongoing research project [7]. The discrete optimization problems of type (P_k) constructed in Section 4 in the framework of this approach are nonstandard in finite-dimensional mathematical programming, especially due to the complicated structure of the underlying geometric constraints arising in the approximation of the sweeping inclusion. However, we can treat them by using advanced tools of variational analysis and generalized differentiation; in particular, their recent second-order developments. The passage to the limit from discrete approximations occurs to be the most challenging issue due to the underlying non-Lipschitzian nature of the sweeping process and the intrinsic presence of state constraints of the inequality and equality types.

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